Subgrid interaction and micro-randomness - novel invariance requirements in infinitesimal gradient elasticity

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Abstract

We present a micromechanically motivated form of the curvature energy in infinitesimal isotropic gradient elasticity. The basis is a homogenization/averaging scheme using a micro-randomness assumption imposed on a directional higher gradient interaction term. These directional interaction terms are matrix-valued allowing to apply the standard orthogonal Cartan Lie-algebra decomposition. Averaging over all (subgrid) directions leads to three quadratic curvature terms, which are conformally invariant when neglecting volumetric effects. Restricted to rotational inhomogeneities we motivate therewith a symmetric couple stress tensor in the infinitesimal indeterminate couple stress model of Koiter-Mindlin-Toupin-type. Relations are established to a novel conformally invariant linear Cosserat model.

Key words: gradient elasticity, strain gradients, invariance conditions, couple stress, microstructure, conformal transformations, symmetry of moment stresses, micromorphic, micropolar, Cosserat.

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1 Introduction

Novel effects like size-dependence and scaling of mechanical laws have attracted considerable attention [8, 61, 10]. In turn, gradient elasticity models [39, 58, 16] have become popular by their inherent possibility to offer a phenomenological description of these size-effects which may become important for very small scale materials, notably in the plastic range. The relevance of size-effects for nano-sized materials is discussed in [36].

Gradient elasticity introduces, through the presence of higher derivatives, a certain non-locality in the model which has to do with an additional long-range force structure present in the material. At larger length scales, the classical (size-independent) elasticity part dominates. A serious drawback of this class of models is that they introduce many additional parameters which are neither easily interpreted, nor easily identified through experiments. We limit ourselves here to the most simplest setting of linear, isotropic, centro-symmetric materials with only second gradients of displacements D^2u. For such a model gradient elasticity means to include, in the variational statement, a quadratic curvature energy of the form $W_{\mathrm{curv}}(D^2u)$. Even within the supposed maximal symmetry assumptions on the macroscale the number of independent terms in a representation of W_{curv} is not entirely obvious: Mindlin [38] gives a five parameter representation while Lam et al [32] and Fleck et al [13, 5] motivate a reduced three parameter setting. We will give special attention to the so called indeterminate couple stress model [20, 50, 1, 27, 40, 57, 53, 21] which is a gradient elasticity model where higher order effects appear only through gradients of macroscopic rotation $\nabla \operatorname{curl} u$. The indeterminate couple stress theory and gradient elasticity is also the basis of strain-gradient extensions of classical plasticity theories [12, 52, 13, 5, 14, 17].

We put particular emphasis on a micromechanical motivation of the conformally invariant curvature measure $\|\operatorname{sym}\nabla \operatorname{curl} u\|^2$ in the infinitesimal indeterminate couple stress model by proposing a specific homogenisation scheme. The method is based on introducing representative volume elements inside a cluster of such volumes which interact on the scale of a superposed subgrid through rotational inhomogeneity along a given direction $h \in \mathbb{R}^3$ only. Since we fix the discrete direction first, it is easy to motivate and interpret various interaction terms on this level. There, we introduce the **novel concept of micro-random material behaviour on the microscale** which leads, after homogenisation (averaging over all subgrid directions), to the symmetry of the moment stresses. This is equivalent to the use of the above mentioned conformally invariant curvature energy [44, 43, 23]. The symmetry of the moment stress in the indeterminate couple stress model has already been proposed in [59, 32] by a different derivation based on point mechanics arguments. Extension of this model to Bernoulli beams and further case studies are documented in [47, 49]. In [46] the size effect for solid polymers is described also with a symmetric moment stress based on the result of [59]. Similarly, in [54, 55, 51] ad hoc additional invariance principles are applied which yield a symmetric moment stress, see also [60]. On the other hand, the usual assumptions of Mindlin and Koiter [40, 27] on the pointwise uniform positive definiteness of the curvature energy exclude the symmetry of the moment stress. Recently, a formal homogenisation scheme towards (essentially) the Koiter-Mindlin model has been given in [6] but excluding the moment stress symmetry. Garikipati motivates a couple stress model for crystalline solids based on three-body interatomic potentials [18]. He arrives as well at a uniform positive curvature expression. Our contribution is thus intended to clarify and delineate under what conditions and for what type of material we may expect a symmetric moment stress in the indeterminate couple stress model. We also touch upon the consequences of our results for gradient elasticity and strain gradient plasticity.

The paper is organised as follows. We start with a general second gradient elasticity model for which we investigate the curvature energy with respect to its induced interaction response. Then we specialise to the well known indeterminate couple stress model. Hereafter we introduce our proper homogenisation scheme and motivate our novel micro-randomness principle. Finally, we draw a connection to the Cosserat model via a least energy extension principle. Thus we motivate the conformally invariant curvature expression $\|\operatorname{sym}\nabla\operatorname{curl}u\|^2$. In the appendix we collect our notation, some relations for infinitesimal conformal mappings as well as some results for spherical averaging. In addition we draw connections to previously given representations of the curvature energy in terms of the third order tensor $\eta_{ijk} = \partial_{ij}u_k$ of second displacement derivatives.

2 Second gradient elasticity

2.1 An isotropic second gradient elasticity model

We are interested in an isotropic, centro-symmetric gradient elasticity model [37, 38, 39] (see [61] for a list of different such models and [32] for a new reduced functional basis for higher order terms) with variable material moduli $\widehat{\mu}(x), \widehat{K}(x)$ and we consider a representative subset $RVE^\sharp \subset \Omega$ where $\Omega \subset \mathbb{R}^3$ is the reference configuration of the body. We refer to $RVE^\sharp$ as the subgrid cluster. The goal is to find the displacement $u : RVE^\sharp \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ minimising the energy

$$I(u) = \int_{RVE^\sharp} \underbrace{\widehat{\mu}(x)\,\|\operatorname{dev}\operatorname{sym}\nabla u\|^2 + \frac{\widehat{K}(x)}{2}\operatorname{tr}[\nabla u]^2}_{\text{size-independent response}} + \underbrace{\mu\,L_c^2\,W_{\text{curv}}(D^2u)}_{\text{subgrid interaction energy}}\ \mathrm{dV} \mapsto \ \min.\ \text{w.r.t.}\ u\,, \tag{2.1}$$

under the homogeneous Dirichlet boundary condition $u_{|_{\partial RVE^\sharp}} = \widehat{B}.x$ for constant, non-symmetric $\widehat{B} \in \mathfrak{gl}(3) = \mathbb{M}^{3\times3}$. If $W_{\text{curv}}(D^2u) = W_{\text{curv}}(D\operatorname{sym}\nabla u)$[1] the model is called a strain gradient model, for its use in regularising strain singularities see [33, 34]. Mindlin [37, 38, 39] is giving a seven parameter (five curvature parameters) energy for the most general quadratic isotropic gradient elasticity model in the third order tensor $\eta_{ijk} = \partial_{ij}u_k$. The second order curvature part can be written as [37, eq.(9.11)]

$$W_{\text{curv}}(D^2u) = a_0\,\eta_{kii}\,\eta_{kjj} + a_1\,\eta_{ijk}\,\eta_{ijk} + a_2\,\eta_{ijk}\,\eta_{jki} + a_3\,\eta_{jji}\,\eta_{kki} + a_4\,\eta_{iik}\,\eta_{kjj}\,, \tag{2.2}$$

where a_i, $i = 0, 1, 2, 3, 4$ are dimensionless weighting parameters. This expression is not easily amenable to mechanical interpretation.[2] In a simpler setting, as an example for a centro-symmetric, isotropic model, in [32, eq.(42)] it is proposed to use a curvature energy depending on the dilational gradient $\nabla\operatorname{Div}u$, the "deviatoric" stretch gradient $\eta^{(1)}_{ijk} = L^{rst}_{ijk}\,\eta_{rst}$ (for the definition of $\eta^{(1)}$ see (6.28)), note already that $\eta^{(1)}$ is not the gradient of deviatoric stretch)[3] and the symmetric part of the rotational gradient $\operatorname{sym}\nabla\operatorname{curl}u$

$$W_{\text{curv}}(D^2u) = a_0\,\|\nabla\operatorname{Div}u\|^2 + a_1'\,\eta^{(1)}_{ijk}\,\eta^{(1)}_{ijk} + a_2'\,\|\operatorname{sym}\nabla\operatorname{curl}u\|^2\,. \tag{2.4}$$

A simplified strain gradient version of (2.2) with

$$W_{\text{curv}}(D^2u) = a_0\,\|\nabla\operatorname{Div}u\|^2 + a_1'\,\|D\operatorname{sym}\nabla u\|^2 \tag{2.5}$$

is considered in [2, 48] and [33, eq.(21)]. In [61, eq.(8)] or [26, eq.(17.82)] for the same purpose

$$W_{\text{curv}}(D^2u) = a_0\,\|\nabla\operatorname{Div}u\|^2 + a_2'\,\|\nabla\operatorname{curl}u\|^2 \tag{2.6}$$

is proposed. In [56, 11] the case $a_1' = a_2' = 0$ is considered and compared to the former setting (2.6) in two space dimensions [56]. Fleck et al [12] take $a_0 = 0$ in (2.6) for simplicity.

[1] All second displacement derivatives D^2u can be expressed as linear combinations of strain gradients $D\operatorname{sym}\nabla u$ [40, eq.11.1]. It holds $u_{k,ij} = \varepsilon_{ik,j} + \varepsilon_{jk,i} - \varepsilon_{ij,k}$. The first appeal to strain gradients is made, apparently, already by Cauchy [7].

[2] The same formal representation applies to strain gradient models [37, eq.(11.3)] in the sense that

$$W_{\text{curv}}(D\operatorname{sym}\nabla u) = a_0\,\kappa_{kii}\,\kappa_{kjj} + a_1\,\kappa_{ijk}\,\kappa_{ijk} + a_2\,\kappa_{ijk}\,\kappa_{jki} + a_3\,\kappa_{jji}\,\kappa_{kki} + a_4\,\kappa_{iik}\,\kappa_{kjj} \tag{2.3}$$

is the most general isotropic, quadratic energy in strain gradients $\kappa_{ijk} := \partial_i[\varepsilon_{jk}]$.

[3] The tensor $\eta^{(1)}$ is that combination of second partial derivatives $\partial_{ij}u_k$ which controls the incompressible, irrotational part of the displacement.

2.2 The indeterminate couple stress model

The infinitesimal, isotropic, centro-symmetric indeterminate couple stress model [20, 50, 1, 27, 40, 57, 28, 53, 21] is a special gradient elasticity formulation in which the higher derivatives only appear through derivatives on the continuum rotation $\operatorname{curl} u$. For the displacement $u : RVE^\sharp \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ we consider the minimization problem

$$I(u) = \int_{RVE^\sharp} W_{\mathrm{mp}}(\nabla u) + \mu\, L_c^2\, W_{\mathrm{curv}}(\nabla \operatorname{curl} u)\, \mathrm{dV} \mapsto \quad \min . \text{ w.r.t. } u, \tag{2.7}$$

under the constitutive requirements and boundary conditions[4]

$$\begin{aligned} W_{\mathrm{mp}}(\nabla u) &= \mu\, \|\operatorname{sym}\nabla u\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}\nabla u\right]^2, \quad u_{|\Gamma} = u_{\mathrm{d}}\,, \\ W_{\mathrm{curv}}(\nabla \operatorname{curl} u) &= a_2'\, \|\operatorname{sym}\nabla \operatorname{curl} u\|^2 + a_3'\, \|\operatorname{skew}\nabla \operatorname{curl} u\|^2 . \end{aligned} \tag{2.8}$$

Grioli [20] initially arrived at $a_2' = a_3'$, meaning that only $\|\nabla \operatorname{curl} u\|^2$ appears in the curvature [12]. In the general model an energy term related to the spherical part of the (higher order) couple stress tensor $m = D_{\nabla \operatorname{curl} u} W_{\mathrm{curv}}(\nabla \operatorname{curl} u)$ remains **indeterminate**, since $\operatorname{tr}[\nabla \operatorname{curl} u] = \operatorname{Div}\operatorname{curl} u = 0$. Following [40, 27], it is usually assumed that $a_3' > 0$ in order to guarantee pointwise uniform positive definiteness (which is, in fact, not needed for well-posedness). For the conformal case $\|\operatorname{sym}\nabla \operatorname{curl} u\|^2$, we have, on the contrary $a_3' = 0$, which makes the couple stress tensor m symmetric and trace free, a choice which has also been used in [59, 32, 46]. This conformal curvature case is indeed well-posed [22].

2.3 Null spaces of the curvature energy

Clearly, the presence of the curvature energy W_{curv} introduces additional long range interaction in the material to which we also refer to as subgrid interaction, see Figure 3. In the next diagram follow different subgrid interaction energies together with the possible form of fluctuations "inside" the $RVE^\sharp$ cluster not giving rise to an (additional) interaction energy. That means we look at those subgrid deformations which do not contribute to the subgrid interaction energy[5]

$$W_{\mathrm{curv}}(D^2 u(x)) = 0\,, \quad u_{|_{\partial RVE^\sharp}} = \widehat{B}.x\,, \tag{2.9}$$

for given homogeneous Dirichlet loading $\widehat{B} \in \mathfrak{gl}(3)$ at the boundary of the $RVE^\sharp$-cluster.[6] In the left column the curvature energy[7] is specified, in the right column the corresponding

[4] It is always possible to include higher order boundary conditions but not strictly necessary in the sense that free Neumann conditions may always apply, thus avoiding arbitrary boundary layer effects.

[5] This question is similar to letting hypothetically $L_c \to \infty$, in which case the subgrid interaction does set a geometrical constraint on the possible response.

[6] A drawback of linear gradient elasticity models with positive definite curvature energy is that they always predict higher levels of energy for inhomogeneous microstructure response than for homogeneous response. The microstructure is always penalised. This is not necessarily the case in e.g., finite strain Cosserat models.

[7] Much more curvature energy terms are, of course, possible. We have chosen representative examples. Note that classical isotropy of the model does not restrict further the curvature energy, since only invariance under a superposition of homogeneous infinitesimal rotations is required, which is trivially satisfied for the curvature. The same remark applies to objectivity requirements.

interaction free displacement is given:

$$\begin{cases}\|D^2u\|^2\\ \|\Delta u\|^2\\ \|D\,\mathrm{sym}\,\nabla u\|^2\\ \|D\,\mathrm{dev}\,\mathrm{sym}\,\nabla u\|^2\\ \|\nabla\,\mathrm{Div}\,u\|^2+\|\nabla\,\mathrm{curl}\,u\|^2\\ \|\nabla\,\mathrm{Div}\,u\|^2+\|\,\mathrm{curl}\,\mathrm{curl}\,u\|^2\\ \|\,\mathrm{skew}\,\nabla\,\mathrm{curl}\,u\|^2\\ \|\,\mathrm{curl}\,\mathrm{curl}\,u\|^2\\ \|\nabla\,\mathrm{Div}\,u\|^2\\ \|\nabla\,\mathrm{curl}\,u\|^2\\ \|\,\mathrm{dev}\,\nabla\,\mathrm{curl}\,u\|^2\\ \|\,\mathrm{sym}\,\nabla\,\mathrm{curl}\,u\|^2\\ \|\,\mathrm{dev}\,\mathrm{sym}\,\nabla\,\mathrm{curl}\,u\|^2\end{cases}\underbrace{\Longrightarrow}_{W_{\mathrm{curv}}\equiv 0}\begin{cases}u(x)=\widehat{B}.x=\mathrm{sym}(\widehat{B}).x+\mathrm{skew}(\widehat{B}).x\\ u(x)=\widehat{B}.x=\mathrm{sym}(\widehat{B}).x+\mathrm{skew}(\widehat{B}).x\\ u(x)=\widehat{B}.x=\mathrm{sym}(\widehat{B}).x+\mathrm{skew}(\widehat{B}).x\\ u(x)=\widehat{B}.x=\mathrm{sym}(\widehat{B}).x+\mathrm{skew}(\widehat{B}).x\\ u(x)=\widehat{B}.x=\mathrm{sym}(\widehat{B}).x+\mathrm{skew}(\widehat{B}).x\\ u(x)=\widehat{B}.x=\mathrm{sym}(\widehat{B}).x+\mathrm{skew}(\widehat{B}).x\\ u(x)=\nabla\zeta(x)+\mathrm{curl}\,v(x)+\mathrm{sym}(\widehat{B}).x+\mathrm{skew}(\widehat{B}).x\\ u(x)=\nabla\zeta(x)+\mathrm{curl}\,v(x)+\mathrm{sym}(\widehat{B}).x+\mathrm{skew}(\widehat{B}).x\\ u(x)=\nabla\zeta(x)+\mathrm{curl}\,v(x)+\mathrm{sym}(\widehat{B}).x+\mathrm{skew}(\widehat{B}).x\\ u(x)=\nabla\zeta(x)+\mathrm{sym}(\widehat{B}).x+\mathrm{skew}(\widehat{B}).x\\ u(x)=\nabla\zeta(x)+\mathrm{sym}(\widehat{B}).x+\mathrm{skew}(\widehat{B}).x\\ u(x)=\nabla\zeta(x)+\phi_C(x)+\mathrm{sym}(\widehat{B}).x+\mathrm{skew}(\widehat{B}).x\\ u(x)=\nabla\zeta(x)+\phi_C(x)+\mathrm{sym}(\widehat{B}).x+\mathrm{skew}(\widehat{B}).x\,.\end{cases}\tag{2.10}$$

Here, $\zeta:\mathbb{R}^3\mapsto\mathbb{R}$ is a displacement potential and $\phi_C:\mathbb{R}^3\mapsto\mathbb{R}^3$ is an infinitesimal conformal mapping having the form (for more information on conformal maps compare to (6.16) and [43])

$$\phi_C(x)=\frac{1}{2}\left(2\langle\mathrm{axl}(\widehat{W}),x\rangle_{\mathbb{R}^3}\,x-\mathrm{axl}(\widehat{W})\,\|x\|^2\right)+[\widehat{p}\,\mathbb{1}+\widehat{A}].x+\widehat{b}\,,\tag{2.11}$$

where $\widehat{W},\widehat{A}\in\mathfrak{so}(3),\widehat{b}\in\mathbb{R}^3,\widehat{p}\in\mathbb{R}$ are arbitrary constants.

Definition 2.1 (Conformal invariance of curvature)
By conformal invariance we mean that the curvature energy vanishes on infinitesimal conformal mappings, i.e.,

$$W_{\mathrm{curv}}(D^2\phi_C)=0\tag{2.12}$$

for the family of mappings ϕ_C given in (2.11) *which infinitesimally preserve angles and shapes of figures. In this sense it can be shown that $\|D\,\mathrm{dev}\,\mathrm{sym}\,\nabla u\|^2$, $\|\,\mathrm{sym}\,\nabla\,\mathrm{curl}\,u\|^2$ and $\|\,\mathrm{dev}\,\mathrm{sym}\,\nabla\,\mathrm{curl}\,u\|^2$ are conformally invariant, as well as $\|\eta^{(1)}\|^2$.*

Remark 2.2 (Conformal invariance and J_2-plasticity)
In connection with gradient plasticity we observe that a conformally displaced material body has zero J_2-deviatoric von Mises invariant regardless how big the conformal displacement is since $\mathrm{dev}\,\sigma(\nabla\phi_C)=2\mu\,\mathrm{dev}\,\mathrm{sym}\,\nabla\phi_C=0$. In this sense, conformal displacements are truly structure preserving, defect free, elastic displacements.

The diagram (2.10) should now be interpreted as (for example the first case): every deformation u inside the subgrid $RVE^\sharp$ cluster which does not have the affine form $u(x)=\widehat{B}.x$ is contributing to the subgrid interaction energy. Similarly, for the last case, every subgrid deformation u not having the form $u(x)=\nabla\zeta(x)+\phi_C(x)+\widehat{B}.x$ contributes to the subgrid interaction energy.

Proof. To see these statements we consider $u_{|_{\partial RVE^\sharp}}=\widehat{B}.x$ and

1. $D^2u(x)=0$ implies that $u(x)=\widehat{B}.x$.
2. $\Delta u(x)=0$ subject to homogeneous Dirichlet conditions has a unique solution, and $u(x)=\widehat{B}.x$ is a solution. The term is not conformally invariant.
3. $D\,\mathrm{sym}\,\nabla u=0$ implies that $\nabla u(x)=\mathrm{sym}(\widehat{G})+A(x)$ for some arbitrary constant matrix $\widehat{G}\in\mathfrak{gl}(3)$ and $A:\mathbb{R}^3\mapsto\mathfrak{so}(3)$. Taking the curl on both sides gives $0=\mathrm{Curl}\,A(x)$ which yields $A(x)=\widehat{A}$, see [45]. Thus $u(x)=\mathrm{sym}(\widehat{G}).x+\widehat{A}.x+\widehat{b}$ and the unique solution is $u(x)=\mathrm{sym}\,\widehat{B}.x+\mathrm{skew}(\widehat{B}).x=\widehat{B}.x$.

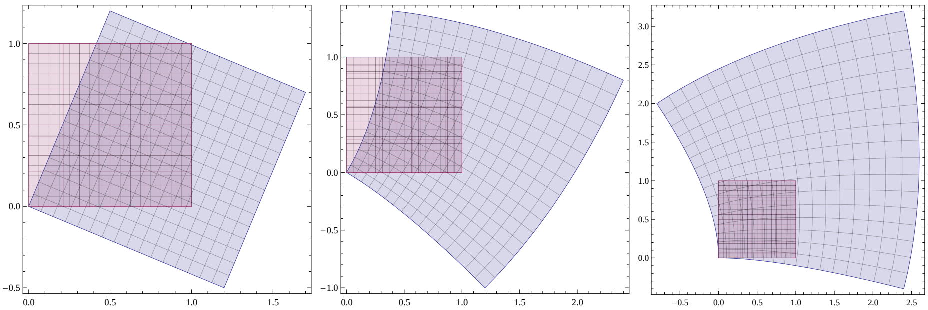

Figure 1: **Infinitesimal conformal mappings**. Mappings that preserve shapes and angles locally are of course constant rotations and dilations (left). But also some specific second order polynomials of the format (2.11). Infinitesimal preservation of angles and shapes does not induce curvature energy for conformally invariant curvature expressions. A conformal map ϕ_C represents a certain long range order.

4. $D\,\mathrm{dev}\,\mathrm{sym}\,\nabla u = 0$ implies that $\nabla u(x) = \mathrm{dev}\,\mathrm{sym}\,\widehat{G} + p(x)\,\mathbb{1} + A(x)$ for some arbitrary constant matrix $\widehat{G} \in \mathfrak{gl}(3)$ and $A : \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ and $p : \mathbb{R}^3 \mapsto \mathbb{R}$. Taking the curl on both sides gives $0 = \mathrm{Curl}[p(x)\mathbb{1}] + \mathrm{Curl}\,A(x)$. This equation has been dealt with in [43, eq.(3.9)]. The general solution is $u(x) = \widehat{H}.x + \phi_C(x)$, for some arbitrary constant matrix $\widehat{H} \in \mathfrak{gl}(3)$. Incorporating the boundary condition implies that $\widehat{H} = \widehat{B}$ and $\phi_C(x)$ must vanish at the boundary. Thus, it follows $\phi_C \equiv 0$, [43, Lem.3.4]. The unique solution is again $u(x) = \widehat{B}.x$. Observe that the energy $\|D\,\mathrm{dev}\,\mathrm{sym}\,\nabla u\|^2$ is trivially conformally invariant (6.16), but does not allow fluctuations inside.
5. We must have $\mathrm{Div}\,u = \mathrm{const}$ and $\mathrm{curl}\,u = \mathrm{const}$. The affine boundary conditions already determine $\mathrm{Div}\,u = \mathrm{tr}\left[\widehat{B}\right]$ and $\mathrm{curl}\,u = 2\,\mathrm{axl}(\mathrm{skew}\,\widehat{B})$. Thus we may still add functions which vanish at the boundary and satisfy $\mathrm{Div}\,v = 0$ and $\mathrm{curl}\,v = 0$. The Div / curl-inequality [19] on the space $H_0^1(\Omega, \mathbb{R}^3)$ implies $v = 0$.
6. In this case, $\nabla\,\mathrm{Div}\,u = 0$ and $\mathrm{curl}\,\mathrm{curl}\,u = 0$ from which follows $\Delta u = 0$. Thus $u = \widehat{B}.x$.
7. $\mathrm{skew}\,\nabla\,\mathrm{curl}\,u = 0$. Note that
$$\|\,\mathrm{skew}\,\nabla\,\mathrm{curl}\,u\|^2 = 2\|\,\mathrm{axl}(\mathrm{skew}\,\nabla\,\mathrm{curl}\,u)\|^2 = \frac{1}{2}\|2\,\mathrm{axl}(\mathrm{skew}\,\nabla\,\mathrm{curl}\,u)\|^2 = \frac{1}{2}\|\,\mathrm{curl}\,\mathrm{curl}\,u\|^2 .$$
Thus the energy coincides with the next case.
8. Since $\Delta u = \nabla\,\mathrm{Div}\,u - \mathrm{curl}\,\mathrm{curl}\,u$ and $\mathrm{curl}\,\mathrm{curl}\,u = 0$ implies $\Delta u = \nabla\,\mathrm{Div}\,u$. Assuming $u(x) = \widehat{B}.x + \nabla\zeta(x) + \mathrm{curl}\,v(x)$ and inserting into the former gives as restriction $\Delta\,\mathrm{curl}\,v = 0$. The boundary condition leads to $\nabla\zeta + \mathrm{curl}\,v = 0$ at $x \in \partial RVE^\sharp$. Thus, once ζ is chosen freely, $\mathrm{curl}\,v$ is uniquely determined. The term is not conformally invariant.
9. $\nabla\,\mathrm{Div}\,u = 0$ implies $\mathrm{Div}\,u = \mathrm{const}$. The displacement can be represented by $u(x) = \nabla\zeta + \mathrm{curl}\,v$ with $\Delta\zeta = \mathrm{const}$. Incorporating the boundary condition leads to the result that the general solution is $u(x) = \nabla\zeta + \mathrm{curl}\,v + \widehat{B}.x$ where $\Delta\zeta = 0$ and $\nabla\zeta + \mathrm{curl}\,v = 0$ at the boundary.
10. The last four cases have been investigated in [43, sect.3].

Note that the symmetric part $\mathrm{sym}(\widehat{B}).x$ can always be realised as $\mathrm{sym}(\widehat{B}).x = \nabla\zeta_1(x)$ for $\zeta_1(x) = \frac{1}{2}\langle\widehat{B}.x, x\rangle$. In the ninth and tenth case we must have $\nabla\zeta(x) = 0$ for $x \in \partial RVE^\sharp$, while in the last two cases the infinitesimal conformal mapping ϕ_C and the potential ζ are such that $\phi_C(x) + \nabla\zeta(x) = 0$ for $x \in \partial RVE^\sharp$. ■

We observe that the first six curvature expressions stipulate **homogeneous response for homogeneous data**, which is consistent with the initial minimization problem (2.1) only if constant elastic moduli are assumed. Thus the first six curvature terms are appropriate only

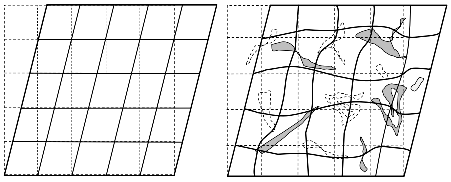

Figure 2: **Left**: Homogeneous deformation of the $RVE^\sharp$ cluster which is assumed homogeneous inside due to homogeneous boundary conditions $y \mapsto \widehat{B}.y$, where $y \in \mathbb{R}^3$ is the local coordinate variable. **Right**: Inhomogeneous response (micro-fluctuations) for same homogeneous boundary conditions due to random, heterogeneously distributed material inside the $RVE^\sharp$.

for **homogeneous Cauchy material inside the** $RVE^\sharp$**:** $\mu(x), K(x) = \text{const.}$, i.e. no microstructural fluctuation is possible.

However, if we think of the cluster $RVE^\sharp$ as consisting of **random isotropic Cauchy material** $\mu(x), K(x)$ the **response to** applied **homogeneous data is**, in general, **not homogeneous**, see Figure 2. The next seven curvature energies allow for such an inhomogeneous response to a different degree: the pointwise positive Mindlin curvature energy $\|\nabla \operatorname{curl} u\|^2$ adds the possibility of an arbitrary irrotational displacement field $\nabla\zeta$ as microstructural fluctuation (in terms of displacement gradients it adds a strain like micro-fluctuation $D^2\zeta$), and the conformal curvature term $\|\operatorname{sym}\nabla \operatorname{curl} u\|^2$ allows in addition for non-irrotational fluctuation with second order polynomials ϕ_C. Of course, the solution for arbitrary random substructure cannot always be written in this form (indeed, it seems that it can never be obtained with only conservative fluctuations if inhomogeneities are present) but we realise, based on Helmholtz decomposition of the displacement u into scalar and vector potential

$$u(x) = \nabla\zeta(x) + \operatorname{curl} w(x)\,, \tag{2.13}$$

that the additional conformal curvature allows at least for the vector potential $w : \mathbb{R}^3 \mapsto \mathbb{R}^3$ to be a second order polynomial since $\operatorname{curl}\phi_C(x) = \widehat{A}.x + \widehat{b}$ for some constant $\widehat{A} \in \mathfrak{so}(3)$ and $\widehat{b} \in \mathbb{R}^3$. In this interpretation, the emergence of the conformal curvature expression for the treatment of random microstructures in a scale-dependent homogenisation framework becomes clear.

3 Homogenisation

3.1 General multiscale setting

What is missing is a micromechanical motivation of the curvature energy in gradient elasticity and the indeterminate couple stress model and of conformal invariance on the continuum level by micro-mechanical considerations. We assume to deal with statistically random Cauchy material in general. Nevertheless, there is a certain scale below which we are not interested in the displacement details. In such a general multiscale situation it seems clear that a unique homogenised medium does not exists. In our approach we cover the body of interest Ω with $RVE^\sharp$-clusters containing a representative microstructure with no preferred directions. The form of the $RVE^\sharp$ could be cube like or sphere like but for the following we use spheres (of diameter $L_c^{\mathrm{RVE}^\sharp}$).

Further, we consider a rudimentary homogenisation method in which we use the word "homogenisation" in a loose sense: the underlying assumption is that there exist two distinct levels in the body of interest: a discontinuous, heterogeneous microscopic one, consisting of matrix material, voids and other inhomogeneities, and a continuous macroscopic one. The representative volume element $RVE^\sharp$ [15] defines the order of the scale of resolution of the envisaged

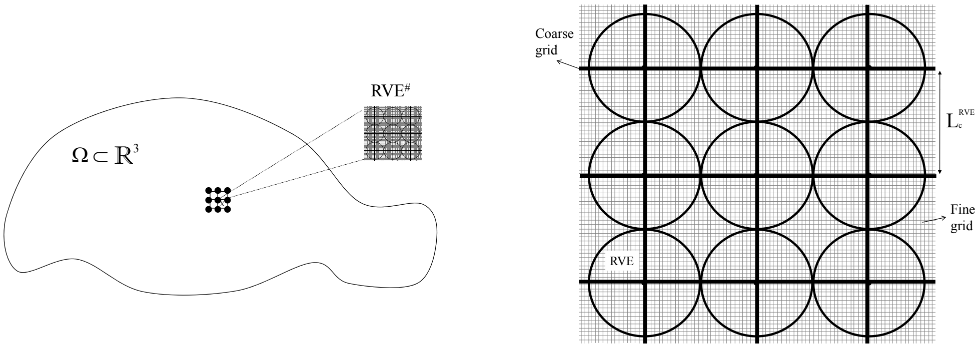

Figure 3: **Left**: The basic situation of our multiscale approach. The black points symbolise the mesoscale constituting the $RVE^\sharp$ cluster. **Right**: In addition to an arbitrary fine grid which is always present and which corresponds to size-independent linear Cauchy elastic response (no length scale associated to this fine grid) we have introduced a large scale structure, the subgrid, from which to extract information on the curvature energy. Size effects are really related to the additional subgrid interaction which we represent through neighbouring $RVE(0)$. The subgrid is not necessarily regular. The question we have to answer is: what kind of elastic properties should the subgrid have? The answer will determine the curvature energy. The absence of the subgrid interaction means size-independent response. If the subgrid was simply a re-inforcing beam structure we would expect a uniformly positive curvature energy.

continuum model, effects below this scale do not appear explicitly in the final model, cf. Figure 3. Summarising, we assume that

- The $RVE^\sharp$-cluster is big enough to be representative of the microstructure in a statistical sense.
- The $RVE^\sharp$-cluster is small enough compared to the actual sample size for it to be considered to be infinitesimal.
- The $RVE^\sharp$-cluster is yet big enough compared to the sample size in order to still influence the macroscopic response. No scale separation applies.

Finally, we introduce one additional preliminary assumption: we focus mainly on rotational inhomogeneity.

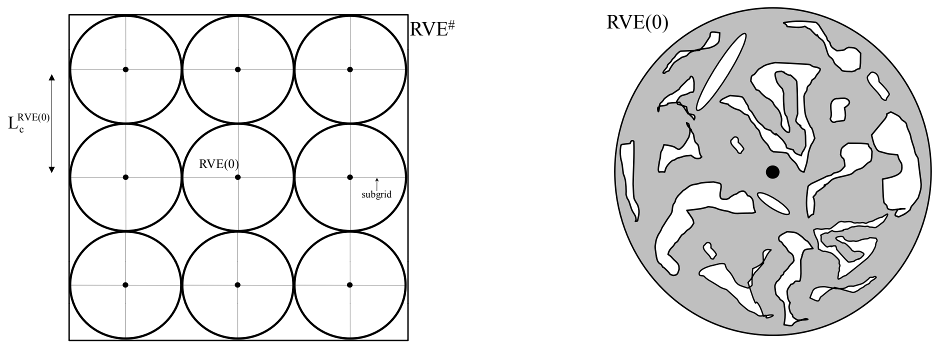

Figure 4: **Left**: The $RVE^\sharp$ cluster fill the body Ω. Each $RVE^\sharp$ represents a cluster of smaller $RVE(0)$ which themselves define the cutoff length (averaging scale) and which interact (inside the $RVE^\sharp$ cluster). **Right**: We assume that each $RVE^\sharp$-cluster consists of micro-heterogeneous, micro-random Cauchy elastic material.

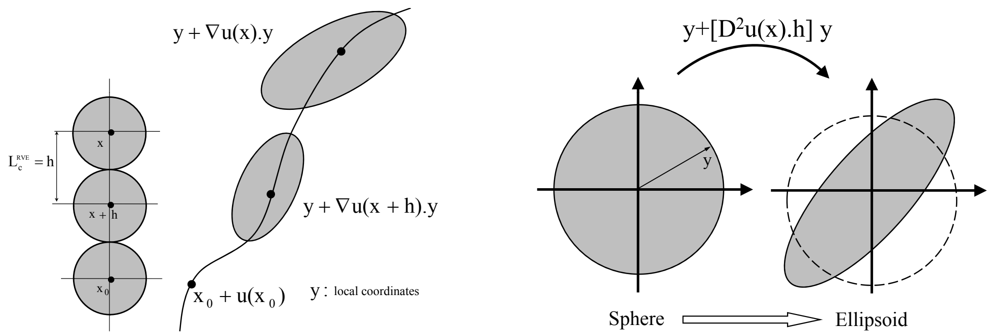

Figure 5: **Left**: The center of mass x, $x+h$ of neighbouring $RVE(0)$ inside the subgrid cluster $RVE^\sharp$ is transformed by the corresponding displacement $u(x)$. The $RVE(0)$ itself is mapped by the displacement gradient $\nabla u(x)$. **Right**: The difference $\nabla u(x+h) - \nabla u(x) = D^2u(x).h + \ldots$ between the mappings of two neighbouring $RVE(0)$ inside the subgrid cluster $RVE^\sharp$ is due to curvature and can be visualised as well by the deformation of a sphere into an ellipsoid. The interaction of $RVE(0)$ determines the energy storage due to curvature and vice versa.

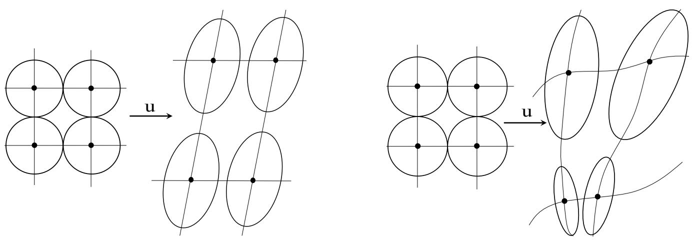

Figure 6: **Left**: Homogeneous deformation of the subgrid generates no subgrid inhomogeneity and therefore no subgrid interaction/curvature. **Right**: Inhomogeneous mapping of the subgrid will generate interaction energy.

3.2 Subgrid interaction modelled by clusters of RVE(0)-interfaces

Whenever $L_c > 0$ is present in the curvature energy we have to deal with the additional subgrid structure inducing an additional energy transfer from the subgrid level onto the resolved/continuum level. This energy transfer onto the resolved scale can be interpreted as describing how neighbouring $RVE(0)$ interact across their interface inside a $RVE^\sharp$-cluster.

Let therefore $h \in \mathbb{R}^3$ be a subgrid direction, orthogonal to the interface. Our idea is to define the h-directional interior subgrid interaction to be a function between neighbouring $RVE(0)$, taking higher order differences into account, see Figures 5 and 6. For these deformation gradient difference we write

$$W^{\text{subgrid}}(1\!\!1 + \nabla u(x+h), 1\!\!1 + \nabla u(x))\,. \tag{3.1}$$

3.3 Micro-randomness and conformal invariance

We assume that the material is micro-random. By this we presuppose that there is no preferred direction at no scale, especially not on the micro-scale. It implies that we are allowed to cut out neighbouring $RVE(0)$ inside the $RVE^\sharp$-cluster, rotate them individually with arbitrary rotation angle, re-insert them back again, without changing the induced subgrid interaction energy, see Figure 7. This is an additional constitutive assumption at the micro and meso-level which is not implied by assuming homogeneous elastic isotropic response at the phenomenological continuum level nor is it related to frame-indifference requirements.

Note that micro-randomness is certainly not satisfied for a regular beam structure. Indeed, it does not make sense to rotate neighbouring beam structural elements against each other without changing the response. Thus the concept of micro-randomness is "orthogonal" to regular lattice type structures, see Figure 8. Moreover, micro-randomness is a notion that is applied

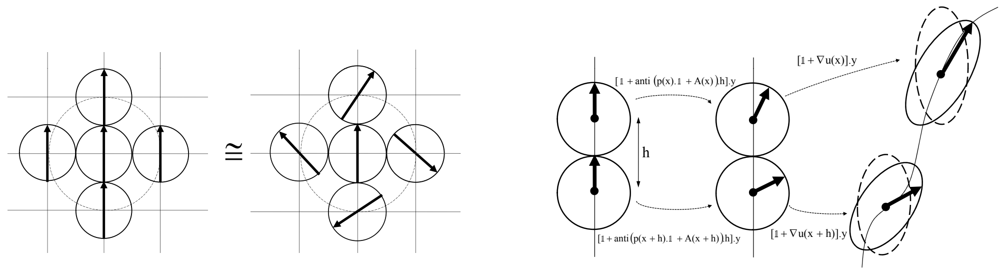

Figure 7: **Left**: Illustration of **micro-randomness**: a re-arrangement of neighbouring $RVE(0)$ inside the subgrid cluster $RVE^\sharp$ in a "discrete" meaning should not change the subgrid response after homogenisation. **Right**: Superposition of local rotation and deformation on the discrete level motivating micro-randomness. Principle of **"first re-arrange then transform"**. The interaction law between neighbouring $RVE(0)$ should be invariant w.r.t. this re-arrangement: the subgrid does not see the "arrows". Note that this is a picture/invariance before homogenisation. After homogenisation (averaging over all unit subgrid directions h) this invariance is "nearly" lost. But it will imply the symmetry of the higher order moment stresses after homogenisation.

"before" homogenisation: it is not meant that at the continuum level "after" homogenisation one could cut out and rotate arbitrary without changing the response. The consequences of micro-randomness on the subgrid interaction will nevertheless be fundamental. What are these consequences? To understand this let us formalise the arbitrary rotational re-arrangement idea. On the "discrete" directional h-level, we may consider the rotational re-arrangement (see Figure 7) to be the effect of first the superposition of an arbitrary purely local infinitesimal rotation of the form[8] $\mathrm{anti}([p(x+h)\,1\!\!1 + A(x+h)].h) \in \mathfrak{so}(3)$ onto the $RVE(0)$ with center of mass $x+h$ and second the application of the subgrid deformation, since (with local coordinates $y \in \mathbb{R}^3$)

$$
\begin{aligned}
y \quad & \overbrace{\mapsto}^{\text{first rotate/re-arrange}} \quad [1\!\!1 + \mathrm{anti}([p(x+h)\,1\!\!1 + A(x+h)].h)].y \quad \overbrace{\mapsto}^{\text{then deform}} \\
& [1\!\!1 + \nabla u(x+h)].[1\!\!1 + \mathrm{anti}([p(x+h)\,1\!\!1 + A(x+h)].h)].y \;\Rightarrow \\
y \mapsto & \; [1\!\!1 + \nabla u(x+h) + \mathrm{anti}([p(x+h)\,1\!\!1 + A(x+h))]h)].y + \dots
\end{aligned}
\tag{3.3}
$$

This will lead us to our definition of discrete micro-randomness:

Definition 3.1 (Discrete micro-randomness before homogenisation)
We call a material to be micro-random whenever the subgrid directional response is invariant under a superposed infinitesimal rotation of the form (3.3), *i.e. it satisfies*

$$
\begin{aligned}
\forall\, h \in \mathbb{R}^3 : \quad & W^{\mathrm{subgrid}}(1\!\!1 + \nabla u(x+h) + \mathrm{anti}([p(x+h)\,1\!\!1 + A(x+h))]h), 1\!\!1 + \nabla u(x)) \\
& = W^{\mathrm{subgrid}}(1\!\!1 + \nabla u(x+h) + \mathrm{anti}(p(x+h)\,1\!\!1.h), 1\!\!1 + \nabla u(x))\,.
\end{aligned}
\tag{3.4}
$$

Remark 3.2
The influence of the remaining term $\mathrm{anti}([p(x+h)\,1\!\!1].h) \in \mathfrak{so}(3)$ *will disappear only after averaging over all directions* h.

With a view towards the indeterminate couple stress model we restrict ourselves to considerations of rotational inhomogeneity (rotational interaction between $RVE(0)$ inside the $RVE^\sharp$-cluster). Thus our quadratic discrete subgrid energy should be expressible as

$$
W^{\mathrm{subgrid}}_{\mathrm{rot}}(1\!\!1 + \nabla u(x+h), 1\!\!1 + \nabla u(x)) = Bilinear(\mathrm{skew}[\nabla u(x+h) - \nabla u(x)])\,, \tag{3.5}
$$

[8]For $n = 3$ it is possible to show that choosing $p \in \mathbb{R}$ and $A \in \mathfrak{so}(3)$ appropriately, we may generate every infinitesimal rotation $W \in \mathfrak{so}(3)$ through $\mathrm{anti}((p\,1\!\!1 + A).h) = W$ for any fixed given direction h. This can be based on the observation that the matrix

$$
\big(h \,|\, \mathrm{anti}(h)\big)_{3\times 4} \tag{3.2}
$$

has full rank three for every direction $h \in \mathbb{R}^3$, $h \neq 0$. Thus $\{\mathrm{anti}([\mathbb{R}\,1\!\!1 + \mathfrak{so}(3)].h)\} = \mathfrak{so}(3)$.

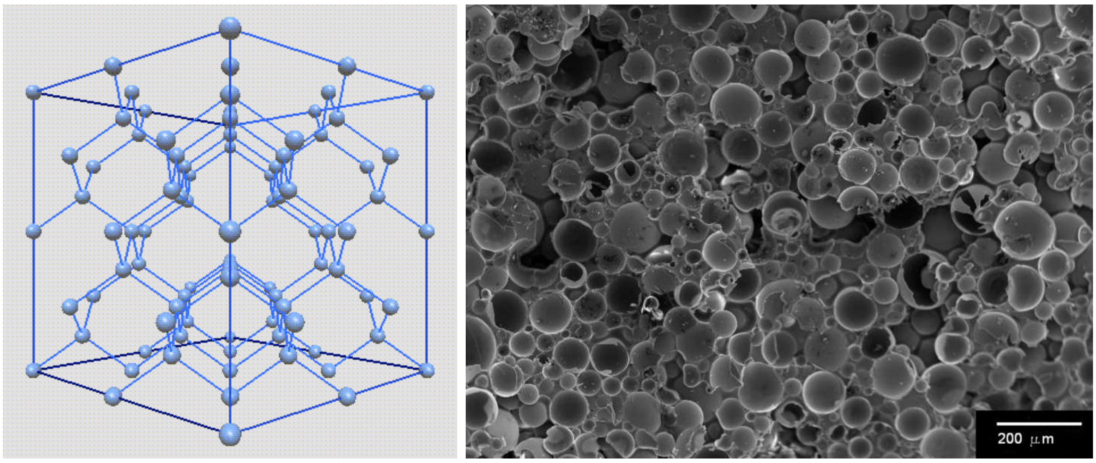

Figure 8: **Left**: A regular diamond lattice structure of an ideal single crystal cannot be considered to be micro-random. **Right**: A syntactic foam structure is to a fair degree micro-random. Is it by chance that Lakes [29, 3, 30, 31] identified foams to have conformal Cosserat curvature [22, 43, 23, 41] consistent with micro-randomness?

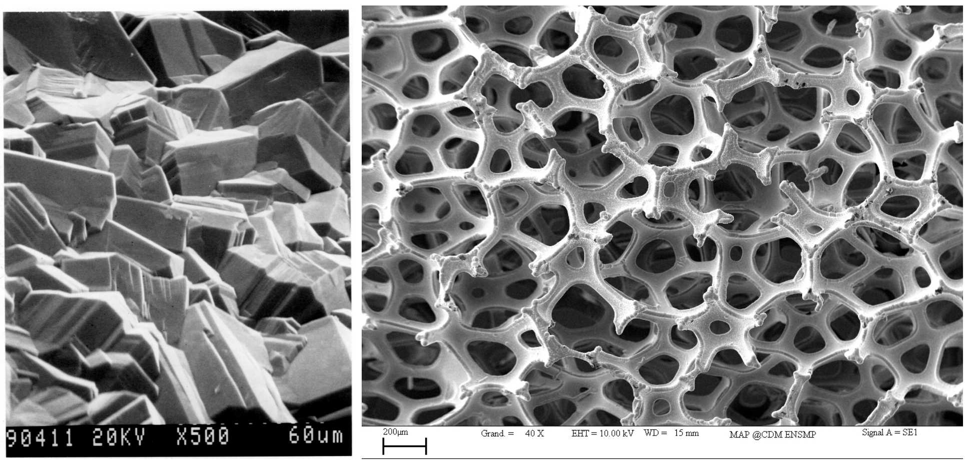

Figure 9: **Left**: A non-textured polycrystalline diamond film might also be considered to be micro-random. The window-size would actually correspond to our $RVE^\sharp$-cluster. **Right**: An open cell foam structure is also micro-random.

where it is understood that ∇u appears quadratically. Further, we base our investigation of rotational inhomogeneity on the subgrid level with spacing h on the specific discrete difference

$$\begin{aligned} W_{\text{rot}}^{\text{subgrid}}(1\!\!1 + \nabla u(x+h), 1\!\!1 + \nabla u(x)) &= \langle \nabla u(x+h) - \nabla u(x), \text{anti}(h)\rangle^2_{\mathbb{M}^{3\times 3}} \\ &= \langle \text{skew}[\nabla u(x+h) - \nabla u(x)], \text{anti}(h)\rangle^2_{\mathfrak{so}(3)} = \langle \text{curl}\, u(x+h) - \text{curl}\, u(x)], h\rangle^2 . \end{aligned} \tag{3.6}$$

Since $\text{anti}(h) \in \mathfrak{so}(3)$, only rotational inhomogeneities of the form $\text{skew}(\nabla u(x+h) - \nabla u(x))$ are seen at all while strain-type inhomogeneities like $\text{sym}(\nabla u(x+h) - \nabla u(x)) = \varepsilon(x+h) - \varepsilon(x)$ are ignored. Let us show that this $W_{\text{rot}}^{\text{subgrid}}$ is indeed micro-random according to Definition 3.1.

Lemma 3.3 ($W_{\text{rot}}^{\text{subgrid}}$ is micro-random)

Proof. Consider (without loss of generality we can drop the dependence of $(p, A) \in \mathbb{R} \times \mathfrak{so}(3)$ on their space position $x \in \mathbb{R}^3$)

$$\begin{aligned} &\langle [\nabla u(x+h) + \text{anti}([p\,1\!\!1 + A].h)] - [\nabla u(x) + \text{anti}([p\,1\!\!1 + A].0)], \text{anti}(h)\rangle^2 \\ &= \langle \nabla u(x+h) + \text{anti}([p\,1\!\!1].h) - \nabla u(x), \text{anti}(h)\rangle^2 . \end{aligned} \tag{3.7}$$

Here, $A \in C^1(\mathbb{R}^3, \mathfrak{so}(3))$, $p \in C^1(\mathbb{R}^3, \mathbb{R})$ and $h \in \mathbb{R}^3$ are otherwise arbitrary. The equivalence holds since

$$\langle \text{anti}(A.h), \text{anti}(h)\rangle = 2\,\langle \text{axl}\,\text{anti}(A.h), \text{axl}\,\text{anti}(h)\rangle_{\mathbb{R}^3} = \langle A.h, h\rangle_{\mathbb{R}^3} = 0 \tag{3.8}$$

for $A \in \mathfrak{so}(3)$ and all $h \in \mathbb{R}^3$. Thus we have shown that (3.6) is in fact invariant under superposed local rotations (3.3). Therefore, (3.6) is micro-random. ■

In order to come up with maniable expressions we simplify our directional subgrid response by switching to second derivatives in the Taylor-expansion $\nabla u(x+h) = \nabla u(x) + D^2u(x).h + \ldots$. Because in our multiscale model, h is not infinitesimally small anyway, there is no reason to neglect the second term. Since

$$\begin{aligned} W_{\text{rot}}^{\text{subgrid}}(1\!\!1 + \nabla u(x+h), 1\!\!1 + \nabla u(x)) &= \langle \text{skew}[\nabla u(x+h) - \nabla u(x)], \text{anti}(h)\rangle^2 \\ &= \langle \text{skew}[D^2u(x).h + \ldots, \text{anti}(h)\rangle^2 = \langle \text{skew}[D^2u(x).h], \text{anti}(h)\rangle^2 + \ldots \\ &= \langle \nabla\, \text{curl}\, u(x).h, h\rangle^2 + \ldots , \end{aligned} \tag{3.9}$$

we replace therefore the difference measure $W_{\text{rot}}^{\text{subgrid}}$ by

$$\widehat{W}_{\text{rot}}^{\text{subgrid}}(D^2u.h) := \langle \text{skew}[D^2u(x).h], \text{anti}(h)\rangle^2 . \tag{3.10}$$

Lemma 3.4 ($\widehat{W}_{\text{rot}}^{\text{subgrid}}$ is micro-random)

Proof. The proof is the same as for Lemma 3.3. We only have to replace the discrete gradient differences by the corresponding linearization

$$\begin{aligned} &[[\nabla u(x+h) + \text{anti}([p\,1\!\!1 + A].h)] - [\nabla u(x) + \text{anti}([p\,1\!\!1 + A].0)]] \\ &= [D^2u(x).h + \text{anti}([p\,1\!\!1 + A].h)] + \ldots \end{aligned}$$

■

Remark 3.5 (Micro-randomness and discrete curvature energy)
Taking instead $\|\text{skew}[D^2u(x).h]\|^2$ *as a discrete quadratic measure for rotational inhomogeneity is possible, but it is not micro-random since* $\|\text{skew}[D^2u(x).h + \text{anti}([p\,1\!\!1].h)]\|^2 \neq \|\text{skew}[D^2u(x).h + \text{anti}([p\,1\!\!1 + A].h])\|^2$. *After averaging over all directions, this term is, up to a multiplicative constant, the pointwise positive definite expression* $\|\nabla\,\text{curl}\,u\|^2$. *Similarly,* $\|\text{dev}\,\text{sym}[D^2u(x).h]\|^2$ *is (trivially) micro-random since the rotational re-arrangement is completely "swallowed" by the symmetrisation operator* sym

$$\|\text{dev}\,\text{sym}[D^2u(x).h]\|^2 = \|\text{dev}\,\text{sym}[D^2u(x).h + \text{anti}([p\,1\!\!1 + A].h])]\|^2 \tag{3.11}$$

(already in the integrand before averaging), but the expression would also couple with strain-like inhomogeneities. See (6.13) for the analytical expression after averaging over all directions. The same remark applies to the micro-random expression $\langle D^2u(x).h, 1\!\!1\rangle^2 = \langle \nabla\,\text{Div}\,u.h, h\rangle^2$ *which does couple with volumetric inhomogeneities.*

3.4 Averaging over all subgrid directions

Since we assume that the **subgrid has no preferred direction** either, we are consequently led to **average** the expression $\widehat{W}_{\mathrm{rot}}^{\mathrm{subgrid}}$ **over all directions** $h \in L_c \cdot \mathbb{S}^2$ in a second step, i.e., we define

$$\begin{aligned} W_{\mathrm{curv}}(D^2u(x)) &:= \frac{1}{4\,\pi\,L_c^2}\int_{h\in L_c\cdot\mathbb{S}^2}\frac{1}{L_c^2}\,\widehat{W}_{\mathrm{rot}}^{\mathrm{subgrid}}(D^2u(x).h)\,\mathrm{d}\mathbb{S}^2_{L_c} \\ &= \frac{1}{4\,\pi\,L_c^2}\int_{h\in L_c\cdot\mathbb{S}^2}\frac{1}{L_c^2}\,\langle\underbrace{D^2u(x).h}_{\text{no units}},\mathrm{anti}(h)\rangle^2\,\mathrm{d}\mathbb{S}^2_{L_c} \\ &= \frac{1}{4\,\pi\,L_c^2}\int_{\tilde{h}\in\mathbb{S}^2}\frac{1}{L_c^2}\,\langle D^2u(x).(L_c\,\tilde{h}),\mathrm{anti}(L_c\,\tilde{h})\rangle^2\,L_c^2\,\mathrm{d}\mathbb{S}^2_1 \\ &= \frac{L_c^4}{4\,\pi\,L_c^2}\int_{\tilde{h}\in\mathbb{S}^2}\langle D^2u(x).\tilde{h},\mathrm{anti}(\tilde{h})\rangle^2\,\mathrm{d}\mathbb{S}^2_1\,. \end{aligned} \tag{3.12}$$

Here, $4\,\pi\,L_c^2$ is the surface measure of the sphere $\mathbb{S}^2_{L_c} = L_c\cdot\mathbb{S}^2$ with radius L_c.[9] The result is trivially independent of the direction h and a quadratic expression in the second partial derivatives D^2u. Moreover, it is conformally invariant, see Appendix 6.2. The spherical integration can be made explicit, see also (6.2). It holds

$$W_{\mathrm{curv}}(D^2u(x)) = \frac{4\,L_c^2}{15}\,\|\,\mathrm{sym}\,\nabla\,\mathrm{curl}\,u(x)\|^2\,, \tag{3.13}$$

i.e., up to a constant factor the conformally invariant curvature energy in (2.7).

It remains to show that superposing the local re-arrangement $\mathrm{anti}([p\,\mathbb{1}+A].\tilde{h}) \in \mathfrak{so}(3)$ leaves the homogenised response invariant up to a (unimportant) global constant. To see this, consider

$$\begin{aligned} &\int_{\tilde{h}\in\mathbb{S}^2}\langle D^2u(x).\tilde{h}+\mathrm{anti}((p\,\mathbb{1}+A).\tilde{h}),\mathrm{anti}(\tilde{h})\rangle^2\,\mathrm{d}\mathbb{S}^2 \\ &= \int_{\tilde{h}\in\mathbb{S}^2}\langle D^2u(x).\tilde{h},\mathrm{anti}(\tilde{h})+p\,\|\,\mathrm{anti}(\tilde{h})\|^2+\langle\mathrm{anti}(A.\tilde{h}),\mathrm{anti}(\tilde{h})\rangle\rangle^2\,\mathrm{d}\mathbb{S}^2 \\ &= \int_{\tilde{h}\in\mathbb{S}^2}\Big(\langle D^2u(x).\tilde{h},\mathrm{anti}(\tilde{h})\rangle+2\,p\Big)^2\,\mathrm{d}\mathbb{S}^2 \\ &= \int_{\tilde{h}\in\mathbb{S}^2}\langle D^2u(x).\tilde{h},\mathrm{anti}(\tilde{h})\rangle^2+4\,p\,\langle D^2u(x).\tilde{h},\mathrm{anti}(\tilde{h})\rangle+4\,p^2\,\mathrm{d}\mathbb{S}^2 \\ &= \int_{\tilde{h}\in\mathbb{S}^2}\langle D^2u(x).\tilde{h},\mathrm{anti}(\tilde{h})\rangle^2+4\,p\,C_1\,\langle\nabla\,\mathrm{curl}\,u(x).\tilde{h},\tilde{h}\rangle+4\,p^2\,\mathrm{d}\mathbb{S}^2 \\ &= \int_{\tilde{h}\in\mathbb{S}^2}\langle D^2u(x).\tilde{h},\mathrm{anti}(\tilde{h})\rangle^2\,\mathrm{d}\mathbb{S}^2+p\,C_2\,\mathrm{tr}\,[\nabla\,\mathrm{curl}\,u(x)]+4\,p^2\,C_3 \\ &= \int_{\tilde{h}\in\mathbb{S}^2}\langle D^2u(x).\tilde{h},\mathrm{anti}(\tilde{h})\rangle^2\,\mathrm{d}\mathbb{S}^2+4\,p^2\,C_3\,. \end{aligned} \tag{3.14}$$

Since we use this result as a curvature energy term, the appearing additive constant (which could even depend on $x \in \Omega$) does not influence the variational formulation. ■

Collecting results we see that, within our micro-randomness assumption, the indeterminate couple stress energy reduces to

$$I(u) = \int_{RVE^\sharp}\mu\,\|\,\mathrm{sym}\,\nabla u\|^2+\frac{\lambda}{2}\,\mathrm{tr}\,[\mathrm{sym}\,\nabla u]^2+\mu\,L_c^2\,\underbrace{\|\,\mathrm{sym}\,\nabla\,\mathrm{curl}\,u\|^2}_{\text{conformal curvature}}\,\mathrm{dV}\,. \tag{3.15}$$

3.5 Gradient elasticity with micro-random subgrid interaction

We do not pursue here in detail the similar question for the general second gradient elasticity model. This would lead to the question: what are the most general isotropic, centro-symmetric curvature terms in D^2u which can be derived starting with discrete directional micro-random

[9] We adhere to the convention that $\mathrm{d}\mathbb{S}^2_1$ is unit free while $\mathrm{d}\mathbb{S}^2_{L_c^2}$ has units $[m^2]$. The additional factor in the integrand is needed for dimensional consistency.

terms. However, for some simple directional subgrid energies (below, left column), which have a clear mechanical interpretation, we can already see that

$$
\begin{aligned}
&\underbrace{\|D^2u(x).h\|^2}_{\text{not micro-random}} &&\overset{\text{averaging}}{\Longrightarrow} &&\underbrace{\|D^2u(x)\|^2}_{\text{not conformally invariant}} \quad , \\
&\underbrace{\|\operatorname{skew}[D^2u(x).h]\|^2}_{\text{not micro-random}} &&\overset{\text{averaging}}{\Longrightarrow} &&\underbrace{\frac{1}{2}\|\nabla \operatorname{curl} u(x)\|^2}_{\text{not conformally invariant}} \quad , \\
&\underbrace{\|\operatorname{sym}[D^2u(x).h]\|^2}_{\text{micro-random}} &&\overset{\text{averaging}}{\Longrightarrow} &&\underbrace{\|D^2u(x)\|^2-\frac{1}{2}\|\nabla \operatorname{curl} u(x)\|^2}_{\text{not conformally invariant}}\, , \qquad (3.16)\\
&\underbrace{\|D[\operatorname{sym}\nabla u(x)].h\|^2}_{\text{micro-random}} &&\overset{\text{averaging}}{\Longrightarrow} &&\underbrace{\|D[\operatorname{sym}\nabla u(x)]\|^2}_{\text{not conformally invariant}} \quad , \quad \text{coincides with the former}\\
&\underbrace{\langle D^2u(x).h, \mathbb{1}\rangle^2}_{\text{micro-random}} &&\overset{\text{averaging}}{\Longrightarrow} &&\underbrace{\|\nabla \operatorname{Div} u(x)\|^2}_{\text{not conformally invariant}} \quad , \\
&\underbrace{\langle \operatorname{skew}[D^2u(x).h], \operatorname{anti}(h)\rangle^2}_{\text{micro-random}} &&\overset{\text{averaging}}{\Longrightarrow} &&\underbrace{\|\operatorname{sym}\nabla \operatorname{curl} u(x)\|^2}_{\text{conformally invariant}}\, , \\
&\underbrace{\|\operatorname{dev}\operatorname{sym}[D^2u(x).h]\|^2}_{\text{micro-random}} &&\overset{\text{averaging}}{\Longrightarrow} &&\underbrace{\|D^2u(x)\|^2_{\mathbb{R}^{27}}-\frac{1}{2}\,\|\nabla \operatorname{curl} u(x)\|^2_{\mathbb{M}^{3\times 3}}-\frac{1}{3}\,\|\nabla \operatorname{Div} u(x)\|^2}_{\text{conformally invariant: }(6.16)_9}\, , \\
&\underbrace{\|D[\operatorname{dev}\operatorname{sym}\nabla u(x)].h\|^2}_{\text{micro-random}} &&\overset{\text{averaging}}{\Longrightarrow} &&\underbrace{\|D[\operatorname{dev}\operatorname{sym}\nabla u(x)]\|^2_{\mathbb{R}^{27}}}_{\text{conformally invariant}}\, , \quad \text{coincides with the former}
\end{aligned}
$$

are individually positive and (some of them in our sense) micro-random, see (6.13) for the detailed proof. We have also checked that $\eta^{(1)}_{ijk}\,\eta^{(1)}_{ijk}$ from (6.28) is conformally invariant as well![10] Based on the orthogonal Cartan Lie-algebra decomposition (for fixed direction $h \in \mathbb{S}^2$) i.e., $\mathfrak{gl}(3) = [\mathfrak{sl}(3) \cap \operatorname{Sym}(3)] \oplus \mathfrak{so}(3) \oplus \mathbb{R}\cdot \mathbb{1}$, we can uniquely decompose

$$
D^2u.h = \operatorname{dev}\operatorname{sym}[D^2u.h] + \operatorname{skew}[D^2u.h] + \frac{1}{3}\operatorname{tr}\left[D^2u.h\right]\mathbb{1}\, , \tag{3.17}
$$

and the three terms $\operatorname{dev}\operatorname{sym}[D^2u.h], \operatorname{skew}[D^2u.h], \operatorname{tr}\left[D^2u.h\right]\mathbb{1}$ can be chosen as an orthogonal basis for the directional subgrid interaction and are therefore naturally privileged candidates on which to base the curvature energy contribution.[11]

Grouping together those terms that arise from micro-randomness and the orthogonality before averaging, we are led to consider curvature energies in the form

$$
\underbrace{a_0\,\|\nabla \operatorname{Div} u\|^2 + a_1\,\|D[\operatorname{dev}\operatorname{sym}\nabla u\|^2}_{\text{strain gradient terms}} \quad + \quad \underbrace{a_2'\,\|\operatorname{sym}\nabla \operatorname{curl} u\|^2}_{\text{symmetric rotational gradient}} \quad . \tag{3.18}
$$

Here, the relevant part for the situation when volumetric gradients are not important

$$
a_1\,\|D[\operatorname{dev}\operatorname{sym}\nabla u\|^2 + a_2'\,\|\operatorname{sym}\nabla \operatorname{curl} u\|^2 \tag{3.19}
$$

is **conformally invariant**. Note that taking $\|D[\operatorname{dev}\operatorname{sym}\nabla u\|^2$ as only strain gradient term determines already a unique solution $u \in H^2(\Omega, \mathbb{R}^3)$ despite loss of pointwise uniform positive definiteness w.r.t. derivatives D^2u, see (6.33). This suggests that (3.19) is a good candidate for further investigations.

[10] Suggesting that $\eta^{(1)}$ might just be an isomorphic mapping of $D[\operatorname{dev}\operatorname{sym}\nabla u]$. But this is not the case: the null-space of $\eta^{(1)}$ as a function of η is 11-dimensional and the 7-dimensional image consists of all second derivative components which appear from irrotational, divergence free displacements, see section 6.7.

[11] The equivalence $\|D[\operatorname{dev}\operatorname{sym}\nabla u(x)]\|^2_{\mathbb{R}^{27}} = \|D^2u(x)\|^2_{\mathbb{R}^{27}} - \frac{1}{2}\,\|\nabla \operatorname{curl} u(x)\|^2_{\mathbb{M}^{3\times 3}} - \frac{1}{3}\,\|\nabla \operatorname{Div} u(x)\|^2_{\mathbb{R}^3}$ is not immediate, see (6.15).

3.6 Example: torsion solution with conformal strain gradient

In order to show the effect of the term $\mu\, L_c^2\, \|D[\operatorname{dev}\operatorname{sym}\nabla u\|^2$ on the size-dependent behaviour let us look at the classical torsion problem of a thin cylindrical bar Ω_T with radius $a > 0$ and length L. We let e_1 be the axis of the bar. Then the classical torsion solution

$$u_\kappa(x_1, x_2, x_3) = \begin{pmatrix} 0 \\ -\kappa x_1\, x_3 \\ \kappa\, x_1\, x_2 \end{pmatrix}, \quad \operatorname{dev}\operatorname{sym}\nabla u_\kappa = \begin{pmatrix} 0 & -\frac{x_3}{2}\kappa & \frac{x_2}{2}\kappa \\ -\frac{x_3}{2}\kappa & 0 & 0 \\ \frac{x_2}{2}\kappa & 0 & 0 \end{pmatrix}, \quad \operatorname{tr}[\nabla u_\kappa] = 0\,, \tag{3.20}$$

where κ is the twist per unit length of the cylindrical bar still satisfies the weak form of equilibrium

$$\begin{aligned} \int_{\Omega_T} & 2\mu\, \langle \operatorname{dev}\operatorname{sym}\nabla u_\kappa, \nabla \delta u\rangle + \lambda \operatorname{tr}[\nabla u_\kappa] \operatorname{tr}[\nabla \delta u] \\ & + 2\mu\, L_c^2 \sum_{i=1}^{3} \langle \partial_{x_i} \operatorname{dev}\operatorname{sym}\nabla u_\kappa, \partial_{x_i} \operatorname{dev}\operatorname{sym}\nabla \delta u\rangle \,\mathrm{dV} = 0 \quad \forall \delta u \in C_0^\infty(\Omega_T, \mathbb{R}^3)\,, \end{aligned} \tag{3.21}$$

and the boundary conditions: traction free boundary conditions at the outer cylindrical surface and applied twist at the horizontal end points. The resultant torque per unit length at the ends of the cylindrical bar can be determined through

$$\begin{aligned} Q_{\mathrm{conf}} &= \frac{d}{d\kappa}\, \frac{1}{L} \int_{\Omega_T} W(\nabla u_\kappa)\,\mathrm{dV} \\ &= \frac{d}{d\kappa}\, \frac{1}{L} \int_0^L \int_0^{2\pi} \int_0^a \left(\mu\, \|\operatorname{dev}\operatorname{sym}\nabla u_\kappa\|^2 + \mu\, L_c^2 \sum_{i=1}^{3} \|\partial_{x_i} \operatorname{dev}\operatorname{sym}\nabla u_\kappa\|^2 \right) r\, dr\, d\phi\, dx_1 \\ &= \frac{\pi\kappa\, \mu\, a^4}{2} \left(1 + \frac{1}{2} \left(\frac{2L_c}{a}\right)^2 \right). \end{aligned} \tag{3.22}$$

As usual, thinner bars are stiffer. This shows that qualitatively the result is similar to the indeterminate couple stress model which results in [32]

$$Q_{\mathrm{couple}} = \frac{\pi\kappa\, \mu\, a^4}{2} \left(1 + 6 \left(\frac{2L_c}{a}\right)^2 \right), \tag{3.23}$$

where the size-independent classical torque per unit length is $Q_0 = \frac{\pi\kappa\,\mu\,a^4}{2}$. Thus, $\mu\, L_c^2\, \|D[\operatorname{dev}\operatorname{sym}\nabla u\|^2$ supports much larger internal length scale parameters L_c for the same set of experimental torsional data, compared to the indeterminate couple stress model with $\mu\, L_c^2 \|\operatorname{sym}\nabla\operatorname{curl} u\|^2$.

4 Relation to the Cosserat/micropolar model

4.1 Least energy lifting

Switching back to rotational inhomogeneity, the infinitesimal Cosserat model [9] can now be seen as a relaxed formulation of the indeterminate couple stress model, in which the higher derivatives act on an independent infinitesimal rotation field $\overline{A} \in \mathfrak{so}(3)$, which itself is coupled energetically to the continuum rotation $\operatorname{curl} u$. Because $\operatorname{curl} u = 2\operatorname{axl}(\operatorname{skew}\nabla u)$, we replace $\operatorname{curl} u$ by $2\operatorname{axl}(\overline{A})$ and the generated coupling term is

$$\mu_c\, \|\operatorname{curl} u - 2\, \operatorname{axl}(\overline{A})\|^2 \tag{4.1}$$

with a new penalty parameter $\mu_c > 0$, called the Cosserat couple modulus. As far as the curvature energy replacement is concerned the Mindlin curvature consisted of the term

$$W^{\mathrm{K}}_{\mathrm{curv}}(\nabla\operatorname{curl} u) = a_2'\, \|\operatorname{sym}\nabla\operatorname{curl} u\|^2 + a_3'\, \|\operatorname{skew}\nabla\operatorname{curl} u\|^2\,, \tag{4.2}$$

which, under micro-randomness assumption reduces to only the symmetric (and trace free) part. We apply a "least replacement principle", i.e. formally lifting the curvature energy from trace free matrices $\mathfrak{sl}(3)$ to $\mathfrak{gl}(3)$ by

$$
\begin{aligned}
W^{\mathrm{K}}_{\mathrm{curv}} : \ \mathfrak{sl}(3) \mapsto \mathbb{R}^+ \quad \rightsquigarrow \quad & W_{\mathrm{curv}} : \mathfrak{gl}(3) \mapsto \mathbb{R}^+ \,, \\
& W_{\mathrm{curv}}(X) := W^{\mathrm{K}}_{\mathrm{curv}}(\mathrm{proj}_{\mathfrak{sl}(3)}(X)) = W^{\mathrm{K}}_{\mathrm{curv}}(\mathrm{dev}\, X)\,,
\end{aligned} \tag{4.3}
$$

where $\mathrm{proj}_{\mathfrak{sl}(3)}(X) = \mathrm{dev}\, X$ is the unique orthogonal projection onto trace-free matrices. Together with the "micro-random" curvature assumption, i.e.,

$$
W^{\mathrm{K}}_{\mathrm{curv}} : \ \mathfrak{sl}(3) \mapsto \mathbb{R}^+ \,, \quad W^{\mathrm{K}}_{\mathrm{curv}}(X) = a_2' \, \| \mathrm{sym}\, X\|^2, \quad X = \nabla \, \mathrm{curl}\, u \,, \tag{4.4}
$$

we obtain altogether the Cosserat curvature energy

$$
W_{\mathrm{curv}}(X) = a_2' \, \| \mathrm{sym}\,\mathrm{dev}\, X\|^2 = a_2' \, \| \mathrm{dev}\,\mathrm{sym}\, X\|^2, \quad X = 2\nabla\, \mathrm{axl}(\overline{A})\,, \tag{4.5}
$$

leading to the conformally invariant Cosserat curvature term

$$
W_{\mathrm{curv}}(\nabla\, \mathrm{axl}(\overline{A})) = 4\, a_2' \, \| \mathrm{dev}\,\mathrm{sym}\, \nabla\, \mathrm{axl}(\overline{A})\|^2 \,, \tag{4.6}
$$

and we arrive at the conformally invariant Cosserat problem [44, 43, 23]:

4.2 The Cosserat model with conformally invariant curvature

For the **displacement** $u : RVE^\sharp \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ and the **skew-symmetric infinitesimal microrotation** $\overline{A} : RVE^\sharp \subset \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ we consider the **two-field** minimization problem

$$
I(u, \overline{A}) = \int_{RVE^\sharp} W_{\mathrm{mp}}(\overline{\varepsilon}) + \mu\, L_c^2 \, \| \mathrm{dev}\,\mathrm{sym}\, \nabla\, \mathrm{axl}\, \overline{A}\|^2 \, \mathrm{dx} \mapsto \quad \min . \text{ w.r.t. } (u, \overline{A}) \tag{4.7}
$$

with

$$
W_{\mathrm{mp}}(\overline{\varepsilon}) = \mu \, \| \mathrm{sym}\, \overline{\varepsilon}\|^2 + \mu_c \, \| \mathrm{skew}\, \overline{\varepsilon}\|^2 + \frac{\lambda}{2} \, \mathrm{tr}\, [\mathrm{sym}\, \overline{\varepsilon}]^2, \quad \overline{\varepsilon} = \nabla u - \overline{A}, \quad u_{|_\Gamma} = u_{\mathrm{d}} \,. \tag{4.8}
$$

The parameter identification of Lakes [29] for a syntactic foam consisting of hollow glass microbubbles embedded in an epoxy matrix has precisely led to this reduced (four parameter) Cosserat formulation. Well-posedness is shown in [22]. The least energy lifting can also be extended to the gradient elasticity model, in which case we obtain a micromorphic model [42] with a specific conformal curvature energy.

5 Conclusion

A major problem of gradient elasticity models is the introduction of many new length scale parameters which are not easily interpreted. In order to compare the predictive power of the new models to some experiments it is therefore necessary to reduce the number of parameters to an absolute minimum. This can be achieved on an ad hoc basis or by formal tensor representations. For the same purpose, we have proposed a homogenisation scheme which takes into account micromechanical structural information. The major new concept is what we call micro-randomness. It represents an additional rotational invariance on the micro-level with consequences on the continuum level which go beyond traditional macroscopic material symmetry requirements like isotropy and centro-symmetry.

Let us first consider models which are based on rotationally interacting $RVE(0)$. If the interaction is micro-random then we always obtain a conformally invariant curvature energy. Thus we have given a physical motivation on the microscale, which leads, upon homogenisation, to a conformal indeterminate couple stress model or Cosserat model.

Micro-randomness is a constitutive assumption which is satisfied by many materials on many scales, but definitely not for regular lattice structures. Comparing with the experimental result of Lakes [29] for a syntactic foam we see that in that case, micro-randomness might actually hold, consistent with the found material parameters. We think that the conformal curvature expression offers thus a fresh departure for the experimental determination of the remaining one length scale in the indeterminate couple stress model and two Cosserat constants μ_c, L_c in the conformal Cosserat model.

A first conclusion is: if a linear elastic micropolar model (or the indeterminate couple stress model) is applicable at all to a material with random microstructure, then one has to use the micro-randomness principle and the homogenised model will have conformal curvature and symmetric moment stresses.

Micro-randomness has also implications for the gradient elastic case which is more general then only rotational interaction. For example the description of grain-size effects become important for polycrystalline materials which are certainly subject to micro-randomness. In this case the more general elastic curvature energy for gradient elasticity in (3.19) might be a well founded choice. The same applies to cellular materials and foam structures. Here, the number of curvature parameters is also reduced from five to three. Restricting then further to conformally invariant strain gradient terms gives a maximal reduction to one additional length scale parameter in front of $\|D[\operatorname{dev}\operatorname{sym}\nabla u]\|^2 = \sum_{i=1}^{3} \|\partial_i[\operatorname{dev}\operatorname{sym}\nabla u]\|^2$ (the norm of the gradient of deviatoric strain) giving a model which still controls completely all second derivatives of the displacement u despite first appearance (6.33). We believe that such a model merits further attention.

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Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times3}$ the set of real 3×3 second order tensors, written with capital letters and the set $\mathrm{Sym}(n)$ denotes all symmetric $n \times n$-matrices. The standard Euclidean scalar product on $\mathbb{M}^{3\times3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times3}} = \mathrm{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times3}$. The identity tensor on $\mathbb{M}^{3\times3}$ will be denoted by $1\!\!1$, so that $\mathrm{tr}\,[X] = \langle X, 1\!\!1\rangle$. We set $\mathrm{sym}(X) = \frac{1}{2}(X^T + X)$ and $\mathrm{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \mathrm{sym}(X) + \mathrm{skew}(X)$. For $X \in \mathbb{M}^{3\times3}$ we set for the deviatoric part $\mathrm{dev}\, X = X - \frac{1}{3}\,\mathrm{tr}\,[X]\, 1\!\!1 \in \mathfrak{sl}(3)$ where $\mathfrak{sl}(3)$ is the Lie-algebra of traceless matrices and $\mathfrak{gl}(3) = \mathbb{M}^{3\times3}$ is the Lie-algebra of $\mathrm{GL}(3)$. The Lie-algebra of $\mathrm{SO}(3) := \{X \in \mathrm{GL}(3)\,|X^TX = 1\!\!1,\ \det[X] = 1\}$ is given by the set $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times3} \mid X^T = -X\}$ of all skew symmetric tensors. The canonical identification of $\mathfrak{so}(3)$ and $\mathbb{R}^3$ is denoted by $\mathrm{axl}\,\overline{A} \in \mathbb{R}^3$ for $\overline{A} \in \mathfrak{so}(3)$. The Curl operator on the three by three matrices acts row-wise, i.e.

$$\mathrm{Curl}\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} = \begin{pmatrix} \mathrm{curl}(X_{11}, X_{12}, X_{13})^T \\ \mathrm{curl}(X_{21}, X_{22}, X_{23})^T \\ \mathrm{curl}(X_{31}, X_{32}, X_{33})^T \end{pmatrix}. \tag{5.1}$$

Moreover, we have

$$\forall\, A \in \mathrm{C}^1(\mathbb{R}^3, \mathfrak{so}(3)): \quad \mathrm{Div}\, A(x) = -\,\mathrm{curl}\,\mathrm{axl}(A(x)). \tag{5.2}$$

Note that $(\mathrm{axl}\,\overline{A}) \times \xi = \overline{A}.\xi$ for all $\xi \in \mathbb{R}^3$, such that

$$\begin{aligned} &\mathrm{axl}\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad \overline{A}_{ij} = \sum_{k=1}^{3} -\varepsilon_{ijk} \cdot \mathrm{axl}\,\overline{A}_k\,, \\ &\|\overline{A}\|^2_{\mathbb{M}^{3\times3}} = 2\,\|\,\mathrm{axl}\,\overline{A}\|^2_{\mathbb{R}^3}\,, \qquad \langle \overline{A}, \overline{B}\rangle_{\mathbb{M}^{3\times3}} = 2\langle \mathrm{axl}\,\overline{A}, \mathrm{axl}\,\overline{B}\rangle_{\mathbb{R}^3}\,, \end{aligned} \tag{5.3}$$

where ε_{ijk} is the totally antisymmetric permutation tensor. Here, $\overline{A}.\xi$ denotes the application of the matrix $\overline{A}$ to the vector ξ and $a \times b$ is the usual cross-product. Moreover, the inverse of axl is denoted by anti and defined by

$$\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \mathrm{anti}\begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad \mathrm{axl}(\mathrm{skew}(a \otimes b)) = -\frac{1}{2}\, a \times b\,, \tag{5.4}$$

and

$$2\,\mathrm{skew}(b \otimes a) = \mathrm{anti}(a \times b) = \mathrm{anti}(\mathrm{anti}(a).b)\,. \tag{5.5}$$

Moreover,

$$\mathrm{curl}\, u = 2\,\mathrm{axl}(\mathrm{skew}\,\nabla u)\,. \tag{5.6}$$

By abuse of notation we denote the differential $D\varphi$ of the deformation $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by $\nabla\varphi$. This implies a transposition in certain comparisons with other literature since here $(\nabla\varphi)_{kj} = \partial_j\varphi_k$ is understood. Differentials of second order matrices are denoted by D, such that strain gradients become $D\varepsilon$. For repeated indices in index notation Einstein summation convention is applied.

6 Appendix

6.1 Second order expansions

Let us gather some expansions and developments which we need in the homogenisation part.

$$\begin{aligned}
[\mathbb{1}+\nabla u(x+h)].y-[\mathbb{1}+\nabla u(x)].y &= [D^2u(x).h].y+\dots\,,\\
\operatorname{tr}[[\mathbb{1}+\nabla u(x+h)]-[\mathbb{1}+\nabla u(x)]] &= \operatorname{tr}\left[D^2u(x).h\right]+\dots\,,\\
\operatorname{tr}[[\mathbb{1}+\nabla u(x+h)]-[\mathbb{1}+\nabla u(x)]] &= \operatorname{Div}u(x+h)-\operatorname{Div}u(x)=\langle\nabla\operatorname{Div}u(x),h\rangle+\dots\\
&= \operatorname{tr}\left[D^2u(x).h\right]+\dots\,, \qquad (6.1)\\
\operatorname{sym}[[\mathbb{1}+\nabla u(x+h)]-[\mathbb{1}+\nabla u(x)]] &= \operatorname{sym}[D^2u(x).h]+\dots\,,\\
\operatorname{dev}\operatorname{sym}[[\mathbb{1}+\nabla u(x+h)]-[\mathbb{1}+\nabla u(x)]] &= \operatorname{dev}\operatorname{sym}[D^2u(x).h]+\dots\,,\\
\operatorname{skew}([\mathbb{1}+\nabla u(x+h)]-[\mathbb{1}+\nabla u(x)]) &= \operatorname{skew}(D^2u(x).h)+\dots\,,\\
2\operatorname{axl}[\operatorname{skew}([\mathbb{1}+\nabla u(x+h)]-[\mathbb{1}+\nabla u(x)])] &= 2\operatorname{axl}[\operatorname{skew}(D^2u(x).h)]+\dots\,,\\
\operatorname{curl}u(x+h)-\operatorname{curl}u(x) &= \nabla\operatorname{curl}u(x).h+\dots=2\operatorname{axl}[\operatorname{skew}(D^2u(x).h)]+\dots\,,\\
\nabla\operatorname{curl}u(x).h &= 2\operatorname{axl}[\operatorname{skew}(D^2u(x).h)]\,.
\end{aligned}$$

Since

$$\begin{aligned}
\operatorname{dev}\operatorname{sym}\nabla[u(x+h)-\nabla u(x)] &= \operatorname{dev}\operatorname{sym}\nabla u(x+h)-\operatorname{dev}\operatorname{sym}\nabla u(x)=D[\operatorname{dev}\operatorname{sym}\nabla u(x)].h+\dots\\
&\overbrace{=}^{(6.1)_5}\operatorname{dev}\operatorname{sym}[D^2u(x).h]+\dots \qquad (6.2)
\end{aligned}$$

we can identify

$$D[\operatorname{dev}\operatorname{sym}\nabla u(x)].h=\operatorname{dev}\operatorname{sym}[D^2u(x).h]\,, \qquad (6.3)$$

giving the gradient of deviatoric stretch another representation.

6.2 Spherical integration inside the subgrid cluster $RVE^\sharp$

We make constantly use of the following simple closed form expressions for integrals over the unit sphere [25, 4, 35] where $X\in\mathfrak{gl}(3)$ and $v\in\mathbb{R}^3$ are given,

$$\begin{aligned}
&\int_{h\in\mathbb{S}^2}\langle X.h,h\rangle^2 d\mathbb{S}^2=\frac{4\pi}{15}\left(2\,\|\operatorname{sym}X\|^2+\operatorname{tr}[X]^2\right)\,,\\
&\int_{h\in\mathbb{S}^2}\langle X.h,h\rangle d\mathbb{S}^2=\frac{4\pi}{3}\operatorname{tr}[X]\,,\qquad \int_{h\in\mathbb{S}^2}\langle h,h\rangle^2 d\mathbb{S}^2=\int_{h\in\mathbb{S}^2}1\,d\mathbb{S}^2=4\pi\,,\\
&\int_{h\in\mathbb{S}^2}\langle v,h\rangle^2 d\mathbb{S}^2=\int_{h\in\mathbb{S}^2}\langle(v\otimes v).h,h\rangle d\mathbb{S}^2=\frac{4\pi}{3}\operatorname{tr}[v\otimes v]=\frac{4\pi}{3}\|v\|^2\,. \qquad (6.4)
\end{aligned}$$

The question is: what energy should we attribute to a rotational inhomogeneity since we are mainly interested in rotationally interacting $RVE(0)$. One basis for the measurement is certainly $\nabla\operatorname{curl}u\in\mathbb{M}^{3\times3}$. Since no subgrid direction $\tilde{h}\in\mathbb{S}^2$ is preferred and previous rearrangements should have no influence, we average the induced strain ellipsoid energy over the unit sphere, which gives

$$\begin{aligned}
\int_{h\in\mathbb{S}^2}\langle\nabla\operatorname{curl}u.h,h\rangle^2\,d\mathbb{S}^2 &= \frac{4\pi}{15}\left(2\,\|\operatorname{sym}\nabla\operatorname{curl}u\|^2+\operatorname{tr}[\operatorname{sym}\nabla\operatorname{curl}u]^2\right)\\
&= \frac{4\pi}{15}\left(2\,\|\operatorname{sym}\nabla\operatorname{curl}u\|^2+(\operatorname{Div}\operatorname{curl}u)^2\right)=\frac{8\pi}{15}\,\|\operatorname{sym}\nabla\operatorname{curl}u\|^2\,. \qquad (6.5)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{h\in\mathbb{S}^2}\langle\nabla\operatorname{curl}u.h,h\rangle^2\,d\mathbb{S}^2 &= \int_{h\in\mathbb{S}^2}\langle 2\operatorname{axl}[\operatorname{skew}(D^2u(x).h)],h\rangle^2\,d\mathbb{S}^2=2\int_{h\in\mathbb{S}^2}\frac{1}{4}\langle\operatorname{skew}(D^2u(x).h),\operatorname{anti}(h)\rangle^2\,d\mathbb{S}^2\\
&= \frac{1}{2}\int_{h\in\mathbb{S}^2}\langle D^2u(x).h,\operatorname{anti}(h)\rangle^2\,d\mathbb{S}^2\,. \qquad (6.6)
\end{aligned}$$

From a different perspective we see that the last expression is conformally invariant, since

$$\langle D^2\phi_C(x).h,\operatorname{anti}(h)\rangle=\langle\operatorname{anti}(\widehat{W}.h)+\langle\operatorname{axl}\widehat{W},h\rangle\mathbb{1},\operatorname{anti}(h)\rangle=\langle\operatorname{anti}(\widehat{W}.h),\operatorname{anti}(h)\rangle=\frac{1}{2}\langle\widehat{W}.h,h\rangle=0\,. \qquad (6.7)$$

The same calculation shows that this expression is re-arrangement invariant (micro-random).

With (6.4) we get as well

$$\int_{h\in\mathbb{S}^2}\operatorname{tr}\left[D^2u(x).h\right]^2\,d\mathbb{S}^2=\int_{h\in\mathbb{S}^2}\langle\nabla\operatorname{Div}u(x),h\rangle^2\,d\mathbb{S}^2=\frac{4\pi}{3}\|\nabla\operatorname{Div}u(x)\|^2\,, \qquad (6.8)$$

(an expression which is micro-random but not conformally invariant) and

$$\begin{aligned}
&\int_{h\in\mathbb{S}^2}\|\operatorname{skew}[D^2u(x).h]\|^2_{\mathbb{M}^{3\times3}}\,d\mathbb{S}^2=\int_{h\in\mathbb{S}^2}2\|\operatorname{axl}[\operatorname{skew}[D^2u(x).h]]\|^2_{\mathbb{R}^3}\,d\mathbb{S}^2=\int_{h\in\mathbb{S}^2}\frac{1}{2}\|2\operatorname{axl}[\operatorname{skew}[D^2u(x).h]]\|^2_{\mathbb{R}^3}\,d\mathbb{S}^2\\
&=\int_{h\in\mathbb{S}^2}\frac{1}{2}\|\nabla\operatorname{curl}u(x).h\|^2_{\mathbb{R}^3}\,d\mathbb{S}^2=\int_{h\in\mathbb{S}^2}\frac{1}{2}\langle[\nabla\operatorname{curl}u(x)]^T[\nabla\operatorname{curl}u(x)].h,h\rangle\,d\mathbb{S}^2\\
&=\frac{4\pi}{6}\,\|\nabla\operatorname{curl}u(x)\|^2\,, \qquad (6.9)
\end{aligned}$$

being neither micro-random nor conformally invariant. Moreover,

$$\int_{h\in\mathbb{S}^2} \|D^2u(x).h\|^2_{\mathbb{M}^{3\times3}}\, d\mathbb{S}^2 = \int_{h\in\mathbb{S}^2} \langle [D^2u(x)]^T[D^2u(x)].h, h\rangle\, d\mathbb{S}^2 = \frac{4\pi}{3}\,\|D^2u(x)\|^2_{\mathbb{R}^{27}}\,. \tag{6.10}$$

Thus, applying the orthogonal Cartan Lie-algebra decomposition

$$D^2u.h = \operatorname{dev}\operatorname{sym}[D^2u.h] + \operatorname{skew}[D^2u.h] + \frac{1}{3}\operatorname{tr}\left[D^2u.h\right]\mathbb{1}\,, \tag{6.11}$$

we observe that (due to orthogonality)

$$\|D^2u(x).h\|^2_{\mathbb{M}^{3\times3}} = \|\operatorname{dev}\operatorname{sym}[D^2u(x).h]\|^2_{\mathbb{M}^{3\times3}} + \|\operatorname{skew}[D^2u(x).h]\|^2_{\mathbb{M}^{3\times3}} + \frac{1}{3}\operatorname{tr}\left[D^2u(x).h\right]^2, \tag{6.12}$$

and we obtain

$$\begin{aligned}
\int_{h\in\mathbb{S}^2} \|\operatorname{dev}\operatorname{sym}[D^2u(x).h]\|^2_{\mathbb{M}^{3\times3}}\, d\mathbb{S}^2 &= \int_{h\in\mathbb{S}^2} \|[D^2u(x).h]\|^2 - \|\operatorname{skew}[D^2u(x).h]\|^2 - \frac{1}{3}\operatorname{tr}\left[D^2u(x).h\right]^2 d\mathbb{S}^2\\
&= \frac{4\pi}{3}\,\|D^2u(x)\|^2_{\mathbb{R}^{27}} - \frac{4\pi}{6}\,\|\nabla\operatorname{curl}u(x)\|^2_{\mathbb{M}^{3\times3}} - \frac{4\,\pi}{9}\,\|\nabla\operatorname{Div}u(x)\|^2 \ge 0\,.
\end{aligned} \tag{6.13}$$

Since $\operatorname{dev}\operatorname{sym}[D^2(u+\phi_C).h] = \operatorname{dev}\operatorname{sym}[D^2u.h]$ (see $(6.16)_8$) we observe as well that (6.13) is conformally invariant (independent of this argument, we may also refer to $(6.16)_9$ combined with $(6.16)_{10}$). On the other hand, consider

$$\begin{aligned}
\int_{h\in\mathbb{S}^2} \|D[\operatorname{dev}\operatorname{sym}\nabla u(x)].h\|^2_{\mathbb{M}^{3\times3}}\, d\mathbb{S}^2 &= \int_{h\in\mathbb{S}^2} \langle (D[\operatorname{dev}\operatorname{sym}\nabla u(x)])^T D[\operatorname{dev}\operatorname{sym}\nabla u(x)].h, h\rangle\, d\mathbb{S}^2\\
&= \frac{4\pi}{3}\,\|D[\operatorname{dev}\operatorname{sym}\nabla u(x)]\|^2_{\mathbb{R}^{27}}\,.
\end{aligned} \tag{6.14}$$

With (6.3) we conclude therefore the representation

$$\|D[\operatorname{dev}\operatorname{sym}\nabla u(x)]\|^2_{\mathbb{R}^{27}} = \|D^2u(x)\|^2_{\mathbb{R}^{27}} - \frac{1}{2}\,\|\nabla\operatorname{curl}u(x)\|^2_{\mathbb{M}^{3\times3}} - \frac{1}{3}\,\|\nabla\operatorname{Div}u(x)\|^2_{\mathbb{R}^3}\,. \tag{6.15}$$

6.3 Infinitesimal conformal mappings (ICT) at a glance

Here we gather some useful formulas for infinitesimal conformal mappings. In the following, $\widehat{W}, \widehat{A} \in \mathfrak{so}(3), \widehat{b} \in \mathbb{R}^3, \widehat{p} \in \mathbb{R}$ are arbitrary constant. Infinitesimal conformal mappings preserve (to first order) angles and shapes of infinitesimal figures. More precisely, $\phi_C : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is infinitesimal conformal if and only if its Jacobian satisfies $\nabla\phi_C(x) \in \mathbb{R}\,\mathbb{1} + \mathfrak{so}(3)$. This implies

$$\begin{aligned}
\phi_C(x) &= \frac{1}{2}\left(2\langle \operatorname{axl}(\widehat{W}), x\rangle\, x - \operatorname{axl}(\widehat{W})\,\|x\|^2\right) + [\widehat{p}\,\mathbb{1} + \widehat{A}].x + \widehat{b}\,,\\
\nabla\phi_C(x) &= [\langle \operatorname{axl}(\widehat{W}), x\rangle + \widehat{p}]\,\mathbb{1} + \operatorname{anti}(\widehat{W}.x) + \widehat{A}\,,\\
\operatorname{tr}[\nabla\phi_C(x)] &= 3\,\left[\langle \operatorname{axl}(\widehat{W}), x\rangle + \widehat{p}\right],\\
\operatorname{skew}\nabla\phi_C(x) &= \operatorname{anti}(\widehat{W}.x) + \widehat{A}\,,\\
\operatorname{sym}\nabla\phi_C(x) &= [\langle \operatorname{axl}(\widehat{W}), x\rangle + \widehat{p}]\,\mathbb{1}\,,\\
\operatorname{dev}\operatorname{sym}\nabla\phi_C(x) &= 0\,,\\
\nabla\operatorname{curl}\phi_C(x) &= 2\,\widehat{W} \in \mathfrak{so}(3)\,,\\
D^2\phi_C(x).h &= \langle \operatorname{axl}\widehat{W}, h\rangle\,\mathbb{1} + \operatorname{anti}(\widehat{W}.h) \qquad \in \mathbb{R}\,\mathbb{1} \oplus \mathfrak{so}(3)\,,\\
\|D^2\phi_C\|^2_{\mathbb{R}^{27}} &= \sum_{i=1}^{3} \|D^2\phi.e_i\|^2_{\mathbb{M}^{3\times3}} = \sum_{i=1}^{3} \|\operatorname{anti}(\widehat{W}.e_i)\|^2 + 3\,\langle \operatorname{axl}(\widehat{W}), e_i\rangle^2 = 7\,\|\operatorname{axl}\widehat{W}\|^2\,,\\
\|\nabla\operatorname{Div}\phi_C\| &= 9\,\|\operatorname{axl}\widehat{W}\|^2\,, \quad \|\nabla\operatorname{curl}\phi_C\| = 8\,\|\operatorname{axl}\widehat{W}\|^2\,.
\end{aligned} \tag{6.16}$$

Using $(6.16)_8$, we observe that $\|\operatorname{dev}\operatorname{sym}[D^2u.h]\|^2$ is not only micro-random but also conformally invariant after homogenisation (6.13). In terms of the third order tensor $\eta = D^2u$ we have for the conformal map ϕ_C

$$\begin{aligned}
\eta_{ijk}(\phi_C) = \partial_{ij}\phi^k_C &= [2\operatorname{sym}(\operatorname{axl}(\widehat{W}) \otimes e_k) - \langle \operatorname{axl}(\widehat{W}), e_k\rangle\,\mathbb{1}]_{ij}\\
&= [\operatorname{axl}(\widehat{W})_i\,\delta_{jk} + \operatorname{axl}(\widehat{W})_j\,\delta_{ik}] - \operatorname{axl}(\widehat{W})_k\,\delta_{ij}\,.
\end{aligned} \tag{6.17}$$

This defines a three-dimensional linear space in the set of all second partial derivative $\eta_{ijk} \in \mathbb{R}^{27}$ (due to symmetry in the first two indices only $\mathbb{R}^{18}$). Moreover,

$$\frac{1}{2}\left(2\langle \operatorname{axl}(\widehat{W}), x\rangle\, x - \operatorname{axl}(\widehat{W})\,\|x\|^2\right) = \begin{pmatrix} (-\widehat{W}_{23}\,x_1 + \widehat{W}_{13}\,x_2 - \widehat{W}_{12}\,x_3)\,x_1\\ (-\widehat{W}_{23}\,x_1 + \widehat{W}_{13}\,x_2 - \widehat{W}_{12}\,x_3)\,x_2\\ (-\widehat{W}_{23}\,x_1 + \widehat{W}_{13}\,x_2 - \widehat{W}_{12}\,x_3)\,x_3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -\widehat{W}_{23}\\ \widehat{W}_{13}\\ -\widehat{W}_{12} \end{pmatrix}\,(x_1^2 + x_2^2 + x_3^2)\,.$$

6.4 Formal invariants of the curvature in indicial notation

Following [32, eq.(17)] we let $\eta_{ijk} = \partial_{ij}u_k = \partial_i[\partial_j u_k]$ be the third order tensor of second derivatives, i.e., the entries of D^2u, which is already symmetric in the first two indices. The formal symmetrisation of this tensor is therefore defined by

$$\eta^S_{ijk} := \frac{1}{6}\left(\eta_{ijk} + \eta_{jik} + \eta_{jki} + \eta_{kji} + \eta_{kij} + \eta_{ikj}\right) = \frac{1}{3}\left(\eta_{ijk} + \eta_{jki} + \eta_{kij}\right) = \frac{1}{3}\left(u_{k,ij} + u_{i,jk} + u_{j,ki}\right)\,, \tag{6.18}$$

i.e. η^S is now symmetric with respect to any permutation of the indices.[12]

An incompressible third order tensor η is characterised by $\eta_{ikk} = 0$, $i = 1, 2, 3$. Since from symmetry in the first two slots η has eighteen independent components, instead of twenty-seven, the three relations in $\eta_{ikk} = 0$, $1 = 1, 2, 3$ reduce the number of independent components of an incompressible tensor to fifteen. Note that η^S might not satisfy $\eta^S_{ikk} = 0$, $i = 1, 2, 3$ even if $\eta_{ikk} = 0$, $i = 1, 2, 3$.

An arbitrary gradient elasticity tensor η can be decomposed into its symmetric and completely "antisymmetric" parts $\eta = \eta^S + \eta^A$ where

$$\eta^A_{ijk} := \frac{1}{3}\left(2\eta_{ijk} - \eta_{kji} - \eta_{ikj}\right) = \frac{1}{3}\left(u_{k,i} - u_{i,k}\right)_{,j} + \frac{1}{3}\left(u_{k,j} - u_{j,k}\right)_{,i} = \frac{2}{3}\varepsilon_{ikp}\,\chi_{pj} + \frac{2}{3}\varepsilon_{jkq}\,\chi_{qi}\,, \tag{6.19}$$

with

$$\chi_{ij} = \theta_{i,j} = \frac{1}{2}\varepsilon_{iqr}u_{r,jq} = \frac{1}{2}\varepsilon_{iqr}\eta_{jqr} \tag{6.20}$$

the curvature tensor $\chi = \nabla \operatorname{curl} u$ which has eight independent components since $\operatorname{tr}[\nabla \operatorname{curl} u] = 0$. It holds that η^S is orthogonal to η^A. Further, η^S can be splitted into a (hydrostatic) trace part $\eta^{(0)}$ and a (deviatoric) traceless part $\eta^{(1)}$ according to $\eta^S_{ijk} = \eta^{(0)}_{ijk} + \eta^{(1)}_{ijk}$, where

$$\eta^{(0)}_{ijk} = \frac{1}{5}\left(\delta_{ij}\,\eta^S_{mmk} + \delta_{jk}\,\eta^S_{mmi} + \delta_{ki}\,\eta^S_{mmj}\right), \quad \eta^{(1)}_{ijk} = \eta^S_{ijk} - \eta^{(0)}_{ijk}, \quad \eta^S_{mmk} = \frac{1}{3}\left(\eta_{mmk} + 2\,\eta_{kmm}\right). \tag{6.21}$$

This decomposition can be traced back at least to [24, eq.(16)]. It is rather easy to see that $\eta \mapsto \eta^{(1)}$ is a projection onto the linear space of trace-free symmetric third order tensors. This space is 7-dimensional. For a projection, the only eigenvalues are 0 and 1, hence the kernel is 11-dimensional and the image is 7-dimensional. The image consists of all tensors $\eta_{ijk} = \partial_{ij}u_k$ which derive from $u = \nabla\zeta$ and $\Delta\zeta = 0$, see section 6.7.

Further decomposition of η^A is done by splitting the curvature tensor χ into symmetric and anti-symmetric parts

$$\chi_{ij} = \chi^S_{ij} + \chi^A_{ij} = \operatorname{sym}\chi + \operatorname{skew}\chi\,. \tag{6.22}$$

As a result the tensor η^A splits into two parts

$$\eta^{AS}_{ijk} := \frac{2}{3}\varepsilon_{ikp}\,\chi^S_{pj} + \frac{2}{3}\varepsilon_{jkq}\,\chi^S_{qi}\,, \qquad \eta^{AA}_{ijk} := \frac{2}{3}\varepsilon_{ikp}\,\chi^A_{pj} + \frac{2}{3}\varepsilon_{jkq}\,\chi^A_{qi}\,, \tag{6.23}$$

and we set $\eta^{(3)}_{ijk} := \eta^{AA}_{ijk} + \eta^{(0)}_{ijk}$. It is clear that

$$\eta^{AS}_{ijk}\,\eta^{AS}_{ijk} \;\sim\; \|\operatorname{sym}\nabla\operatorname{curl} u\|^2\,, \qquad \eta^{AA}_{ijk}\,\eta^{AA}_{ijk} \;\sim\; \|\operatorname{skew}\nabla\operatorname{curl} u\|^2\,. \tag{6.24}$$

The tensor η^{AS}_{ijk} has five independent entries ($\operatorname{sym}\nabla\operatorname{curl} u$ is symmetric and trace free) and η^{AA}_{ijk} has three independent entries ($\operatorname{skew}\nabla\operatorname{curl} u$), while $\eta^{(0)}_{ijk}$ has three independent entries. In [52] then the orthogonal decomposition

$$\eta_{ijk} = \eta^{(1)}_{ijk} + \eta^{AS}_{ijk} + \eta^{(3)}_{ijk} \tag{6.25}$$

is proposed which leads to a strain energy of the type

$$a_0'\,\|\eta^{(3)}_{ijk}\|^2 + a_1'\,\|\eta^{(1)}_{ijk}\|^2 + a_2'\,\|\operatorname{sym}\nabla\operatorname{curl} u\|^2\,. \tag{6.26}$$

In [32] the tensors $\eta^{(1)}_{ijk}$ and η^{AS}_{ijk} are retained but $\eta^{(3)}$ is replaced by $\nabla\operatorname{Div} u$. Their curvature energy reads therefore

$$a_0\,\|\nabla\operatorname{Div} u\|^2 + a_1'\,\|\eta^{(1)}_{ijk}\|^2 + a_2'\,\|\operatorname{sym}\nabla\operatorname{curl} u\|^2\,. \tag{6.27}$$

Pointwise positive definiteness in the components η_{ijk} requires a_0, a_1', $a_2' > 0$. The symmetric triad η^S has ten independent entries which reduce to seven for incompressibility due to the former three linear relations. Accordingly, the traceless symmetric tensor $\eta^{(1)}_{ijk}$ has seven independent components and is called **deviatoric stretch gradient tensor** [32]. We have checked that

$$\sum_{i,j,k=1}^{3} \eta^{(1)}_{ijk}\,\eta^{(1)}_{ijk} = \|\eta^{(1)}_{ijk}\|^2 \tag{6.28}$$

(see (6.27)) vanishes for $\eta^{(1)}_{ijk}(\phi_C) = 0$, meaning that (6.28) is a conformally invariant curvature expression as well.

6.5 A natural orthogonal representation

For $u : \mathbb{R}^3 \mapsto \mathbb{R}^3$ consider $\partial_i u = (\partial_i u_1, \partial_i u_2, \partial_i u_3)^T \in \mathbb{R}^3$. Thus $\nabla\partial_i u = \partial_i\nabla u \in \mathbb{M}^{3\times3}$ and we may write

$$\nabla\partial_i u = \operatorname{dev}\operatorname{sym}\nabla\partial_i u + \operatorname{skew}\nabla\partial_i u + \frac{1}{3}\operatorname{tr}[\nabla\partial_i u]\,\mathbb{1}\,,$$
$$\nabla\partial_i u = \partial_i\nabla u = \operatorname{dev}\operatorname{sym}\nabla\partial_i u + \frac{1}{2}\operatorname{anti}(\operatorname{curl}\partial_i u) + \frac{1}{3}(\partial_i\operatorname{Div} u)\,\mathbb{1}\,. \tag{6.29}$$

[12] The third order strain gradient tensor $\kappa_{ijk} := \partial_i\varepsilon_{jk}$ is, in this sense, not symmetric.

Let us define accordingly three third order tensors

$$
\begin{aligned}
N^{(1)}_{ijk} &:= \operatorname{dev}\operatorname{sym}[\nabla \partial_i u] = \operatorname{dev}\operatorname{sym}[\partial_i \nabla u] = \partial_i[\operatorname{dev}\operatorname{sym}\nabla u] := \widehat{L}^{ijk}_{rst}\,\eta_{rst} \quad \sim \quad D[\operatorname{dev}\operatorname{sym}\nabla u]\,,\\
N^{(2)}_{ijk} &:= \operatorname{skew}[\nabla \partial_i u] = \operatorname{skew}[\partial_i \nabla u] = \partial_i[\operatorname{skew}\nabla u] = \partial_i \frac{1}{2}\operatorname{anti}(\operatorname{curl} u) \quad \sim \quad \nabla \operatorname{curl} u\,, \qquad (6.30)\\
N^{(3)}_{ijk} &:= \frac{1}{3}\operatorname{tr}[\nabla \partial_i u]\,1\!\!1 = \frac{1}{3}(\partial_i \operatorname{Div} u)\,1\!\!1 \quad \sim \quad \nabla \operatorname{Div} u\,.
\end{aligned}
$$

They are mutually orthogonal and from the foregoing it is clear that

$$
\eta_{ijk} = N^{(1)}_{ijk} + N^{(2)}_{ijk} + N^{(3)}_{ijk}\,. \qquad (6.31)
$$

The representation is therefore complete. Counting the number of independent entries in each tensor we have that $N^{(2)}$ has eight (corresponds to χ) and $N^{(3)}$ has three independent entries such that $N^{(1)}$ must have seven in order to sum up to the eighteen independent components in η_{ijk}. The tensor $N^{(1)}$ is really the **gradient of deviatoric stretch**. In index notation $N^{(1)}$ has the representation

$$
N^{(1)}_{ijk}\, e_i \otimes e_j \otimes e_k = \left(\frac{1}{2}(u_{k,ji} + u_{j,ki}) - \frac{1}{3}\, u_{l,li}\,\delta_{jk}\right) e_i \otimes e_j \otimes e_k = \left(\frac{1}{2}(\eta_{ijk} + \eta_{ikj}) - \frac{1}{3}\,\eta_{ill}\,\delta_{jk}\right) e_i \otimes e_j \otimes e_k\,. \qquad (6.32)
$$

It is instructive to note that

$$
\begin{aligned}
\int_\Omega \|N^{(1)}\|^2 + \sum_{i=1}^{3} \|\partial_i u\|^2 \,\mathrm{dV} &= \int_\Omega \sum_{i=1}^{3} \|\partial_i[\operatorname{dev}\operatorname{sym}\nabla u]\|^2_{\mathbb{M}^{3\times 3}} + \sum_{i=1}^{3}\|\partial_i u\|^2 \,\mathrm{dV}\\
&= \int_\Omega \sum_{i=1}^{3} \|\operatorname{dev}\operatorname{sym}\nabla[\partial_i u]\|^2_{\mathbb{M}^{3\times 3}} + \sum_{i=1}^{3}\|\partial_i u\|^2 \,\mathrm{dV}\\
&\geq C^+(\Omega)\sum_{i=1}^{3}\|\partial_i u\|^2_{H^1(\Omega)} \geq C^+(\Omega)\,\|D^2 u\|^2_{L^2(\Omega)}\,, \qquad (6.33)
\end{aligned}
$$

where we have used a novel coercive inequality given in [22]. Observe that pointwise positive definiteness in the components η_{ijk}, i.e.

$$
\|N^{(1)}(\eta)\|^2 := \sum_{i,j,k=1}^{3} |N^{(1)}_{ijk}(\eta)| = \sum_{i,j,k=1}^{3} |\widehat{L}^{ijk}_{rst}\,\eta_{rst}|^2 \geq C^+ \sum_{i,j,k=1}^{3} |\eta_{ijk}|^2 = C^+\,\|\eta\|^2 \qquad (6.34)
$$

does not hold (and is not needed)!

6.6 Fleck's earlier representation for incompressibility

In [13] different curvature energies are introduced. There, a **third order deviatoric part** η' **of the strain gradient tensor** η is defined as

$$
\eta'_{ijk} := \eta_{ijk} - \underbrace{\frac{1}{4}\left(\delta_{ik}\,\eta_{jpp} + \delta_{jk}\,\eta_{ipp}\right)}_{:=\eta^H_{ijk},\ \eta^H_{ijk}=0 \text{ if } \operatorname{Div} u = 0} = u_{k,ij} - \frac{1}{4}\left(\delta_{ik}\,u_{p,jp} + \delta_{jk}\,u_{p,ip}\right) \quad \not\sim \quad N^{(1)}_{ijk}\,, \qquad (6.35)
$$

$$
\eta^H_{ijk} = \eta^H_{jik}\,, \quad \eta^H_{ikk} = \eta_{ikk}\,, \quad i = 1,2,3\,,
$$

having the property $\eta'_{ikk} = 0$, $i = 1,2,3$ (formally coming from $\eta_{ikk} = 0$, $i = 1,2,3$ for $\operatorname{Div} u = \mathit{const} = 0$). They further show [52] that η' admits a unique orthogonal decomposition

$$
\eta'_{ijk} = \eta^{(I)}_{ijk} + \eta^{(II)}_{ijk} + \eta^{(III)}_{ijk}\,, \qquad \eta^{(*)}_{ijk}\,\eta^{(**)}_{ijk} = 0\,, \qquad (6.36)
$$

such that $\eta^{(*)}_{ijk} = \eta^{(*)}_{jik}$ and $\eta^{(*)}_{ikk} = 0$. Here, $\eta^{(I)} := \eta'^{(1)}$, which means to use the definition for the calculation of $\eta^{(1)}$ in (6.21) applied to η' instead of η. The most general isotropic, quadratic dependence of the curvature energy depending only on η' can be written as

$$
a_0\,\|\eta^{(I)}_{ijk}\|^2 + a'_2\,\|\eta^{(II)}_{ijk}\|^2 + a'_3\,\|\eta^{(III)}_{ijk}\|^2\,. \qquad (6.37)
$$

Since, without going into details [13]

$$
\begin{aligned}
\|\eta^{(II)}_{ijk}\|^2 &= \frac{4}{3}\,\left(\|\nabla \operatorname{curl} u\|^2 + \langle \nabla \operatorname{curl} u, (\nabla \operatorname{curl} u)^T\rangle\right) = \frac{8}{3}\,\|\operatorname{sym}\nabla \operatorname{curl} u\|^2\,,\\
\|\eta^{(III)}_{ijk}\|^2 &= \frac{8}{5}\,\left(\|\nabla \operatorname{curl} u\|^2 - \langle \nabla \operatorname{curl} u, (\nabla \operatorname{curl} u)^T\rangle\right) = \frac{16}{5}\,\|\operatorname{skew}\nabla \operatorname{curl} u\|^2\,, \qquad (6.38)
\end{aligned}
$$

the curvature energy can equivalently be expressed in [5, eq.(6)]

$$
a'_0\,\|\eta^{(I)}_{ijk}\|^2 + a'_2\,\|\operatorname{sym}\nabla \operatorname{curl} u\|^2 + a'_3\,\|\operatorname{skew}\nabla \operatorname{curl} u\|^2\,. \qquad (6.39)
$$

While the mechanical interpretation of $\eta^{(I)}$ is not immediate due to the involved formal tensor operations, it is stated [52] that $\|\eta^{(I)}_{ijk}\|^2$ depends on both stretch and rotation gradients.

6.7 Additional observations

We have calculated $\eta^{(1)}$ for our conformal map ϕ_C and it turns out that $\eta^{(1)}$ is, incidentally, also conformally invariant. It is clear that $\eta^{(1)} \neq N^{(1)}$ since $\eta^{(1)}_{ijk}$ is by definition completely symmetric in (i,j,k) while $N^{(1)}_{ijk}$ is not. Moreover, considering $\tilde{u}(x) = \frac{1}{2}\left(2\langle \vec{v}, x\rangle\, x - \vec{b}\,\|x\|^2\right)$, which is not a conformal map for $\vec{v} \neq \vec{b}$, it can be seen that $\eta^{(1)}(D^2\tilde{u}) = 0$ such that an estimate of the type

$$\forall\, \eta \in \mathbb{R}^{27}: \quad \|N^{(1)}(\eta)\|^2 \leq C_2\,\|\eta^{(1)}_{ijk}(\eta)\|^2 \tag{6.40}$$

is impossible, since $N^{(1)}$ vanishes only for $\eta_{ijk} = \partial_{ij}\phi^k_C$ coming from conformal maps. As a consequence, using $\|\eta^{(1)}\|^2$ as only curvature term is not sufficient in order to arrive at a coercive problem in $H^2(\Omega)$. As regards Fleck's tensor $\eta^{(I)}$ we have calculated that it is not conformally invariant.

In order to understand better the third order tensor $\eta^{(1)}$ we consider arbitrary second order homogeneous polynomials

$$\begin{pmatrix}\overline{u}_1(x_1,x_2,x_3)\\ \overline{u}_2(x_1,x_2,x_3)\\ \overline{u}_3(x_1,x_2,x_3)\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\langle S^1.x, x\rangle\\ \frac{1}{2}\langle S^2.x, x\rangle\\ \frac{1}{2}\langle S^2.x, x\rangle\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\left(S^1_{11}x_1^2 + S^1_{22}x_2^2 + S^1_{33}x_3^2 + 2S^1_{12}x_1x_2 + 2S^1_{13}x_1x_3 + 2S^1_{23}x_2x_3\right)\\ \frac{1}{2}\left(S^2_{11}x_1^2 + S^2_{22}x_2^2 + S^2_{33}x_3^2 + 2S^2_{12}x_1x_2 + 2S^2_{13}x_1x_3 + 2S^2_{23}x_2x_3\right)\\ \frac{1}{2}\left(S^3_{11}x_1^2 + S^3_{22}x_2^2 + S^3_{33}x_3^2 + 2S^3_{12}x_1x_2 + 2S^3_{13}x_1x_3 + 2S^3_{23}x_2x_3\right)\end{pmatrix}, \tag{6.41}$$

where $S^1, S^2, S^3 \in \operatorname{Sym}(3)$ are given. Then $\eta_{ijk} = \partial_{ij}\overline{u}_k = S^k_{ij}$ and each $\eta = D^2u$ can be realised in this way. Let us calculate the curl and Div of this polynomial. It holds

$$\begin{aligned}
\operatorname{curl}\overline{u} &= \begin{pmatrix} [S^3_{12} - S^2_{13}]\,x_1 + [S^3_{22} - S^2_{23}]\,x_2 + [S^3_{32} - S^2_{33}]\,x_3\\ -\left([S^3_{11} - S^1_{13}]\,x_1 + [S^3_{21} - S^1_{23}]\,x_2 + [S^3_{31} - S^1_{33}]\,x_3\right)\\ [S^2_{11} - S^1_{12}]\,x_1 + [S^2_{21} - S^1_{22}]\,x_2 + [S^2_{31} - S^1_{32}]\,x_3\end{pmatrix}\\
&= \begin{pmatrix} [S^3_{12} - S^2_{13}]\,x_1 + [S^3_{22} - S^2_{23}]\,x_2 + [S^3_{23} - S^2_{33}]\,x_3\\ -\left([S^3_{11} - S^1_{13}]\,x_1 + [S^3_{12} - S^1_{23}]\,x_2 + [S^3_{13} - S^1_{33}]\,x_3\right)\\ [S^2_{11} - S^1_{12}]\,x_1 + [S^2_{12} - S^1_{22}]\,x_2 + [S^2_{13} - S^1_{23}]\,x_3\end{pmatrix}, \qquad (6.42)\\
\operatorname{Div}\overline{u} &= x_1\,[S^1_{11} + S^2_{12} + S^3_{13}] + x_2\,[S^1_{12} + S^2_{22} + S^3_{23}] + x_3\,[S^1_{13} + S^2_{23} + S^3_{33}]\,.
\end{aligned}$$

Imposing $\operatorname{curl}\overline{u} = 0$ as a side condition gives, in fact, 8-independent conditions on the 18 parameters in S^1, S^2, S^3 since $\operatorname{tr}[\operatorname{curl}\overline{u}] = 0$ is automatically satisfied. Thus the space of curl-free homogeneous polynomials of second degree can be represented with 10 independent parameters. The same argument shows that $\eta^S = \eta$ for a function u if and only if $u = \nabla\zeta$, i.e. u is irrotational.
A fully symmetric third order tensor η^S arises naturally from a displacement potential $\widehat{u}(x) = \nabla\zeta$, $\zeta: \mathbb{R}^3 \mapsto \mathbb{R}$. Then $\eta^S_{ijk} = D^2\widehat{u} = \partial_{ijk}\zeta$. In order to generate all possible forms of η^S we consider ζ as being given by homogeneous polynomials of order three (with ten degrees of freedom
$(a_{111},\, a_{222},\, a_{333},\, a_{112},\, a_{113},\, a_{122},\, a_{133},\, a_{223},\, a_{233},\, a_{123})$), i.e.,

$$\begin{aligned}
\zeta(x_1,x_2,x_3) := {}& a_{111}\,x_1^3 + a_{222}\,x_2^3 + a_{333}\,x_3^3 + a_{112}\,x_1^2\,x_2 + a_{113}\,x_1^2\,x_3 + a_{122}\,x_1\,x_2^2\\
&+ a_{133}\,x_3^2\,x_1 + a_{223}\,x_2^2\,x_3 + a_{233}\,x_3^2\,x_2 + a_{123}\,x_1\,x_2\,x_3\,.
\end{aligned} \tag{6.43}$$

Then

$$\begin{aligned}
\nabla\zeta(x_1,x_2,x_3) &= \begin{pmatrix} 3\,a_{111}\,x_1^2 + 2\,a_{112}\,x_1\,x_2 + 2\,a_{113}\,x_1\,x_3 + a_{122}\,x_2^2 + a_{133}\,x_3^2 + a_{123}\,x_2\,x_3\\ 3\,a_{222}\,x_2^2 + a_{112}\,x_1^2 + 2\,a_{122}\,x_1\,x_2 + 2\,a_{223}\,x_2\,x_3 + a_{233}\,x_3^2 + a_{123}\,x_1\,x_3\\ 3\,a_{333}\,x_3^2 + a_{113}\,x_1^2 + 2\,a_{133}\,x_3\,x_1 + a_{223}\,x_2^2 + 2\,a_{233}\,x_3\,x_2 + a_{123}\,x_1\,x_2\end{pmatrix},\\
\Delta\zeta(x_1,x_2,x_3) &= [6\,a_{111}\,x_1 + 2\,a_{112}\,x_2 + 2\,a_{113}\,x_3] + [6\,a_{222}\,x_2 + 2\,a_{122}\,x_1 + 2\,a_{223}\,x_3] \qquad (6.44)\\
&\quad + [6\,a_{333}\,x_3 + 2\,a_{133}\,x_1 + 2\,a_{233}\,x_2]\\
&= 2\,x_1\,[3\,a_{111} + a_{122} + a_{133}] + 2\,x_2\,[3\,a_{222} + a_{112} + a_{233}] + 2\,x_3\,[3\,a_{333} + a_{113} + a_{223}]\,.
\end{aligned}$$

Next, we want to calculate $\eta^{(1)}$ for $D^3\zeta$. Since $\eta^S(D^3\zeta) = \eta(D^3\zeta)$ it remains to calculate $\eta^{(0)}$. Since

$$\begin{aligned}
\eta^{(0)}_{ijk} &= \frac{1}{5}\left(\delta_{ij}\,\eta^S_{mmk} + \delta_{jk}\,\eta^S_{mmi} + \delta_{ki}\,\eta^S_{mmj}\right) = \frac{1}{5}\left(\delta_{ij}\sum_{m=1}^{3}\partial_{mmk}\zeta + \delta_{jk}\sum_{m=1}^{3}\partial_{mmi}\zeta + \delta_{ki}\sum_{m=1}^{3}\partial_{mmj}\zeta\right)\\
&= \frac{1}{5}\left(\delta_{ij}\,\partial_k\Delta\zeta + \delta_{jk}\,\partial_i\Delta\zeta + \delta_{ki}\,\partial_j\Delta\zeta\right),
\end{aligned} \tag{6.45}$$

we observe that $\eta^{(0)} = 0$ if $\Delta\zeta = \mathit{const}$ or $\operatorname{Div}\widehat{u} = \mathit{const.}$ and $\Delta\zeta = \mathit{const}$ is satisfied if and only if

$$0 = [3\,a_{111} + a_{122} + a_{133}]\,, \quad 0 = [3\,a_{222} + a_{112} + a_{233}]\,, \quad 0 = [3\,a_{333} + a_{113} + a_{223}]\,. \tag{6.46}$$

This suggests to define finally an irrotational displacement

$$\begin{aligned}
\widehat{u}_\nabla(x_1,x_2,x_3) &:= \nabla\zeta(x_1,x_2,x_3)\\
&= \begin{pmatrix} -[a_{122} + a_{133}]\,x_1^2 + a_{122}\,x_2^2 + a_{133}\,x_3^2 + 2\,a_{112}\,x_1\,x_2 + 2\,a_{113}\,x_1\,x_3 + a_{123}\,x_2\,x_3\\ -[a_{112} + a_{233}]\,x_2^2 + a_{112}\,x_1^2 + a_{233}\,x_3^2 + 2\,a_{122}\,x_1\,x_2 + 2\,a_{223}\,x_2\,x_3 + a_{123}\,x_1\,x_3\\ -[a_{113} + a_{223}]\,x_3^2 + a_{223}\,x_2^2 + a_{113}\,x_1^2 + 2\,a_{133}\,x_3\,x_1 + 2\,a_{233}\,x_3\,x_2 + a_{123}\,x_1\,x_2\end{pmatrix},
\end{aligned} \tag{6.47}$$

with seven independent parameters $(a_{122},\, a_{133},\, a_{112},\, a_{233},\, a_{113},\, a_{223},\, a_{123})$, satisfying $\eta^{(1)}(D^2\widehat{u}) = D^2\widehat{u}$. The same argument shows also that $\eta^{(1)} = \eta$ for a function u if and only if $u = \nabla\zeta$, $\Delta\zeta = 0$.