Local and global regularity in time dependent viscoplasticity

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Summary. In this note we discuss results recently obtained by the authors on local and global regularity for quasistatic initial-boundary value problems from viscoplasticity and prove a refinement of these results. The initial-boundary value problems considered belong to a general class with monotone constitutive equations modelling inelastic materials showing kinematic hardening. A standard example is the Melan-Prager model. Our result is that the strain/stress/internal variable fields have $H^{1+1/3-\delta}/H^{1/3-\delta}/H^{1/3-\delta}$ regularity up to the boundary. After discussion of this result we prove that in the case of generalized standard materials the same regularity can be obtained under weaker assumptions on the regularity in time of the given data.

1 Introduction and setting of the problem

In this note we present interior and boundary regularity results for solutions of quasistatic initial-boundary value problems from viscoplasticity. The models we study use constitutive equations with internal variables to describe the deformation behavior of inelastic metals at small strain.
We consider constitutive equations of monotone type introduced in [1], which generalizes the class of generalized standard materials introduced by Halphen and Nguyen Quoc Son [12]. The class includes the well known models of Prandtl-Reuss, Norton-Hoff and Melan-Prager [15, 16, 20, 22], to mention a few. In this work we deal only with models of monotone type, for which the associated free energy is a positive definite quadratic form. Materials showing linear kinematic hardening can be described by such models. This excludes the models of Prandtl-Reuss and Norton-Hoff, but includes the model of Melan-Prager. For a larger number of examples of constitutive equations used in engineering and for details on the monotone type class we refer to [1].

Setting of the problem

Let $\Omega \subseteq \mathbb{R}^3$ be an open bounded set, the set of material points of the solid body. If not otherwise stated we assume that Ω has C^1–boundary. By T_e we denote a positive number (time of existence), which can be chosen arbitrarily large. $\mathcal{S}^n$ denotes the set of symmetric $n \times n$–matrices. Unknown are the displacement $u(x,t) \in \mathbb{R}^3$ of the material point x at time t, the Cauchy stress tensor $T(x,t) \in \mathcal{S}^3$ and the vector $z(x,t) \in \mathbb{R}^N$ of internal variables. The model equations of the problem are

$$-\mathrm{div}_x T(x,t) = b(x,t), \tag{1}$$

$$T(x,t) = \mathcal{D}[x]\big(\varepsilon(\nabla_x u(x,t)) - Bz(x,t)\big), \tag{2}$$

$$\begin{aligned}\partial_t z(x,t) &\in g\big(x, -\nabla_z \psi\big(x, \varepsilon(\nabla_x u(x,t)), z(x,t)\big)\big)\\ &= g\big(x, B^T T(x,t) - L[x]\, z(x,t)\big),\end{aligned} \tag{3}$$

which must be satisfied in $\Omega \times (0, T_e)$. The initial condition and the Dirichlet boundary condition are

$$z(x,0) = z^{(0)}(x), \quad x \in \Omega, \tag{4}$$

$$u(x,t) = \gamma(x,t), \quad (x,t) \in \partial\Omega \times [0, T_e)\,. \tag{5}$$

Here $\nabla_x u(x,t)$ denotes the 3×3–matrix of first order partial derivatives. The strain tensor is

$$\varepsilon(\nabla_x u(x,t)) = \frac{1}{2}\big(\nabla_x u(x,t) + (\nabla_x u(x,t))^T\big) \in \mathcal{S}^3,$$

with the transposed matrix $(\nabla_x u)^T$. For every $x \in \Omega$, the elasticity tensor $\mathcal{D}[x] : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric mapping, which is positive definite, uniformly with respect to x. The linear mapping $B : \mathbb{R}^N \to \mathcal{S}^3$ assigns to the vector $z(x,t)$ the plastic strain tensor $\varepsilon_p(x,t) = Bz(x,t)$. The free energy is

$$\psi(x, \varepsilon, z) = \frac{1}{2}\big(\mathcal{D}[x](\varepsilon - Bz)\big) \cdot (\varepsilon - Bz) + \frac{1}{2}(L[x]z) \cdot z, \tag{6}$$

where $L[x]$ denotes a symmetric $N \times N$–matrix, which is positive definite, uniformly with respect to $x \in \Omega$. The assumptions for $\mathcal{D}$ and L imply that ψ is a positive definite quadratic form with respect to (ε, z). Finally, we require that the nonlinear mapping $g : \Omega \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ satisfies

$$0 \in g(x, 0), \tag{7}$$

$$0 \le (z_1 - z_2, y_1 - y_2), \tag{8}$$

for all $x \in \Omega,\ z_i \in \mathbb{R}^N,\ y_i \in g(x, z_i),\ i = 1, 2$. This means that g is monotone with respect to z. Given are the volume force $b(x,t) \in \mathbb{R}^3$, the boundary data $\gamma(x,t) \in \mathbb{R}^3$ and the initial data $z^{(0)}(x) \in \mathbb{R}^N$.

This completes the formulation of the initial-boundary value problem. (2) and (3) are the constitutive equations, which assign the stress $T(x,t)$ to the strain history $s \mapsto \varepsilon(\nabla_x u(x,s))$, $s \le t$, and which model the viscoelastic material behavior of the solid body.

Under suitable regularity assumptions on the volume force and the boundary data we show that the time derivative $\partial_t u$ belongs to $L^\infty(0,\infty;H^1(\Omega))$ and the space derivatives $\nabla_x u$ to $L^\infty(0,\infty;H^1_{\mathrm{loc}}(\Omega))$. Concerning derivatives at the boundary we prove that the tangential derivatives $\partial_\tau u$ belong to $L^\infty(0,\infty;H^1(\Omega))$, whereas for the normal derivatives we can only show a weaker result. Namely, we show that $\nabla_x u$ belongs to $L^\infty(0,\infty;H^{1/3-\delta}(\Omega))$ for every $\delta > 0$. The stress field T and the vector of internal variables z have the same regularity as the $\nabla_x u$–field.

For the time dependent problem to the Norton-Hoff and Prandtl-Reuss laws it was shown in [6] that the stress field T belongs to $L^\infty(0,\infty;H^1_{\mathrm{loc}}(\Omega))$. In [10] this result is proved again using other methods and under different assumptions on the data. We are not aware of regularity results for the normal derivatives up to the boundary in the time dependent case.

Such results exist for time independent problems. In [21] it is shown for the stationary problem of elasto-plasticity with linear hardening in two space dimensions that the strain and stress fields belong to $H^2(\Omega)$ and $H^1(\Omega)$, respectively. For a stationary power-law model in the full three-dimensional case it is proved in [13, 14] that these fields belong to $H^{3/2-\delta}(\Omega)$ and $H^{1/2-\delta}(\Omega)$, whereas in [17] it is shown for a class of time discrete models, which includes a Cosserat model, that the displacement is in $H^2(\Omega)$ and the stress field in $H^1(\Omega)$. For local regularity results in the time independent case we refer to [5, 7, 9, 23, 24, 25, 26] and to [11] for a survey on other results.

We consider coefficients and constitutive functions, which depend on x. Our results thus generalize and extend the local regularity results for constant coefficients in the time dependent case obtained in [18].

2 Regularity for materials with monotone constitutive equations

We use the following notations. For functions w defined on $\Omega \times [0,\infty)$ we denote by $w(t)$ the mapping $x \mapsto w(x,t)$, which is defined on Ω. The space $W^{m,p}(\Omega,\mathbb{R}^k)$ with $p \in [1,\infty]$ consists of all functions in $L^p(\Omega,\mathbb{R}^k)$ with weak derivatives in $L^p(\Omega,\mathbb{R}^k)$ up to order m. We set $H^m(\Omega,\mathbb{R}^k) = W^{m,2}(\Omega,\mathbb{R}^k)$. For the space of linear, symmetric mappings from a vector space V to itself we write $\mathcal{LS}(V,V)$.

The basis for our regularity results is the existence theorem for the initial-boundary value problem (1) – (5), which is proved in [2] in the case where the coefficient functions $\mathcal{D}$, L and the constitutive function g are independent of x. It is shown in [19] that the proof generalizes immediately to x–dependent coefficient and constitutive functions satisfying some natural conditions. In the statement of this general existence theorem given below we use that for fixed t the equations (1), (2) and (5) together form an elliptic boundary value problem, the Dirichlet problem of linear elasticity theory. The data of this problem are $b(t)$, $z(t)$ and $\gamma(t)$. For $(b(t), z(t), \gamma(t)) \in L^2(\Omega) \times L^2(\Omega) \times H^1(\Omega)$ this problem has a unique weak solution $(u(t), T(t)) \in H^1(\Omega) \times L^2(\Omega)$. The existence theorem is

Theorem 1 (Existence). *Assume that the coefficient functions satisfy $L \in L^\infty(\Omega, \mathcal{S}^N)$, $\mathcal{D} \in L^\infty(\Omega, \mathcal{LS}(\mathcal{S}^3, \mathcal{S}^3))$, and that there is a constant $c > 0$ such that*

$$(\zeta, L[x]\,\zeta) \geq c|\zeta|^2, \quad (\sigma, \mathcal{D}[x]\,\sigma) \geq c|\sigma|^2, \quad \text{for all } x \in \Omega,\ \zeta \in \mathbb{R}^N,\ \sigma \in \mathcal{S}^3. \tag{9}$$

Let the mapping $g : \Omega \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ satisfy the following three conditions:

- $0 \in g(x, 0)$,
- $z \mapsto g(x, z) : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ *is maximal monotone,*
- *the mapping $x \mapsto j_\lambda(x, z) : \mathbb{R}^3 \to \mathbb{R}^N$ is measurable for all $\lambda > 0$, where $z \mapsto j_\lambda(\cdot, z)$ is the inverse of $z \mapsto z + \lambda g(\cdot, z)$.*

Suppose that $b \in W^{2,1}(0, T_e; L^2(\Omega, \mathbb{R}^3))$ and $\gamma \in W^{2,1}(0, T_e; H^1(\Omega, \mathbb{R}^3))$. Finally, assume that $z^{(0)} \in L^2(\Omega, \mathbb{R}^N)$ and that there exists $\zeta \in L^2(\Omega, \mathbb{R}^N)$ such that

$$\zeta(x) \in g(x, B^T T^{(0)}(x) - L z^{(0)}(x)), \quad \text{a.e. in } \Omega, \tag{10}$$

with the weak solution $(u^{(0)}, T^{(0)}) \in H^1(\Omega) \times L^2(\Omega)$ of the Dirichlet problem (1), (2), (5) of linear elasticity theory to the given data $b(0)$, $z(0) = z^{(0)}$, $\gamma(0)$.

Then to every $T_e > 0$ there is a unique solution

$$(u, T) \in W^{1,\infty}(0, T_e; H^1(\Omega, \mathbb{R}^3)) \times W^{1,\infty}(0, T_e; L^2(\Omega, \mathcal{S}^3)), \tag{11}$$

$$z \in W^{1,\infty}(0, T_e; L^2(\Omega, \mathbb{R}^N)) \tag{12}$$

of the initial-boundary value problem (1) – (5).

Now we are in a position to state our main results.

2.1 Interior regularity

Theorem 2 (Interior regularity). *Let all conditions of Theorem 1 be satisfied. Assume further that there are constants C, C_1, C_2 such that for every $x \in \Omega$ and every $y \in \mathbb{R}^3$ with $x + y \in \Omega$, for every $z \in \mathbb{R}^N$ and all $\lambda > 0$ the Yosida approximation $z \mapsto g^\lambda(x, z)$ of $z \mapsto g(x, z)$ and the mappings $\mathcal{D}$, L satisfy*

$$|g^\lambda(x + y, z) - g^\lambda(x, z)| \leq C|y||g^\lambda(x, z)|, \tag{13}$$

$$\|\mathcal{D}[x + y] - \mathcal{D}[x]\|_{\mathcal{LS}(\mathcal{S}^3, \mathcal{S}^3)} \leq C_1|y|, \tag{14}$$

$$\|L[x + y] - L[x]\|_{\mathcal{S}^N} \leq C_2|y|. \tag{15}$$

Suppose that $b \in W^{2,1}(0,T_e;L^2(\Omega,\mathbb{R}^3))$, $\gamma \in W^{2,1}(0,T_e;H^1(\Omega,\mathbb{R}^3))$ *and* $z^{(0)} \in H^1(\Omega,\mathbb{R}^N)$.

Then in addition to (11), (12), *the solution of the problem* (1) – (5) *satisfies*

$$(u,T) \in L^\infty\big(0,T_e;H^2_{\mathrm{loc}}(\Omega,\mathbb{R}^3)\big) \times L^\infty\big(0,T_e;H^1_{\mathrm{loc}}(\Omega,\mathcal{S}^3)\big), \tag{16}$$

$$z \in L^\infty\big(0,T_e;H^1_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\big). \tag{17}$$

Remark 1. If the function g is univalued, then (13) is equivalent to

$$|g(x+y,z) - g(x,z)| \le C|y||g(x,z)|.$$

This follows directly from the relation $g^\lambda(y,z) = g(y,j_\lambda(y,z))$, which holds in this case. In general we only have $g^\lambda(y,z) \subseteq g(y,j_\lambda(y,z))$.

Of course, (14), (15) mean that $\mathcal{D}$ and L are Lipschitz continuous.

2.2 Boundary regularity

At the boundary the tangential derivatives are as regular as all derivatives in the interior. This is shown by the next theorem.

Theorem 3 (Boundary regularity, tangential derivatives). *Let all conditions of Theorem 2 be satisfied. Assume additionally that* $\partial\Omega \in C^2$ *and* $\gamma \in W^{2,1}(0,T_e;H^2(\Omega,\mathbb{R}^3))$.

Then, for any vector field $\tau \in C^1(\overline{\Omega},\mathbb{R}^3)$, *which is tangential at the boundary* $\partial\Omega$, *the solution of the problem* (1) – (5) *satisfies*

$$(\partial_\tau u, \partial_\tau T) \in L^\infty(0,T_e;H^1(\Omega,\mathbb{R}^3)) \times L^\infty(0,T_e;L^2(\Omega,\mathcal{S}^3)), \tag{18}$$

$$\partial_\tau z \in L^\infty(0,T_e;L^2(\Omega,\mathbb{R}^N)), \tag{19}$$

where ∂_τ *denotes derivation in the direction of the vector field.*

The next regularity result for normal derivatives at the boundary holds in Besov spaces. Here we recall only the definition of Besov spaces. A detailed exposition can be found in [8], for example.

For $h \in \mathbb{R}^n$ and an open set $\Omega \subseteq \mathbb{R}^n$ we define

$$\Omega_h = \bigcap_{j=0}^{1} \{x \in \Omega \mid x + jh \in \Omega\}.$$

Definition 1. *Let* $1 \le p,\theta \le \infty$, $s \ge 0$ *and let* $\ell \in \mathbb{N}$ *with* $\ell > s$. *The function* f *belongs to the Besov (Nikol'skii–Besov) space* $B^s_{p,\theta}(\Omega) = B^s_{p,\theta}(\Omega,\mathbb{R}^n)$ *with order of smoothness* s, *if* f *is measurable on* Ω *and satisfies*

$$\|f\|_{B^s_{p,\theta}(\Omega)} = \|f\|_{L^p(\Omega)} + \|f\|_{b^s_{p,\theta}(\Omega)} < \infty\,,$$

where

$$\|f\|_{b^s_{p,\theta}(\Omega)} = \begin{cases} \left(\int_{\mathbb{R}^n} \left(\frac{\|\triangle_h^\ell f\|_{L^p(\Omega_h)}}{|h|^s} \right)^\theta \frac{dh}{|h|^n} \right)^{1/\theta}, & \textit{for } 1 \le \theta < \infty\,, \\ \sup_{h\in\mathbb{R}^n, h\neq 0} \frac{\|\triangle_h^\ell f\|_{L^p(\Omega_h)}}{|h|^s}, & \textit{for } \theta = \infty\,. \end{cases}$$

The ℓ–th order difference operator $\triangle_h^\ell$ is defined by $\triangle_h f(x) = f(x+h) - f(x)$ and $\triangle_h^\ell f(x) = \triangle_h(\triangle_h^{\ell-1} f(x))$. Of course, the norm $\|f\|_{b^s_{p,\theta}(\Omega)}$ depends on the choice of ℓ, but for $\ell > s$ all norms are equivalent, cf. [8]. There exist other equivalent norms on the space $B^s_{p,\theta}(\Omega)$, but this one is the most convenient for our purposes.

Theorem 4 (Boundary regularity, all derivatives). *Under the conditions of Theorem 3 the solution of the problem* (1) – (5) *satisfies*

$$(u, T) \in L^\infty(0, T_e; B^{5/4}_{2,\infty}(\Omega, \mathbb{R}^3)) \times L^\infty(0, T_e; B^{1/4}_{2,\infty}(\Omega, \mathcal{S}^3)), \tag{20}$$

$$z \in L^\infty(0, T_e; B^{1/4}_{2,\infty}(\Omega, \mathbb{R}^N)). \tag{21}$$

For $H^s = B^s_{2,2}$ and for $\delta > 0$ we have

$$(u, T) \in L^\infty(0, T_e; H^{4/3-\delta}(\Omega, \mathbb{R}^3)) \times L^\infty(0, T_e; H^{1/3-\delta}(\Omega, \mathcal{S}^3)), \tag{22}$$

$$z \in L^\infty(0, T_e; H^{1/3-\delta}(\Omega, \mathbb{R}^N)). \tag{23}$$

3 Regularity for generalized standard materials

For all our regularity results we need that the data b and γ have time derivatives of second order; the solutions obtained are in general only differentiable. We do not know whether this regularity requirement for the data is optimal and whether this loss of a derivative can be avoided. Yet, we show that in the special case when $g(x, z) = \partial_z \chi(x, z)$ with a convex function $z \mapsto \chi(x, z)$, that is for a generalized standard material, it suffices when the data have one time derivative. Indeed, to see that we first reduce the initial boundary value problem (1) – (5) to an evolution equation in a Hilbert space. To this end, we employ the standard reduction procedure from [2, 3] (see also [4]). Assume first that $b \in W^{1,2}(0, T_e; L^2(\Omega, \mathbb{R}^3))$, $\gamma \in W^{1,2}(0, T_e; H^1(\Omega, \mathbb{R}^3))$ and $z^{(0)} \in L^2(\Omega, \mathbb{R}^N)$. Suppose that the functions $\hat{\varepsilon}_p \in L^2(\Omega, \mathcal{S}^3)$, $\hat{b} \in H^{-1}(\Omega, \mathbb{R}^3)$ and $\hat{\gamma} \in H^1(\Omega, \mathbb{R}^3)$ are given and consider the problem

$$-\mathrm{div}_x \tilde{T}(x) = \hat{b}(x), \tag{24}$$

$$\tilde{T}(x) = \mathcal{D}\big(\varepsilon(\nabla_x \tilde{u}(x)) - \hat{\varepsilon}_p(x)\big), \tag{25}$$

$$\tilde{u}(x) = \hat{\gamma}(x), \qquad x \in \partial\Omega, \tag{26}$$

which has a unique solution in virtue of ellipticity theory. We drop the x–dependence of $\mathcal{D}$, L and χ ($g(z) = \partial_z \chi(z)$) for simplicity.

Definition 2. *Let the linear operator* $P : L^2(\Omega, \mathcal{S}^3) \to L^2(\Omega, \mathcal{S}^3)$ *be defined by*

$$P\hat{\varepsilon}_p = \varepsilon(\nabla_x \tilde{u}),$$

where $(\tilde{u}, \tilde{T})$ *is the solution of* (24) – (26) *to* $\hat{b} = 0$, $\hat{\gamma} = 0$ *and* $\hat{\varepsilon}_p \in L^2(\Omega, \mathcal{S}^3)$. *With the identity operator* I *on* $L^2(\Omega, \mathcal{S}^3)$ *set* $Q = I - P$.

Lemma 1. *(i) The operators* P *and* Q *are projectors on* $L^2(\Omega, \mathcal{S}^3)$, *which are orthogonal with respect to the scalar product* $[\xi, \zeta]_\Omega := (\mathcal{D}\xi, \zeta)_\Omega$.
(ii) The operator $B^T\mathcal{D}QB : L^2(\Omega, \mathbb{R}^N) \to L^2(\Omega, \mathbb{R}^N)$ *is selfajoint and non-negative with respect to the scalar product* $(\xi, \zeta)_\Omega$.

Assume that the function $\chi : \mathbb{R}^N \to \bar{\mathbb{R}}$ is proper, convex, lower semi-continuous and define $G : L^2(\Omega, \mathbb{R}^N) \to \bar{\mathbb{R}}$ by

$$H(z) = \begin{cases} \int_\Omega \chi(z(x))dx & \text{if } \ \chi(z) \in L^1(\Omega) \\ +\infty & \text{otherwise} \end{cases}.$$

It is well known [27, p. 85] that H is proper, convex, lower semi-continuous, and $z^* \in \partial H(z)$ if and only if

$$z \in L^2(\Omega, \mathbb{R}^N), \ z^* \in L^2(\Omega, \mathbb{R}^N) \ \text{ and } \ z^*(x) \in \partial\chi(z(x)), \ \text{a.e. } x \in \Omega.$$

Note that ∂H is maximal monotone as a subdifferential of a proper, convex and lower semi-continuous function.
Assume that $z(t)$ is known. Then the component $(u(t), T(t))$ of the solution is obtained as unique solution of the boundary value problem (1), (2), (5). Due to the linearity we have

$$(u(t), T(t)) = (\tilde{u}(t), \tilde{T}(t)) + (v(t), \sigma(t)), \tag{27}$$

where $(v(t), \sigma(t))$ is the solution of (24) – (26) to the data $\hat{b} = b(t)$, $\hat{\gamma} = \gamma(t)$, $\hat{\varepsilon}_p = 0$, and $(\tilde{u}(t), \tilde{T}(t))$ is the solution of (24) – (26) to the data $\hat{b} = \hat{\gamma} = 0$, $\hat{\varepsilon}_p = Bz(t)$. By definition of Q we have that $\tilde{T}(t) = -\mathcal{D}QBz(t)$. Insertion of this equation into (3) yields

$$\frac{\partial}{\partial t} z(t) \in \partial H\big(-Mz(t) + B^T\sigma(t)\big), \tag{28}$$

with the mappings $M : L^2(\Omega, \mathbb{R}^N) \to L^2(\Omega, \mathbb{R}^N)$ defined by

$$M = B^T\mathcal{D}QB + L. \tag{29}$$

The operator M is symmetric and positive definite. Since σ is determined from the boundary value problem (24) – (26) to the data b, γ, it can be considered to be known. Therefore (28) is a non-autonomous evolution equation for z. We transform this equation to an autonomous equation with a maximal monotone evolution operator, since strong existence and perturbation theorems are mainly available for such equations. To this end, define a function $d : [0, T_e] \to L^2(\Omega, \mathbb{R}^N)$ by

$$d = -Mz + B^T \sigma. \tag{30}$$

We insert this function into (28) and use the initial condition (4) to obtain the initial boundary value problem

$$\frac{d}{dt} d(t) + A\, d(t) \ni B^T \sigma_t(t), \tag{31}$$

$$d(0) = -Mz^{(0)} + B^T \sigma(0), \tag{32}$$

for d, where the operator A is given by

$$A = M \partial H.$$

The relation between z and d given in (30) is one-to-one, and the evolution equation (31) is equivalent to the equation (28).
Since M^{-1} is selfadjoint and positive definite, the scalar product

$$\langle \xi, \zeta \rangle_\Omega := (M^{-1}\xi, \zeta)_\Omega$$

is well defined in $L^2(\Omega, \mathbb{R}^N)$. Let us denote by $\mathcal{L}^2(\Omega, \mathbb{R}^N)$ the Hilbert space $L^2(\Omega, \mathbb{R}^N)$ endowed with the scalar product $\langle \xi, \zeta \rangle_\Omega$. Then the operator A is a subdifferential of a proper, convex and lower semi-continuous function $\mathcal{H} : \mathcal{L}^2(\Omega, \mathbb{R}^N) \to \bar{\mathbb{R}}$ given by

$$\mathcal{H}(z) := H(z).$$

Indeed, one sees that from the equivalence

$$z^* \in M\partial H(z) \Leftrightarrow (M^{-1}z^*, y - z)_\Omega \le H(y) - H(z)$$

$$\Leftrightarrow \langle z^*, y - z \rangle_\Omega \le \mathcal{H}(y) - \mathcal{H}(z) \Leftrightarrow z^* \in \partial\mathcal{H}(z)$$

for all $y \in \mathcal{L}^2(\Omega, \mathbb{R}^N)$. Thus, due to Theorem 4.3 [27, p. 186] the problem (31) - (32) considered in $\mathcal{L}^2(\Omega, \mathbb{R}^N)$

$$\frac{d}{dt} d(t) + \partial\mathcal{H}\, d(t) \ni B^T \sigma_t(t),$$

$$d(0) = -Mz^{(0)} + B^T \sigma(0) \in D(\mathcal{H})$$

has a solution $d \in W^{1,2}(0, T_e, L^2(\Omega, \mathbb{R}^N))$ for $\sigma_t \in L^2(0, T_e, L^2(\Omega, \mathbb{R}^N))$. Note that the norms in $\mathcal{L}^2(\Omega, \mathbb{R}^N)$ and $L^2(\Omega, \mathbb{R}^N)$ are equivalent. Using the one-to-one relation between z and d we obtain that $z \in W^{1,2}(0, T_e, L^2(\Omega, \mathbb{R}^N))$.
In virtue of ellipticity theory,

$$(u, T) \in W^{1,2}(0, T_e; H^1(\Omega, \mathbb{R}^3)) \times W^{1,2}(0, T_e; L^2(\Omega, \mathcal{S}^3)).$$

Thus we have the following existence result in the case of generalized standard materials.

Theorem 5 (Existence, $g = \partial\chi$). *Let the mapping $\chi : \mathbb{R}^N \to \mathbb{R}$ be proper, convex, lower semi-continuous and satisfy the condition* (33)*. Suppose that $b \in W^{1,2}(0,T_e;L^2(\Omega,\mathbb{R}^3))$, $\gamma \in W^{1,2}(0,T_e;H^1(\Omega,\mathbb{R}^3))$ and $z^{(0)} \in L^2(\Omega,\mathbb{R}^N)$. Assume further that the function $\chi : \mathbb{R}^N \to \bar{\mathbb{R}}$ satisfies the condition*

$$\int_\Omega \chi(B^T T^{(0)}(x) - Lz^{(0)}(x))dx < \infty, \tag{33}$$

where $(u^{(0)}, T^{(0)}) \in H^1(\Omega) \times L^2(\Omega)$ is the weak solution of the Dirichlet problem (1), (2), (5) *of linear elasticity theory to the given data $b(0)$, $z(0) = z^{(0)}$, $\gamma(0)$. Then to every $T_e > 0$ there is a unique solution*

$$(u,T) \in W^{1,2}(0,T_e;H^1(\Omega,\mathbb{R}^3)) \times W^{1,2}(0,T_e;L^2(\Omega,\mathcal{S}^3)), \tag{34}$$

$$z \in W^{1,2}(0,T_e;L^2(\Omega,\mathbb{R}^N)) \tag{35}$$

of the initial-boundary value problem (1) – (5) *with $g = \partial\chi$.*

Remark 2. We note that (33) is equivalent to the condition

$$B^T T^{(0)} - Lz^{(0)} \in D(H).$$

The proof of Theorem 4 undergoes no essential changes in the case of generalized standard materials ($g = \partial\chi$), although the solution is of slightly weaker regularity compared to the general case. We formulate the result.

Theorem 6 (Boundary regularity, $g = \partial\chi$). *Let all conditions of Theorem 5 be satisfied. Assume additionally that $\gamma \in W^{1,2}(0,T_e;H^2(\Omega,\mathbb{R}^3))$ and $\partial\Omega \in C^2$. Then the solution of the problem* (1) – (5) *with $g = \partial\chi$ satisfies*

$$(u,T) \in L^\infty(0,T_e;B^{5/4}_{2,\infty}(\Omega,\mathbb{R}^3)) \times L^\infty(0,T_e;B^{1/4}_{2,\infty}(\Omega,\mathcal{S}^3)), \tag{36}$$

$$z \in L^\infty(0,T_e;B^{1/4}_{2,\infty}(\Omega,\mathbb{R}^N)). \tag{37}$$

For $H^s = B^s_{2,2}$ and $\delta > 0$ we have

$$(u,T) \in L^\infty(0,T_e;H^{4/3-\delta}(\Omega,\mathbb{R}^3)) \times L^\infty(0,T_e;H^{1/3-\delta}(\Omega,\mathcal{S}^3)), \tag{38}$$

$$z \in L^\infty(0,T_e;H^{1/3-\delta}(\Omega,\mathbb{R}^N)). \tag{39}$$

Remark 3. The model of Melan-Prager either with Prandtl-Reuss or Norton-Hoff flow rules meets all assumptions of Theorem 6.

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