Essential spectra and exponential estimates of eigenfunctions of lattice operators of quantum mechanics

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Abstract

This paper is devoted to estimates of the exponential decay of eigenfunctions of difference operators on the lattice $\mathbb{Z}^n$ which are discrete analogs of the Schrödinger, Dirac and square-root Klein-Gordon operators. Our investigation of the essential spectra and the exponential decay of eigenfunctions of the discrete spectra is based on the calculus of the so-called pseudodifference operators (i.e., pseudodifferential operators on the group $\mathbb{Z}^n$ with analytic symbols, as developed in the paper [29]), and the limit operators method (see [34] and the references cited there). We obtain a description of the location of the essential spectra and estimates of the eigenfunctions of the discrete spectra of the main lattice operators of quantum mechanics, namely: matrix Schrödinger operators on $\mathbb{Z}^n$, Dirac operators on $\mathbb{Z}^3$, and square root Klein-Gordon operators on $\mathbb{Z}^n$.

Key words: lattice Schrödinger, Dirac and Klein-Gordon operators, essential spectra, exponential estimates of eigenfunctions
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1 Introduction

The main goal of this paper is estimates of the exponential decay of eigenfunctions of difference operators on the lattice $\mathbb{Z}^n$. Particular attention is paid to the discrete analogs of the Schrödinger, Dirac and square-root Klein-Gordon operators. Schrödinger operators on the lattice $\mathbb{Z}^n$ appear at several places, so in the tight binding model in solid state physics, in the propagation of spin waves and waves in quasi-crystals, and in mathematical models of nano-structure, to mention only a few (see, [21, 22, 35, 44]). There is an extensive bibliography devoted to different aspects of the spectral theory of discrete Schrödinger operators. Let us only mention the papers [3, 4, 13, 14, 39, 44, 45], and see also the references cited there. In our recent papers [30, 31], we study the essential spectra of discrete Schrödinger operators with variable magnetic and electric potentials on the lattice $\mathbb{Z}^n$ and on periodic combinatorial graphs by means of the limit operators method (for the latter see also [34]).

In the last time, also discrete relativistic operators attracted many attention. They were used, e.g., in comparative studies of relativistic and nonrelativistic electronlocalization phenomena [5], in relativistic investigations of electrical conduction in disordered systems [38], in the construction of supertransparent models with supersymmetric structures [41], and in relativistic tunnelling problems [37].

The problem of exponential estimates of solutions of the elliptic partial differential equations with applications to the Schrödinger operator is a classical one. There are an extensive bibliography devoted to this problem (see for instance [1, 2, 7, 9, 10, 11]). Exponential estimates of solutions of pseudodifferential equations are considered in [15, 19, 20, 23, 26, 27]. Note also the our recent papers [32, 33] where we proposed a new approach to exponential estimates for partial differential and pseudodifferential operators, based on the limit operators method (see [34]).

Our approach to study essential spectra and the exponential decay of eigenfunctions is based on the calculus of pseudodifference operators (i.e., pseudodifferential operators on the group $\mathbb{Z}^n$) with analytic symbols as developed in [29], and on the limit operators method (see [34] and the references cited there).

The paper is organized as follows. In Section 2 we recall some auxiliary facts on the pseudodifference operators with analytic symbols on $\mathbb{Z}^n$, limit operators, essential spectra and the behavior of solutions of pseudodifference equations at infinity.

In Section 3 we consider the discrete Schrödinger operators on $l^2(\mathbb{Z}^n, \mathbb{C}^N)$ of the form

$$(Hu)(x) = \sum_{k=1}^{n} \left(V_{e_k} - e^{ia_k(x)} \right) \left(V_{-e_k} - e^{-ia_k(x)} \right) u(x) + \Phi(x)u(x)$$

where V_{e_k} is the operator of shift by e_k, the a_k are real-valued bounded slowly oscillating functions on $\mathbb{Z}^n$, and Φ is a Hermitian slowly oscillating and bounded matrix function on $\mathbb{Z}^n$. We show that the essential spectrum $\mathrm{sp}_{ess}\, H$ of H is the interval

$$\mathrm{sp}_{ess}\, H = \bigcup_{j=1}^{n} [\lambda_j^{\inf}, \, \lambda_j^{\sup} + 4n]$$

where

$$\lambda_j^{\inf} := \liminf_{x \to \infty} \lambda_j(\Phi(x)), \qquad \lambda_j^{\sup} := \limsup_{x \to \infty} \lambda_j(\Phi(x))$$

and where the $\lambda_j(\Phi(x))$ are the increasingly ordered eigenvalues of the matrix $\Phi(x)$, i.e.

$$\lambda_1(\Phi(x)) < \lambda_2(\Phi(x)) < \ldots < \lambda_N(\Phi(x))$$

for $x \in \mathbb{Z}^n$ large enough. Note that $\mathrm{sp}_{ess}\, H$ does not depend on the exponents a_k, and that there is a gap $(\lambda_j^{\sup} + 4n, \, \lambda_{j+1}^{\inf})$ in the essential spectrum of H if $\lambda_j^{\sup} + 4n < \lambda_{j+1}^{\inf}$.

We also obtain the following estimates of eigenfunctions belonging to points in the discrete spectrum of H. In each of the cases

- $\lambda \in (\lambda_j^{\sup} + 4n,\, \lambda_{j+1}^{\inf})$ is an eigenvalue of H and
$$0 < r < \cosh^{-1}\left(\frac{\min\{\lambda - \lambda_j^{\sup} - 2n,\, \lambda_{j+1}^{\inf} - \lambda + 2n\}}{2n}\right),$$
- $\lambda > \lambda_N^{\sup} + 4n$ is an eigenvalue of H and
$$0 < r < \cosh^{-1}\left(\frac{\lambda - \lambda_N^{\sup} - 2n}{2n}\right),$$
- $\lambda < \lambda_1^{\inf}$ is an eigenvalue of H and
$$0 < r < \cosh^{-1}\left(\frac{\lambda_1^{\inf} - \lambda + 2n}{2n}\right),$$

every λ-eigenfunction u of H has the property that $e^{r|x|}u \in l^p(\mathbb{Z}^n,\, \mathbb{C}^N)$ for every $1 < p < \infty$.

In Section 4 we introduce self-adjoint Dirac operators on the lattice $\mathbb{Z}^3$ with variable slowly oscillating electric potentials. In accordance with the general properties of Dirac operators on $\mathbb{R}^3$ (see for instance [6, 42]), the corresponding discrete Dirac operator on $\mathbb{Z}^3$ should be a self-adjoint system of the first order difference operators. We thus let
$$\mathcal{D} := \mathcal{D}_0 + e\Phi E_4 \tag{1}$$
where
$$\mathcal{D}_0 := c\hbar d_k \gamma^k + c^2 m\, \gamma^0,$$
E_N is the $N \times N$ unit matrix, the γ^k with $k = 0, 1, 2, 3$ refer to the 4×4 Dirac matrices, the
$$d_k := I - V_{e_k}, \quad k = 1, 2, 3$$
are difference operators of the first order, $\hbar$ is Planck's constant, c is the speed of light, m and e are the mass and the charge of the electron and, finally, Φ is the real electric potential. The operator $\mathcal{D}$, acting on $l^2(\mathbb{Z}^3,\, \mathbb{C}^4)$, can be considered as the direct discrete analog of the Dirac operator on $\mathbb{R}^3$, but note that $\mathcal{D}$ is not *self-adjoint* on $l^2(\mathbb{Z}^3,\, \mathbb{C}^4)$. To force the self-adjointness, we consider the "symmetrization" $\mathbb{D} := \mathbb{D}_0 + e\Phi I$ of $\mathcal{D}$ with
$$\mathbb{D}_0 := \begin{pmatrix} 0 & \mathcal{D}_0 \\ \mathcal{D}_0^* & 0 \end{pmatrix}, \tag{2}$$
which acts on $l^2(\mathbb{Z}^3,\, \mathbb{C}^8)$. The operator $\mathbb{D}$ is self-adjoint, and
$$\mathbb{D}_0^2 = \begin{pmatrix} (\hbar^2 c^2 \Gamma + m^2 c^4) E_4 & 0 \\ 0 & (\hbar^2 c^2 \Gamma + m^2 c^4) E_4 \end{pmatrix},$$

where $\hbar^2 c^2 \Gamma + m^2 c^4$ is the lattice Klein-Gordon Hamiltonian with Laplacian

$$\Gamma := \sum_{k=1}^{3} d_k^* d_k = \sum_{k=1}^{3} (2I - V_{e_k} - V_{e_k}^*).$$

Note that one-dimensional Dirac operators of the form (2) on $\mathbb{Z}$ were considered in [24, 25].

We prove that the essential spectrum of $\mathbb{D}$ is the union

$$\begin{aligned} \operatorname{sp}_{ess} \mathbb{D} \quad = \quad & [e\Phi^{\inf} - \sqrt{12\hbar^2 c^2 + m^2 c^4},\, e\Phi^{\sup} - mc^2] \\ & \cup [e\Phi^{\inf} + mc^2,\, e\Phi^{\sup} + \sqrt{12\hbar^2 c^2 + m^2 c^4}], \end{aligned}$$

where

$$\Phi^{\inf} := \liminf_{x \to \infty} \Phi(x), \qquad \Phi^{\sup} := \limsup_{x \to \infty} \Phi(x).$$

Again we observe that if $e\Phi^{\sup} - e\Phi^{\inf} < 2mc^2$, then the essential spectrum of $\mathbb{D}$ has the gap $(e\Phi^{\sup} - mc^2,\, e\Phi^{\inf} + mc^2)$.

We also obtain the following estimates of eigenfunctions of the discrete spectrum. Let λ be a point of the discrete spectrum, and let λ and $r > 0$ satisfy one of the conditions

- $\lambda \in (e\Phi^{\sup} - mc^2,\, e\Phi^{\inf} + mc^2)$ and

$$0 < r < \cosh^{-1} \left(\frac{m^2 c^4 - \max\left\{(e\Phi^{\inf} - \lambda)^2,\, (e\Phi^{\sup} - \lambda)^2\right\} + 6\hbar^2 c^2}{6\hbar^2 c^2} \right);$$

- $\lambda > e\Phi^{\sup} + \sqrt{12\hbar^2 c^2 + m^2 c^4}$ and

$$0 < r < \cosh^{-1} \left(\frac{(e\Phi^{\sup} - \lambda)^2 - m^2 c^4 - 6\hbar^2 c^2}{6\hbar^2 c^2} \right);$$

- $\lambda < e\Phi^{\inf} - \sqrt{12\hbar^2 c^2 + m^2 c^4}$ and

$$0 < r < \cosh^{-1} \left(\frac{(e\Phi^{\inf} - \lambda)^2 - m^2 c^4 - 6\hbar^2 c^2}{6\hbar^2 c^2} \right).$$

Then every λ-eigenfunction u of the operator $\mathbb{D}$ satisfies $e^{r|x|} u \in l^p(\mathbb{Z}^3, \mathbb{C}^8)$ for every $p \in (1, \infty)$.

In Section 5, we consider the lattice model of the relativistic square root Klein-Gordon operator as the pseudodifference operator of the form

$$\mathcal{K} := \sqrt{c^2 \hbar^2 \Gamma + m^2 c^4} + e\Phi$$

on $l^2(\mathbb{Z}^n)$. We determine the essential spectrum of $\mathcal{K}$ and obtain exact estimates of the exponential decay at infinity of eigenfunctions of the discrete spectrum.

2 Pseudodifference operators, essential spectra, and exponential estimates

2.1 Some function spaces

For each Banach space X, $\mathcal{B}(X)$ refers to the Banach algebra of all bounded linear operators acting on X. For $1 \le p \le \infty$, we let $l^p(\mathbb{Z}^n, \mathbb{C}^N)$ denote the Banach space of all functions on $\mathbb{Z}^n$ with values in $\mathbb{C}^N$ with the norm

$$\|f\|^p_{l^p(\mathbb{Z}^n,\,\mathbb{C}^N)} := \sum_{x\in\mathbb{Z}^n} \|f(x)\|^p_{\mathbb{C}^N} < \infty \quad \text{if } p < \infty,$$

$$\|f\|_{l^\infty(\mathbb{Z}^n,\,\mathbb{C}^N)} := \sup_{x\in\mathbb{Z}^n} \|f(x)\|_{\mathbb{C}^N} < \infty.$$

The choice of the norm on $\mathbb{C}^N$ is not of importance in general; only for $p = 2$ we choose the Euclidean norm (such that $l^2(\mathbb{Z}^n, \mathbb{C}^N)$ becomes a Hilbert space and $\mathcal{B}(\mathbb{C}^N)$ a C^*-algebra in the usual way). Given a positive function w on $\mathbb{Z}^n$, which we will call a weight, let $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ stand for the Banach space of all functions on $\mathbb{Z}^n$ with values in $\mathbb{C}^N$ such that

$$\|u\|_{l^p(\mathbb{Z}^n,\,\mathbb{C}^N,\,w)} := \|wu\|_{l^p(\mathbb{Z}^n,\,\mathbb{C}^N)} < \infty.$$

Similarly, we write $l^\infty(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$ for the Banach algebra of all bounded functions on $\mathbb{Z}^n$ with values in $\mathcal{B}(\mathbb{C}^N)$ and the norm

$$\|f\|_{l^\infty(\mathbb{Z}^n,\,\mathcal{B}(\mathbb{C}^N))} := \sup_{x\in\mathbb{Z}^n} \|f(x)\|_{\mathcal{B}(\mathbb{C}^N)} < \infty.$$

Finally, we call a function $a \in l^\infty(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$ *slowly oscillating* if

$$\lim_{x\to\infty} \|a(x+y) - a(x)\|_{\mathcal{B}(\mathbb{C}^N)} = 0$$

for every point $y \in \mathbb{Z}^n$. We denote the class of all slowly oscillating functions by $SO(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$ and write simply $SO(\mathbb{Z}^n)$ in case $N = 1$.

2.2 Pseudodifference operators

Consider the n-dimensional torus $\mathbb{T}^n$ as a multiplicative group and let

$$d\mu := \left(\frac{1}{2\pi i}\right)^n \frac{dt_1 \cdot \ldots \cdot dt_n}{t_1 \cdot \ldots \cdot t_n} = \left(\frac{1}{2\pi i}\right)^n \frac{dt}{t}$$

denote the corresponding normalized Haar measure on $\mathbb{T}^n$.

Definition 1 *Let $\mathcal{S}(N)$ denote the class of all matrix-valued functions $a = (a_{ij})^n_{i,j=1}$ on $\mathbb{Z}^n \times \mathbb{T}^n$ with*

$$\|a\|_k := \sup_{(x,t)\in\mathbb{Z}^n\times\mathbb{T}^n,\,|\alpha|\le k} \|\partial^\alpha_t a(x,t)\|_{\mathcal{B}(\mathbb{C}^N)} < \infty \tag{3}$$

for every non-negative integer k, provided with the convergence defined by the semi-norms $|a|_k$. To each function $a \in \mathcal{S}(N)$, we associate the pseudodifference operator

$$(\mathrm{Op}\,(a)u)(x) := \int_{\mathbb{T}^n} a(x,t)\hat{u}(t)t^x \, d\mu(t), \quad x \in \mathbb{Z}^n, \tag{4}$$

which is defined on vector-valued functions with finite support. Here, $\hat{u}$ refers to the discrete Fourier transform of u, i.e.,

$$\hat{u}(t) := \sum_{x \in \mathbb{Z}^n} u(x)t^x, \quad t \in \mathbb{T}^n.$$

We denote the class of all pseudodifference operators by $OP\mathcal{S}(N)$.

Pseudodifference operators on $\mathbb{Z}^n$ can be thought of as the discrete analog of pseudodifferential operators on $\mathbb{R}^n$ (see for instance [40, 43]); they can be also interpreted as (abstract) pseudodifferential operators with respect to the group $\mathbb{Z}^n$. For another representation of pseudodifference operators, we need the operator V_α of shift by $\alpha \in \mathbb{Z}^n$, i.e. the operator V_α on $l^p(\mathbb{Z}^n, \mathbb{C}^N)$ which acts via

$$(V_\alpha u)(x) = u(x - \alpha), \quad x \in \mathbb{Z}^n.$$

Then the operator $\mathrm{Op}\,(a)$ can be written as

$$\mathrm{Op}\,(a) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha V_\alpha \tag{5}$$

where

$$a_\alpha(x) := \int_{\mathbb{T}^n} a(x,t)t^\alpha \, d\mu(t).$$

Integrating by parts we obtain

$$\|a_\alpha\|_{l^\infty(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))} \le C|a|_2(1 + |\alpha|)^{-2}, \tag{6}$$

whence

$$\|\mathrm{Op}\,(a)\|_{W(\mathbb{Z}^n, \mathbb{C}^N)} := \sum_{\alpha \in \mathbb{Z}^n} \|a_\alpha\|_{l^\infty(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))} < \infty. \tag{7}$$

We thus obtain that the pseudodifference operator $\mathrm{Op}\,(a)$ belongs to the Wiener algebra $W(\mathbb{Z}^n, \mathbb{C}^N)$ which, by definition, consists of all operators of the form (5) with norm (7). It is an immediate consequence of this fact that all operators $\mathrm{Op}\,(a)$ in $OP\mathcal{S}(N)$ are bounded on $l^p(\mathbb{Z}^n, \mathbb{C}^N)$ for all $p \in [1, \infty]$. Moreover, since the algebra $W(\mathbb{Z}^n, \mathbb{C}^N)$ is inverse closed in $\mathcal{B}(l^p(\mathbb{Z}^n, \mathbb{C}^N))$, the spectrum of $\mathrm{Op}\,(a) \in OP\mathcal{S}(N)$ is independent of the underlying space $l^p(\mathbb{Z}^n, \mathbb{C}^N)$.

Note that the operator (4) can be also written as

$$\mathrm{Op}\,(a)u(x) = \sum_{y \in \mathbb{Z}^n} \int_{\mathbb{T}^n} a(x,t)t^{x-y}u(y) \, d\mu(t),$$

which leads to the following generalization of pseudodifference operators. Let a be a function on $\mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{T}^n$ with values in $\mathcal{B}(\mathbb{C}^N)$ which is subject to the estimates

$$|a|_k =: \sup_{(x,y,t)\in\mathbb{Z}^n\times\mathbb{Z}^n\times\mathbb{T}^n,\, |\alpha|\le k} \|\partial_t^\alpha a(x,y,t)\|_{\mathcal{B}(\mathbb{C}^N)} < \infty \tag{8}$$

for every non-negative integer k. Let $\mathcal{S}_d(N)$ denote the set of all functions with these properties. To each function $a \in \mathcal{S}_d(N)$, we associate the *pseudodifference operator with double symbol*

$$(\mathrm{Op}_d\,(a)u)(x) := \sum_{y\in\mathbb{Z}^n} \int_{\mathbb{T}^n} a(x,y,t)u(y)t^{x-y}\, d\mu(t) \tag{9}$$

where $u : \mathbb{Z}^n \to \mathbb{C}^N$ is a function with finite support. The right-hand side of (9) has to be understood as in (5.6) in [34], which is in analogy with the definition of an oscillatory integral (see [40] and also Section 4.1.2 in [34]). The class of all operators of this form is denoted by $OP\mathcal{S}_d(N)$.

The representation of operators on $\mathbb{Z}^n$ as pseudodifference operators is very convenient due to the fact that one has explicit formulas for products and adjoints of such operators. The basic results are as follows.

Proposition 2 *(i) Let $a, b \in \mathcal{S}(N)$. Then the product* $\mathrm{Op}\,(a)\mathrm{Op}\,(b)$ *is an operator in $OP\mathcal{S}(N)$, and* $\mathrm{Op}\,(a)\mathrm{Op}\,(b) = \mathrm{Op}\,(c)$ *with*

$$c(x,t) = \sum_{y\in\mathbb{Z}^n} \int_{\mathbb{T}^n} a(x,t\tau)b(x+y,\tau)\tau^{-y}\, d\mu(\tau), \tag{10}$$

with the right-hand side understood as an oscillatory integral.
(ii) Let $a \in \mathcal{S}(N)$ and consider $\mathrm{Op}\,(a)$ *as acting on $l^p(\mathbb{Z}^n, \mathbb{C}^N)$ with $p \in (1,\infty)$. Then the adjoint operator of* $\mathrm{Op}\,(a)$ *belongs to $OP\mathcal{S}(N)$, too, and it is of the form* $\mathrm{Op}\,(a)^* = \mathrm{Op}\,(b)$ *with*

$$b(x,t) = \sum_{y\in\mathbb{Z}^n} \int_{\mathbb{T}^n} a^*(x+y,t\tau)\tau^{-y}\, d\mu(\tau), \tag{11}$$

where $a^*(x,t)$ is the usual adjoint (i.e., transposed and complex conjugated) matrix.
(iii) Let $a \in \mathcal{S}_d(N)$. Then $\mathrm{Op}_d\,(a) \in OP\mathcal{S}(N)$, *and* $\mathrm{Op}_d\,(a) = \mathrm{Op}\,(a^\#)$ *where*

$$a^\#(x,t) = \sum_{y\in\mathbb{Z}^n} \int_{\mathbb{T}^n} a(x+y,t\tau)\tau^{-y}\, d\mu(\tau).$$

2.3 Limit operators and the essential spectrum

Recall that an operator $A \in \mathcal{B}(X)$ is a *Fredholm operator* if its kernel $\ker A = \{x \in X : Ax = 0\}$ and its cokernel $\operatorname{coker} A = X/(AX)$ are finite-dimensional linear spaces. The essential spectrum of A consists of all points $\lambda \in \mathbb{C}$ such that the operator $A-\lambda I$ is not a Fredholm operator. We denote the (usual) spectrum and the essential spectrum of A by $\mathrm{spec}_X A$ and $\mathrm{sp}_{ess\ X} A$, respectively.

Our main tool to study the Fredholm property is limit operators. The following definition is crucial in what follows.

Definition 3 *Let $A \in \mathcal{B}(l^p(\mathbb{Z}^n, \mathbb{C}^N))$ with $p \in (1, \infty)$, and let $h : \mathbb{N} \to \mathbb{Z}^n$ be a sequence which tends to infinity in the sense that $|h(n)| \to \infty$ as $n \to \infty$. An operator $A^h \in \mathcal{B}(l^p(\mathbb{Z}^n, \mathbb{C}^N))$ is called a limit operator of A with respect to the sequence h if*

$$\text{s-lim}_{m\to\infty} V_{-h(m)} A V_{h(m)} = A^h \quad \textit{and} \quad \text{s-lim}_{m\to\infty} V_{-h(m)} A^* V_{h(m)} = (A^h)^*,$$

where s-lim *refers to the strong limit. Clearly, every operator has at most one limit operator with respect to a given sequence. We denote the set of all limit operators of A by $op(A)$.*

Let aI be the operator of multiplication by the function $a \in l^\infty(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$. A standard Cantor diagonal argument shows that every sequence h tending to infinity possesses a subsequence g such that, for every $x \in \mathbb{Z}^n$, the limit

$$\lim_{m\to\infty} a(x + g(m)) =: a^g(x)$$

exists. Clearly, a^g is again in $l^\infty(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$. Hence, all limit operators of aI are of the form $a^g I$. In particular, if $a \in SO(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$, then it follows easily from the definition of a slowly oscillating function that all limit operators of aI are of the form $a^g I$ where now $a^g \in \mathcal{B}(\mathbb{C}^N)$ is a constant function.

Let $\mathrm{Op}\,(a) \in OP\mathcal{S}(N)$, and let $h : \mathbb{N} \to \mathbb{Z}^n$ be a sequence tending to infinity. Then $V_{-h(m)} A V_{h(m)} = \mathrm{Op}\,(a_m)$ with $a_m(x) := a(x+h(m), t)$. It follows as above that the sequence h has a subsequence g such that $a(x + g(m), t)$ converges to a limit $a^g(x, t)$ for every $x \in \mathbb{Z}^n$ uniformly with respect to $t \in \mathbb{T}^n$. One can prove that the so-defined function a^g belongs to $\mathcal{S}(N)$ and the associated operator $\mathrm{Op}\,(a^g)$ is the limit operator of $\mathrm{Op}\,(a)$ with respect to g.

The following theorem gives a complete description of the essential spectrum of pseudodifference operators in terms of their limit operators.

Theorem 4 *Let $a \in \mathcal{S}(N)$. Then, for every $p \in (1, \infty)$,*

$$\mathrm{sp}_{ess\ l^p} \mathrm{Op}\,(a) = \bigcup_{\mathrm{Op}\,(a^g) \in op(A)} \mathrm{spec}_{l^r} \mathrm{Op}\,(a^g) \tag{12}$$

where $r \in [1, \infty]$ is arbitrary.

Since $\mathrm{spec}_{l^r} \mathrm{Op}\,(a^g)$ does not depend on the underlying space, the essential spectrum $\mathrm{sp}_{ess\ l^p} \mathrm{Op}\,(a)$ is independent of $p \in (1, \infty)$. Hence, in what follows we will omit the explicit notation of the underlying space in the spectrum and the essential spectrum.

2.4 Pseudodifference operators with analytic symbols and exponential estimates of eigenfunctions

For $r > 1$ let $\mathbb{K}_r$ be the annulus $\{t \in \mathbb{C} : r^{-1} < |t| < r\}$, and let $\mathbb{K}_r^n$ be the product $\mathbb{K}_r \times \ldots \times \mathbb{K}_r$ of n factors.

Definition 5 *Let $\mathcal{S}(N,\, \mathbb{K}_r^n)$ denote the set of all functions*

$$a : \mathbb{Z}^n \times \mathbb{K}_r^n \to \mathcal{B}(\mathbb{C}^N)$$

which are analytic with respect to t in the domain $\mathbb{K}_r^n$ and satisfy the estimates

$$|a|_k := \sum_{|\alpha| \le k} \sup_{x \in \mathbb{Z}^n,\, t \in \mathbb{K}_r^n} \|\partial_t^\alpha a(x,t)\|_{\mathcal{B}(\mathbb{C}^N)} < \infty$$

for every non-negative integer k. With every function $a \in \mathcal{S}(N,\, \mathbb{K}_r^n)$, we associate a pseudodifference operator defined on vector-valued functions with finite support via (4)*, and we denote the corresponding class of pseudodifference operators by $OP\mathcal{S}(N,\, \mathbb{K}_r^n)$.*

Definition 6 *For $r > 1$, let $\mathcal{W}(\mathbb{K}_r^n)$ denote the class of all exponential weights $w = \exp v$, where v is the restriction onto $\mathbb{Z}^n$ of a function $\tilde{v} \in C^{(1)}(\mathbb{R}^n)$ with the property that, for every point $x \in \mathbb{R}^n$ and every $j = 1, ..., n$,*

$$-\log r < \frac{\partial \tilde{v}(x)}{\partial x_j} < \log r. \tag{13}$$

In what follows we will denote both the function $\tilde{v}$ on $\mathbb{R}^n$ and its restriction onto $\mathbb{Z}^n$ by v. Note that it is an immediate consequence of Definition 6 that if $w \in \mathcal{W}(\mathbb{K}_r^n)$, then $w^\mu \in \mathcal{W}(\mathbb{K}_r^n)$ for every $\mu \in [-1,\, 1]$.

Proposition 7 *Let $A := \mathrm{Op}\,(a) \in OP\mathcal{S}(N, \mathbb{K}_r^n)$ and $w \in \mathcal{W}(\mathbb{K}_r^n)$. Then the operator $A_w := wAw^{-1}$, defined on vector-valued functions with finite support, belongs to the class $OP\mathcal{S}_d(N)$, and $A_w = \mathrm{Op}_d\,(b)$ with*

$$b(x, y, t) = a(x, e^{-\theta_w(x,y)} \cdot t)$$

where

$$e^{-\theta_w(x,y)} \cdot t := \left(e^{-\theta_{w,1}(x,y)} t_1,\, e^{-\theta_{w,2}(x,y)} t_2,\, \ldots,\, e^{-\theta_{w,n}(x,y)} t_n\right)$$

and

$$\theta_{w,j}(x, y) := \int_0^1 \frac{\partial v((1-\gamma)x + \gamma y)}{\partial x_j}\, d\gamma.$$

Proposition 2 and estimate (15) imply the following theorem.

Theorem 8 *Let $a \in \mathcal{S}(N, \mathbb{K}_r^n)$ and $w \in \mathcal{W}(\mathbb{K}_r^n)$. Then $\mathrm{Op}\,(a)$ is a bounded operator on each of the spaces $l^p(\mathbb{Z}^n,\, \mathbb{C}^N, w)$ with $1 \le p \le \infty$.*

Next we consider essential spectra of pseudodifference operators on weighted spaces. Let a, A and A_w be as in Proposition 7. One can easily check that for $h \in \mathbb{Z}^n$

$$V_{-h} A_w V_h = \mathrm{Op}_d\,(b_h) \quad \text{with} \quad b_h(x, y, t) = a(x + h, e^{-\theta_w(x+h,y+h)} \cdot t)).$$

Let now $h : \mathbb{N} \to \mathbb{Z}^n$ be a sequence tending to infinity. Then there exists a subsequence g of h such that the limit operator of A_w with respect to g exists and

$$A_w^g = \mathrm{Op}_d\,(b^g) \quad \text{with} \quad b^g(x, y, t) = a^g(x, e^{\theta_w^g(x,y)} \cdot t) \tag{14}$$

where

$$a^g(x, t) := \lim_{m\to\infty} a(x + g(m), t) \tag{15}$$

and

$$\theta_w^g(x, y) := \lim_{m\to\infty} \int_0^1 \nabla v((1 - \gamma)x + \gamma y + g(m))\, d\gamma. \tag{16}$$

The limits in (15) and (16) are understood as pointwise with respect to $x, y \in \mathbb{Z}^n$ and uniform with respect to $t \in \mathbb{T}^n$.

Theorem 9 *Let $a \in \mathcal{S}(N, \mathbb{K}_r^n)$ and $w \in \mathcal{W}(\mathbb{K}_r^n)$, set $A := \mathrm{Op}\,(a)$ and $A_w := wAw^{-1}$, and consider A as operating from $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ to $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ where $p \in (1, \infty)$. Then*

$$\mathrm{sp}_{ess}\,\mathrm{Op}\,(a) = \bigcup_{\mathrm{Op}_d\,(b^g) \in op(A_w)} \mathrm{spec}\,\mathrm{Op}_d\,(b^g)$$

with b^g as in (14).

Remark 10 *Note that the essential spectrum of an operator in $OP\mathcal{S}(N, \mathbb{K}_r^n)$, considered as acting on $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$, is independent of $p \in (1, \infty)$, but it can depend on the weight in general. But if the weight $w = e^v$ has the property that*

$$\lim_{x\to\infty} \nabla v(x) = 0, \tag{17}$$

then the symbol a_w^g does not depend on the weight and, hence, the essential spectrum of $\mathrm{Op}\,(a) \in \mathcal{B}(l^p(\mathbb{Z}^n, \mathbb{C}^N, w))$ with $a \in \mathcal{S}(N, \mathbb{K}_r^n)$ is independent both on $p \in (1, \infty)$ and on the weight w. Important examples of weights satisfying (17) are the power weights $w(x) = (1 + |x|^2)^{s/2} = e^{\frac{s}{2} log(1+|x|^2)}$ with $s > 0$ and the subexponential weights $w(x) = e^{\alpha|x|^\beta}$ where $\alpha > 0$ and $\beta \in (0, 1)$.

The next theorem provides exponential estimates of solutions of pseudodifference equations.

Theorem 11 *Let $A = Op(a) \in OP\mathcal{S}(N, \mathbb{K}_r^n)$ and $w \in \mathcal{W}(\mathbb{K}_r^n)$. Suppose that $\lim_{x\to\infty} w(x) = +\infty$ and that 0 is not in the essential spectrum of A_{w^μ} : $l^p(\mathbb{Z}^n, \mathbb{C}^N) \to l^p(\mathbb{Z}^n, \mathbb{C}^N)$ for some $p \in (1, \infty)$ and every $\mu \in [-1, 1]$. If $u \in l^p(\mathbb{Z}^n, \mathbb{C}^N, w^{-1})$ is a solution of the equation $Au = f$ with $f \in l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$, then $u \in l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$.*

Theorem 11 has some important corollaries.

Theorem 12 *Let a, A and w be as in the previous theorem, and let λ be an eigenvalue of A which is not in the essential spectrum of $A_{w^\mu} : l^p(\mathbb{Z}^n, \mathbb{C}^N) \to l^p(\mathbb{Z}^n, \mathbb{C}^N)$ for some $p \in (1, \infty)$ and every $\mu \in [0, 1]$. Then every λ-eigenfunction belongs to $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ for every $p \in (1, \infty)$.*

Corollary 13 *Let $A = Op(a) \in OP\mathcal{S}(N, \mathbb{K}_r^n)$ and let λ be an eigenvalue of A which is not in the essential spectrum of A. Then every λ-eigenfunction $u = (u_1, ..., u_N)$ satisfies the sub-exponential estimate*

$$\sup |u_i(x)| \leq C_i e^{-\alpha|x|^\beta}, \quad x \in \mathbb{Z}^n,\, i = 1, ..., N \tag{18}$$

for arbitrary $\alpha > 0$ and $0 < \beta < 1$.

Proof. Let $w(x) = e^{v(x)}$ where $v(x) = \alpha|x|^\beta$ with $\alpha > 0$ and $0 < \beta < 1$. Then $\lim_{x\to\infty} \nabla v(x) = 0$, whence $A^g_{w^\mu} = A^g$ for every limit operator A^g. Let λ be an eigenvalue of A which is not in the essential spectrum of A. Then λ is not in the essential spectrum of A_{w^μ} for every $\mu \in [0, 1]$. Hence, by Theorem 12, every λ-eigenfunction belongs to each of the spaces $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ with $p \in (1, \infty)$. Applying the Hölder inequality we obtain estimate (18). ■

We are now going to specialize these results to the context of slowly oscillating symbols and slowly oscillating weights.

Definition 14 *The symbol $a \in \mathcal{S}(N, \mathbb{K}_r^n)$ is said to be* slowly oscillating *if*

$$\lim_{x\to\infty} \sup_{t\in\mathbb{T}^n} \|a(x + y, t) - a(x, t)\|_{\mathcal{B}(\mathbb{C}^N)} = 0$$

for every $y \in \mathbb{Z}^n$. We write $\mathcal{S}^{sl}(N, \mathbb{K}_r^n)$ for the class of all slowly oscillating symbols and $OP\mathcal{S}^{sl}(N, \mathbb{K}_r^n)$ for the corresponding class of pseudodifference operators.

Definition 15 *The weight $w = e^v \in \mathcal{W}(\mathbb{K}_r)$ is* slowly oscillating *if the partial derivatives $\frac{\partial v}{\partial x_j}$ are slowly oscillating for $j = 1, \ldots, n$. We denote the class of all slowly oscillating weights by $\mathcal{W}^{sl}(\mathbb{K}_r)$.*

Example 16 *If $v(x) = \gamma|x|$, then $\frac{\partial v(x)}{\partial x_j} = \gamma\frac{x_j}{|x|}$ for $j = 1, \ldots, n$. Thus, $w := e^v$ is in $\mathcal{W}^{sl}(\mathbb{K}_r)$ if $\gamma < r$.*

The next theorem describes the structure of the limit operators of the operator $A_w = wAw^{-1}$ if $A \in OP\mathcal{S}^{sl}(N, \mathbb{K}_r^n)$ and $w \in \mathcal{W}^{sl}(\mathbb{K}_r)$.

Theorem 17 *Let $A = \mathrm{Op}\,(a) \in OP\mathcal{S}^{sl}(N, \mathbb{K}_r^n)$ and $w \in \mathcal{W}^{sl}(\mathbb{K}_r)$. Then the limit operator A^g_w of A_w with respect to the sequence g tending to infinity exists if the limits*

$$a_g(t) = \lim_{m\to\infty} a(g(m), t), \qquad \theta^g_w = \lim_{m\to\infty} (\nabla v)(g(m)) \tag{19}$$

exist. In this case, it is of the form

$$A^g_w = \mathrm{Op}\,(c_g) \quad \textit{with} \quad c_g(x, t) = a_g(\theta^g_w \cdot t). \tag{20}$$

Consequently, if A and w are as in this theorem, then the limit operators $A^g_{w^\mu}$ are invariant with respect to shifts. This fact implies the following explicit description of their essential spectra. Let $\{\lambda_j(A^g_{w^\mu})(t)\}_{j=1}^n$ denote the eigenvalues of the matrix $a_g(\theta^g_{w^\mu} \cdot t)$. Then

$$\mathrm{spec}_{l^p(\mathbb{Z}^n,\mathbb{C}^N)} A^g_{w^\mu} = \bigcup_{j=1}^N \{\lambda_j(A^g_{w^\mu})(t) : t \in \mathbb{T}^n \text{ and } j = 1, \ldots, n\},$$

whence

$$\mathrm{sp}_{ess\ l^p(\mathbb{Z}^n,\mathbb{C}^N)} A_{w^\mu} = \bigcup_{A^g_{w^\mu} \in op(A_{w^\mu})} \bigcup_{j=1}^N \{\lambda_j(A^g_{w^\mu})(t) : t \in \mathbb{T}^n \text{ and } j = 1, \ldots, n\}.$$

3 Matrix Schrödinger operators

3.1 Essential spectrum

In this section we consider the essential spectrum and the behavior at infinity of eigenfunctions of general discrete Schrödinger operators acting on $u \in l^2(\mathbb{Z}^n, \mathbb{C}^N)$ by

$$(Hu)(x) = \sum_{k=1}^n (V_{e_k} - e^{ia_k(x)})\,(V_{-e_k} - e^{-ia_k(x)})u(x) + \Phi(x) \tag{21}$$

where $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 standing at the kth place, the $a_k \in SO(\mathbb{Z}^n)$ are real-valued, and $\Phi \in SO(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$ is Hermitian. The vector $a := (a_1, \ldots, a_n)$ is the discrete analog of the magnetic potential, whereas Φ can be viewed of as a discrete analog of the electric potential. Since the essential spectrum of H is independent of $p \in (1, \infty)$, we consider the case $p = 2$ only. Note that our assumptions guarantee that H is a self-adjoint operator on $l^2(\mathbb{Z}^n, \mathbb{C}^N)$.

The limit operators H_g of H are of form

$$\begin{aligned} H_g &= \sum_{k=1}^n (V_{e_k} - e^{ia^g_k} I)\,(V_{-e_k} - e^{-ia^g_k} I) + \Phi^g I \\ &= \sum_{k=1}^n (2I - e^{-ia^g_k} V_{-e_k} - e^{ia^g_k} V_{e_k}) + \Phi^g I \end{aligned}$$

with the constant functions

$$a^g_k = \lim_{m\to\infty} a_k(x + g(m)) \quad \text{and} \quad \Phi^g = \lim_{m\to\infty} \Phi(x + g(m)).$$

Let $U : l^2(\mathbb{Z}^n, \mathbb{C}^N) \to l^2(\mathbb{Z}^n, \mathbb{C}^N)$ be the unitary operator

$$(Uu)(x) = e^{-i\langle a^g, x\rangle} u(x), \quad a^g = (a^g_1, \ldots, a^g_n).$$

Then

$$U^* H_g U = \sum_{k=1}^{n} (2I - V_{-e_k} - V_{e_k}) + \Phi^g.$$

Further, the operator $H'_g := U^* H_g U$ is unitarily equivalent to the operator of multiplication by the function

$$\tilde{H}_g(\psi_1, \ldots, \psi_n) := 4 \sum_{k=1}^{n} \sin^2 \frac{\psi_k}{2} + \Phi^g, \quad \psi_k \in [0, 2\pi],$$

acting on $L^2([0, 2\pi]^n, \mathbb{C}^N)$. Hence,

$$\operatorname{spec} H_g = \operatorname{spec} H'_g = \bigcup_{j=1}^{N} [\lambda_j(\Phi^g), \lambda_j(\Phi^g) + 4n],$$

where the $\lambda_j(\Phi^g)$ refer to the eigenvalues of the matrix Φ^g. Applying formula (12) we obtain

$$\mathrm{sp}_{ess}\, H = \bigcup_{g} \bigcup_{j=1}^{N} [\lambda_j(\Phi^g), \lambda_j(\Phi^g) + 4n] \tag{22}$$

where the first union is taken over all sequences g for which the limit operator of H exists. Let $\lambda_j(\Phi(x))$, $j = 1, \ldots, N$, denote the eigenvalues of the matrix $\Phi(x)$. We suppose that these eigenvalues are simple for x large enough and that they are increasingly ordered,

$$\lambda_1(\Phi(x)) < \lambda_2(\Phi(x)) < \ldots < \lambda_N(\Phi(x)).$$

Then one can show that the functions $x \mapsto \lambda_j(\Phi(x))$ belong to $SO(\mathbb{Z}^n)$. Let

$$\lambda_j^{\inf} := \liminf_{x \to \infty} \lambda_j(\Phi(x)), \qquad \lambda_j^{\sup} := \limsup_{x \to \infty} \lambda_j(\Phi(x)).$$

Since the set of the partial limits of a slowly oscillating function on $\mathbb{Z}^n$ is connected for $n > 1$ (see [34], Theorem 2.4.7), we conclude from (22) that

$$\mathrm{sp}_{ess}\, H = \bigcup_{j=1}^{N} [\lambda_j^{\inf}, \lambda_j^{\sup} + 4n] \tag{23}$$

for $n > 1$. Note that if $\lambda_j^{\sup} + 4n < \lambda_{j+1}^{\inf}$, then there is the gap $(\lambda_j^{\sup} + 4n, \lambda_{j+1}^{\inf})$ in the essential spectrum of H.

In case $n = 1$, the set of the partial limits of a slowly oscillating function on $\mathbb{Z}$ consists of two connected components, which collect the partial limits as $x \to -\infty$ and $x \to +\infty$, respectively. Accordingly, in this case we set

$$\lambda_j^{\inf,\pm} := \liminf_{x \to \pm\infty} \lambda_j(\Phi(x)), \qquad \lambda_j^{\sup,\pm} := \limsup_{x \to \pm\infty} \lambda_j(\Phi(x)).$$

and obtain

$$\mathrm{sp}_{ess}\, H = \bigcup_{j=1}^{N} \left([\lambda_j^{\inf,-}, \lambda_j^{\sup,-} + 4] \cup [\lambda_j^{\inf,+}, \lambda_j^{\sup,+} + 4] \right).$$

3.2 Exponential estimates of eigenfunctions

Our next goal is to apply Theorem 11 to eigenfunctions of (discrete) eigenvalues of the operator H with slowly oscillating potentials. We will formulate the results for $n > 1$ only; for $n = 1$ the non-connectedness of the set of the partial limits requires some evident modifications. According to (23), the discrete spectrum of H is located outside the set $\mathrm{sp}_{ess}\, H = \bigcup_{j=1}^{N} [\lambda_j^{\inf},\, \lambda_j^{\sup} + 4n]$ if $n > 1$.

Let $\cosh^{-1} : [1, +\infty) \to [0, +\infty)$ refer to the function inverse to $\cosh : [0, +\infty) \to [1, +\infty)$, i.e.,

$$\cosh^{-1} \mu = \log(\mu + \sqrt{\mu^2 - 1}).$$

Further let $\mathcal{R}^{sl} := \bigcup_{r>1} \mathcal{W}(\mathbb{K}_r^n)$.

Theorem 18 *Let $w = e^v$ be a weight in $\mathcal{R}^{sl}$ with $\lim_{x\to\infty} v(x) = \infty$. Further let λ be an eigenvalue of H such that $\lambda \notin \mathrm{sp}_{ess}\, H$ and assume that one of the following conditions is satisfied:*

(i) there is a $j \in \{1, \ldots, N\}$ such that $\lambda \in (\lambda_j^{\sup} + 4n,\, \lambda_{j+1}^{\inf})$ and

$$\limsup_{x\to\infty} \left| \frac{\partial v(x)}{\partial x_k} \right| < \cosh^{-1} \left(\frac{\min\{\lambda - \lambda_j^{\sup} - 2n,\, \lambda_{j+1}^{\inf} - \lambda + 2n\}}{2n} \right) \tag{24}$$

for every $k = 1, \ldots, n$;

(ii) $\lambda > \lambda_N^{\sup} + 4n$ and

$$\limsup_{x\to\infty} \left| \frac{\partial v(x)}{\partial x_k} \right| < \cosh^{-1} \left(\frac{\lambda - \lambda_N^{\sup} - 2n}{2n} \right)$$

for every $k = 1, \ldots, n$;

(iii) $\lambda < \lambda_1^{\inf}$ and

$$\limsup_{x\to\infty} \left| \frac{\partial v(x)}{\partial x_k} \right| < \cosh^{-1} \left(\frac{\lambda_1^{\inf} - \lambda + 2n}{2n} \right)$$

for every $k = 1, \ldots, n$.

Then every λ-eigenfunction of H belongs to each of the spaces $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ with $p \in (1, \infty)$.

Proof. For $\mu \in [0, 1]$, let $H'_{w^\mu} := w^\mu H' w^{-\mu}$. The limit operators $H'^g_{w^\mu} - \lambda E$ are unitarily equivalent to the operator of multiplication by the matrix-function

$$\mathcal{H}^g_{w^\mu}(\psi) = \left(-2 \sum_{j=1}^{n} \cos(\psi_j + i\mu\theta_j^g) + 2n - \lambda \right) E + \Phi^g$$

where

$$\psi = (\psi_1, \ldots, \psi_n) \in [0, 2\pi]^n \quad \text{and} \quad \theta_j^g := \lim_{m\to\infty} \frac{\partial v(g(m))}{\partial x_j}.$$

Note that

$$\Re(\mathcal{H}^g_{w^\mu}(\psi)) = \left(-2 \sum_{j=1}^{n} \cos\psi_j \cosh\mu\theta^g_{w,j} + 2n - \lambda \right) E + \Phi^g,$$

where $\theta^g_{w,j} := \left(\frac{\partial v}{\partial x_j}\right)^g$. It is easy to check that condition (24) implies that $\lambda \notin \operatorname{spec} H^g_{w^\mu}$ for every limit operator $H^g_{w^\mu}$ of H_{w^μ} and every $\mu \in [0, 1]$. Hence, by Theorem 12, every λ-eigenfunction belongs to $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ for every $1 < p < \infty$. ■

Corollary 19 *In each of the following cases*
(i) $\lambda \in (\lambda_j^{\sup} + 4n,\, \lambda_{j+1}^{\inf})$ *for some* $j \in \{1, \ldots, N\}$ *and*

$$0 < r < \cosh^{-1}\left(\frac{\min\{\lambda - \lambda_j^{\sup} - 2n,\, \lambda_{j+1}^{\inf} - \lambda + 2n\}}{2n}\right),$$

(ii) $\lambda > \lambda_N^{\sup} + 4n$ *and* $0 < r < \cosh^{-1}\left(\frac{\lambda - \lambda_N^{\sup} - 2n}{2n}\right)$,
(iii) $\lambda < \lambda_1^{\inf}$ *and* $0 < r < \cosh^{-1}\left(\frac{\lambda_1^{\inf} - \lambda + 2n}{2n}\right)$,
every λ*-eigenfunction of* H *belongs to* $l^p(\mathbb{Z}^n, \mathbb{C}^N, e^{r|x|)})$ *for each* $p \in (1, \infty)$.

Remark 20 *In the case of the scalar Schrödinger operator* (21) *with* $\Phi \in SO(\mathbb{Z}^n)$*, we have*

$$\mathrm{sp}_{ess}\, H = [\Phi^{\inf},\, \Phi^{\sup} + 4n]$$

with $\Phi^{\inf} = \liminf_{x\to\infty} \Phi(x)$ *and* $\Phi^{\sup} = \limsup_{x\to\infty} \Phi(x)$*. If one of the following conditions holds for an eigenvalue* λ *of* H*:*
(i) $\lambda > \Phi^{\sup} + 4n$ *and* $0 < r < \cosh^{-1}\left(\frac{\lambda - \Phi^{\sup} - 2n}{2n}\right)$*, or*
(ii) $\lambda < \Phi^{\inf}$ *and* $0 < r < \cosh^{-1}\left(\frac{\Phi^{\inf} + 2n - \lambda}{2n}\right)$,
then every λ*-eigenfunction of* H *belongs to* $l^p(\mathbb{Z}^n, \mathbb{C}^N, e^{r|x|})$ *for each* $p \in (1, \infty)$.

4 The discrete Dirac operator

4.1 The essential spectrum

On $l^2(\mathbb{Z}^3, \mathbb{C}^4)$, we consider the Dirac operators

$$\mathcal{D} := \mathcal{D}_0 + e\Phi I \quad \text{and} \quad \mathcal{D}_0 := c\hbar d_k \gamma^k + c^2 m \gamma^0 \tag{25}$$

where the γ^k, $k = 0, 1, 2, 3$, refer to the 4×4 Dirac matrices, i.e., they satisfy

$$\gamma^j \gamma^k + \gamma^k \gamma^j = 2\delta_{jk} E_4 \tag{26}$$

for all choices of $j, k = 0, 1, 2, 3$ where E_4 stands for the 4×4 identity matrix. Further,

$$d_k := I - V_{e_k}, \quad k = 1, 2, 3$$

are difference operators of the first order, $\hbar$ is Planck's constant, c the light speed, m and e are the mass and the charge of the electron, and Φ is the electric potential. We suppose that the function Φ is real-valued and belongs to the space $SO(\mathbb{Z}^3)$.

It turns out that the operator $\mathcal{D}$ is *not self-adjoint* on $l^2(\mathbb{Z}^3, \mathbb{C}^4)$. Therefore we introduce self-adjoint Dirac operators as the matrix operators

$$\mathbb{D} := \mathbb{D}_0 + e\Phi I \quad \text{with} \quad \mathbb{D}_0 := \begin{pmatrix} 0 & \mathcal{D}_0 \\ \mathcal{D}_0^* & 0 \end{pmatrix},$$

acting on the space $l^2(\mathbb{Z}^3, \mathbb{C}^8)$ (i.e. I refers now to the identity operator on that space). First we are going to determine the spectrum of $\mathbb{D}_0$. It is

$$(\mathbb{D}_0 - \lambda I)(\mathbb{D}_0 + \lambda I) = \begin{pmatrix} \mathcal{L}(\lambda) & 0 \\ 0 & \mathcal{L}(\lambda) \end{pmatrix} \tag{27}$$

where $\mathcal{L}(\lambda) = \hbar^2 c^2 \Gamma + (m^2 c^4 - \lambda^2) I$, and

$$\Gamma := \sum_{k=1}^{3} d_k^* d_k = \sum_{k=1}^{3} (2I - V_{e_k} - V_{e_k}^*)$$

is the discrete Laplacian with symbol

$$\hat{\Gamma}(\varphi) = \hat{\Gamma}(\varphi_1, \varphi_2, \varphi_3) = \sum_{k=1}^{3} (2 - 2\cos\varphi_k), \quad \varphi_k \in [0, 2\pi].$$

Similarly, we denote by $\hat{\mathbb{D}}_0(\varphi)$ and $\hat{\mathcal{L}}(\lambda, \varphi)$ the symbols of the operators $\mathbb{D}_0$ and $\mathcal{L}(\lambda)$, respectively. Then

$$(\hat{\mathbb{D}}_0(\varphi) - \lambda E_8)(\hat{\mathbb{D}}_0(\varphi) + \lambda E_8) = \hat{\mathcal{L}}(\lambda, \varphi) E_8 \tag{28}$$

with the scalar-valued function

$$\hat{\mathcal{L}}(\lambda, \varphi) = \hbar^2 c^2 \sum_{k=1}^{3} (2 - 2\cos\varphi_k) + m^2 c^4 - \lambda^2.$$

We claim that $\lambda \in \operatorname{spec} \mathbb{D}_0$ if and only if there exists a $\varphi_0 \in [0, 2\pi]^3$ such that $\hat{\mathcal{L}}(\lambda, \varphi_0) = 0$. Indeed, let $\lambda \in \operatorname{spec} \mathbb{D}_0$. Then there exists a $\varphi_0 \in [0, 2\pi]^3$ such that $\det(\hat{\mathbb{D}}_0(\varphi_0) - \lambda E_8) = 0$. Hence by (28) $\hat{\mathcal{L}}(\lambda, \varphi_0) = 0$. Conversely, if $\hat{\mathcal{L}}(\lambda, \varphi_0) = 0$, then it follows from (28) that

$$(\hat{\mathbb{D}}_0(\varphi_0) - \lambda E_8)(\hat{\mathbb{D}}_0(\varphi_0) + \lambda E_8) = 0.$$

Hence, $\det(\hat{\mathbb{D}}_0(\varphi_0) - \lambda E_8) = 0$, whence $\lambda \in \operatorname{spec} \mathbb{D}_0$.

Since the equation $\hat{\mathcal{L}}(\lambda, \varphi) = 0$ has two branches of solutions (spectral curves), namely

$$\lambda_{\pm}(\varphi) = \pm\sqrt{\hbar^2 c^2 \hat{\Gamma}(\varphi) + m^2 c^4}, \quad \varphi \in [0, 2\pi]^3,$$

the spectrum of $\mathbb{D}_0$ is the union

$$\operatorname{spec} \mathbb{D}_0 = [-\sqrt{12\hbar^2 c^2 + m^2 c^4},\, -mc^2] \cup [mc^2,\, \sqrt{12\hbar^2 c^2 + m^2 c^4}].$$

Our next goal is to determine the essential spectrum of $\mathbb{D} = \mathbb{D}_0 + e\Phi I$. All limit operators of $\mathbb{D}$ are of the form $\mathbb{D}^g = \mathbb{D}_0 + e\Phi^g I$ where $\Phi^g = \lim_{j\to\infty} \Phi(g(j))$ is the partial limit of Φ corresponding to the sequence $g : \mathbb{N} \to \mathbb{Z}^3$ tending to infinity. By what we have just seen, this gives

$$\begin{aligned} \operatorname{spec} \mathbb{D}^g &= [e\Phi^g - \sqrt{12\hbar^2 c^2 + m^2 c^4},\, e\Phi^g - mc^2] \\ &\qquad \cup [e\Phi^g + mc^2,\, e\Phi^g + \sqrt{12\hbar^2 c^2 + m^2 c^4}]. \end{aligned}$$

Since $\operatorname{sp}_{ess} \mathbb{D} = \cup_g \operatorname{spec} \mathbb{D}^g$ we obtain

$$\begin{aligned} \operatorname{sp}_{ess} \mathbb{D} &= [e\Phi^{\inf} - \sqrt{12\hbar^2 c^2 + m^2 c^4},\, e\Phi^{\sup} - mc^2] \\ &\qquad \cup [e\Phi^{\inf} + mc^2,\, e\Phi^{\sup} + \sqrt{12\hbar^2 c^2 + m^2 c^4}], \end{aligned}$$

where

$$\Phi^{\inf} := \liminf_{x\to\infty} \Phi(x) \quad \text{and} \quad \Phi^{\sup} := \limsup_{x\to\infty} \Phi(x).$$

In particular, if $e(\Phi^{\sup} - \Phi^{\inf}) < 2mc^2$, then the interval $(e\Phi^{\sup} - mc^2,\, e\Phi^{\inf} + mc^2)$ is a gap in the essential spectrum of $\mathbb{D}$.

4.2 Exponential estimates of eigenfunctions

The following is the analog of Theorem 18.

Theorem 21 *Let $\lambda \notin \operatorname{sp}_{ess} \mathbb{D}$ be an eigenvalue of $\mathbb{D} : l^p(\mathbb{Z}^3, \mathbb{C}^8) \to l^p(\mathbb{Z}^3, \mathbb{C}^8)$ with $p \in (1, \infty)$. Assume further that the weight $w = e^v$ is in $\mathcal{R}^{sl}$ and that $\lim_{x\to\infty} v(x) = \infty$. If one of the conditions*
(i) $\lambda \in (e\Phi^{\sup} - mc^2,\, e\Phi^{\inf} + mc^2)$ and, for every $j = 1, 2, 3$,

$$\begin{aligned} &\limsup_{x\to\infty} \left| \frac{\partial v(x)}{\partial x_j} \right| \\ &\qquad < \cosh^{-1}\left(\frac{m^2 c^4 - \max\left\{(e\Phi^{\inf} - \lambda)^2,\, (e\Phi^{\sup} - \lambda)^2\right\} + 6\hbar^2 c^2}{6\hbar^2 c^2} \right), \end{aligned} \tag{29}$$

(ii) $\lambda > e\Phi^{\sup} + \sqrt{12\hbar^2 c^2 + m^2 c^4}$ and, for every $j = 1, 2, 3$,

$$\limsup_{x\to\infty} \left| \frac{\partial v(x)}{\partial x_j} \right| < \cosh^{-1}\left(\frac{(e\Phi^{\sup} - \lambda)^2 - m^2 c^4 - 6\hbar^2 c^2}{6\hbar^2 c^2} \right), \tag{30}$$

(iii) $\lambda < e\Phi^{\inf} - \sqrt{12\hbar^2 c^2 + m^2 c^4}$ *and, for every* $j = 1, 2, 3$,

$$\limsup_{x \to \infty} \left| \frac{\partial v(x)}{\partial x_j} \right| < \cosh^{-1} \left(\frac{(e\Phi^{\inf} - \lambda)^2 - m^2 c^4 - 6\hbar^2 c^2}{6\hbar^2 c^2} \right), \tag{31}$$

is satisfied, then every λ*-eigenfunction of the operator* $\mathbb{D}$ *belongs to the space* $l^p(\mathbb{Z}^3, \mathbb{C}^8, w)$ *for each* $p \in (1, \infty)$.

Proof. We will prove the assertion in case condition (i) is satisfied. The other cases follow similarly. Further, since the essential spectrum of $\mathbb{D}$ and the spectra of the associated limit operators do not depend on p, we can assume that $p = 2$ in this proof.

Let condition (29) hold, and let λ be an eigenvalue in the gap $(e\Phi^{\sup} - mc^2,\, e\Phi^{\inf} + mc^2)$ of the essential spectrum. In order to apply Theorem 11 to determine the decaying behavior of the associated eigenfunction u_λ, we need estimates of the spectrum of the limit operators $(\mathbb{D}_{w^\mu})^g$ of $\mathbb{D}_{w^\mu} := w^\mu \mathbb{D} w^{-\mu}$ for $\mu \in [0, 1]$. The limit operator $(w^\mu V_{e_k} w^{-\mu})^g$ of $w^\mu V_{e_k} w^{-\mu}$ is of the form

$$(w^\mu V_{e_k} w^{-\mu})^g = e^{-\mu \left(\frac{\partial v}{\partial x_k} \right)^g} V_{e_k}.$$

Hence,

$$(\mathcal{D}_{w^\mu})^g = \sum_{k=1}^{3} c\gamma^k (I - e^{-\mu \left(\frac{\partial v}{\partial x_k} \right)^g} V_{e_k}) + mc^2 \gamma^0 + e\Phi^g E_4 \tag{32}$$

where $\left(\frac{\partial v}{\partial x_k} \right)^g = \lim_{m \to \infty} \frac{\partial v(g(m))}{\partial x_k}$.

Let $\mathbb{D}' = \mathbb{D}_0 - e\Phi I$. The identity (32) implies that $(\mathbb{D}'^g_{w^\mu} - \lambda I)(\mathbb{D}'^g_{w^\mu} + \lambda I)$ is the diagonal matrix $\operatorname{diag}(F,\, F)$ with

$$F := \hbar^2 c^2 \Gamma^g_{w^\mu} + (m^2 c^4 - (e\Phi^g - \lambda)^2) I$$

and

$$\Gamma^g_{w^\mu} = \sum_{k=1}^{3} \left(2I - e^{-\left(\frac{\partial v}{\partial x_k} \right)^g} V_{e_k} - e^{\left(\frac{\partial v}{\partial x_k} \right)^g} V^*_{e_k} \right).$$

The operator $\Gamma^g_{w^\mu}$ is unitarily equivalent to the operator of multiplication by the function

$$\hat{\Gamma}^g_{w^\mu}(\varphi) = \hat{\Gamma}^g_{w^\mu}(\varphi_1,\, \varphi_2,\, \varphi_3) = \sum_{k=1}^{3} \left(2 - 2\cos\left(\varphi_k + i \left(\frac{\partial v}{\partial x_k} \right)^g \right) \right)$$

acting on the space $L^2([0,\, 2\pi]^3)$. Note that

$$\Re(\hat{\Gamma}^g_{w^\mu}(\varphi)) = 6 - 2\sum_{j=1}^{3} \cos\varphi_k \cosh\left(\frac{\partial v}{\partial x_k} \right)^g.$$

Hence, and by condition (29),

$$\Re(\hbar^2 c^2 \hat{\Gamma}^g_{w^\mu}(\varphi) + m^2 c^4 - (e\Phi^g - \lambda)^2) \neq 0 \tag{33}$$

for every sequence g defining a limit operator and for every $\mu \in [0, 1]$. The property (33) implies that $\lambda \notin \operatorname{spec} \mathbb{D}^g_{w^\mu}$ for every limit operator $\mathbb{D}^g_{w^\mu}$ and every $\mu \in [0, 1]$. By Theorem 11, every λ-eigenfunction belongs to $l^p(\mathbb{Z}^3, \mathbb{C}^8, w)$ for every $p \in (1, \infty)$. ■

For the important case of the symmetric weight $w(x) = e^{r|x|}$, we obtain the following corollary of Theorem 21.

Corollary 22 *Let λ be an eigenvalue of $\mathbb{D} : l^p(\mathbb{Z}^3, \mathbb{C}^8) \to l^p(\mathbb{Z}^3, \mathbb{C}^8)$. If one of the conditions*

(i) $\lambda \in (e\Phi^{\sup} - mc^2, e\Phi^{\inf} + mc^2)$ *and*

$$0 < r < \cosh^{-1}\left(\frac{m^2c^4 + 6\hbar^2c^2 - \max\left\{(e\Phi^{\inf} - \lambda)^2, (e\Phi^{\sup} - \lambda)^2\right\}}{6\hbar^2c^2}\right),$$

(ii) $\lambda > e\Phi^{\sup} + \sqrt{12\hbar^2c^2 + m^2c^4}$ *and*

$$0 < r < \cosh^{-1}\left(\frac{(e\Phi^{\sup} - \lambda)^2 - m^2c^4 - 6\hbar^2c^2}{6\hbar^2c^2}\right),$$

(iii) $\lambda < e\Phi^{\inf} - \sqrt{12\hbar^2c^2 + m^2c^4}$ *and*

$$0 < r < \cosh^{-1}\left(\frac{(e\Phi^{\inf} - \lambda)^2 - m^2c^4 - 6\hbar^2c^2}{6\hbar^2c^2}\right),$$

is satisfied, then every λ-eigenfunction of the operator $\mathbb{D}$ belongs to the space $l^p(\mathbb{Z}^3, \mathbb{C}^8, e^{r|x|})$ for every $p \in (1, \infty)$.

5 The square-root Klein-Gordon operator

5.1 The essential spectrum

Here we consider the square-root Klein-Gordon operator on $l^2(\mathbb{Z}^n)$, that is the operator

$$K = \sqrt{c^2\hbar^2\Gamma + m^2c^4} + e\Phi$$

where $m > 0$ is the mass of the particle, $\hbar > 0$ is Planck's constant, $c > 0$ the light speed, $\Phi \in SO(\mathbb{Z}^n)$ a scalar potential, and

$$\Gamma = \sum_{j=1}^{n} (2I - V_{e_j} - V^*_{e_j})$$

is the discrete Laplacian on $\mathbb{Z}^n$. The operator $K_0 := \sqrt{c^2\hbar^2\Gamma + m^2c^4}$ is understood as the pseudodifference operator with symbol

$$k(\tau) = \sqrt{c^2\hbar^2\hat{\Gamma}(\tau) + m^2c^4} \in \mathcal{S},$$

where $\hat{\Gamma}(\tau) = \sum_{j=1}^{n}(2 - \tau_j - \tau_j^{-1})$ at $\tau = (\tau_1, \ldots, \tau_n)$. Let

$$\tilde{\Gamma}(\varphi) := \hat{\Gamma}(e^{i\varphi}) = \sum_{j=1}^{n}(2 - 2\cos\varphi_j), \quad \varphi = (\varphi_1, \ldots, \varphi_n) \in [0, 2\pi]^n.$$

Every limit operator of K is unitarily equivalent to an operator of multiplication by a function of the form

$$\tilde{K}^g(\varphi) = \sqrt{c^2\hbar^2\tilde{\Gamma}(\varphi) + m^2c^4} + e\Phi^g \quad \text{with} \quad \Phi^g \in \mathbb{R}$$

acting on $L^2([0, 2\pi]^n)$. Thus,

$$\operatorname{spec} K^g = \bigcup_g [mc^2 + e\Phi^g, \sqrt{4nc^2\hbar^2 + m^2c^4} + e\Phi^g],$$

where the union is taken with respect to all sequences g tending to infinity such that the partial limit $\Phi^g := \lim_{m\to\infty} \Phi(g(m))$ exists. Consequently,

$$\operatorname{sp}_{ess} K = [mc^2 + e\Phi^{\inf}, \sqrt{4nc^2\hbar^2 + m^2c^4} + e\Phi^{\sup}]).$$

5.2 Exponential estimates of eigenfunctions

Theorem 23 *Let λ be an eigenvalue of the square-root Klein-Gordon operator K such that $\lambda \notin \operatorname{sp}_{ess} K$, and let $w = e^v$ be a weight in $\mathcal{R}^{sl}$ with $\lim_{x\to\infty} v(x) = \infty$. If one of the conditions*

(i) $\lambda > e\Phi^{\sup} + \sqrt{4n\hbar^2c^2 + m^2c^4}$ *and*

$$\limsup_{x\to\infty} \left|\frac{\partial v(x)}{\partial x_j}\right| < \cosh^{-1}\left(\frac{m^2c^4 - (e\Phi^{\sup} - \lambda)^2 + 2n\hbar^2c^2}{2n\hbar^2c^2}\right), \qquad (34)$$

(ii) $\lambda < e\Phi^{\inf} - \sqrt{4n\hbar^2c^2 + m^2c^4}$ *and*

$$\limsup_{x\to\infty} \left|\frac{\partial v(x)}{\partial x_j}\right| < \cosh^{-1}\left(\frac{m^2c^4 - (e\Phi^{\inf} - \lambda)^2 + 2n\hbar^2c^2}{2n\hbar^2c^2}\right), \qquad (35)$$

is satisfied, then every λ-eigenfunction of K belongs to $l^p(\mathbb{Z}^n, w)$ for every $p \in (1, \infty)$.

Proof. The proof proceeds similarly to the proof of Theorem 21. It is based on the following construction. Let $w = e^v \in \mathcal{R}^{sl}$. Then the limit operator $K^g_{w^\mu}$ is unitarily equivalent to the operator of multiplication by the function

$$\tilde{K}^g_{w^\mu}(\varphi) = \sqrt{c^2\hbar^2\tilde{\Gamma}(\varphi + i(\nabla v))^g + m^2c^4} + e\Phi^g$$

acting on $L^2([0, 2\pi]^n$. Hence,

$$\begin{aligned}\mathcal{L}^g_{w^\mu}(\varphi, \lambda) &:= (\tilde{K}^g_{w^\mu}(\varphi) - \lambda)\left(\sqrt{c^2\hbar^2\tilde{\Gamma}(\varphi + i(\nabla v))^g + m^2c^4} - (e\Phi^g - \lambda)\right) \\ &= c^2\hbar^2\tilde{\Gamma}(\varphi + i(\nabla v))^g + m^2c^4 - (e\Phi^g - \lambda)^2,\end{aligned}$$

and

$$\Re(\mathcal{L}^g_{w^\mu}(\varphi,\,\lambda)) = c^2\hbar^2\sum_{j=1}^{n}\left(2-\cos\varphi_j\cosh(\frac{\partial v}{\partial x_j})^g\right)+m^2c^4-(e\Phi^g-\lambda)^2.$$

Note that $\Re(\mathcal{L}^g_{w^\mu}(\varphi,\,\lambda)) \neq 0$ for every λ satisfying condition (i) or (ii). Hence, $\lambda \notin \mathrm{sp}_{ess}\, K_{w^\mu}$ for every $\mu \in [0,\,1]$. Thus, by Theorem 11, every λ-eigenfunction belongs to the space $l^p(\mathbb{Z}^n,\,w)$ for all $p \in (1,\infty)$. ∎

Specifying the weight in the previous theorem as $w(x) = e^{r|x|}$, we obtain the following.

Theorem 24 *Let λ be an eigenvalue of K such that $\lambda \notin \mathrm{sp}_{ess}\, K$. If one of the conditions*
(i) $\lambda > e\Phi^{\sup} + \sqrt{4n\hbar^2c^2+m^2c^4}]$ and

$$0 < r < \cosh^{-1}\left(\frac{m^2c^4-(e\Phi^{\sup}-\lambda)^2+2n\hbar^2c^2}{2n\hbar^2c^2}\right),$$

(ii) $\lambda < e\Phi^{\inf} - \sqrt{4n\hbar^2c^2+m^2c^4}$ and

$$0 < r < \cosh^{-1}\left(\frac{m^2c^4-(e\Phi^{\inf}-\lambda)^2+2n\hbar^2c^2}{2n\hbar^2c^2}\right)$$

is satisfied, then every λ-eigenfunction of K belongs to the space $l^p(\mathbb{Z}^n,\,e^{r|x|})$ for every $p \in (1,\,\infty)$.

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