A Large Class of Solutions for the Instationary Navier-Stokes System

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We investigate very weak solutions to the instationary Navier-Stokes system being contained in $L^r(0,T;L^q(\Omega))$ where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and $\frac{2}{r}+\frac{n}{q} \leq 1$. The chosen space of data is small enough to guarantee uniqueness of solutions and existence in case of small data or short time intervals. On the other hand, the data space is large enough that every vector field in $L^r(0,T;L^q_\sigma(\Omega))$ is a very weak solution for appropriate data. The solutions and the data depend continuously on each other.

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1 Introduction and Main Results

We consider the Navier-Stokes equations with inhomogeneous data

$$\begin{aligned} \partial_t u - \Delta u + u\nabla u + \nabla p &= F && \text{in } (0,T)\times\Omega \\ \operatorname{div} u &= 0 && \text{in } (0,T)\times\Omega \\ u &= 0 && \text{on } (0,T)\times\partial\Omega \\ u(0) &= u_0 && \text{in } \Omega \end{aligned} \tag{1}$$

on a bounded C^2-domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and a time interval $[0,T)$ with $T \in (0,\infty]$. For simplicity we assume without loss of generality that the coefficient of viscosity is equal to 1.

According to the fundamental paper [15] by Serrin a large class, where uniqueness and regularity of solutions to (1) can be guaranteed, is the so-called Serrin's class

$$L^r(0,T;L^q(\Omega)) \quad \text{with} \quad \frac{2}{r}+\frac{n}{q} \leq 1. \tag{2}$$

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Thus, we aim at constructing a class of solutions to (1) that is contained in Serrin's class (2) but that fulfills no further restrictions. In particular, solutions in this class need not fulfill any differentiability assumptions. Consequently the notion of weak solutions is no longer suitable in this context. Hence an appropriate formulation of the problem is needed, the so-called *very weak solutions* to the Navier-Stokes equations. To come to this formulation one multiplies (1) with a sufficiently smooth test function ϕ with $\phi(t)|_{\partial\Omega} = 0$ and $\operatorname{div}\phi(t) = 0$ for every t and with $\operatorname{supp}\phi \subset [0,T) \times \overline{\Omega}$. Then one applies formal integration by parts and obtains

$$-\langle u, \phi_t\rangle_{\Omega,T} - \langle u, \Delta\phi\rangle_{\Omega,T} = \langle F, \phi\rangle_{\Omega,T} + \langle u_0, \phi(0)\rangle_{\Omega} + \langle uu, \nabla\phi\rangle_{\Omega,T} \tag{3}$$

using the identity $u \cdot \nabla u = \operatorname{div}(uu) - (\operatorname{div} u)u$. Applying the same procedure to the second equation in (1) with a test function ψ, which does not necessarily vanish on the boundary, we get

$$-\langle u(t), \nabla\psi\rangle_{\Omega} = 0 \tag{4}$$

for almost every t. Now, u is called a very weak solution to the Navier-Stokes equations if (3) and (4) are fulfilled for all test functions ϕ and ψ. Note that the information about the boundary values is preserved because $\nabla\phi$ and ψ do not necessarily vanish on the boundary. This or similar formulations have been introduced by Amann in [3], by Amrouche and Girault in [4] and by Galdi, Simader and Sohr in [11]. In these articles as well as by Farwig, Galdi and Sohr in [7], [8], [9] and by Giga in [12] solvability with low-regularity data has been shown.

However, the notion of very weak solutions that is used in this paper is even more general then the ones used in the aforementioned papers. More precisely, the space that is used for the data is much larger, so that, for example, we do not even distinguish between initial data and external forces. In fact, we even allow data which cannot be decomposed in any reasonable way into two parts corresponding to an external force and an initial datum, respectively. As a consequence of the generality of the data class the resulting space of solutions is so large that every $u \in L^r(0,T; L^q_\sigma(\Omega))$, cf. (15), can be understood as a very weak solution with respect to appropriate data, cf. Theorem 5.2. This makes it the largest possible class of data for our notion of very weak solutions to the Navier-Stokes equations. However, the space of data is small enough to guarantee uniqueness of solutions. It is obvious that in such a general context the boundary conditions $u|_{\partial\Omega}$ are not well-defined in the usual sense, see [14] or [10] for further discussions about the boundary values of very weak solutions.

It turns out that the space of data that belongs to solutions in Serrin's class (2), cf. (14), consists of functionals that in general cannot be understood as distributions on the space-time cylinder $\Omega \times (0,T)$. However, this space is the natural one in this context since it is shown in Theorem 5.2 that there exist neighborhoods of zero in this space of data and in Serrin's class such that the (nonlinear) solution operator of the Navier-Stokes equations is continuous and one-to-one.

To be more precise we now give the definition of a very weak solution and state the main results:

Definition 1.1. Let Ω be a bounded C^2-domain, $0 < T \leq \infty$ and $1 < r, q < \infty$. Furthermore let $f \in \left(\mathbb{W}_0^{1,r',q'}([0,T);\Omega)\right)'$. Then a function $u \in L^r(0,T;L^q(\Omega))$ is called *very weak solution to the Navier Stokes system with data* f, if

$$-\langle u, \phi_t\rangle_{T,\Omega} - \langle u, \Delta\phi\rangle_{T,\Omega} = \langle f, \phi\rangle + \langle uu, \nabla\phi\rangle_{T,\Omega} \quad \text{and} \tag{5}$$

$$\langle u(t), \nabla\psi\rangle_\Omega = 0 \tag{6}$$

for all $\phi \in \mathbb{W}_{0,\sigma}^{1,r',q'}([0,T),\Omega)$, all $\psi \in W^{1,q'}(\Omega)$ and almost all t with $0 \leq t \leq T$.

Here the spaces $\mathbb{W}_0^{1,r',q'}([0,T);\Omega)$ and $\mathbb{W}_{0,\sigma}^{1,r',q'}([0,T),\Omega)$ are those which are defined in (11), (12) and (13) below. Now our main theorem on existence reads as follows:

Theorem 1.2. *Let Ω be a bounded C^2-domain in $\mathbb{R}^n$ and $0 < T \leq \infty$. Furthermore assume that r and q satisfy the Serrin conditions $\frac{2}{r} + \frac{n}{q} \leq 1$, $2 < r < \infty$, $n < q < \infty$.*

Then there is a number $\delta > 0$ not depending on T such that for every functional

$$f \in \left(\mathbb{W}_0^{1,r',q'}([0,T),\Omega)\right)'$$

with

$$\|f\|_{\left(\mathbb{W}_0^{1,r',q'}([0,T),\Omega)\right)'} \leq \delta \tag{7}$$

there exists a very weak solution $u \in L^r(0,T;L^q(\Omega))$ to the instationary Navier-Stokes system with data f. The estimate

$$\|u\|_{L^r(0,T;L^q(\Omega))} \leq C \cdot \|f\|_{\left(\mathbb{W}_0^{1,r',q'}([0,T),\Omega)\right)'} \tag{8}$$

holds with a constant $C > 0$ depending only on n, Ω, r and q but not on T.

By Proposition 5.1 the smallness condition (7) can be guaranteed by choosing a sufficiently short time interval. Moreover, we have a result concerning uniqueness of very weak solutions, where no smallness condition on the data is needed:

Theorem 1.3. *Let Ω be a bounded C^2-domain in $\mathbb{R}^n$ and $0 < T \leq \infty$. Furthermore assume that r and q satisfy the Serrin conditions $2 < r < \infty$, $n < q < \infty$, $\frac{2}{r} + \frac{n}{q} \leq 1$ and let $f \in \left(\mathbb{W}_0^{1,r',q'}([0,T),\Omega)\right)'$.*

Then there is at most one very weak solution to the instationary Navier-Stokes system with data f.

In Theorem 5.2 a reformulation of the main results is given, emphasizing that the class of data considered here is actually the largest one.

2 Preliminaries and General Notation

For a Banach space X, $1 \le q \le \infty$ and a domain Ω we will denote by $L^q(\Omega; X)$ or $L^q(X)$ the usual Bochner space of all equivalence classes of strongly measurable functions $f : \Omega \to X$ such that

$$\|f\|_q := \|f\|_{L^q(\Omega,X)} := \left(\int_\Omega \|f(x)\|_X^q dx \right)^{\frac{1}{q}} < \infty$$

if $q < \infty$ and

$$\|f\|_\infty := \|f\|_{L^\infty(\Omega,X)} := \operatorname*{ess\,sup}_{x\in\Omega} \|f(x)\|_X < \infty$$

if $q = \infty$. In case $X = \mathbb{R}^3$ we only write $L^q(\Omega)$ or L^q and in case $\Omega = (0,T)$ is a subinterval of $\mathbb{R}$ we abbreviate $L^q((0,T); X)$ to $L^q(0,T;X)$. As an important special case we will deal with spaces of the form $L^r(0,T;L^q(\Omega))$, where $0 < T \le \infty$, Ω is a subdomain of $\mathbb{R}^n$ and $1 \le r, q \le \infty$. Note that in this case we write

$$\|f\|_{r,q} := \|f\|_{L^r(0,T;L^q(\Omega))}.$$

If additionally $k \in \mathbb{N}$ we denote by $W^{k,q}(\Omega, X)$ or $W^{k,q}(X)$ the space of equivalence classes of strongly measurable functions $f : \Omega \to X$ such that all distributional partial derivatives $D^\alpha f$ with order $|\alpha| \le k$ are contained in $L^q(\Omega, X)$. It is normed by

$$\|f\|_{W^{k,q}(\Omega,X)} := \sum_{|\alpha|\le k} \|D^\alpha f\|_{L^q(\Omega,X)} < \infty$$

which makes it a Banach space. We refer to theses spaces as *Sobolev spaces*. Again we write $W^{k,q}(\Omega)$ for $W^{k,q}(\Omega, \mathbb{R}^3)$ and $W^{k,q}(0,T;X)$ for $W^{k,q}((0,T), X)$.

It is well-known that for reflexive spaces X and $1 \le q < \infty$ we have the duality relation

$$(L^q(\Omega; X))' = L^{q'}(\Omega; X')$$

where q' is the conjugate exponent defined by the relation $\frac{1}{q} + \frac{1}{q'} = 1$, where $\frac{1}{\infty} := 0$. In particular,

$$(L^r(0,T;L^q(\Omega)))' = L^{r'}(0,T;L^{q'}(\Omega)) \tag{9}$$

for $0 < T \le \infty$, $1 \le r < \infty$ and $1 < q < \infty$. If $f \cdot g \in L^1(0,T;L^1(\Omega))$ we use the notation

$$\langle f, g \rangle_{T,\Omega} := \int_0^T \int_\Omega f(t)(x) \cdot g(t)(x) dx dt.$$

We also have a duality result for Sobolev spaces with values in reflexive spaces.

Lemma 2.1. *Let $1 \le q < \infty$, let $k \in \mathbb{N}_0$ and $0 < T \le \infty$. Furthermore let X be a reflexive space. Then for every functional $f \in \big(W^{k,q}(0,T;X)\big)'$ there are functions*

$g_j \in L^{q'}(0,T;X')$, $j \in \{0,\dots,k\}$, *such that for all* $\phi \in W^{k,q}(0,T;X)$

$$\langle f, \phi \rangle = \sum_{j=0}^{k} \int_0^T \langle g_j(t), \phi^{(j)}(t) \rangle_{X',X} dt,$$
$$\|f\|_{(W^{k,q}(0,T;X))'} \le \sum_{j=0}^{k} \|g_j\|_{L^{q'}(0,T;X')}.$$

Proof. The space $W^{k,q}(0,T;X)$ is isometrically isomorphic to a closed subspace of the product $(L^q(0,T;X))^{k+1}$. Now a Hahn-Banach argument and the fact that the dual of $(L^q(0,T;X))^{k+1}$ is $(L^{q'}(0,T;X'))^{k+1}$ yields the result. □

3 The Concept of Very Weak Solutions

3.1 Definitions

Since we intend to construct solutions in the large class $L^r(0,T;L^q(\Omega))$ where neither the derivatives nor the evaluations in (1) are well-defined, we need to define some appropriate test function and data spaces to be able to formulate what we mean by a solution.

Let $1 < r, q < \infty$ and $0 < T \le \infty$. Furthermore let $\Omega \subseteq \mathbb{R}^n$ be a bounded C^1-domain and E_0 and E_1 be Banach spaces. We set

$$\mathbb{W}^{1,r'}(0,T;E_0,E_1) := W^{1,r'}(0,T;E_0) \cap L^{r'}(0,T;E_1).$$

The space $\mathbb{W}^{1,r'}(0,T;E_0,E_1)$ is normed by

$$\|\phi\|_{\mathbb{W}^{1,r'}(0,T;E_0,E_1)} = \|\phi\|_{W^{1,r'}(0,T;E_0)} + \|\phi\|_{L^{r'}(0,T;E_1)} \tag{10}$$

which makes it a Banach space. As a special case we define

$$\mathbb{W}^{1,r',q'}(0,T;\Omega) := \mathbb{W}^{1,r'}(0,T;L^{q'}(\Omega), W^{2,q'}(\Omega)).$$

We also set for $0 < T < \infty$

$$\mathbb{W}_0^{1,r',q'}([0,T);\Omega) := \{\phi \in \mathbb{W}^{1,r',q'}(0,T;\Omega) \colon \phi(t)|_{\partial\Omega} = 0 \text{ for almost all } t \in [0,T] \text{ and } \phi(T) = 0\} \tag{11}$$

and, for $T = \infty$,

$$\mathbb{W}_0^{1,r',q'}([0,\infty);\Omega) := \{\phi \in \mathbb{W}^{1,r',q'}(0,\infty;\Omega) \colon \phi(t)|_{\partial\Omega} = 0 \text{ for almost all } t \in [0,\infty) \text{ and } \operatorname{supp}\phi \text{ is compact in } [0,\infty)\}. \tag{12}$$

For the latter spaces we will also need the divergence free variant

$$\mathbb{W}_{0,\sigma}^{1,r',q'}([0,T),\Omega) := \{\phi \in \mathbb{W}_0^{1,r',q'}([0,T);\Omega) \colon \operatorname{div}\phi(t) = 0 \text{ for almost all } t \in [0,T]\}. \tag{13}$$

All these spaces defined in (11), (12) and (13) are equipped with the subspace norms, i.e. they all have the norm $\|\cdot\|_{\mathbb{W}^{1,r',q'}(0,T;\Omega)}$ as in (10). We will refer to $\mathbb{W}^{1,r',q'}_{0,\sigma}([0,T);\Omega)$ as the *space of test functions* and to the dual space

$$\left(\mathbb{W}^{1,r',q'}_{0}([0,T);\Omega)\right)' \tag{14}$$

as *space of data.*

Finally we will need the space $L^q_\sigma(\Omega)$, $1<q<\infty$ of solenoidal L^q-vector fields on Ω defined by

$$L^q_\sigma(\Omega) := \left\{u \in L^q(\Omega)\colon \langle u, \nabla\psi\rangle_\Omega = 0 \text{ for all } \psi \in W^{1,q'}(\Omega)\right\}, \tag{15}$$

which is a closed subspace of $L^q(\Omega)$. Its dual $(L^q_\sigma(\Omega))'$ equals $L^{q'}_\sigma(\Omega)$.

Again, if Ω or T is fixed and confusion seems to be unlikely we will omit the domains and write $\mathbb{W}^{1,r',q'}_{0}([0,T))$, $\mathbb{W}^{1,r',q'}_{0}(\Omega)$ or simply $\mathbb{W}^{1,r',q'}_{0}$ for $\mathbb{W}^{1,r',q'}_{0}([0,T);\Omega)$. Similarly, for the divergence free variants, $\mathbb{W}^{1,r',q'}_{0,\sigma}([0,T))$, $\mathbb{W}^{1,r',q'}_{0,\sigma}(\Omega)$ or $\mathbb{W}^{1,r',q'}_{0,\sigma}$ stands for $\mathbb{W}^{1,r',q'}_{0,\sigma}([0,T);\Omega)$ if Ω and T are clear from the context. Of course, L^q_σ abbreviates $L^q_\sigma(\Omega)$.

Some basic facts, which will we be frequently used in the sequel, are summarized in the following Lemma, which follows easily from the definitions.

Lemma 3.1. *Let $0<T\le\infty$, let $\Omega\subseteq\mathbb{R}^n$ be a bounded C^1-domain and let $1<r,q<\infty$. Then*

1. *$\mathbb{W}^{1,r',q'}_{0}$ is a closed subspace of $\mathbb{W}^{1,r',q'}$ if $T<\infty$.*
2. *if $0<T_1\le T_2\le\infty$ every $\phi\in\mathbb{W}^{1,r',q'}_{0}([0,T_1))$ can be extended to a function $\tilde\phi\in\mathbb{W}^{1,r',q'}_{0}([0,T_2))$ by just setting $\tilde\phi(t):=\phi(t)$ if $0\le t<T_1$, and $\tilde\phi(t):=0$ if $T_1\le t<T_2$. Moreover, $\|\tilde\phi\|_{\mathbb{W}^{1,r',q'}_{0}([0,T_2))} = \|\phi\|_{\mathbb{W}^{1,r',q'}_{0}([0,T_1))}$.*

Note that by Lemma 3.1 (2) a very weak solution to the Navier-Stokes system with data f on the time interval $[0,T)$ is also a very weak solution to the Navier-Stokes system with data $f|_{\mathbb{W}^{1,r',q'}_{0}([0,T'))}$ on the shorter time interval $[0,T')$, $0<T'\le T$. This is not only a reasonable property for a solution but will also be important for some estimates later on.

We focus now again on the definition of a very weak solution to the Navier-Stokes system, cf. 1.1. While the first and the last equation in (1) and the tangential part of the third equation of the formal system (1) have been replaced by the first equation (5) in Definition 1.1, the second equation (6) represents the second equation of (1) and the normal part of the third equation in (1). Indeed, if an external force F and an initial velocity u_0 are given in reasonable spaces, then any classical solution u of (1) satisfies (5) with f being the functional

$$f := [\phi \mapsto \langle F,\phi\rangle + \langle u_0,\phi(0)\rangle_\Omega]\,. \tag{16}$$

Both the external force and the initial condition are hence included in the functional f. Moreover, the classical solution u also satisfies (6). This justifies the name "very weak solution" for functions satisfying (5) and (6).

We want to emphasize that the space of data is so large that we do not distinguish between external force and initial data, but we consider them as only one functional on the space of test functions in the sense just described. We even allow data, which cannot be decomposed as in (16) so that it is, in that case, senseless to talk about external force or initial data. The data space we use has the advantage that it guarantees uniqueness of very weak solutions as well as existence in case of small data, see Theorems 1.2 and 1.3; furthermore, every vector field in $L^r(0,T;L^q_\sigma)$ is a very weak solution to some appropriately chosen data in the data space, cf. Theorem 5.2. This has important consequences. Note for example that we cannot, in general, expect a very weak solution u to attain zero boundary values, apart from the fact that $u(t) \in L^q_\sigma$ for almost all $0 \le t \le T$. However, if the data are more regular, one obtains more regular solutions, which do respect boundary values, see [14] and [10] for more details.

3.2 The Linearized Case - Very Weak Solutions to the Stokes System

Our analysis of very weak solutions of the Navier-Stokes system is very essentially based on the following result on very weak solutions for the Stokes system, cf. [13, Theorem 4.3]. Because of the importance of this Theorem to our work, we will sketch its proof here as well, which uses duality techniques.

Theorem 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded C^2-domain, $0 < T \le \infty$ and $1 < r, q < \infty$. Furthermore let $f \in \left(\mathbb{W}_0^{1,r',q'}([0,T),\Omega)\right)'$. Then there exists a unique function $u \in L^r(0,T;L^q(\Omega))$ such that*

$$\begin{aligned} -\langle u, \phi_t\rangle_{T,\Omega} - \langle u, \Delta\phi\rangle_{T,\Omega} &= \langle f, \phi\rangle \quad \textit{and} \\ \langle u(t), \nabla\psi\rangle_\Omega &= 0 \end{aligned} \tag{17}$$

for all $\phi \in \mathbb{W}_{0,\sigma}^{1,r',q'}([0,T),\Omega)$, all $\psi \in W^{1,q'}(\Omega)$ and almost all t with $0 \le t \le T$. The function u satisfies the a priori estimate

$$\|u\|_{L^r(0,T;L^q(\Omega))} \le C\|f\|_{\left(\mathbb{W}_0^{1,r',q'}([0,T),\Omega)\right)'}$$

where $C > 0$ is a constant depending on n, r, q and Ω but not on T.

A function u satisfying (17) is called a *very weak solution to the Stokes system with data f.*

Proof. Assume first that $T < \infty$. For $v \in L^{r'}(0,T;L^{q'}_\sigma(\Omega))$ we find functions $\phi_v \in \mathbb{W}_{0,\sigma}^{1,r',q'}([0,T);\Omega)$ and $\psi_v \in L^{r'}(W^{1,q'}(\Omega)$ with $\int_\Omega \psi_v(t)dx = 0$ for almost all $t \in (0,T)$ being the unique solution pair to

$$-(\phi_v)_t - \Delta\phi_v + \nabla\psi_v = v$$

and satisfying the estimate

$$\|\phi_v\|_{\mathbb{W}^{1,r',q'}_{0,\sigma}} + \|\nabla\psi_v\|_{r',q'} \leq C\|v\|_{r',q'},$$

where C does not depend on T. This is possible by the following considerations.

We define $\tilde{v} := v(T-\cdot)$ and find by [10, Lemma 1.12] unique functions $\tilde{\phi}_v \in \mathbb{W}^{1,r,q}$ and $\tilde{\psi}_v \in L^{r'}(W^{1,q'})$ with $\int_\Omega \tilde{\psi}_v dx = 0$ satisfying

$$(\tilde{\phi}_v)_t - \Delta\tilde{\phi}_v + \nabla\tilde{\psi}_v = \tilde{v}, \quad \tilde{\phi}_v|_{\partial\Omega} = 0, \quad \tilde{\phi}_v(0) = 0, \quad \operatorname{div}\tilde{\phi}_v(t) = 0$$

for almost all $0 \leq t \leq T$ and the estimate

$$\|\tilde{\phi}_v\|_{\mathbb{W}^{1,r,q}} + \|\nabla\tilde{\psi}_v\|_{r',q'} \leq C\|\tilde{v}\|_{r,q},$$

where C does not depend on T. The functions $\tilde{\phi}_v$ and $\tilde{\psi}_v$ depend linearly on $\tilde{v}$. Now the functions $\phi_v := \tilde{\phi}_v(T-\cdot)$ and $\psi_v := \tilde{\psi}_v(T-\cdot)$ are the functions we were looking for.

To find a very weak solution to the Stokes system with data f define $u \in L^r(0,T;L^q_\sigma)$ via the duality $L^r(0,T;L^q_\sigma) = (L^{r'}(0;T;L^{q'}_\sigma))'$ by

$$\langle u, v\rangle := \langle f, \phi_v\rangle$$

for every $v \in L^{r'}(L^{q'})$. The right hand side is well defined by existence and uniqueness of ϕ_v as just discussed.

The estimate

$$\langle u, v\rangle = \langle f, \phi_v\rangle \leq \|f\|_{\left(\mathbb{W}^{1,r',q'}_0\right)'}\|\phi_v\|_{\mathbb{W}^{1,r',q'}_{0,\sigma}} \leq C\|f\|_{\left(\mathbb{W}^{1,r',q'}_0\right)'}\|v\|_{r',q'}$$

where the constant C is the constant from above and hence independent of T, shows that indeed $u \in L^r(0,T;L^q)$. It also implies the asserted a priori estimate.

To prove that u is a very weak solution to the Stokes system with data f let $\phi \in \mathbb{W}^{1,r',q'}_{0,\sigma}$ be an arbitrary test function and find a function $\psi \in L^{r'}(W^{1,q'})$, using the Helmholtz decomposition, cf. [16], such that

$$v := -\phi_t - \Delta\phi + \nabla\psi \in L^{r'}(0,T;L^{q'}_\sigma).$$

Then

$$-\langle u, \phi_t\rangle_{T,\Omega} - \langle u, \Delta\phi\rangle_{T,\Omega} = -\langle u, \phi_t\rangle_{T,\Omega} - \langle u, \Delta\phi\rangle_{T,\Omega} + \langle u, \nabla\psi\rangle_{T,\Omega} = \langle u, v\rangle_{T,\Omega} = \langle f, \phi\rangle$$

by the definition of u and hence u is indeed a very weak solution.

To show uniqueness assume that u is a very weak solution to the Stokes system with data $f = 0$. Let furthermore $v \in L^{r'}(L^{q'}_\sigma)$ be arbitrary. Then, as above, we find a solution pair $\phi_v \in \mathbb{W}^{1,r',q'}_{0,\sigma}$ and $\psi_v \in L^{r'}(W^{1,q'})$ satisfying $v = -(\phi_v)_t - \Delta\phi_v + \nabla\psi_v$. We thus get

$$\langle u, v\rangle_{T,\Omega} = -\langle u, (\phi_v)_t\rangle_{T,\Omega} - \langle u, \Delta\phi_v\rangle_{T,\Omega} = \langle f, \phi_v\rangle = 0$$

and this proves uniqueness of very weak solutions to the Stokes system.

For the proof of the case $T = \infty$ consider the restriction $f_N := f|_{\mathbb{W}_0^{1,r',q'}([0,N))}$, $N \in \mathbb{N}$, of f to functions being zero on $[N, \infty)$ in the sense of Lemma 3.1. Then, by the above, there is a very weak solution $u_N \in L^r(0,N;L^q)$ to the Stokes system with data f_N on the bounded interval $[0,N)$ which can be estimated by

$$\|u_N\|_{L^r(0,N;L^q)} \le C\|f_N\|_{\mathbb{W}_0^{1,r',q'}([0,N))} \le C\|f\|_{W_0^{1,r',q'}([0,\infty))}. \tag{18}$$

Note that C does not depend on time and hence the last quantity is independent of N. Note that furthermore $u_{N_2}|_{[0,N_1)} = u_{N_1}$, for $0 < N_1 \le N_2 < \infty$ using again Lemma 3.1. This allows us to construct a function u on the whole interval $[0,\infty)$ with the property $u|_{[0,N)} = u_N$. The uniform estimate (18) shows that $u \in L^r(0,\infty;L^q_\sigma)$. Now clearly u is a very weak solution to the Stokes system with data f, the a priori estimate holds true and u is unique.

□

4 An Embedding Theorem

4.1 Tools

We need the theory of Bessel potential spaces, see e.g. Bergh and Löfström, [5, Chapter 6] for Bessel potential spaces of complex valued functions. Here we will need results for $\mathbb{R}^n$-valued Bessel potential spaces, but standard complexification arguments and componentwise application of the results will allow us to pass over to the desired context.

Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n, \mathbb{R}^n)$ denote the Schwartz space of rapidly decreasing functions and let $\mathcal{S}'$ denote its dual, the space of $\mathbb{R}^n$-valued tempered distributions on $\mathbb{R}^n$. Furthermore let $\mathcal{F}$ be the Fourier transform on $\mathcal{S}$ or on $\mathcal{S}'$.

For $\beta \in \mathbb{R}$ and $1 \le q \le \infty$ the spaces $H^{\beta,q}(\mathbb{R}^n)$, consisting of all $f \in \mathcal{S}'$ such that

$$\|f\|_{H^{\beta,q}(\mathbb{R}^n)} := \left\| \mathcal{F}^{-1}\left[(1+|\cdot|^2)^{\frac{\beta}{2}}\mathcal{F}f(\cdot)\right]\right\|_q < \infty,$$

are called *Bessel potential spaces*. The Bessel potential spaces $H^{\beta,q}(\mathbb{R}^n)$ are Banach spaces and for $k \in \mathbb{N}_0$ it holds that

$$H^{k,q}(\mathbb{R}^n) = W^{k,q}(\mathbb{R}^n) \tag{19}$$

for $1 < q < \infty$ with equivalent norms. Moreover, for $0 < \theta < 1$, $\beta_1, \beta_2 \in \mathbb{R}$ and $1 < q_1, q_2 < \infty$ one has the interpolation property

$$\left[H^{\beta_1,q_1}(\mathbb{R}^n), H^{\beta_2,q_2}(\mathbb{R}^n)\right]_\theta = H^{\beta,q}(\mathbb{R}^n) \tag{20}$$

with $\beta = (1-\theta)\beta_1 + \theta\beta_2$ and $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$, where the norms are equivalent. Here $[\cdot,\cdot]_\theta$ denotes the complex interpolation functor, cf. [5, Chapter 4].

Theorem 4.1. *Let $\beta_1, \beta_2 \in \mathbb{R}$ and $1 < q_1 \le q_2 < \infty$ satisfy $\beta_1 - \frac{n}{q_1} = \beta_2 - \frac{n}{q_2}$. Then*

$$H^{\beta_1,q_1}(\mathbb{R}^n) \hookrightarrow H^{\beta_2,q_2}(\mathbb{R}^n).$$

Before we finally come to the proof of the embedding we are heading for, we need another tool, which is a special case of [3, Theorem 1.3].

Theorem 4.2. *Let $1 < q < \infty$, $2 < r < \infty$ and $0 < T \le \infty$. Then the following embedding holds:*

$$\mathbb{W}^{1,r',q'}(0,T;\mathbb{R}^n) \hookrightarrow L^{\left(\frac{r}{2}\right)'}(0,T;H^{\frac{2}{r'},q'}(\mathbb{R}^n)).$$

Proof. In [3, Theorem 1.3] take $I = [0,T)$, $A := 1 - \Delta$, $E_0 := L^{q'}(\mathbb{R}^n)$, $E_1 := W^{2,q'}(\mathbb{R}^n)$, $r := r'$, $p := \left(\frac{r}{2}\right)'$ and $\alpha := \frac{1}{r'}$. We note that $L^{q'}(\mathbb{R}^n)$ is a UMD space, see H. Amann, [2, Theorem III.4.5.2], and the operator $1 - \Delta$ in $L^{q'}(\mathbb{R}^n)$ has bounded imaginary powers, cf. R. Denk, M. Hieber and J. Prüss, [6, Theorem 5.5]. Furthermore, it is well-known that $1 - \Delta$ generates an analytic semigroup. This shows that

$$\mathbb{W}^{1,r',q'}(0,T;\mathbb{R}^n) \hookrightarrow L^{\left(\frac{r}{2}\right)'}(0,T;[L^{q'},W^{2,q'}]_{\frac{1}{r'}}).$$

Now (19) and (20) show that

$$[L^{q'},W^{2,q'}]_{\frac{1}{r'}} = H^{\frac{2}{r'},q'},$$

which finishes the proof. □

4.2 The embedding theorem

Finally we have collected all tools to be able to prove the embedding which will give us bounds for the nonlinear term of the Navier-Stokes system.

Theorem 4.3. *If $\Omega \subset \mathbb{R}^n$ is a bounded C^2-domain and $0 < T \le \infty$, then*

$$\mathbb{W}_0^{1,r',q'}([0,T),\Omega) \hookrightarrow L^{\left(\frac{r}{2}\right)'}(0,T;W^{1,\left(\frac{q}{2}\right)'}(\Omega)), \tag{21}$$

$$2 < r < \infty, \quad n < q < \infty, \quad \frac{2}{r} + \frac{n}{q} \le 1. \tag{22}$$

Moreover, one can choose a constant C, depending only on n, Ω, r and q but not on T, which bounds the embedding (21) for any T.

Proof. Let E be an extension operator for Ω satisfying $E \in \mathcal{L}(W^{k,q'}(\Omega), W^{k,q'}(\mathbb{R}^n))$ for all $k \in \{0,1,2\}$ and $(E\psi)|_\Omega = \psi$ for all $\psi \in L^{q'}(\Omega)$. Such an operator exists by e.g. Adams and Fournier, [1, Theorem 5.22].

Let $\phi \in \mathbb{W}_0^{1,r',q'}([0,T),\Omega)$ and let $\tilde{\phi} := E\phi \in \mathbb{W}^{1,r',q'}([0,T),\mathbb{R}^n)$ be the spatial extension of ϕ. Moreover, let s be defined by

$$\frac{2}{r'} - \frac{n}{q'} = 1 - \frac{n}{s} \qquad \text{or equivalently} \qquad \frac{1}{(q/2)'} - \frac{1}{s} = \frac{1}{n}\left(1 - \left(\frac{2}{r} + \frac{n}{q}\right)\right).$$

It is exactly the condition $\frac{2}{r} + \frac{n}{q} \leq 1$ which then gives $\left(\frac{q}{2}\right)' \leq s$ and hence

$$\|\phi\|_{L^{(\frac{r}{2})'}(W^{1,(\frac{q}{2})'}(\Omega))} \leq C_1 \|\phi\|_{L^{(\frac{r}{2})'}(W^{1,s}(\Omega))}, \tag{23}$$

since Ω is bounded. Furthermore we have

$$\|\phi\|_{L^{(\frac{r}{2})'}(W^{1,s}(\Omega))} \leq \|\tilde{\phi}\|_{L^{(\frac{r}{2})'}(W^{1,s}(\mathbb{R}^n))}.$$

By Theorem 4.1 and (19) we get $H^{\frac{2}{r'},q'}(\mathbb{R}^n) \hookrightarrow H^{1,s}(\mathbb{R}^n) = W^{1,s}(\mathbb{R}^n)$ by definition of s and hence

$$\|\tilde{\phi}\|_{L^{(\frac{r}{2})'}(W^{1,s}(\mathbb{R}^n))} \leq C_2 \|\tilde{\phi}\|_{L^{(\frac{r}{2})'}(H^{\frac{2}{r'},q'}(\mathbb{R}^n))}.$$

Now Theorem 4.2 yields

$$\|\tilde{\phi}\|_{L^{(\frac{r}{2})'}(H^{\frac{2}{r'},q'}(\mathbb{R}^n))} \leq C_3 \|\tilde{\phi}\|_{\mathbb{W}^{1,r',q'}(0,T;\mathbb{R}^n)} \tag{24}$$

for $2 < r < \infty$ and $1 < q < \infty$. Finally the inequality

$$\|\tilde{\phi}\|_{\mathbb{W}^{1,r',q'}(0,T;\mathbb{R}^n)} \leq C_4 \|\phi\|_{\mathbb{W}^{1,r',q'}(0,T;\Omega)} = C_4 \|\phi\|_{\mathbb{W}_0^{1,r',q'}([0,T);\Omega)}$$

implies the embedding (21) for all T, since neither of the positive constants C_1, C_2, C_3 and C_4 depends on ϕ.

We still need to show that the constant C, which bounds the above embedding, can be chosen independently of T. To this end let C_∞ denote the minimal constant for which the estimate

$$\|\phi\|_{L^{(\frac{r}{2})'}(0,\infty;W^{1,(\frac{q}{2})'}(\Omega))} \leq C_\infty \|\phi\|_{\mathbb{W}_0^{1,r',q'}([0,\infty),\Omega)}$$

holds for all $\phi \in \mathbb{W}_0^{1,r',q'}([0,\infty))$ and let $0 < T \leq \infty$ be arbitrary. Then we have for $\phi \in \mathbb{W}_0^{1,r',q'}([0,T))$

$$\begin{aligned}\|\phi\|_{L^{(\frac{r}{2})'}(0,T;W^{1,(\frac{q}{2})'}(\Omega))} &= \|\phi\|_{L^{(\frac{r}{2})'}(0,\infty;W^{1,(\frac{q}{2})'}(\Omega))} \\ &\leq C_\infty \|\phi\|_{\mathbb{W}_0^{1,r',q'}([0,\infty),\Omega)} = C_\infty \|\phi\|_{\mathbb{W}_0^{1,r',q'}([0,T),\Omega)}\end{aligned}$$

using that ϕ can be extended by zero to a function in $\mathbb{W}_0^{1,r',q'}([0,\infty))$ by Lemma 3.1. Hence with C_∞, depending on n, Ω, r and q, we have indeed found a constant which bounds the embedding (21) for all T. This finishes the proof. □

This embedding result allows us to consider the mapping $\phi \mapsto \langle uu, \nabla\phi\rangle$ as a bounded linear functional on $\mathbb{W}_0^{1,r',q'}([0,T),\Omega)$ if the Serrin conditions are satisfied. The next corollary makes this more precise.

Corollary 4.4. *Let r and q satisfy the Serrin conditions $\frac{2}{r} + \frac{n}{q} \leq 1$, $2 < r < \infty$, $n < q < \infty$, let $\Omega \subseteq \mathbb{R}^n$ be a bounded C^2-domain and $0 < T \leq \infty$. If u and v are* *elements of* $L^r(0,T;L^q(\Omega))$ *then* $W(u,v) := [\phi \mapsto \langle uv, \nabla\phi\rangle]$ *is a bounded linear functional on* $\mathbb{W}_0^{1,r',q'}([0,T);\Omega)$ *with norm*

$$\|W(u,v)\|_{\left(\mathbb{W}_0^{1,r',q'}([0,T),\Omega)\right)'} \le C\|uv\|_{L^{\frac{r}{2}}(0,T;L^{\frac{q}{2}}(\Omega))} \le C\|u\|_{L^r(0,T;L^q(\Omega))}\|v\|_{L^r(0,T;L^q(\Omega))}.$$

Moreover, we have the estimate

$$\|W(u,u) - W(v,v)\|_{\left(\mathbb{W}_0^{1,r',q'}\right)'} \le C\left(\|u\|_{r,q} + \|v\|_{r,q}\right)\|u-v\|_{r,q}.$$

In both inequalities $C > 0$ *is the constant from Theorem 4.3 and can hence be chosen independently of* T*.*

Proof. To prove well-definedness and continuity of $W(u,v)$ let $\phi \in \mathbb{W}_0^{1,r',q'}$. Then, by Theorem 4.3, $\phi \in L^{\left(\frac{r}{2}\right)'}(W^{1,\left(\frac{q}{2}\right)'})$ and hence $\nabla\phi \in L^{\left(\frac{r}{2}\right)'}(L^{\left(\frac{q}{2}\right)'})$. On the other hand Hölder's inequality implies that $uv \in L^{\frac{r}{2}}(L^{\frac{q}{2}})$ with $\|uv\|_{\frac{r}{2},\frac{q}{2}} \le \|u\|_{r,q}\|v\|_{r,q}$. We thus get

$$\langle uv, \nabla\phi\rangle \le \|uv\|_{\frac{r}{2},\frac{q}{2}}\|\phi\|_{L^{\left(\frac{r}{2}\right)'}(W^{1,\left(\frac{q}{2}\right)'})} \le \|uv\|_{\frac{r}{2},\frac{q}{2}}C\|\phi\|_{\mathbb{W}_0^{1,r',q'}} \le C\|u\|_{r,q}\|v\|_{r,q}\|\phi\|_{\mathbb{W}_0^{1,r',q'}}.$$

by Theorem 4.3. This proves the well-definedness of $W(u,v)$, its continuity and the estimate of its norm.

For the second assertion observe that $W(u,u) - W(v,v) = W(u-v,v) + W(u,u-v)$. Hence, by what we just proved,

$$\begin{aligned}\|W(u,u) - W(v,v)\|_{\left(\mathbb{W}_0^{1,r',q'}\right)'} &\le \|W(u-v,v)\|_{\left(\mathbb{W}_0^{1,r',q'}\right)'} + \|W(u,u-v)\|_{\left(\mathbb{W}_0^{1,r',q'}\right)'} \\ &\le C\left(\|u\|_{r,q} + \|v\|_{r,q}\right)\|u-v\|_{r,q}.\end{aligned}$$

This finishes the proof. □

5 Proofs and Reformulation of the Main Results

Proof of Theorem 1.2. Let $C_1 = C_1(n,\Omega,r,q) > 0$ denote the constant from the estimate on the solution of the instationary Stokes system from Theorem 3.2. Moreover, let $C_2 = C_2(n,\Omega,r,q) > 0$ be the constant from Theorem 4.3 bounding the embedding

$$\mathbb{W}_0^{1,r',q'}([0,T),\Omega) \hookrightarrow L^{\left(\frac{r}{2}\right)'}(0,T;W^{1,\left(\frac{q}{2}\right)'}(\Omega)).$$

Furthermore, assume that

$$\|f\|_{\left(\mathbb{W}_0^{1,r',q'}\right)'} \le \delta := \frac{3}{16C_1^2C_2}. \tag{25}$$

Indeed the number δ is positive and independent of T since both C_1 and C_2 are.

For $v \in L^r(L^q)$ let $Sv \in L^r(L^q)$ be the solution of

$$- \langle Sv, \phi_t \rangle_{T,\Omega} - \langle Sv, \Delta\phi \rangle_{T,\Omega} = \langle f, \phi \rangle + \langle vv, \nabla\phi \rangle_{T,\Omega} \tag{26}$$
$$\langle Sv(t), \nabla\psi \rangle_{\Omega} = 0 \tag{27}$$

for all $\phi \in \mathbb{W}^{1,r',q'}_{0,\sigma}$, all $\psi \in W^{1,q'}(\Omega)$ and almost all $t \in [0,T]$. Since the right hand side of (26) is a linear functional on $\mathbb{W}^{1,r',q'}_{0}$ by Corollary 4.4, the solution Sv is well defined by Theorem 3.2 and can be estimated by

$$\begin{aligned} \|Sv\|_{r,q} &\le C_1 \|f\|_{\left(\mathbb{W}^{1,r',q'}_0\right)'} + C_1 \|W(v,v)\|_{\left(\mathbb{W}^{1,r',q'}_0\right)'} \\ &\le C_1 \|f\|_{\left(\mathbb{W}^{1,r',q'}_0\right)'} + C_1 C_2 \|v\|^2_{r,q} \end{aligned} \tag{28}$$

using again the notation $W(u,v) := [\phi \mapsto \langle uv, \nabla\phi \rangle]$.

Since obviously the fixed points of the mapping $S\colon L^r(L^q) \to L^r(L^q)$ defined by $v \mapsto Sv$ correspond exactly to the very weak solutions of the Navier-Stokes system with data f, the strategy will be to use Banach's Fixed Point Theorem.

Assume that $\|v\|_{r,q} \le \rho := \frac{1}{4C_1C_2}$. Then

$$\|Sv\|_{r,q} \le C_1 \|f\|_{\left(\mathbb{W}^{1,r',q'}_0\right)'} + C_1 C_2 \|v\|^2_{r,q} \le C_1 \frac{3}{16 C_1^2 C_2} + C_1 C_2 \left(\frac{1}{4C_1C_2} \right)^2 = \rho$$

by (25) and (28). This proves that S is a self-map of the closed ball $\overline{B}_\rho(0)$. We still have to show, that S is a contraction on $\overline{B}_\rho(0)$. So let $u,v \in \overline{B}_\rho(0)$. Then clearly $Su - Sv$ solves the equations

$$- \langle Su - Sv, \phi_t \rangle_{T,\Omega} - \langle Su - Sv, \Delta\phi \rangle_{T,\Omega} = \langle W(u,u) - W(v,v), \phi \rangle_{T,\Omega}$$
$$\langle (Sv - Su)(t), \nabla\psi \rangle_{\Omega} = 0$$

for all $\phi \in \mathbb{W}^{1,r',q'}_{0,\sigma}$, all $\psi \in W^{1,q'}$ and almost all $0 \le t \le T$. By the a priori estimate of Theorem 3.2, Corollary 4.4 and the definition of ρ it follows that

$$\|Su - Sv\|_{r,q} \le C_1 \|W(u,u) - W(v,v)\|_{\left(\mathbb{W}^{1,r',q'}_0\right)'} \le C_1 C_2 2\rho \|u - v\|_{r,q} = \frac{1}{2} \|u - v\|_{r,q}.$$

Consequently, S is a contraction on $\overline{B}_\rho(0)$ with constant of contraction $\frac{1}{2}$.

Banach's Fixed Point Theorem now yields a fixed point u of S which is a very weak solution to the instationary Navier-Stokes system. This fixed point u lies in the ball $\overline{B}_\rho(0)$ and hence $C_1C_2\|u\|^2_{r,q} \le \frac{1}{4}\|u\|_{r,q}$. Using this, the a priori estimate (8) follows from

$$\|u\|_{r,q} = \|Su\|_{r,q} \le C_1 \|f\|_{\left(\mathbb{W}^{1,r',q'}_0\right)'} + C_1 C_2 \|u\|^2_{r,q} \le C_1 \|f\|_{\left(\mathbb{W}^{1,r',q'}_0\right)'} + \frac{1}{4} \|u\|_{r,q}.$$

The choice $C := \frac{4}{3} C_1$ implies that C is independent of T, since C_1 already was. □

Given a functional f in the data space, the smallness condition (7) can be achieved by restricting f to functions which are zero outside a small time interval $[0,T')$ where $0 < T' \le T$. This is specified in the next proposition.

Proposition 5.1. *Let $1 < r,q < \infty$ and $0 < T \le \infty$ as well as $\Omega \subset \mathbb{R}^n$ be a bounded C^1-domain. Given a functional $f \in \left(\mathbb{W}_0^{1,r',q'}([0,T);\Omega)\right)'$ and a number $\epsilon > 0$, there exists a number T' with $0 < T' \le T$ such that*

$$\|f\|_{\left(\mathbb{W}_0^{1,r',q'}([0,T'),\Omega)\right)'} \le \epsilon.$$

Proof. First consider the case $T < \infty$. Since in this case $\mathbb{W}_0^{1,r',q'}([0,T))$ is a closed subspace of

$$\mathbb{W}^{1,r',q'}([0,T),\Omega) = L^{r'}(0,T;W^{2,q'}(\Omega)) \cap W^{1,r'}(0,T;L^{q'}(\Omega))$$

by Lemma 3.1, the functional f is the restriction of a functional

$$F \in \left(L^{r'}(0,T;W^{2,q'}) \cap W^{1,r'}(0,T;L^{q'})\right)' = L^{r}(0,T;(W^{2,q'})') + (W^{1,r'}(0,T;L^{q'}))'$$

with

$$\|f\|_{\left(\mathbb{W}_0^{1,r',q'}([0,T)\right)'} = \|F\|_{\left(\mathbb{W}^{1,r',q'}([0,T))\right)'}$$

by the Hahn-Banach theorem. Consider a typical decomposition $F = F_1 + F_2$ where $F_1 \in L^r(0,T;(W^{2,q'})')$ and $F_2 \in \left(W^{1,r'}(0,T;L^{q'})\right)'$ and let $0 < \tilde{T} \le T$. Then we have

$$\|F\|_{\left(\mathbb{W}^{1,r',q'}([0,\tilde{T}))\right)'} \le \|F_1\|_{L^r(0;\tilde{T};(W^{2,q'})')} + \|F_2\|_{(W^{1,r'}(0,\tilde{T};L^{q'}))'}\,.$$

Now we find by Lemma 2.1 functions $g_0, g_1 \in L^r(0,T;L^q)$ with

$$\langle F_2, \phi\rangle = \langle g_0, \phi\rangle_{T,\Omega} + \langle g_1, \phi_t\rangle_{T,\Omega}$$

for all $\phi \in \mathbb{W}^{1,r',q'}([0,T))$ and

$$\|F_2\|_{(W^{1,r'}(0,\tilde{T};L^{q'}))'} \le \|g_0\|_{L^r(0,\tilde{T};L^q)} + \|g_1\|_{L^r(0,\tilde{T};L^q)}\,.$$

Consequently,

$$\begin{aligned}\|f\|_{\left(\mathbb{W}_0^{1,r',q'}([0,\tilde{T}))\right)'} &= \|F\|_{\left(\mathbb{W}^{1,r',q'}([0,\tilde{T}))\right)'}\\ &\le \|F_1\|_{L^r(0;\tilde{T};(W^{2,q'})')} + \|g_0\|_{L^r(0,\tilde{T};L^q)} + \|g_1\|_{L^r(0,\tilde{T};L^q)}\,.\end{aligned}$$

By Lebesgue's Theorem on Dominated Convergence we can find a T', $0 < T' \le T$, with $\|F_1\|_{L^r(0,T';(W^{2,q'})')} + \|g_0\|_{L^r(0,T';L^q)} + \|g_1\|_{L^r(0,T';L^q)} \le \epsilon$ which implies the assertion for $T < \infty$.

Now consider the case $T = \infty$. Restricting the functional f to $\mathbb{W}_0^{1,r',q'}([0,1))$, which is possible by Lemma 3.1, we are in the situation from above. □

This Proposition allows us to replace the smallness condition (7) by a condition on the length of the time interval. We now prove uniqueness of very weak solutions, which – in contrast to the problem of existence – does not need any assumption on the smallness of the data.

Proof of Theorem 1.3. As in the proof of Theorem 1.2 let C_1 be the constant from the a priori estimate of Theorem 3.2 and C_2 be the constant bounding the embedding from Theorem 4.3. They can both be chosen independently of T. Assume that u and $v \in L^r(0,T;L^q)$ are both very weak solutions to the Navier-Stokes system with data f and let $[0,T_{max})$ be the maximal half-open interval on which u and v coincide almost everywhere. If no such interval exists, which is a priori possible, let T_{max} be zero. Suppose $T_{max} < T$.

Let $\tilde{T}$ satisfy $T_{max} < \tilde{T} \le T$ and note that, by Lemma 3.1, we can consider every $\phi \in \mathbb{W}^{1,r',q'}_{0,\sigma}([0,\tilde{T}))$ also as an element of $\mathbb{W}^{1,r',q'}_{0,\sigma}([0,T))$ by just extending it by zero. Thus u and v also satisfy

$$\begin{gathered} -\langle u-v,\phi_t\rangle_{\tilde{T},\Omega} - \langle u-v,\Delta\phi\rangle_{\tilde{T},\Omega} = \langle (uu-vv,\nabla\phi\rangle_{\tilde{T},\Omega} \\ \langle (u-v)(t),\nabla\psi\rangle_\Omega = 0 \end{gathered}$$

for all $\phi \in \mathbb{W}^{1,r',q'}_{0,\sigma}([0,\tilde{T}))$, all $\psi \in W^{1,q'}$ and almost all $0 < t < \tilde{T}$. By Theorem 3.2 and Theorem 4.3 we then get with $\tilde{u} := u|_{[0,\tilde{T})}$ and $\tilde{v} := v|_{[0,\tilde{T})}$

$$\begin{aligned} \|u-v\|_{L^r(0,\tilde{T};L^q)} &\le C_1 \|W(\tilde{u},\tilde{u}) - W(\tilde{v},\tilde{v})\|_{\left(\mathbb{W}^{1,r',q'}_0([0,\tilde{T}))\right)'} \\ &\le C_1 \|W(\tilde{u},\tilde{u}-\tilde{v})\|_{\left(\mathbb{W}^{1,r',q'}_0([0,\tilde{T}))\right)'} + C_1 \|W(\tilde{u}-\tilde{v},\tilde{v})\|_{\left(\mathbb{W}^{1,r',q'}_0([0,\tilde{T}))\right)'} \\ &\le C_1C_2 \left(\|u(u-v)\|_{L^{\frac{r}{2}}(0,\tilde{T};L^{\frac{q}{2}})} + \|(u-v)v\|_{L^{\frac{r}{2}}(0,\tilde{T};L^{\frac{q}{2}})}\right) \\ &= C_1C_2 \left(\|u(u-v)\|_{L^{\frac{r}{2}}(T_{max},\tilde{T};L^{\frac{q}{2}})} + \|(u-v)v\|_{L^{\frac{r}{2}}(T_{max},\tilde{T};L^{\frac{q}{2}})}\right) \\ &\le C_1C_2 \left(\|u\|_{L^r(T_{max},\tilde{T};L^q)} + \|v\|_{L^r(T_{max},\tilde{T};L^q)}\right) \|u-v\|_{L^r(T_{max},\tilde{T};L^q)} \\ &= C_1C_2 \left(\|u\|_{L^r(T_{max},\tilde{T};L^q)} + \|v\|_{L^r(T_{max},\tilde{T};L^q)}\right) \|u-v\|_{L^r(0,\tilde{T};L^q)}. \end{aligned}$$

for all $T_{max} < \tilde{T} \le T$ since $u-v=0$ almost everywhere on $[0,T_{max})$. Since C_1 and C_2 were independent of T or $\tilde{T}$, by Lebesgue's Theorem on Dominated Convergence there exists a T', $T_{max} < T' \le T$ such that

$$C_1C_2 \left(\|u\|_{L^r(T_{max},T';L^q)} + \|v\|_{L^r(T_{max},T';L^q)}\right) < 1,$$

which leads to

$$\|u-v\|_{L^r(0,T';L^q(\Omega))} = 0.$$

and hence $u = v$ almost everywhere on $[0,T')$. This contradicts the maximality of the interval $[0,T_{max})$. Thus $T_{max} = T$ – in particular the case $T_{max} = 0$ is impossible – which proves that u and v coincide almost everywhere on the whole interval $[0,T)$. □

Finally we will show that every function $u \in L^r(0,T;L^q_\sigma(\Omega))$ is a very weak solution to some data. This is a consequence of our choice of a large data space. In the next theorem this is formulated more precisely and the main results as presented in Theorem 1.2 and Theorem 1.3 are reformulated. Furthermore it is proved that solutions and data depend continuously on each other at least for the small data ball from Theorem 1.2.

Theorem 5.2. *Let Ω be a bounded C^2-domain and $0 < T \leq \infty$ and let r and q satisfy the Serrin conditions $2 < r < \infty$, $n < q < \infty$ and $\frac{2}{r} + \frac{n}{q} \leq 1$. Let*

$$\mathcal{G}\colon L^r(0,T;L^q_\sigma(\Omega)) \to \left(\mathbb{W}^{1,r',q'}_{0,\sigma}([0,T),\Omega)\right)'$$

be defined by

$$\langle \mathcal{G}(u), \phi \rangle := -\langle u, \phi_t \rangle - \langle u, \Delta\phi \rangle - \langle uu, \nabla\phi \rangle$$

for all $\phi \in \mathbb{W}^{1,r',q'}_{0,\sigma}([0,T),\Omega)$.

Then $\mathcal{G}$ is injective and continuous. Moreover, there exist open neighborhoods of zero

$$U \subseteq L^r(0,T;L^q_\sigma(\Omega)) \quad \text{and} \quad V \subseteq \left(\mathbb{W}^{1,r',q'}_{0,\sigma}([0,T),\Omega)\right)'$$

such that $\tilde{\mathcal{G}} := \mathcal{G}|_U^V\colon U \to V$ is bijective and continuous and its inverse map $L := \tilde{\mathcal{G}}^{-1}\colon V \to U$ is continuous as well.

Proof. The mapping $\mathcal{G}$ is well defined by Corollary 4.4. Let C_1, C_2, ρ and δ be chosen as in the proof of Theorem 1.2.

The uniqueness theorem 1.3 yields the injectivity of the mapping $\mathcal{G}$. Indeed, we can extend a functional f on $\mathbb{W}^{1,r',q'}_{0,\sigma}$ to a functional F on $\mathbb{W}^{1,r',q'}_{0}$ preserving its norm by the Hahn-Banach theorem. Then, by the uniqueness of very weak solutions, there can at most be one function $u \in L^r(0,T;L^q_\sigma(\Omega))$ such that $\langle \mathcal{G}(u), \phi \rangle = \langle F, \phi \rangle = \langle f, \phi \rangle$ for all $\phi \in \mathbb{W}^{1,r',q'}_{0,\sigma}([0,T),\Omega)$ and this proves the injectivity of $\mathcal{G}$.

Furthermore, $\mathcal{G}$ is continuous: For $u, v \in L^r(0,T;L^q_\sigma(\Omega))$ and $\phi \in \mathbb{W}^{1,r',q'}_{0,\sigma}([0,T),\Omega)$ it holds that

$$\begin{aligned}\langle \mathcal{G}(u) - \mathcal{G}(v), \phi \rangle &= \langle v-u, \phi_t \rangle_{T,\Omega} + \langle v-u, \Delta\phi \rangle_{T,\Omega} + \langle vv - uu, \nabla\phi \rangle_{T,\Omega} \\ &\leq 2\|u-v\|_{r,q}\|\phi\|_{\mathbb{W}^{1,r',q'}_{0,\sigma}} + C_2\left(\|u\|_{r,q} + \|v\|_{r,q}\right)\|u-v\|_{r,q}\|\phi\|_{\mathbb{W}^{1,r',q'}_{0,\sigma}}\end{aligned}$$

and hence

$$\|\mathcal{G}(u) - \mathcal{G}(v)\|_{\left(\mathbb{W}^{1,r',q'}_{0,\sigma}\right)'} \leq \left[2 + C_2\left(\|u\|_{r,q} + \|v\|_{r,q}\right)\right]\|u-v\|_{r,q}.$$

This implies the continuity of $\mathcal{G}$.

Now we choose $V := B_\delta(0)$ to be the open ball in $\left(\mathbb{W}^{1,r',q'}_{0,\sigma}\right)'$ around 0 with radius δ. Furthermore we let $U := \mathcal{G}^{-1}(V) \subseteq L^r(0,T;L^q_\sigma(\Omega))$ be the inverse image of V under $\mathcal{G}$. Then, by continuity of $\mathcal{G}$, U is open and it contains 0 since $\mathcal{G}(0) = 0 \in V$. It is contained in $B_\rho(0)$ by Theorem 1.2.

Now $\tilde{\mathcal{G}} := \mathcal{G}|_U^V$ is continuous and injective. To see that it is surjective let $f \in V$. By the Hahn-Banach Theorem we can extend f to a functional F on $\mathbb{W}_0^{1,r',q'}$ with the same norm. Then, by Theorem 1.2, there is a function $u \in L^r(L^q_\sigma)$ with $\langle \mathcal{G}(u), \phi \rangle = \langle F, \phi \rangle = \langle f, \phi \rangle$ for all $\phi \in \mathbb{W}_{0,\sigma}^{1,r',q'}$. This implies that $\tilde{\mathcal{G}}$ is surjective. Consequently, $\tilde{\mathcal{G}}$ is bijective and continuous.

Finally we show that $L := \tilde{\mathcal{G}}^{-1}$ is continuous. This is proved as follows. Let $f, g \in V$ and $u = L(f)$, $v = L(g)$. Then we have

$$-\langle u - v, \phi_t \rangle_{T,\Omega} - \langle u - v, \Delta\phi \rangle_{T,\Omega} = \langle f - g, \phi \rangle + \langle uu - vv, \nabla\phi \rangle_{T,\Omega}$$

for all $\phi \in \mathbb{W}_{0,\sigma}^{1,r',q'}$ and $u - v \in L^r(L^q_\sigma)$. This implies by the a priori estimate from Theorem 3.2 – using that $f - g$ can be extended to a functional on $\mathbb{W}_0^{1,r',q'}$ again by the Hahn-Banach theorem – that

$$\begin{aligned}
\|L(f) - L(g)\|_{r,q} = \|u - v\|_{r,q} &\leq C_1 \|f - g\|_{\left(\mathbb{W}_{0,\sigma}^{1,r',q'}\right)'} + C_1 \|W(u,u) - W(v,v)\|_{\left(\mathbb{W}_0^{1,r',q'}\right)'} \\
&\leq C_1 \|f - g\|_{\left(\mathbb{W}_{0,\sigma}^{1,r',q'}\right)'} + C_1 C_2 \left(\|u\|_{r,q} + \|v\|_{r,q}\right) \|u - v\|_{r,q} \\
&\leq C_1 \|f - g\|_{\left(\mathbb{W}_{0,\sigma}^{1,r',q'}\right)'} + \frac{1}{2} \|L(f) - L(g)\|_{r,q},
\end{aligned}$$

using that $\|u\|_{r,q} + \|v\|_{r,q} \leq 2\rho = \frac{1}{2C_1C_2}$ and Corollary 4.4. This estimate yields

$$\|L(f) - L(g)\|_{r,q} \leq 2C_1 \|f - g\|_{\left(\mathbb{W}_{0,\sigma}^{1,r',q'}\right)'}$$

which finally implies the continuity of L. □

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