Mathematical Models and Polyhedral Studies for Integral Sheet Metal Design

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Abstract

Using the new manufacturing technique linear flow splitting it is possible to produce branched sheet metal products containing several chambers out of one piece of sheet metal. This yields a wide variety of possible products. Even if the geometry of a profile is given, it is open how to produce it, that is how to unroll it. With every additional branch the number of possible ways to produce a profile increases exponentially. In this paper we present how the problem of finding a valid unrolling that approximates a given geometry as close as possible can be modelled as a discrete optimization problem. In order to be able to solve the model in reasonable time, further modifications are necessary. In this context we give the complete convex hull description of some substructures of the underlying polyhedron. Moreover, we introduce a new class of facet-defining inequalities that represent connectivity constraints for the profile. Finally, we will show how these inequalities can be separated in polynomial time.

1 Introduction

We deal with the optimization of the production of branched sheet metal products, as a new forming technique for sheet metal gives rise to a wide variety of possible products and possible ways to produce them.

In many everyday products as well as industry applications sheet metal is used as underlying material. It is one of the most commonly used semi-finished products in metalworking. One of the main advantages is the ability to be formed and shaped up to high deformation degrees. In order to give sheet metal the needed rigidity, branches like stringers, which are often used in aviation industry, can be employed.

Up to now, branched sheet metal components were obtained either by joining processes like glueing or welding, or by heating the material. These techniques have some disadvantages, for instance heating up sheet metal changes the properties of the material, and branched components obtained by joining techniques are always instable at the connecting piece. To overcome these disadvantages, a new technique for forming sheet metal, the so-called linear flow splitting, has been developed recently. It uses two different pairs of rolls, one pair of obtuse angled splitting rolls and a pair of supporting rolls. The latter apply high pressure on the sheet metal so that the splitting rolls can increase the surface of the band edge, thus forcing the band edge to develop branches, which we will call flanges. The splitting process is depicted in Figure 1. The flanges can be further proceeded in order to get a more

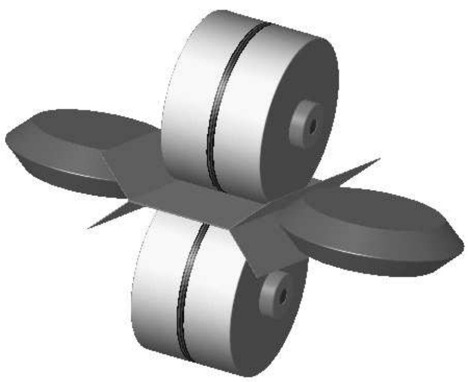


Figure 1: Linear flow splitting process

complex profile. For more information on the technical background on linear flow splitting technique, see [GVJ07]. We will also not give a detailed explanation of other sheet forming techniques which will appear in this paper, but we will try to point out the basic ideas that are necessary to follow the paper, and give references for the interested reader. To the best of our knowledge, there exists no literature on the topic of how to optimize the production of branched sheet metal products. This is due to the fact that the technology of linear flow splitting itself is very new.

The aim of this paper is to show how the problem of finding the optimal (we will specify what we mean by optimal in the following section) way to produce a branched sheet metal profile using linear flow splitting can be modeled as a mixed integer problem (MIP), and point out how it can be solved in reasonable time.

To this end, we proceed as follows. In Section 2 we will explain in more detail what possibilities there are for producing branched sheet metal product, and we will state the problem formally. The resulting MIP model is presented in Section 3. In order to be able to solve the model for problems in realistic size, we improve some of the formulations of the model. This is done in Section 4. Here the complete convex hull description of some substructures of the underlying polyhedron is given. Moreover, we introduce a new class of facet-defining inequalities that represent connectivity constraints for the profile. Since there are exponentially many of these inequalities, we will show they can be separated in polynomial time. We conclude the paper by indicating in which direction further research can go in Section 5

2 Design plans for branched sheet metal products - where discrete optimization comes into play

Before setting up the model, we want to give a more detailed insight on what the problem is about.

The input

We are given a profile consisting of one or more channels and either no or some free flanges, see Figure 2. We assume that the profile is optimized in terms of some properties, e.g. in terms of weight, stiffness or heat transfer. Thus each line segment of the profile comes with a predefined length and a suggested thickness. The problem of how such a profile can be obtained is an interesting topic on its own which we will not deal within the scope of this paper. However, the interested reader is encouraged to consult [FF08] and [B06] for more information. Moreover, we know what the product is used for in practice, that is, we know the load of each part of the profile. Thus, using analysis from mechanical engineering, we can deduce which production technologies are preferable for which part of the profile, where we will now explain what production technologies exist.

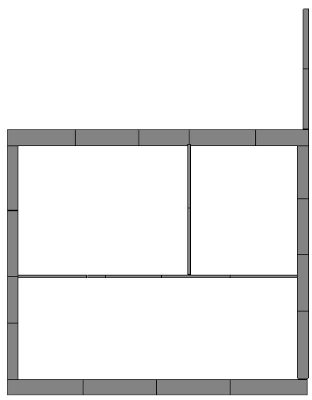


Figure 2: Profile with three channels and one free flange

Technologies

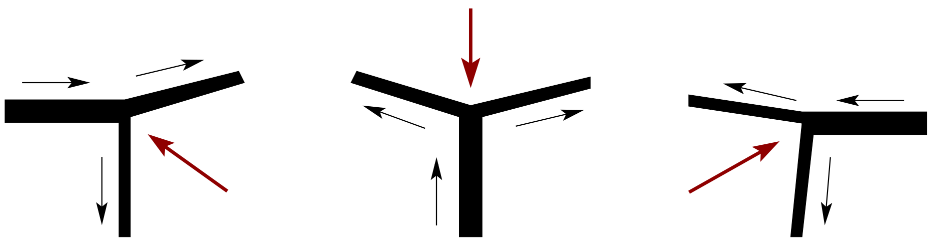
Each intersection point of the profile can be obtained by a number of technologies. In the course of this paper, we will only consider corners and t-junctions, but our results can easily be generalized to intersection points of higher degree. Two ends of sheet metal can be joined by various techniques, for example by laser weldings or by glueing, [S03]. A piece of sheet metal can be processed by a roll forming process where rolls are used to bend sheet metal to a given angle [T06]. Finally, the newly invented technique of linear flow splitting can be used to obtain two flanges as explained in the previous section. Combining these techniques, we obtain the following possibilities for producing line segments, corners and t-junctions.

T-junctions

There are ten different possibilities to obtain a construction where three line segments meet. One can perform

- linear flow splitting, from any of the three sides, see Figure 3,
- roll forming and one joining operation, see Figure 4,
- one joining operation, see Figure 5(a) and 5(b), or
- two joining operations, see Figure 5(c)

Note that we distinguish between technologies where the direction of the flow of the sheet metal differs even though the technologies look alike (for instance in the joining left and bending counterclockwise resp. clockwise, see Figure 4(a) resp. 4(b)). The reason for it is that it does indeed make a difference for the product, as the processed material behaves differently depending on where the roll is applied and where the forces are operating.



(a) Splitting from the right hand side

(b) Splitting from above

(c) Splitting from the left hand side

Figure 3: Producing t-junctions using linear flow splitting

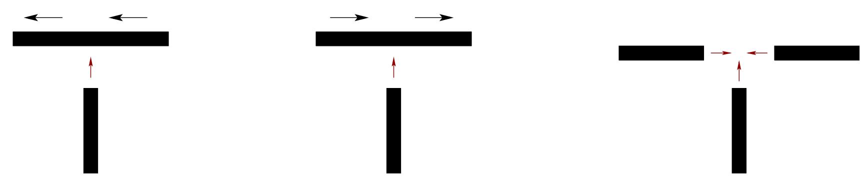
(a) Joining left, bending counterclockwise

(b) Joining left, bending clockwise

(c) Joining right, bending counterclockwise

(d) Joining right, bending clockwise

Figure 4: Bending and Joining



(a) Flow left, join down

(b) Flow right, join down

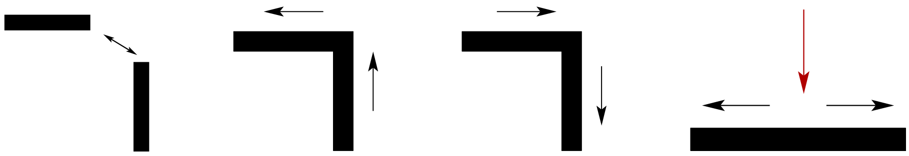
(c) Join three segments

Figure 5: Joining

Corners

Corners can be constructed by

- joining two ends , see Figure 6(a), or by
- a roll forming process, see Figure 6(b) and 6(c).



(a) Joining

(b) Bending counterclockwise

(c) Bending clockwise

(d) Starting point

Figure 6: Corners and straight line segments

Straight line segments

Similarly, any part of a straight line segment can be obtained by joining two pieces of sheet metal together, or by letting the sheet metal flow. Therefore we introduce a discrete number of points at every line segment where a technology decision has to take place. A special case is the case where the sheet metal "starts flowing", that is, the part of the sheet metal that is not processed. We illustrate this circumstance by supposing that this is the point of the profile where the sheet metal flow starts, see Figure 6(d). Later on we will see that in the mixed integer model, corners and points on straight line segments can be treated the same.

Goal of optimization

Deciding on a technology for an intersection point (where from now on we also count the points we introduced on straight line segments as intersection points) has an influence on the thickness of the neighboring line segments, as bending conserves the thickness, but linear flow splitting changes it. As an example, two different unrollings for the profile given in Figure 2 are shown in Figure 7.

So what we are looking for is a way to produce the component such that the suggested thickness is best approximated. Moreover, we want to chose the technologies in such a way that they agree as far as possible with the analysis from the

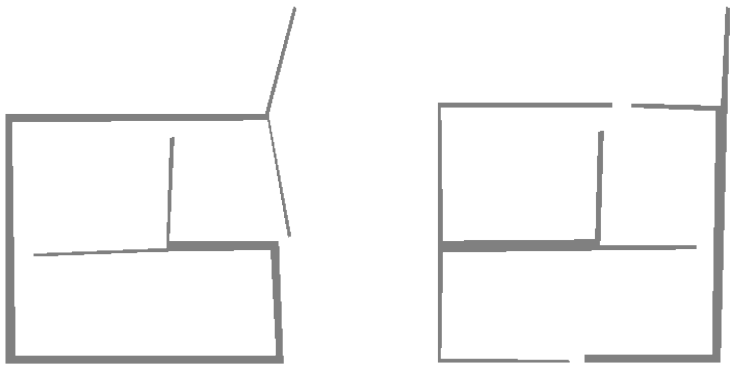


Figure 7: Two unrollings

mechanical engineers. More precise, for each intersection point and each technology, we are given an estimation on how appropriate the technology is for the intersection point. In general, the splitting techniques will be preferred over joining operations due to the disadvantages of the latter as mentioned in the introduction, but we willingly leave the precise evaluation of the technologies in a specific loading case to the engineers and just use them as an input for the optimization process.

3 Model

We will now show how the problem defined in the previous section can be modeled as a mixed integer program. To this end, Section 3.1 introduces a graph which we will use to set up the model. Section 3.2 then gives the mixed integer formulation modeling the problem of finding an unrolling. The model resembles a directed steiner tree formulation with some additional constraints.

3.1 Construction of the graph

The profile can be described as a set of intersection points $IP = \{i_1, \ldots, i_n\}$ and line segments $L = \{l_1, \ldots, l_m\}$ where each $l = \{u, v\}, u, v \in IP$ for all $l \in L$. Moreover, for each line segment $i = \{u, v\}$ we have a length l_i and a desired thickness t_i. We can interpret this as an undirected graph G with nodes set IP and edge set L.

Let $D = (V, E)$ denote the graph we will use for the optimization. It is directed and can be deduced from G as follows. The node set V consist of

- a source node q and a sink node s,
- intersection nodes $I = IP$, and
- joining nodes $J = \{u_l : u \in I, \quad l \in \delta_G(u)\}$.

The set of arcs A is given by

- starting arcs $Q := \{(q, i) : i \in I \text{ with } \delta_G(i) = 2\}$,
- sink arcs $S := \{(j, s) : j \in J\}$
- technology arcs $N := \{(u, u_l), (u_l, u) : u \in I, l \in \delta_G(u)\}$, and
- material arcs $M := \{(u_l, v_l),\ (v_l, u_l) : u, v \in I, \exists l \in L \text{ with } l = (u, v)\}$.

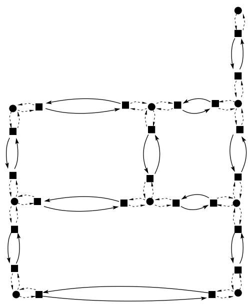


Figure 8: Graph

In Figure 8, the graph corresponding to the profile introduced in the previous section is given, omitting starting arcs, sink arcs, source and sink node due to readability. Note that for each line segment l in the profile there are two joining nodes that are associated with it. Thus, for a subset $K \subseteq L$ of line segments we can define $J(K)$, the set of all joining nodes that are associated with some $l \in K$, or more formally, $J(K) := \{j \in J \mid \exists u \in U, l \in K \text{ with } j = u_l\}$. Later on we will also need, for a given set of line segments, the set of all intersection nodes v such that v corresponds to an intersection points that is adjacent to one or more of the line segments. Thus, for a subset $K \subseteq L$ we define $I(K) := \{v \in I \mid \exists l = \{i, j\} \in K \text{ with } v = i\}$. If necessary, we indicate that a graph G is based on the set of line segments L by $G_L = (V_L, A_L)$ If it is unambiguous, we omit the index.

We now have to find an arborescence A in D rooted at s with the following additional properties:

- The arborescence has to connect all joining nodes J, i.e., for all $j \in J$, there is a directed path from s to J.
- For each line segment $l = (u, v) \in L$ either the material arc (u_l, v_l) or (v_l, u_l) is in A_T, but not both.

Note that the first condition implies that the resulting arborescence is a Steiner arborescence with terminal set J. From this arborescence, we can deduce what the unrolling looks like by interpreting the arc configurations for each intersection node as shown in Figure 9 and 10. In addition, we need a flow on the arcs that are contained in the solution to indicate the thickness of the corresponding line segments.

For each material arc we have functions $l : M \longrightarrow \mathbb{Q}$ and $t : M \longrightarrow \mathbb{Q}$ with length and desired thickness of the corresponding line segment. We can define $l(\cdot)$ and $t(\cdot)$ for all arcs of A by setting $l(a) = t(a) = 0$ whenever $a \in S \cup N \cup Q$.

3.2 MIP model

The model introduces binary variables x_a for each arc $a \in A$, denoting whether a is contained in the solution or not. To measure the actual thickness of an arc $a \in A$, continuous variables f_a are used. Finally, another class of continuous variables s_a gives us the discrepancy between actual thickness and suggested thickness. More formally, for each $l = \{i, j\}$, $\quad s^-_{i,j} = (|f(i,j) - t_{i,j}| - f(i,j) + t_{i,j})/2$, which is greater than zero if the if arc $(i, j) \in M$ is too thin, and $s^+_{i,j} = (|f(i,j) - t_{i,j}| + f(i,j) - t_{i,j})/2$,

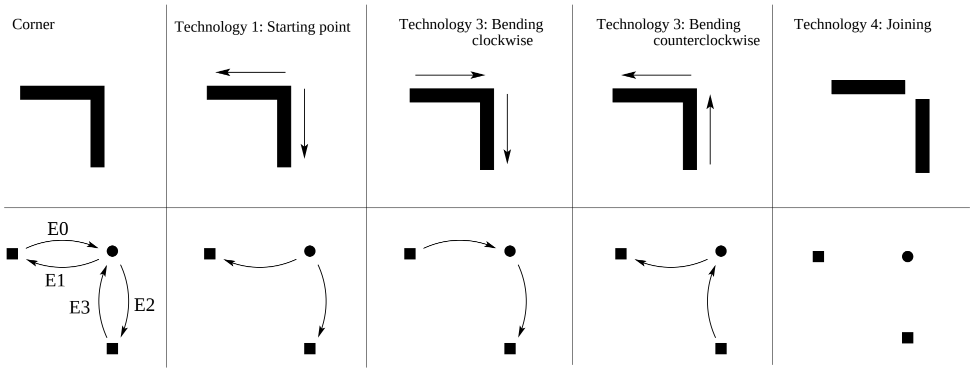


Figure 9: Technologies and arc configurations for corners

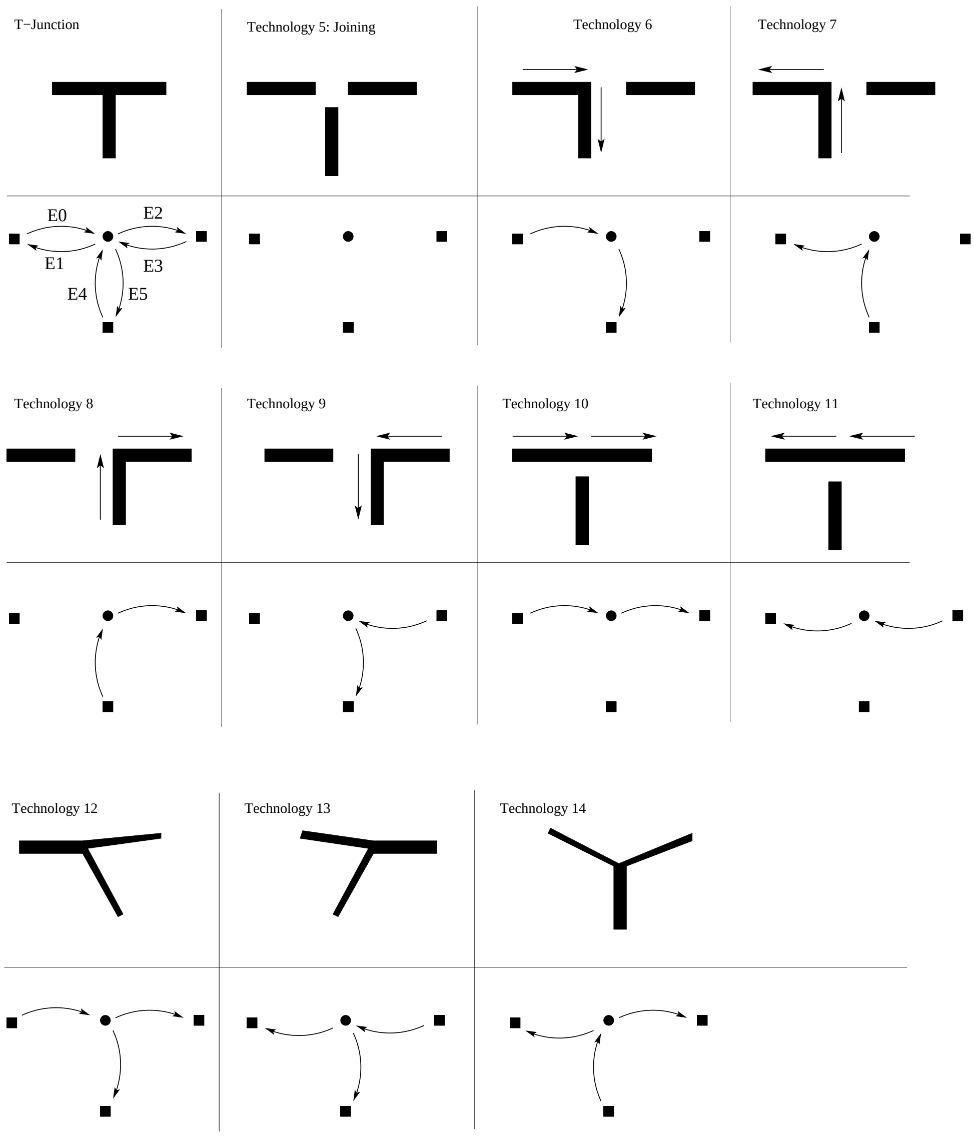


Figure 10: Technologies and arc configurations for t-junctions

which is greater than zero if it is too thick. If the sole aim of optimization is to approximate the desired thicknesses, (1) can be used as an objective function:

$$\text{minimize} \sum_{m \in M} s_m^+ + s_m^-. \tag{1}$$

We will soon see how to include the rating of the technologies in the objective function. For the moment, we want to show how we can make sure that the solution of the mixed integer program will represent a steiner arborescence with the desired properties.

On every line segment there has to be sheet metal. Thus,

$$\forall (i,j) \in M : x_{i,j} + x_{j,i} = 1 \tag{2}$$

As that the resulting graph is an arborescence, we have to ensure that each node has at most on predecessor (3) and that it doesn't contain any cycles (4).

$$\forall v \in V : \sum_{a \in \delta^-(v)} x_a \leq 1 \tag{3}$$

$$\forall W \subseteq V \setminus \{q\}, W \cap J \neq \emptyset : \sum_{a \in \delta^-(W)} x_a \geq 1 \tag{4}$$

The inequalities (5) ensure that sheet metal flows on arcs if and only if they are contained in the solution, where $\mu := \max_{e \in E} t_e$ is a big M constant. Flow is conserved on account of inequality (6).

$$\begin{aligned} &\forall a \in A : x_a \leq f_a \\ &\forall a \in A : \mu x_a \geq f_a \end{aligned} \tag{5}$$

$$\forall v \in I \cup J : \sum_{a \in \delta^-(v)} f_a - \sum_{b \in \delta^+(v)} f_b = 0 \tag{6}$$

Moreover, we need to assure that the s-variables measure the distance between actual and suggested thickness. This is done via equation (7).

$$\forall (i,j) \in M : f_{i,j} + f_{j,i} + s_{i,j}^+ + s_{i,j}^- = t_{i,j} \tag{7}$$

Note that either $f_{i,j}$ or $f_{j,i}$ equals 0. Eventually, the profile is not allowed to be build off more than c components:

$$\sum_{a \in S} x_a \leq c \tag{8}$$

To solve this model, we can make use of numerous works on polyhedra that have a similar structure. Consider for example equations (5) - (6). Together with integrality constraints for x they model a special case of the well-studied Fixed-Charge-Flow network problem whose exploration was initiated by Padberg et al in 1985 [PRW85]. A good survey on this topic is given in [LW03], where more references can be found. Another set of famous inequalities is given in (4), the so-called subtour elimination inequalities, which appear for example in the classical mixed integer formulation of the Traveling Salesman Problem. For a detailed instruction on how to cope with subtour elimination inequalities [PR91] and the references therein can be consulted. Finally note that as we are looking for a steiner arborescence, we can benefit from studies on that matter as well. An extensive study on the undirected and directed steiner tree polyhedron is given in [CR93a] and [CR93b].

Technology variables

Coming back to our original problem, it still remains to find a way to incorporate an evaluation of the different technologies in the objective function as well as to find a way to forbid or to force the use of certain technologies for certain parts of the profile. To this end, we introduce a new type of binary variables. In Section 2 we have presented all possible ways to produce corners and t-junctions. Now, for every intersection node v, we will define a set $B(v)$ of technologies that can be used for producing the part of the profile that corresponds to v. In the following we will encode the technologies according to the numbers given in Figure 9 and 10. Then clearly $B(v) \subseteq \{1, \ldots, 4\}$ if $|\delta^+(v)| = 2$ and $B(v) \subseteq \{5, \ldots, 14\}$ if $|\delta^+(v)| = 3$. The technology variables denote whether a certain technology is used at a certain point, or more formal, $y_v^t = 1$ if technology $t \in B(v)$ is used for producing the intersection point that corresponds to v, it is 0 otherwise. As we have mentioned in the previous section, the mechanical engineers provide us with an estimation on how suitable a certain technology is for a certain intersection point. We will denote this estimation by λ_v^t for every $v \in I$ and $t \in B(v)$. Let $\mathcal{B} := \{(v,t) \mid v \in I, t \in B(v)\}$ and $\mathcal{B}_K := \{(v,t) \mid v \in I(K), t \in B(v)\}$. Then the second objective function can be formulated as

$$\text{minimize} \sum_{(v,t)\in\mathcal{B}} \lambda_v^t y_v^t. \tag{9}$$

For each intersection node $v \in I$ and each technology $t \in B(v)$ some of the adjacent arcs are contained in the solution and the others are not, see Figure 9 and 10. For given $v \in I$ and $t \in B(v)$, let $N_{v,t}^1 \subset N \cap \delta(v)$ denote the set of arcs that have to be in the solution if technology t is chosen for node v, and let $N_{v,t}^2 = N \cap \delta(v) \setminus N_{v,t}^1$ denote the set of arcs that are not allowed to be in the solution if technology t is chosen for node v. Then we can make sure that the variable classes x and y are consistent by the following set of inequalities:

$$\forall (v,t) \in \mathcal{B}, a \in N_{v,t}^1 : y_v^t \leq x_a. \tag{10}$$

$$\forall (v,t) \in \mathcal{B}, a \in N_{v,t}^2 : x_a \leq 1 - y_v^t. \tag{11}$$

Finally, every intersection point is obtained by using exactly one technology:

$$\forall v \in I : \sum_{t\in B(v)} y_v^t = 1 \tag{12}$$

Now the problem of finding an optimal unrolling can be written as a mixed integer program as follows:

$$\begin{aligned}
\text{minimize } & (1) + (9) && && \text{(P)}\\
\text{subject to } & (2) - (8) \\
& (10) - (12) \\
& x_a \in \{0,1\} && \forall a \in A \\
& y_v^t \in \{0,1\} && \forall (v,t) \in \mathcal{B} \\
& s_{i,j}, f_{i,j} \in \mathbb{R}_0^+ && \forall \{i,j\} \in L
\end{aligned}$$

4 Polyhedral investigations

In this section we study substructures of the model described in Section 3 from a polyhedral point of view. We will give the convex hull description of some substructures and introduce a new class of facets that will express connectivity constraints in terms of the technology variables, and show how to separate them.

As mentioned before, we will concentrate on those parts of the model where technology variables come into play. This is done for two reasons. Firstly most of the other parts have been studied in the literature. Moreover, we are optimistic that improving those parts of the formulation were the technology variables will help to increase the overall mip solving time, as the technology variables appear directly in the objective function.

Recall that the variables x_a denoting whether arc a is in the solution or not and the technology variables have to be in accordance with each other. This is done using the inequalities given in (10) and (11). We find that for each intersection node v representing a corner we obtain $4*4 = 16$ inequalities, since there are four adjacent arcs and four possible technologies for v. Similarly, we obtain $10*6 = 6$ inequalities for each intersection node v with $|\delta^+(v)| = 3$. Note that the inequalities are weak in the sense that they only contain 2 variables. In the following we will rigorously reduce the number of inequalities by giving the complete convex hull description for intersection points.

4.1 Convex hull description for corners and straight line segments

Consider the polytope modeling the technologies for intersection points of degree 2. It is given by

$$\begin{aligned}
& (10) - (12) \\
& x_a \in \{0,1\} && \forall a \in A && \text{(P2)}\\
& y_v^t \in \{0,1\} && \forall v \in I \text{ with } |\delta^+(v)| = 2, t \in B(v)
\end{aligned}$$

Recall that the sets $N_{v,t}^1$ and $N_{v,t}^2$, i.e., the definition of which arcs have to be resp. are not allowed to be in the solution when a certain technology is chosen as well as the labeling of arcs are given in Figure 9 and 10.

Proposition 4.1 *The following inequalities give a complete description of the convex hull of (P2).*

$$\begin{aligned} \sum_{i=1}^{4} y_v^i &= 1 \\ y_v^i &\geq 0 \qquad \forall i \in \{1, \dots, 4\} \\ x_0 &= y_v^2 \\ x_1 &= y_v^1 + y_v^3 \\ x_2 &= y_v^1 + y_v^2 \\ x_3 &= y_v^3 \end{aligned} \tag{13}$$

Proof:
The only integer vertices of (P2) are given by

$$v_1 := \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 := \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_3 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } v_4 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

We have to choose exactly one technology for every corner. The edge variables x can then be deduced as indicated by (13). □

4.2 Convex hull description for t-junction

The idea for t-junctions are similar. Here, the polyhedron is given by

$$\begin{aligned} &(10) - (12) \\ &x_a \in \{0, 1\} \qquad \forall a \in A \\ &y_v^t \in \{0, 1\} \qquad \forall v \in I \text{ with } |\delta^+(v)| = 3, t \in B(v) \end{aligned} \tag{P3}$$

and the labeling of the technologies and arcs can be recalled in Figure 10.

Proposition 4.2 *The following inequalities give a complete description of the convex hull of (P3).*

$$\begin{aligned} \sum_{i=5}^{14} y_v^i &= 1 \\ y_v^i &\geq 0 \qquad \forall i \in \{5, \dots, 14\} \\ x_0 &= y_v^6 + y_v^{10} + y_v^{12} \\ x_1 &= y_v^7 + y_v^{11} + y_v^{13} + y_v^{14} \\ x_2 &= y_v^8 + y_v^{10} + y_v^{12} + y_v^{14} \\ x_3 &= y_v^9 + y_v^{11} + y_v^{13} \\ x_4 &= y_v^7 + y_v^8 + y_v^{14} \\ x_5 &= y_v^6 + y_v^9 + y_v^{12} + y_v^{13} \end{aligned} \tag{14}$$

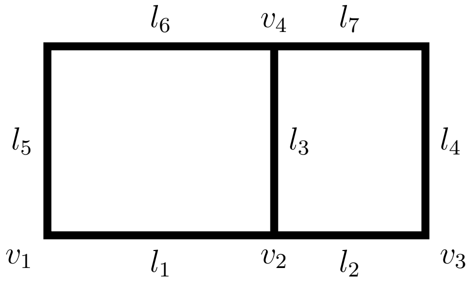


Figure 11: Profile with two channels

Proof:
The only vertices of (P3) are given by the rows of the matrix $\mathcal{W}$,

$$\mathcal{W} := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

We have to choose exactly one technology for every t-junction. The edge variables x can then be deduced as indicated by (14). $\square$

Connectivity Constraints

In Section 3 we have seen how the problem of finding a feasible unrolling can be modeled as a directed steiner tree problem with certain additional constraints. Thus, given the graph corresponding to a profile, we need to find a substructure that is connected. In our model, this has been ensured via the classical subtour elimination constraints (4) so far. In this section, we will describe a new way of formulating connectivity constraints by making use of the technology variables, and show that the corresponding inequalities are facet-defining. Thinking in terms of our application, connectivity simply means that for each part of the profile, sheet metal has to flow into it. Thus we can make use of the fact that the technology variables indicate whether or not sheet metal flows into a certain line segment or not. By formulating connectivity constraints via technology variables we hope to decrease the solution time for those instances where certain technologies are rated in the objective function.

As an example, consider the profile given in Figure 11.

The line segments l_1, l_2 and l_3 must be made out of sheet metal. Hence sheet metal has to flow into node v_1, v_3 or v_4 (by definition, v_2 cannot be a starting point), either from the source node or from another part of the profile. Thus, $y^1_{v_1} = 1$ or $y^1_{v_3} = 1$ (v_1 resp. v_3 is the staring node), or $y^3_{v_1} = 1$ or $y^2_{v_3} = 1$ (sheet metal flows into v_1 resp. v_3 from the other part of the profile, or $y^6_{v_4}, y^9_{v_4}, y^{12}_{v_4},$ or $y^{13}_{v_4} = 1$ (sheet metal flows into v_4 from the other part of the profile). This yields the inequality

$$y^1_{v_1} + y^3_{v_1} + y^1_{v_3} + y^2_{v_3} + y^6_{v_4} + y^9_{v_4} + y^{12}_{v_4} + y^{13}_{v_4} \geq 1.$$

Note that if the profile is supposed to be build out of one piece of sheet metal, i.e., $c = 1$, the inequality becomes an equality.

To formulate this class of inequalities in general, we need to define the set of all technologies that, for a given intersection node v and a given set of line segments, contains all technologies that induce flow from outside the line segments into v:

$$\tilde{T}(K, v) := \{t \in B(v) \mid y^t_v = 1 \implies x_{w,v} = 1 \text{ for some } w \in \{s\} \cup J \setminus J(K)\}.$$

Now a valid inequality is given by

$$\sum_{v \in I(K)} \sum_{t \in \tilde{T}(K,v)} y^t_v \geq 1. \tag{15}$$

However, we can obtain an even stronger inequality. To illustrate this, consider the profile given in Figure 11 again, except for that this time, $K = \{l_1, l_2\}$. Then, if the sheet metal is send into K only via v_2, it has to flow into l_1 and l_4 at the same time, i.e., we have to use the splitting technology that splits from below. In this example, we have to claim

$$y^1_{v_1} + y^3_{v_1} + y^1_{v_3} + y^2_{v_3} + y^{14}_{v_2} \geq 1.$$

In general, we need to define the set $T(K, v)$ that, given an intersection node v and a set of line segments K, sends flow into each connected component of $G_K \setminus \{v\}$, where $G \setminus \{v\}$ is defined as $G \setminus \{v\} := (V \setminus \{v\}, A \setminus \{(v,w) \mid w \in J\} \cup \{(w,v) \mid w \in J \cup \{s\}\}$. To this end, let $(C^i_{K,v})_{i \in I_{K,v}} = (VC^i_{K,v}, AC^i_{K,v})_{i \in I_{K,v}}$ denote the set of all connected components of $G \setminus \{v\}$, where $I_{K,v}$ denotes the index set of all connected components of $G \setminus \{v\}$. Then $T(K, v)$ can be written as

$$\begin{aligned} T(K, v) := \{t \in B(v) \mid y^t_v = 1 \implies x_{w,v} = 1 \text{ for some } w \in \{s\} \cup J \setminus J(K) \\ \text{and for all } i \in I_{K,v} : x_{v,w} = 1 \text{ for some } w \in J(K) \cap VC^i_{K,v}\} \end{aligned}$$

Proposition 4.3 *For each subset $K \subseteq L$ of line segments, the inequality*

$$\sum_{v \in I(K)} \sum_{t \in T(K,v)} y^t_v \geq 1 \tag{16}$$

is valid for (P).

Proof:
Assume that for a given set of line segments K, the inequality is not valid, i.e., there is a feasible solution $\hat{z} := (\hat{x}, \hat{y}, \hat{f}, \hat{s})$ of (P) with

$$\sum_{v \in I(K)} \sum_{t \in T(K,v)} \hat{y}^t_v < 1 \implies \sum_{v \in I(K)} \sum_{t \in T(K,v)} \hat{y}^t_v = 0. \tag{17}$$

By the definition of $T(K, v)$, for all $v \in I(K)$ there is at least one connected component $C^{i_0}_{K,v}$ such that $\hat{z}$ does not send flow into from the outside of K. Let $v_1 \in I(K)$

denote any intersection node, and let $C^{i_0}_{K,v_1}$ denote the corresponding connected component, i.e., $\sum_{w\in J(K)\cap VC^{i_0}_{K,v_1}} x_{v_1,w} = 0$. Then there must exist another intersection node $v_2 \in I(K)\setminus\{v_1\}$ that send flow into $C^{i_0}_{K,v_1}$, i.e., $\sum_{w\in J(K)\cap VC^{i_0}_{K,v_1}} x_{v_2,w} \geq 1$, because otherwise (4) would be violated for $W = VC^{i_0}_{K,v_1}$. Note that $G_K \setminus \{v_2\}$ has to consist of at least two connected components, as otherwise $\sum_{t\in\tilde{T}(K,v_2)} \hat{y}^t_{v_2} = \sum_{t\in T(K,v_2)} \hat{y}^t_{v_2} \geq 1$ and (17) would not hold. Let $C^{v_2}_K$ denote one of the connected components that v_2 does not send flow into. Then again, there has to exist an intersection node v_3 that sends flow into $C^{v_2}_K$ as otherwise (4) would be violated for $W = VC^{i_0}_{K,v_2}$. Now $v_1 \neq v_3$ because if v_1 did send flow into $C^{v_2}_K$, it would also send flow into $C^{v_1}_K$ (via v_2) which contradicts the definition of $C^{v_1}_K$. If we continue this train of thought, we obtain an infinite chain $v_1 \neq v_2 \neq v_3 \neq \dots$ of intersection nodes which is a contradiction to the fact that the profile is finite. □

4.3 Facets of P_y

Let P_y denote the projection of P onto the space of technology variables, i.e.,

$$P_y := \{y \in \{0,1\}^{|\mathcal{B}|} : \exists x, f, s \in \{0,1\}^m \times \mathbb{R}^m \times \mathbb{R}^m \text{ such that } (x,y,f,s) \in (P)\},$$

and define the technology polyhedron as $P^+_y := P_y + \mathbb{R}^m_+ \cup [0,1]^m$.

Now we identify a class of facets of P^+_y:

Proposition 4.4 *If $K \subseteq L$ is connected, then*

$$\sum_{v\in I(K)} \sum_{t\in T(K,v)} y^t_v \geq 1$$

is a facet-defining inequality of P^+_y.

Proof

Assume that the face induced by (16) is contained in a non-trivial face induced by $b^T y \geq \beta$. We will prove that in this case $b^T y \geq \beta$ must be a non-negative multiple of (16), i.e., the faces are in fact identical.

To this end, we will first show that for every $K \subseteq L$ and for every $v_0 \in I(K), t_0 \in T(K, v_0)$ we can always find a feasible solution $\hat{y} \in P_y$ such that $\hat{y}^{t_0}_{v_0} = 1$ and $\hat{y}^t_v = 0$ for all $v \in I(K) \setminus \{v_0\}, t \in T(K,v) \setminus \{t_0\}$. This will imply that

i) $\emptyset \neq \{y \in P^+_y : \sum\limits_{v\in I(K)} \sum\limits_{t\in T(K,v)} y^t_v = 1\} \subseteq \{y \in P^+_y : b^T y = \beta\}$, and

ii) $b^{t_i}_{v_i} = b^{t_j}_{v_j}$ for $v_i, v_j \in I(K),\ t_i \in T(K,v_i),\ t_j \in T(K,v_j)$,

as $b^{t_i}_{v_i} = b^T e_{t_i,v_i} = \beta = b^T e_{t_j,v_j} = b^{t_j}_{v_j}$.

Indeed, let $\tilde{y} \in P^+_y$ be a solution with $\tilde{y}^{t_0}_{v_0} = 1$ and $\tilde{y}^{t_k}_{v_k} = 1$ for some $v_0, v_k \in I(K),\ t_0 \in T(K,v_0),\ t_k \in T(K,v_k)$ From this, we can construct a feasible solution $\hat{y} \in P^+_y$ with $\hat{y}^{t_k}_{v_k} = 0$ as follows. Let $l_0 = (v_0, v_1)$ and $l_k = (v_{k-1}, v_k)$ denote the line segments that are adjacent to v_0 resp. v_k in G such that there is a path $P :=\ l_0{=}(v_0,v_1),\ l_1{=}(v_1,v_2),\ \dots,\ l_k{=}(v_{k-1},v_k)$ in G with $v_1, v_2, \dots, v_{k-1} \neq v_0$ and $v_1, v_2, \dots, v_{k-1} \neq v_k$. Such a path exists as K is connected and by the definition of $T(K, v_0)$ resp. $T(K, v_k)$.

Now for every intersection node on the path, $v_i \in \{v_1, v_2, \ldots, v_{k-1}\}$, choose the technology as follows. If v_i represents a corner, we set $y_{v_i}^{t_i} = 1$ for the technology t_i that sends flow from l_i to l_{i+1}. If v_i represents a t-junction let l_3 denote the third line segment that is adjacent to v_i apart from l_{i-1} and l_i. If $\tilde{y}_{v_i}^{t_i} = 1$, where t_i sends flow into l_3, we set $\hat{y}_{v_i}^{t_j} = 1$, where t_j is the splitting technology that sends flow from l_{i-1} into l_i and l_3. Otherwise we set $\hat{y}_{v_i}^{t_l} = 1$, where t_l is the technology that sends flow from l_{i-1} into l_i and joins l_3 to it. Finally, we set $\hat{y}_{v_k}^{\hat{t_k}} = 1$ where $\hat{t_k}$ denotes the technology that sends flow from the connected component of $G \setminus \{v_k\}$ that contains l_k into every connected component of $G \setminus \{v_k\}$. All other technologies remain unchanged, i.e., $\hat{y}_v^t = \tilde{y}_v^t$ for $v \notin P$. Note that $\tilde{y}$ was a feasible solution and we ensured that we didn't cut off any parts that were connected in $\tilde{y}$. Therefore, by choosing the technologies in the suggested way, we obtain an arborescence rooted at v_0 that has v_k as a leave and spans all joining nodes in $J(K)$. Moreover, we are only setting those $\hat{y}_v^t = 1$ were $t \notin T(K, v)$, and $\hat{y}_{v_k}^{t_k} = 0$, thus

$$\sum_{v \in I(K)} \sum_{t \in T(K,v)} \hat{y}_v^t \leq \sum_{v \in I(K)} \sum_{t \in T(K,v)} \tilde{y}_v^t - 1.$$

By applying this procedure to every $v \in I(K) \backslash \{v_0\},\ t \in T(K, v) \backslash \{t_0\}$, we obtain the above claim.

Finally we will show that

iii) $b_{v_i}^{t_i} = 0$ for all (v_i, t_i) with $t_i \notin T(K, v_i)$ for $v_i \in I(K)$.

This follows from the fact that for any $\hat{y} \in F_a^K$ (which exists by i), $\hat{y} + e_{v_i,t_i}$ is also contained in F_a^K since $(v_i, t_i) \notin I(K) \times T(K, v_i)$. Thus $b^T \hat{y} = \beta = b^T(\hat{y} + e_{v_i,t_i}) = b^T \hat{y} + b^T e_{v_i,t_i}$, implying that $b_{v_i}^{t_i} = 0$, which completes the proof. □

4.4 Separation

Since there are exponentially many of the inequalities of (16), we need to find a suitable way to separate them. Discarding the connectivity constraint (4) and the integrality constraints, let (x^*, y^*, f^*, s^*) be an optimal solution of the relaxed problem. Here we can use the same idea that is used to solve the standard separation problem of finding violated subtour inequality constraints, namely to compute minimum cuts [L86]. To this end, we need a new graph that, roughly speaking, is a directed version of the line graph of G with some additional staring arcs

More formally, the *vertices* consists of

- the source node s, and
- $\tilde{V} = L$, i.e., for each line segment l of the profile, we introduce a node.

The set of arcs $\tilde{A}$ consists of

- $\tilde{A}_1 = \{(s, l) \mid l = (\{u, v\} \in L \text{ and } |\delta(u)| = 2 \text{ or } |\delta(v)| = 2\}$
- $\tilde{A}_2 = \{(l_1, l_1) \mid l_1, l_2 \in L \text{ and } l_1 \text{ and } l_2 \text{ are adjacent in } G\}$

The graph corresponding to the profile given in Figure 11 is given in Figure 12.

For $\tilde{a} = (l_1, l_2) \in \tilde{A}_2$, let v_{l_1,l_2} denote the intersection node that is associated with their common intersection point. Moreover, let $T(l_i, l_j)$ denote the set of all

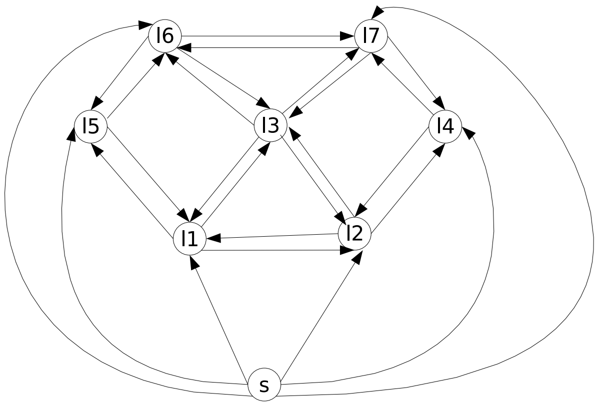


Figure 12: Separation graph

technologies that send flow from l_i to l_j for two adjacent line segments l_i and l_j, i.e.,

$$T(l_i, l_j) := \{t \in B(v_{l_i,l_j}) \mid y^t_{v_{l_i,l_j}} = 1 \implies x_{v_{l_i,l_j},w_j} = x_{w_i,v_{l_i,l_j}} = 1 \text{ for } w_i \in J(\{l_i\}), w_j \in J(\{l_j\})\}$$

Then the weight of $\tilde{a} = (l_1, l_2) \in \tilde{A}_2$ is defined as

$$c_{\tilde{a}} := \sum_{t \in T(l_1,l_2)} y^{t*}_{v_{l_1,l_2}}$$

For $\tilde{a} = (s, l = (u_l, v_l)) \in \tilde{A}_1$, the weight is set to $c_{\tilde{a}} := y^{1^*}_{u_l} + y^{1^*}_{v_l}$. Now an inequality of the type (16) is violated for some $K \subseteq L$ if and only if the (s, K) cut has weight smaller than one.
In order to find such a cut, minimal cut algorithms as described for instance in [CGKLS97], [HO94] or [S03] can be consulted. If the minimal cut has a value of greater or equal to one, we know that (16) is not violated. Otherwise the set of nodes K of the (s, K)-cut represents a set of line segments for which (16) is violated.

5 Conclusions and outlook

In this article, we showed how a new problem arising from the world of mechanical engineering can be modeled by means of mixed integer programming. Some parts of the model we developed have been well-studied in the literature. However, certain requests from mechanical engineering necessitates the development of further variables which led to a new kind of facet-defining inequalities representing connectivity constraints.

In our ongoing research, we want to concentrate on integrating manufacturing restrictions into the model. As the profiles have to be produced on existing machines, there are a number of manufacturing restrictions that have to be taken into account, a fact that we neglected in this article. For instance, the machine can only process sheet metal of certain width, which will result in diameter restrictions on the tree. Other restrictions induce bounds on the degree and the depth of certain subgraphs of the tree. Here it will be interesting to analyze the altered polyhedron, but also to study how algorithms from graph theory can be incorporated in order to to solve the problem.

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