Justification of homogenization in viscoplasticity: From convergence on two scales to an asymptotic solution in $\boldsymbol{L}^2(\Omega)$

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Abstract. A homogenized material model can be used effectively for simulation, if the difference of the solutions of this model and the microscopic model converges to zero in a strong norm when the microstructure is scaled. The second author recently showed for the quasistatic initial-boundary value problem with internal variables modelling an inelastic solid body Ω at small strain that this convergence holds in an averaged sense over phase shifts of the microstructure. Based on this result we construct an asymptotic solution, which converges to the solution of the microscopic problem in the $L^2(\Omega)$–norm, thus avoiding the averaging.

Key words: homogenization, plasticity, viscoplasticity, strong two-scale convergence, two-scale convergence, maximal monotone operator, microstructure.

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1 Introduction and setting of the problem

The numerical simulation of inelastic material behavior is expensive, since the dependence of the material properties on the defomation history must be taken into account. This is all the more true when the material properties vary on a small spatial scale, since in this case a fine space discretization is required. Therefore material models based on homogenization are widely used to reduce the numerical expense. To determine the reliability of such a model it is necessary to compare its solutions with solutions of a faithful material model, in which the small scale variation of the material properties, the microstructure, is resolved. We call this the microscopic model. A homogenized model can be used effectively for simulation, if the difference of a solution of the microscopic model and of a corresponding solution of the homogenized model converges to zero rapidely in a strong norm when the length scale of the microstructure tends to zero.

In [24] this convergence was studied for the quasistatic initial-boundary value problem modelling the inelastic behavior of a solid body with a peri-

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odic microstructure at small strain. The main result was later also published in [25]. The constitutive equations in this model govern the evolution of internal variables. They can be rate dependent or rate independent. Under the assumption that the material behavior shows linear kinematic hardening, it is shown that when the scaling parameter η of the periodic microstructure tends to zero, then the difference of the solutions of the microscopic model and the homogenized model converges to zero in a special sense, which we call phase shift convergence, and which involves the $L^2(\Omega \times Y)$–norm, where $\Omega \subseteq \mathbb{R}^3$ is the solid body and Y is the periodicity cell. Though this type of convergence is different from strong two scale convergence as defined in [13, 14], it is related. The exact result is recorded below in Theorem 2.1.

However, the convergence with respect to the $L^2(\Omega \times Y)$–norm is not satisfactory, since, as in the classical theory of homogenization, one wants to construct by homogenization an approximate solution, which asymptotically converges to the solution of the microscopic problem in the $L^2(\Omega)$–norm for $\eta \to 0$. Approximation in the $L^2(\Omega)$–norm is also needed in the investigation of the convergence of numerical solutions. The obstacle, which prevents the construction of such an approximation in the classical way is the low regularity of the solutions of the highly nonlinear homogenized problem. Here we show that, starting from the result of [25], such an approximate solution, which is asymptotic to the exact solution in $L^2(\Omega)$, can be constructed by averaging.

In the remainder of the introduction we first formulate the model equations and, for completeness, state an existence theorem for this system proved in [3]. Subsequently we formulate the homogenized system, explain the construction of the asymptotic solution and state our main convergence result in Theorem 1.2. We end the introduction by a short discussion of the literature. The proof of Theorem 1.2 is given in Section 3. This proof is based on the convergence result from [24, 25], which we review in Section 2 for completeness.

The initial value problem. Suppose that Ω is an open bounded set, the set of material points of the body, with Lipschitz boundary $\partial\Omega$. By T_e we denote a positive number (time of existence), which can be chosen to be arbitrarily large. The unknowns, which we want to determine, are the displacement $u(x,t) \in \mathbb{R}^3$ of the material point $x \in \Omega$ at time t, the Cauchy stress tensor $T(x,t) \in \mathcal{S}^3$, where $\mathcal{S}^3$ is the set of symmetric 3×3-matrices, and the vector $z(x,t) \in \mathbb{R}^N$ of internal variables. With a small positive parameter η these unknowns must satisfy the equations

$$\begin{aligned}
-\mathrm{div}_x T(x,t) &= b(x,t), && (1)\\
T(x,t) &= \mathcal{D}[\tfrac{x}{\eta}]\Big(\varepsilon\big(\nabla_x u(x,t)\big) - Bz(x,t)\Big), && (2)\\
\frac{\partial}{\partial t} z(x,t) &\in g\Big(\frac{x}{\eta}, -\nabla_z \psi\big(\frac{x}{\eta}, \varepsilon\big(\nabla_x u(x,t)\big), z(x,t)\big)\Big)\\
&= g\Big(\frac{x}{\eta}, B^T T(x,t) - L[\tfrac{x}{\eta}]\, z(x,t)\Big), && (3)
\end{aligned}$$

for $x \in \Omega$ and $t \in [0, T_e]$. The initial and boundary conditions are

$$\begin{aligned}
z(x,0) &= z_\eta^{(0)}(x) \ := \ z^{(0)}\Big(x, \frac{x}{\eta}\Big), \quad x \in \Omega\,, && (4)\\
u(x,t) &= 0, \quad (x,t) \in \partial\Omega \times [0, T_e]\,, && (5)
\end{aligned}$$

with a given function $z^{(0)}$. To reduce the technical difficulties we only consider the homogeneous Dirichlet boundary condition. The notations are as follows:

$$\varepsilon(\nabla_x u(x,t)) = \frac{1}{2}\big(\nabla_x u(x,t) + (\nabla_x u(x,t))^T\big) \in \mathcal{S}^3$$

is the linear strain tensor and $B : \mathbb{R}^N \to \mathcal{S}^3$ is a linear mapping, which assigns to the vector $z(x,t)$ the plastic strain tensor $\varepsilon_p(x,t) = Bz(x,t)$. For every $y \in \mathbb{R}^3$ the elasticity tensor $\mathcal{D}[y] : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric mapping, which is positive definte, uniformly with respect to y. This means that there are constants $0 < \alpha \le \beta$ satisfying

$$\alpha|\xi|^2 \le \xi \cdot \mathcal{D}[y]\xi \le \beta|\xi|^2, \quad \text{for all } \xi \in \mathcal{S}^3.$$

We assume that the mapping $y \to \mathcal{D}[y]$ is measurable and periodic with a rectangular periodicity cell $Y \subset \mathbb{R}^3$. Without restriction of generality we suppose that the measure $|Y|$ of Y is equal to 1. The given function $b : \Omega \times [0, T_e] \to \mathbb{R}^3$ is the volume force, and $\psi : \mathbb{R}^3 \times \mathcal{S}^3 \times \mathbb{R}^N \to \mathbb{R}$ denotes the free energy. Since we consider small deformations, we assume that ψ is a quadratic form

$$\psi(y, \varepsilon, z) = \frac{1}{2}\big(\mathcal{D}[y](\varepsilon - Bz)\big) \cdot (\varepsilon - Bz) + \frac{1}{2}(L[y]z) \cdot z, \tag{6}$$

where $L[y]$ is for every $y \in \mathbb{R}^3$ a symmetric, positive semi-definite $N \times N-$ matrix, and where for matrices $A, B \in \mathcal{S}^3$ we write $A \cdot B = \sum_{i,j=1}^3 A_{ij} B_{ij}$. We also assume that the function $y \mapsto L[y]$, the given initial data

$$(x,y) \mapsto z^{(0)}(x,y) : \Omega \times \mathbb{R}^3 \to \mathbb{R}^N$$

and the constitutive function

$$(y,z) \mapsto g(y,z) : \mathbb{R}^3 \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$$

are periodic with respect to y and have periodicity cell Y. The second law of thermodynamics implies that the function g must in addition satisfy

$$\xi \cdot z \geq 0, \quad \text{for all } (y,z) \in D(g) \subseteq \mathbb{R}^3 \times \mathbb{R}^N \text{ and all } \xi \in g(y,z). \tag{7}$$

This completes the formulation of the initial boundary value problem. (2), (3) are the constitutive equations, which determine the material behavior. In the appendix of [5] it is explained how these equations are obtained from the second law of thermodynamics. For many engineering models from plasticity and viscoplasticity it is shown in [1] that they can be written in this abstract form, which reflects their essential mathematical properties. We surmise, in fact, that virtually all such models can be written in this form. A similar formulation of the constitutive equations is also used in [16, 21].

(7) is a very general condition, and the known existence theorems for the initial-boundary value problem (1) – (5) hold under more restrictive assumptions for g. Here we assume that g is monotone and that $L[y]$ in (6) is positive definite. The latter assumption means that the material shows linear kinematic hardening. Under these conditions a strong existence and uniqueness theorem was proved in [3]. Though existence theorems are available under more general conditions, see [5, 9, 10, 11, 17, 28] for example, we need perturbation estimates, which at present seem to be available only under these strong conditions. For reference we state the existence theorem below. To formulate this theorem we need some notations and definitions, which we introduce now, and which we use throughout.

Let $E = \Omega$ or $E = \Omega \times Y$ and let X be a Banach space. For a function $w : E \times [0, T_e] \to X$ we write $w(t)$ to denote the function defined by $\xi \mapsto w(\xi, t) : E \to X$. For an open set $\Gamma \subseteq \mathbb{R}^k$ let $W^{m,p}(\Gamma, \mathbb{R}^n)$ be the Sobolev space of functions with weak derivatives in $L^p(\Gamma, \mathbb{R}^n)$ up to order m. The norm of this space is denoted by $\|\cdot\|_{m,p,\Gamma}$. The scalar product in the Hilbert space $H^m(\Gamma, \mathbb{R}^n) = W^{m,2}(\Gamma, \mathbb{R}^n)$ is $(\cdot, \cdot\cdot)_{m,\Gamma}$, the corresponding norm is $\|\cdot\|_{m,\Gamma}$. We also write $(\cdot, \cdot\cdot)_\Gamma = (\cdot, \cdot\cdot)_{0,\Gamma}$ and $\|\cdot\|_\Gamma = \|\cdot\|_{0,\Gamma}$. The space $H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^n)$ of traces of functions in $H^1(\Omega, \mathbb{R}^n)$ is equiped with the factor norm

$$\|w\|_{H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^n)} = \inf \{\|v\|_{1,\Omega} \mid v \in H^1(\Omega, \mathbb{R}^n),\ \mathrm{trace}_{\partial\Omega}(v) = w\}.$$

If we fix t in the equations (1), (2), (5) we obtain a linear boundary value problem, which slightly extends the classical boundary value problem of linear elasticity theory. If $z(t)$ is known, then $(u(t), T(t))$ can be uniquely determined from this problem by the standard solution theory in $L^2(\Omega)$, since $b(t)$ is given. To define an operator $\mathcal{G}_\eta : L^2(\Omega, \mathbb{R}^N) \to 2^{L^2(\Omega, \mathbb{R}^N)}$, let $v \in L^2(\Omega, \mathbb{R}^N)$ be given, let $T[v]$ be the stress in the solution of this boundary value problem to the data $b(0) \in L^2(\Omega, \mathbb{R}^3)$, $z(0) = v$, and set

$$\mathcal{G}_\eta v = \{\zeta \in L^2(\Omega, \mathbb{R}^N) \mid \zeta(x) \in g\big(\frac{x}{\eta}, B^T T[v](x) - L\big[\frac{x}{\eta}\big]\, v(x)\big),\ \text{a.e. in } \Omega\}.$$

The domain of definition $D(\mathcal{G}_\eta)$ consists of all $v \in L^2(\Omega, \mathbb{R}^N)$ with $\mathcal{G}_\eta(v) \neq \emptyset$. Now we can formulate the existence and uniqueness theorem proved in [3].

Theorem 1.1. *Let $T_e > 0$. Assume that $L \in L^\infty(Y, \mathbb{R}^{N\times N})$ and that the matrix $L[y]$ is positive definite, uniformly with respect to $y \in \mathbb{R}^3$. Assume further that the mapping $g : \mathbb{R}^3 \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ satisfies the conditions*

1. *$0 \in g(y, 0)$ for almost all $y \in \mathbb{R}^3$,*
2. *$z \mapsto g(y, z)$ is maximal monotone for almost all $y \in \mathbb{R}^3$,*
3. *the mapping $y \mapsto j_\lambda(y, z) : \mathbb{R}^3 \to \mathbb{R}^N$ is measurable for all $\lambda > 0$, where $j_\lambda(y, z)$ is the inverse of $z \mapsto z + \lambda g(y, z)$.*

Suppose that $b \in W^{2,1}(0, T_e; L^2(\Omega, \mathbb{R}^3))$ and $z_\eta^{(0)} \in D(\mathcal{G}_\eta)$. Then there is a unique solution (u, T, z) of the initial-boundary value problem (1) – (5) with

$$\begin{aligned} (u, T) &\in W^{1,1}(0, T_e; H^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)), &(8)\\ z &\in W^{1,\infty}(0, T_e; L^2(\Omega, \mathbb{R}^N)). &(9)\end{aligned}$$

Moreover, there is a constant K, independent of η, b and $z_\eta^{(0)}$, such that

$$\|z\|_{W^{1,\infty}(0,T_e;L^2(\Omega))} \leq K\big(|\mathcal{G}_\eta z_\eta^{(0)}|_\Omega + \|b\|_{W^{2,1}(0,T_e;L^2(\Omega))}\big), \qquad (10)$$

where $|\mathcal{G}_\eta z_\eta^{(0)}|_\Omega = \inf\{\|\zeta\|_\Omega \mid \zeta \in \mathcal{G}_\eta z_\eta^{(0)}\}$.

We remark that the mapping $z \mapsto j_\lambda(y, z)$ is single valued and well-defined, since $z \mapsto g(y, z)$ is assumed to be maximal monotone.

Homogenization. We want to construct an approximate solution of (1) – (5), which is close to the exact solution (u_η, T_η, z_η) for small values of $\eta > 0$. Since for small η the initial data $x \mapsto z^{(0)}(x, \frac{x}{\eta})$ are close to a periodic function with periodicity cell ηY, and since $x \mapsto \mathcal{D}[\frac{x}{\eta}]$ and $x \mapsto g(\frac{x}{\eta}, z)$ are periodic with this periodicity cell, one expects that also (u_η, T_η, z_η) will be close to a quasiperiodic function $(\overline{u}_\eta, \overline{T}_\eta, \overline{z}_\eta)$ of the form

$$\begin{aligned} \overline{u}_\eta(x, t) &= u_0(x, t) + \eta u_1\Big(x, \frac{x}{\eta}, t\Big), &(11)\\ \overline{T}_\eta(x, t) &= T_0\Big(x, \frac{x}{\eta}, t\Big), &(12)\\ \overline{z}_\eta(x, t) &= z_0\Big(x, \frac{x}{\eta}, t\Big), &(13)\end{aligned}$$

where the function $(x, y, t) \mapsto (u_1, T_0, z_0)(x, y, t) : \Omega \times \mathbb{R}^3 \times [0, T_e] \to \mathbb{R}^3 \times \mathcal{S}^3 \times \mathbb{R}^N$ is required to be periodic with respect to y and to have periodicity cell Y. In [2] it has been shown that if $(\overline{u}_\eta, \overline{T}_\eta, \overline{z}_\eta)$ is asymptotically equal to the solution (u_η, T_η, z_η) for $\eta \to 0$, then (u_0, u_1, T_0, z_0) must satisfy the homogenized initial-boundary value problem formed by the equations

$$
\begin{aligned}
-\mathrm{div}_x T_\infty(x,t) &= b(x,t), &(14)\\
T_\infty(x,t) &= \frac{1}{|Y|}\int_Y T_0(x,y,t)dy, &(15)\\
-\mathrm{div}_y T_0(x,y,t) &= 0, &(16)\\
T_0(x,y,t) &= \mathcal{D}[y]\Big(\varepsilon(\nabla_y u_1(x,y,t)) - Bz_0(x,y,t) &\\
&\qquad + \varepsilon(\nabla_x u_0(x,t))\Big), &(17)\\
\frac{\partial}{\partial t} z_0(x,y,t) &\in g\big(y, B^T T_0(x,y,t) - L[y]z_0(x,y,t)\big), &(18)\\
z_0(x,y,0) &= z^{(0)}(x,y), &(19)
\end{aligned}
$$

which hold for $(x,y,t) \in \Omega \times \mathbb{R}^3 \times [0,T_e]$, and by the boundary condition

$$u_0(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T_e]. \tag{20}$$

For fixed x the equations (16) – (19) together with the periodicity condition for $y \mapsto (u_1, T_0)(x,y,t)$, which can be considered to be a boundary condition, define an initial-boundary problem in $Y \times [0,T_e)$, the cell problem. The functions u_0 and u_1 can be interpreted as macro- and microdisplacement, respectively, T_0 as the microstress; the macrostress T_∞ is obtained by averaging of T_0 over the representative volume element.

If the above reasoning can be reversed, then $(\overline{u}_\eta, \overline{T}_\eta, \overline{z}_\eta)$ defined by (11) – (13) is an approximate solution of the microscopic initial-boundary value problem (1) – (5) for small values of $\eta > 0$, provided that (u_0, u_1, T_0, z_0) solves the homogenized problem (14) – (20). The goal is therefore to show that for solutions (u_0, u_1, T_0, z_0) of (14) – (20) the difference of $(\overline{u}_\eta, \overline{T}_\eta, \overline{z}_\eta)$ and of the solution (u_η, T_η, z_η) of (1) – (5) converges to zero for $\eta \to 0$, preferably in the norm of the space $L^\infty(0,T_e; L^2(\Omega))$.

Yet, the approximate solution can not be defined by (11) – (13) because of the low regularity of the solution of the homogenized problem. To explain this, note that in [4, Theorem 2] it is proved that under conditions stated below in Theorem 1.2 the homogenized initial-boundary value problem (14) – (20) has a unique solution $(u_0, u_1, T_\infty, T_0, z_0)$ with

$$
\begin{aligned}
(u_0, T_\infty) &\in W^{1,1}\big(0,T_e; H^1(\Omega,\mathbb{R}^3) \times L^2(\Omega,\mathcal{S}^3)\big), &(21)\\
(u_1, T_0) &\in W^{1,1}\big(0,T_e; L^2(\Omega, H^1(Y,\mathbb{R}^3)) \times L^2(\Omega\times Y,\mathcal{S}^3)\big), &(22)\\
z_0 &\in W^{1,\infty}\big(0,T_e; L^2(\Omega\times Y,\mathbb{R}^N)\big). &(23)
\end{aligned}
$$

These functions thus have one weak time derivative; from the investigations of the regularity of the solution in [6, 24] one can expect that $(T_0(t), z_0(t)) \in H^{1,\mathrm{loc}}(\Omega \times Y)$, but that these functions do in general not have higher space derivatives, and that these functions are even less regular at the boundary $\partial\Omega$. The reason for the low regularity is that the function g can be highly nonlinear. For example, it can be the subdifferential of an indicator function. Yet, with this low regularity the right hand sides of (12) and (13) are not well defined, since the argument $(x, \frac{x}{\eta})$ varies on a three-dimensional subset of the six-dimensional set $\Omega \times Y$, and therefore T_0 and z_0 do not have traces on the low dimensional subset.

Therefore the construction of the asymptotic solution must be modified. The main result of this paper, which we state in the following theorem, shows that an asymptotic solution can be obtained by applying a suitable averaging operator to the functions $\overline{T}_\eta$ and $\overline{z}_\eta$. To state this result we need some more definitions.

Consider the linear boundary value problem

$$\begin{aligned}
-\mathrm{div}_x \tilde{T}(x,y) &= \tilde{b}(x), && (24)\\
\tilde{T}(x,y) &= \mathcal{D}\big[\tfrac{x}{\eta}+y\big]\big(\varepsilon(\nabla_x \tilde{u}(x,y)) - B\tilde{v}(x,y)\big), && (25)\\
\tilde{u}(x,y) &= 0, \qquad (x,y)\in\partial\Omega\times Y, && (26)
\end{aligned}$$

where the first two equations must be satisfied for $(x,y)\in\Omega\times Y$. Note that for every fixed y this is an elliptic boundary value problem of the same type as the boundary value problem (1), (2), (5). The only difference is that the phase of the coefficient $\mathcal{D}$ is shifted by y. We can therefore consider y to be a parameter of phase shift. For the right hand side in (24) we choose $\tilde{b} = b(0) \in L^2(\Omega,\mathbb{R}^3)$ with b from (1). For given $\tilde{v} \in L^2(\Omega\times Y, \mathcal{S}^3)$ let $(\tilde{u}[\tilde{v}], \tilde{T}[\tilde{v}])$ be the solution of the resulting boundary value problem. Set

$$\begin{aligned}
\tilde{\mathcal{G}}_\eta \tilde{v} = \{\zeta \in L^2(\Omega\times Y,\mathbb{R}^N) \mid \\
\zeta(x,y) \in g\big(\tfrac{x}{\eta}+y, B^T\tilde{T}[\tilde{v}](x,y) - L\big[\tfrac{x}{\eta}+y\big]\,\tilde{v}(x,y)\big),\ \text{a.e. in } \Omega\times Y\}.
\end{aligned}$$

This defines an operator $\tilde{\mathcal{G}}_\eta : L^2(\Omega\times Y,\mathbb{R}^N) \to 2^{L^2(\Omega\times Y,\mathbb{R}^N)}$. To define another operator $\hat{\mathcal{G}} : L^2(\Omega\times Y,\mathbb{R}^N) \to 2^{L^2(\Omega\times Y,\mathbb{R}^N)}$ note that if we fix $t=0$, then the equations (14) – (17), (20) define a linear boundary value problem, which differs from the homogenized problem of elasticity theory only by the additional term Bz_0. For b from (1) and for $z_0(0)=v$ with $v \in L^2(\Omega\times Y,\mathbb{R}^N)$ this problem has a unique solution $(u_0,u_1,T_0)[v] \in H^1(\Omega)\times L^2(\Omega,H^1(Y))\times L^2(\Omega\times Y)$, see [4]. Now set

$$\begin{aligned}
\hat{\mathcal{G}} v = \{\zeta \in L^2(\Omega\times Y,\mathbb{R}^N) \mid \\
\zeta(x,y) \in g\big(y, B^T T_0[v](x,y) - L[y]\,v(x,y)\big),\ \text{a.e. in } \Omega\times Y\}.
\end{aligned}$$

Our main result is

Theorem 1.2. *Suppose that L, g and b satisfy the assumptions of Theorem 1.1. Assume that the function $z^{(0)} \in L^2(\Omega, C(Y,\mathbb{R}^N))$ belongs to the* *domain of definition* $D(\hat{\mathcal{G}})$ *and that the functions* $z_\eta^{(0)}$, $\tilde{z}_\eta^{(0)}$ *defined by*

$$z_\eta^{(0)}(x) = z^{(0)}\Big(x, \frac{x}{\eta}\Big), \qquad \tilde{z}_\eta^{(0)}(x,y) = z^{(0)}\Big(x, \frac{x}{\eta} + y\Big)$$

satisfy $z_\eta^{(0)} \in D(\mathcal{G}_\eta)$, $\tilde{z}_\eta^{(0)} \in D(\tilde{\mathcal{G}}_\eta)$ *for* $\eta > 0$, *and*

$$\sup_{\eta>0} |\mathcal{G}_\eta z_\eta^{(0)}|_\Omega < \infty, \qquad \sup_{\eta>0} |\tilde{\mathcal{G}}_\eta \tilde{z}_\eta^{(0)}|_{\Omega\times Y} < \infty. \tag{27}$$

Then for all $\eta > 0$ *there is a unique solution* (u_η, T_η, z_η) *of* (1) – (5), *which satisfies* (8), (9), *and there is a unique solution* (u_0, u_1, T_0, z_0) *of* (14) – (20), *which satisfies* (21) – (23). *For these solutions we have*

$$\lim_{\eta\to 0} \big(\|u_\eta(t) - u_0(t)\|_\Omega + \|T_\eta(t) - T_\eta^*(t)\|_\Omega + \|z_\eta(t) - z_\eta^*(t)\|_\Omega\big) = 0, \tag{28}$$

for all $t \in [0, T_e]$, *where*

$$\begin{aligned} T_\eta^*(x,t) &= \int_{Y_{\eta,x}} T_0\Big(x - \eta y, \frac{x}{\eta}, t\Big)dy, \\ z_\eta^*(x,t) &= \int_{Y_{\eta,x}} z_0\Big(x - \eta y, \frac{x}{\eta}, t\Big)dy, \end{aligned}$$

and where

$$Y_{\eta,x} = \{y \in Y \mid x - \eta y \in \Omega\} = Y \cap \frac{1}{\eta}(x - \Omega).$$

This theorem is proved in Section 3.

The engineering literature devoted to homogenization in plasticity and viscoplasticity is large, and we are not able to give an overview. A few of the more recent publications are [8, 19, 20, 23, 27], which, as we hope, can be used by the interested reader as a starting point.

The investigations devoted to the rigorous justification of the homogenization of the microscopic model (1) – (5) began with [2, 4, 15]. In [4] the energy method is used to show for local smooth solutions of the homogenized problem (14) – (20) that the functions $(\overline{u}_\eta, \overline{T}_\eta, \overline{z}_\eta)$ defined with these smooth solutions in (11) – (13) approximate the solutions of the microscopic model in $L^2(\Omega)$ for small η.

For the special case of rate independent problems a similar result as in [24, 25] was obtained in [22] shortly afterwards by the method of energetic solutions. It is shown there that the approximate solution converges to the exact solution in the sense of strong two-scale convergence as defined in [13, 14]. Strong two-scale convergence differs from the type of convergence used in Theorem 2.1, but as we show in Remark 3.4 below, there are clear similarities. In particular, both types of convergence are based on the double integration over $\Omega \times Y$. In fact, we surmise that starting from the convergence result in [22] we can construct an asymptotic solution converging in $L^2(\Omega)$ by a modification of the proof of Proposition 3.2 given below.

For the model of viscoplasticity with special rate dependent constitutive equations weak two-scale convergence to the homogenized solution was shown to hold in [29, 30]. Here weak two-scale convergence is meant in the original sense as defined in [18, 26]. In [29, 30] it is also shown that the cell problem, which in the homogenized problem serves to determine the macroscopic stress from the macroscopic displacement, can be replaced by a variational problem, in which the fast variables do not appear explicitly.

Homogenization of a time independent boundary value problem in plasticity with the Hencky law as the constitutive relation is studied in [12].

2 Phase shift convergence

In this section we review the result from [24, 25], which is used to prove Theorem 1.2. Assume that (u_0, u_1, T_0, z_0) is a solution of the homogenized problem (14) – (20), which satisfies (21) – (23), and consider the function $(x, y, t) \mapsto (\hat{u}_\eta, \hat{T}_\eta, \hat{z}_\eta)$ defined by

$$\hat{u}_\eta(x,y,t) = u_0(x,t) + \eta u_1\Big(x, \frac{x}{\eta} + y, t\Big), \tag{29}$$

$$\hat{T}_\eta(x,y,t) = T_0\Big(x, \frac{x}{\eta} + y, t\Big), \tag{30}$$

$$\hat{z}_\eta(x,y,t) = z_0\Big(x, \frac{x}{\eta} + y, t\Big), \tag{31}$$

for $(x, y, t) \in \Omega \times \mathbb{R}^3 \times [0, T_e]$. By (21) – (23) this function is square integrable over $\Omega \times Y$ for almost every $t \in [0, T_e]$. For $y = 0$ it formally coincides with the function $(\overline{u}_\eta, \overline{T}_\eta, \overline{z}_\eta)$. If we assume for the moment that the latter function approximates the solution (u_η, T_η, z_η) of the initial-boundary value problem (1) – (5) for small values of η, then the function $(\hat{u}_\eta, \hat{T}_\eta, \hat{z}_\eta)$ will approximate the solution $(\tilde{u}_\eta, \tilde{T}_\eta, \tilde{z}_\eta)$ of the initial-boundary value problem

$$-\mathrm{div}_x \tilde{T}_\eta(x,y,t) = b(x,t), \tag{32}$$

$$\tilde{T}_\eta(x,y,t) = \mathcal{D}\big[\tfrac{x}{\eta} + y\big]\big(\varepsilon(\nabla_x \tilde{u}_\eta(x,y,t)) - B\tilde{z}_\eta(x,y,t)\big), \tag{33}$$

$$\frac{\partial}{\partial t}\tilde{z}_\eta(x,y,t) \in g\big(\tfrac{x}{\eta} + y, B^T \tilde{T}_\eta(x,y,t) - L\big[\tfrac{x}{\eta} + y\big]\,\tilde{z}_\eta(x,y,t)\big), \tag{34}$$

$$\tilde{z}_\eta(x,y,0) = \tilde{z}_\eta^{(0)}(x,y) := z^{(0)}\big(x, \tfrac{x}{\eta} + y\big), \tag{35}$$

$$\tilde{u}_\eta(x,y,t) = 0, \quad (x,y,t) \in \partial\Omega \times Y \times [0, T_e], \tag{36}$$

where the equations (32) – (34) must hold for $(x, y, t) \in \Omega \times Y \times [0, T_e]$. For fixed y these equations define an initial-boundary value problem of the same type as the problem (1) – (5). y is therefore a parameter, which shifts the phase of the periodic microstructure and of the quasi-periodic inital data. In [24, 25] the following theorem is proved, which shows that $(\hat{u}_\eta(t), \hat{T}_\eta(t), \hat{z}_\eta(t))$

in fact approximates the function $(\tilde{u}_\eta(t), \tilde{T}_\eta(t), \tilde{z}_\eta(t))$ in $L^2(\Omega\times Y)$ for almost all t:

Theorem 2.1. *Let $T_e > 0$, let L, g and b satisfy the same assumptions as in Theorem 1.1, let $z^{(0)} \in D(\hat{\mathcal{G}})$ and suppose that $\tilde{z}_\eta^{(0)} \in D(\tilde{\mathcal{G}}_\eta)$ satisfies the second inequality of* (27).

Then there is a unique solution $(u_0, u_1, T_\infty, T_0, z_0)$ of the homogenized initial-boundary value problem (14) – (20)*, which satisfies* (21) – (23)*. Moreover, for every $\eta > 0$ there is a unique solution $(\tilde{u}_\eta, \tilde{T}_\eta, \tilde{z}_\eta)$ of the initial-boundary value problem* (32) – (36) *with*

$$(\tilde{u}_\eta, \tilde{T}_\eta) \in W^{1,1}\big(0, T_e; L^2(Y, H^1(\Omega, \mathbb{R}^3)) \times L^2(\Omega \times Y, \mathcal{S}^3)\big), \tag{37}$$
$$z_0 \in W^{1,\infty}\big(0, T_e; L^2(\Omega \times Y, \mathbb{R}^N)\big). \tag{38}$$

Also, the estimate

$$\sup_{\eta>0} \|\tilde{z}_\eta\|_{W^{1,\infty}(0,T_e;L^2(\Omega\times Y))} < \infty \tag{39}$$

holds. Furthermore, for all $0 \le t \le T_e$ these solutions satisfy the limit relations

$$\lim_{\eta\to 0} \Big(\|u_0(t) - \tilde{u}_\eta(t)\|_{\Omega\times Y} + \|\hat{T}_\eta(t) - \tilde{T}_\eta(t)\|_{\Omega\times Y} + \|\hat{z}_\eta(t) - \tilde{z}_\eta(t)\|_{\Omega\times Y} \Big) = 0 \tag{40}$$

and, with $\hat{u}_{1\eta}(x, y, t) = u_1(x, \frac{x}{\eta} + y, t)$,

$$\lim_{\eta\to 0} \|(\nabla_x u_0(t) + \nabla_y \hat{u}_{1\eta}(t)) - \nabla_x \tilde{u}_\eta(t)\|_{\Omega\times Y} = 0. \tag{41}$$

Remark 2.2. Condition (27) in the last theorem is more general than the one imposed in [25]. It is immediately seen from the proof that the convergence result from [25] holds also under this more general assumption.

3 Asymptotics of the solution

3.1 Proof of Theorem 1.2

Here we show that Theorem 1.2 is an immediate consequence of Theorem 2.1 and of Proposition 3.2 stated below. The proof of Proposition 3.2 is postponed to Section 3.2. We start with a definition:

Definition 3.1. *Let $\eta > 0$ and let v be a function defined on $\Omega \times Y \times [0, T_e]$. We define another function $\mathcal{T}_\eta(v)$ on $\Omega \times Y \times [0, T_e]$ by*

$$\mathcal{T}_\eta(v)(x, y, t) = \begin{cases} v(x - \eta y, y, t), & x \in \Omega_{\eta y} := \{x \in \Omega \mid x - \eta y \in \Omega\}, \\ 0, & x \in \Omega \setminus \Omega_{\eta y}\,. \end{cases}$$

$\mathcal{T}_\eta$ is a linear mapping acting on functions. Since $\Omega_{\eta y} = \Omega \cap (\Omega + \eta y)$, we clearly have for $v(t) \in L^2(\Omega \times Y, X)$ with $X = \mathbb{R}^n$ or $X = \mathcal{S}^3$ that

$$\begin{aligned} \|\mathcal{T}_\eta(v)(t)\|^2_{\Omega\times Y} &= \int_Y \int_{\Omega\cap(\Omega+\eta y)} |v_\eta(x-\eta y, y, t)|^2 \, dx dy \\ &= \int_Y \int_{(\Omega-\eta y)\cap\Omega} |v_\eta(\xi, y, t)|^2 \, d\xi dy \;\leq\; \|v_\eta(t)\|^2_{\Omega\times Y}. \end{aligned} \tag{42}$$

Proposition 3.2. *Let all requirements of Theorem 1.2 be fulfilled. Then the solutions (u_η, T_η, z_η) of (1) – (5) and $(\tilde{u}_\eta, \tilde{T}_\eta, \tilde{z}_\eta)$ of (32) – (36) satisfy*

$$\begin{aligned} \lim_{\eta\to 0} \big(\|u_\eta(t) - \mathcal{T}_\eta(\tilde{u}_\eta)(t)\|_{\Omega\times Y} + \|T_\eta(t) - \mathcal{T}_\eta(\tilde{T}_\eta)(t)\|_{\Omega\times Y} \\ + \|z_\eta(t) - \mathcal{T}_\eta(\tilde{z}_\eta)(t)\|_{\Omega\times Y} \big) = 0 \end{aligned} \tag{43}$$

for all $0 \leq t \leq T_e$.

Here we consider u_η, T_η and z_η, which are defined on $\Omega \times [0, T_e]$, to be functions on the domain $\Omega \times Y \times [0, T_e]$, which are constant with respect to $y \in Y$. We shall often use this convention in the following without mentioning.

Corollary 3.3. *Let all requirements of Theorem 1.2 be fulfilled. With the solution (u_0, u_1, T_0, z_0) of the homogenized initial-boundary value problem (14) – (20) let $(\hat{T}_\eta, \hat{z}_\eta)$ be defined by (30), (31), and let (u_η, T_η, z_η) be the solution of (1) – (5). Then the limit relation*

$$\begin{aligned} \lim_{\eta\to 0} \big(\|u_\eta(t) - \mathcal{T}_\eta(u_0)(t)\|_{\Omega\times Y} + \|T_\eta(t) - \mathcal{T}_\eta(\hat{T}_\eta)(t)\|_{\Omega\times Y} \\ + \|z_\eta(t) - \mathcal{T}_\eta(\hat{z}_\eta)(t)\|_{\Omega\times Y} \big) = 0 \end{aligned} \tag{44}$$

holds for all $0 \leq t \leq T_e$.

Proof of Corollary 3.3. From (42) and the limit relations (40), (43) we obtain

$$\begin{aligned} &\|z_\eta(t) - \mathcal{T}_\eta(\hat{z}_\eta)(t)\|_{\Omega\times Y} \\ &\quad \leq \|z_\eta(t) - \mathcal{T}_\eta(\tilde{z}_\eta)(t)\|_{\Omega\times Y} + \|\mathcal{T}_\eta(\hat{z}_\eta - \tilde{z}_\eta)(t)\|_{\Omega\times Y} \\ &\quad \leq \|z_\eta(t) - \mathcal{T}_\eta(\tilde{z}_\eta)(t)\|_{\Omega\times Y} + \|\hat{z}_\eta(t) - \tilde{z}_\eta(t)\|_{\Omega\times Y} \to 0, \end{aligned}$$

for $\eta \to 0$. In the same way we conclude that $\|u_\eta(t) - \mathcal{T}_\eta(u_0)(t)\|_{\Omega\times Y} + \|T_\eta(t) - \mathcal{T}_\eta(\hat{T}_\eta)(t)\|_{\Omega\times Y}$ converges to zero. □

Remark 3.4. To compare the type of convergence given in (44) with strong two-scale convergence note that after making the transformation $x \mapsto x + \eta y$ in (44) this equation becomes

$$\begin{aligned} \lim_{\eta\to 0} \int_{\Omega\times Y} |u_\eta(x+\eta y, t) - u_0(x,t)|^2 + |T_\eta(x+\eta y, t) - T_0(x, \tfrac{x}{\eta} + y, t)|^2 \\ + |z_\eta(x+\eta y, t) - z_0(x, \tfrac{x}{\eta} + y, t)|^2 d(x,y) = 0, \end{aligned} \tag{45}$$

where we extended the functions u_η, T_η and z_η by 0 outside of Ω. Assume now for the moment that $Y = [0,1]^3$. For every $\xi \in \mathbb{R}^3$ we denote by $\lfloor \xi \rfloor \in \mathbb{R}^3$ the point obtained from ξ by replacing all components by their integer parts. Strong two-scale convergence as defined in [14] would then be

$$\lim_{\eta\to 0} \int\limits_{\Omega\times Y} |u_\eta(\eta\lfloor \tfrac{x}{\eta}\rfloor + \eta y, t) - u_0(x,t)|^2 + |T_\eta(\eta\lfloor \tfrac{x}{\eta}\rfloor + \eta y, t) - T_0(x,y,t)|^2$$
$$+ |z_\eta(\eta\lfloor \tfrac{x}{\eta}\rfloor + \eta y, t) - z_0(x,y,t)|^2 d(x,y) = 0. \quad (46)$$

Formally this equation is obtained from (45) by inserting $\eta\lfloor \frac{x}{\eta}\rfloor + \eta y$ everywhere for $x + \eta y$, noting that the periodicity implies

$$T_0(x, \lfloor \tfrac{x}{\eta}\rfloor + y, t) = T_0(x,y,t), \quad z_0(x, \lfloor \tfrac{x}{\eta}\rfloor + y, t) = z_0(x,y,t).$$

The difference of (45) and (46) seems to be mainly of technical nature.

Proof of Theorem 1.2. Note first that by Definition 3.1 and by definition of $\hat{T}_\eta$, $\hat{z}_\eta$ in (30), (31) we have that

$$T_\eta^*(t) = \int_Y \mathcal{T}_\eta(\hat{T}_\eta)(t)dy\,, \quad z_\eta^*(t) = \int_Y \mathcal{T}_\eta(\hat{z}_\eta)(t)dy\,.$$

Noting that $|Y| = 1$, by assumption, we conclude from the second of these equations, from Hölder's inequality and from (44) that

$$\begin{aligned}
\|z_\eta(t) - z_\eta^*(t)\|_\Omega^2 &= \int_\Omega \Big| z_\eta(x,t) - \int_Y \mathcal{T}_\eta(\hat{z}_\eta)(x,y,t)dy\Big|^2 dx \\
&\le \int_\Omega \Big| \int_Y z_\eta(x,y,t) - \mathcal{T}_\eta(\hat{z}_\eta)(x,y,t)dy\Big|^2 dx \\
&\le \int_\Omega \int_Y |z_\eta(x,y,t) - \mathcal{T}_\eta(\hat{z}_\eta)(x,y,t)|^2 dydx \\
&= \|z_\eta(t) - \mathcal{T}_\eta(\hat{z}_\eta)(t)\|_{\Omega\times Y}^2 \to 0, \quad \text{for } \eta \to 0. \quad (47)
\end{aligned}$$

We see in the same way that

$$\|u_\eta(t) - u_\eta^*(t)\|_\Omega + \|T_\eta(t) - T_\eta^*(t)\|_\Omega \to 0, \quad \text{for } \eta \to 0, \quad (48)$$

where $u_\eta^*(x,t) = \int_{Y_{\eta,x}} u_0(x - \eta y, t)dy = \int_Y \mathcal{T}_\eta(u_0)(x,y,t)dy$. With this definition we obtain by a similar computation as in (47) that

$$\begin{aligned}
\|u_0(t) - u_\eta^*(t)\|_\Omega^2 &\le \int_{\Omega\times Y} |u_0(x,t) - \mathcal{T}_\eta(u_0)(x,y,t)|^2 d(x,y) \\
&= \int_Y \int_{\Omega_{\eta y}} |u_0(x,t) - u_0(x-\eta y, t)|^2 dxdy \quad (49) \\
&\quad + \int_Y \int_{\Omega\setminus(\Omega+\eta y)} |u_0(x,t)|^2 dxdy \to 0, \quad \text{for } \eta \to 0.
\end{aligned}$$

To get the convergence to zero we use standard arguments from integration theory. Relation (28) results by combination of (47) – (49). The proof of Theorem 1.2 is complete. □

3.2 Proof of Proposition 3.2

In this section we prove Proposition 3.2. To simplify the notation we set

$$(u_{[\eta]}, T_{[\eta]}, z_{[\eta]}) = (\mathcal{T}_\eta(\tilde{u}_\eta), \mathcal{T}_\eta(\tilde{T}_\eta), \mathcal{T}_\eta(\tilde{z}_\eta)), \tag{50}$$

where $(\tilde{u}_\eta, \tilde{T}_\eta, \tilde{z}_\eta)$ is the solution of the problem (32) – (36). From (37), (38) and from (42) we infer that $(T_{[\eta]}, z_{[\eta]}) \in L^\infty(0, T_e; L^2(\Omega \times Y))$. We also have

$$u_{[\eta]} \in L^\infty(0, T_e; L^2(Y, H^1(\Omega))). \tag{51}$$

To see this note that the function $x \mapsto \tilde{u}_\eta(x - \eta y, y, t)$ is defined on $\Omega + \eta y$ and vanishes on the boundary of this set, by the boundary condition (36). We extend this function from $\Omega + \eta y$ to $\mathbb{R}^3$ by zero. The extended function is defined on $\mathbb{R}^3 \times Y \times [0, T_e]$, belongs to $L^\infty(0, T_e; L^2(Y, H^1(\mathbb{R}^3)))$, by (37), and coincides with $u_{[\eta]}$ on the set $\Omega \times Y \times [0, T_e]$, from which we get (51).

Step one: initial-boundary value problem for $(u_{[\eta]}, T_{[\eta]}, z_{[\eta]})$. Using the equations (32) – (36), (51), the definition of $\mathcal{T}_\eta$ and the property $0 \in g(y, 0)$, we get that $(u_{[\eta]}, T_{[\eta]}, z_{[\eta]})$ satisfies the equations

$$\begin{aligned}
-\mathrm{div}_x T_{[\eta]}(x,y,t) &= b_{[\eta]}(x,y,t), && (52)\\
T_{[\eta]}(x,y,t) &= \mathcal{D}\big[\tfrac{x}{\eta}\big]\big(\varepsilon(\nabla_x u_{[\eta]}(x,y,t)) - B z_{[\eta]}(x,y,t)\big), && (53)\\
\frac{\partial}{\partial t} z_{[\eta]}(x,y,t) &\in g\Big(\tfrac{x}{\eta}, B^T T_{[\eta]}(x,y,t) - L[\tfrac{x}{\eta}] z_{[\eta]}(x,y,t)\Big), && (54)
\end{aligned}$$

for $(x, y, t) \in \Omega \times Y \times [0, T_e)$, and the initial and boundary conditions

$$\begin{aligned}
z_{[\eta]}(x,y,0) &= z^{(0)}_{[\eta]}(x,y), \quad x \in \Omega \times Y, && (55)\\
u_{[\eta]}(x,y,t) &= \gamma_{[\eta]}(x,y,t), \quad (x,y,t) \in \partial\Omega \times Y \times [0, T_e], && (56)
\end{aligned}$$

where $b_{[\eta]} \in L^2(Y \times [0, T_e], H^{-1}(\Omega, \mathbb{R}^3))$ is defined by

$$[b_{[\eta]}(y,t), \phi]_\Omega = (T_{[\eta]}(y,t), \nabla\phi)_\Omega, \quad \phi \in H^1_0(\Omega, \mathbb{R}^3), \tag{57}$$

and where for $(y, t) \in Y \times [0, T_e]$

$$\begin{aligned}
\gamma_{[\eta]}(x,y,t) &= u_{[\eta]}(x,y,t), \quad x \in \partial\Omega, && (58)\\
z^{(0)}_{[\eta]}(x,y) &= \begin{cases} z^{(0)}_0(x - \eta y, \frac{x}{\eta}), & x \in \Omega_{\eta y}, \\ 0, & x \in \Omega \setminus \Omega_{\eta y}. \end{cases} && (59)
\end{aligned}$$

We consider the function $u_{[\eta]}$ on the right hand side of (58) to be known, since it is obtained from $\tilde{u}_\eta$ by shifting of the x–variable by ηy.

Step two: reduction to evolution equations. We next reduce the problems (52) – (56) and (1) – (5) to evolution equations in a Hilbert space. To this end we follow the procedure proposed in [3, 25]. Here we only sketch this procedure and refer to these papers for details.

Consider the linear boundary value problem

$$\begin{aligned} -\mathrm{div}_x T(x,y) &= \hat{b}(x,y), &\quad (60)\\ T(x,y) &= \mathcal{D}\big[\tfrac{x}{\eta}\big](\varepsilon(\nabla_x u(x,y)) - \hat{\varepsilon}_p(x,y)), &\quad (61)\\ u(x,y) &= \hat{\gamma}(x,y), \quad (x,y) \in \partial\Omega \times Y, &\quad (62) \end{aligned}$$

where the first two equations must hold for $(x,y) \in \Omega \times Y$. We can also consider (60) – (62) to be a family of linear boundary value problems with respect to x depending on the parameter y. Noting that the boundary value problem with respect to x is elliptic and corresponds to the boundary value problem of linear elasticity theory, we obtain from the classical theory that for every $\hat{\varepsilon}_p \in L^2(\Omega \times Y, \mathcal{S}^3)$, $\hat{b} \in L^2(Y, H^{-1}(\Omega, \mathbb{R}^3))$ and $\hat{\gamma} \in L^2(Y, H^1(\Omega, \mathbb{R}^3))$ there is a unique solution $(u,T) \in L^2(\Omega \times Y, \mathbb{R}^3 \times \mathcal{S}^3)$ with $\varepsilon(\nabla_x u) \in L^2(\Omega \times Y, \mathcal{S}^3)$.

Definition 3.5. *Let the linear operator $P_\eta : L^2(\Omega \times Y, \mathcal{S}^3) \mapsto L^2(\Omega \times Y, \mathcal{S}^3)$ be defined by*

$$P_\eta \hat{\varepsilon}_p = \varepsilon(\nabla_x u),$$

where (u,T) is the solution of (60) – (62) *to $\hat{b} = \hat{\gamma} = 0$ and to $\hat{\varepsilon}_p \in L^2(\Omega \times Y, \mathcal{S}^3)$. Furthermore, we define the operator $Q_\eta = I - P_\eta$. Here I is the identity operator.*

The classical theory to (60) – (62) also implies that the operators P_η and Q_η are uniformly bounded with respect to η.

By assumption the elasticity tensor $\mathcal{D}[\frac{x}{\eta}] : \mathcal{S}^3 \to \mathcal{S}^3$ is positive definite and bounded, uniformly with respect to $x \in \Omega$. Therefore we can associate to the elasticity tensor a linear, bounded, selfadjoint, positive definite mapping $\mathcal{D}_\eta : L^2(\Omega \times Y, \mathcal{S}^3) \to L^2(\Omega \times Y, \mathcal{S}^3)$ given by

$$(\mathcal{D}_\eta \xi)(x,y) = \mathcal{D}[\tfrac{x}{\eta}]\xi(x,y), \quad \xi \in L^2(\Omega \times Y, \mathcal{S}^3), \quad (x,y) \in \Omega \times Y.$$

With this mapping we can define a new scalar product on $L^2(\Omega \times Y, \mathcal{S}^3)$ by

$$(\xi, \zeta)_{\mathcal{D}_\eta, \Omega\times Y} = (\mathcal{D}_\eta \xi, \zeta)_{\Omega\times Y}\,.$$

The norm associated to the new scalar product is equivalent to the standard norm $\|\cdot\|_{\Omega\times Y}$. By [4, Lemma 6] we have

Lemma 3.6. *(i) The operators P_η and Q_η are projectors on $L^2(\Omega \times Y, \mathcal{S}^3)$, which are orthogonal with respect to the scalar product $(\xi,\zeta)_{\mathcal{D}_\eta,\Omega\times Y}$.*
(ii) The operator $B^T \mathcal{D}_\eta Q_\eta B : L^2(\Omega \times Y, \mathbb{R}^N) \to L^2(\Omega \times Y, \mathbb{R}^N)$ is selfajoint and non-negative with respect to the scalar product $(\xi,\zeta)_{\Omega\times Y}$.

Since by assumption $L[y]$ is positive definite, uniformly with respect to y, we can define a linear, bounded, selfadjoint, positive definite mapping $L_\eta : L^2(\Omega \times Y, \mathbb{R}^N) \to L^2(\Omega \times Y, \mathbb{R}^N$ by

$$(L_\eta \xi)(x,y) = L[\tfrac{x}{\eta}]\xi(x,y), \quad \xi \in L^2(\Omega \times Y, \mathcal{S}^3), \quad (x,y) \in \Omega \times Y.$$

The second statement of Lemma 3.6 implies that

$$M_\eta = L_\eta + B^T \mathcal{D}_\eta Q_\eta B : L^2(\Omega \times Y, \mathbb{R}^N) \to L^2(\Omega \times Y, \mathbb{R}^N)$$

is a selfadjoint, positive definite mapping. Therefore another scalar product on $L^2(\Omega \times Y, \mathbb{R}^N)$ is given by

$$\langle \xi, \zeta \rangle_{\Omega \times Y, \eta} = (M_\eta \xi, \zeta)_{\Omega \times Y} \, .$$

The associated norm $\|\xi\|_{\Omega \times Y, \eta} = \langle \xi, \xi \rangle_{\Omega \times Y, \eta}^{1/2}$ satisfies

$$c_1 \|\xi\|_{\Omega \times Y} \le \|\xi\|_{\Omega \times Y, \eta} \le c_2 \|\xi\|_{\Omega \times Y}, \tag{63}$$

where c_1, c_2 are positive constants, which can be chosen independent of η.

Now we are able to reduce the initial-boundary value problem (52) – (56) to an evolution equation in a Hilbert space. If $z_{[\eta]}(t)$ is known, then the component $(u_{[\eta]}(t), T_{[\eta]}(t))$ of the solution of this initial-boundary value problem is obtained as unique solution of the boundary value problem (52), (53), (56). Due to the linearity we have

$$(u_{[\eta]}(t), T_{[\eta]}(t)) = (\tilde{v}_{[\eta]}(t), \tilde{\sigma}_{[\eta]}(t)) + (v_{[\eta]}(t), \sigma_{[\eta]}(t)),$$

where $(v_{[\eta]}(t), \sigma_{[\eta]}(t))$ is the solution of (60) – (62) to the data $\hat{b} = b_{[\eta]}(t)$, $\hat{\gamma} = \gamma_{[\eta]}(t)$, $\hat{\varepsilon}_p = 0$, and $(\tilde{v}_{[\eta]}(t), \tilde{\sigma}_{[\eta]}(t))$ solves the problem (60) – (62) to the data $\hat{b} = \hat{\gamma} = 0$, $\hat{\varepsilon}_p = Bz_{[\eta]}(t)$. By definition of Q_η we have that $\tilde{\sigma}_{[\eta]}(t) = -\mathcal{D}_\eta Q_\eta B z_{[\eta]}(t)$. Insertion of this equation into (54) yields

$$\frac{\partial}{\partial t} z_{[\eta]}(t) \in G_\eta\big(-M_\eta z_{[\eta]}(t) + B^T \sigma_{[\eta]}(t)\big), \tag{64}$$

where the mapping $G_\eta : L^2(\Omega \times Y, \mathbb{R}^N) \mapsto 2^{L^2(\Omega \times Y, \mathbb{R}^N)}$ is defined by

$$G_\eta(\xi) = \big\{\zeta \in L^2(\Omega \times Y, \mathbb{R}^N) \;\big|\; \zeta(x,y) \in g(\tfrac{x}{\eta}, \xi(x,y)) \text{ a.e.}\big\}.$$

Since $\sigma_{[\eta]}$ is determined by the boundary value problem (60) – (62) to the data $b_{[\eta]}$, $\gamma_{[\eta]}$, it can be considered to be known. Therefore (64) is a non-autonomous evolution equation for $z_{[\eta]}$. We combine (64) with the initial condition (55) and obtain the initial value problem

$$\frac{\partial}{\partial t} z_{[\eta]}(t) + A_{[\eta]}(t) z_{[\eta]}(t) \ni 0, \tag{65}$$

$$z_{[\eta]}(0) = z_{[\eta]}^{(0)}\,, \tag{66}$$

where for brevity we introduced the notation

$$A_{[\eta]}(t)w = -G_\eta\big(-M_\eta w + B^T\sigma_{[\eta]}(t)\big).$$

The operator $A_{[\eta]}(t) : D(A_{[\eta]}(t)) \subseteq L^2(\Omega \times Y, \mathbb{R}^N) \to L^2(\Omega \times Y, \mathbb{R}^N)$ is maximal monotone with respect to the scalar product $\langle \xi, \zeta\rangle_{\Omega\times Y,\eta}$, cf. [25, Lemma 2.3]. In exactly the same way we reduce the problem (1) – (5) to the initial value problem

$$\frac{\partial}{\partial t} z_\eta(t) + A_\eta(t) z_\eta(t) \ni 0, \tag{67}$$

$$z_\eta(0) = z_\eta^{(0)}, \tag{68}$$

with the operator

$$A_\eta(t)w = -G_\eta\big(-M_\eta w + B^T\sigma_\eta(t)\big),$$

where the function $(v_\eta(t), \sigma_\eta(t))$ is the solution of (60) – (62) to the data $\hat{b} = b(t)$, $\hat{\gamma} = 0$, $\hat{\varepsilon}_p = 0$. The operator $A_\eta(t)$ is also maximal monotone on $L^2(\Omega \times Y, \mathbb{R}^N)$ with respect to the scalar product $\langle \xi, \zeta\rangle_{\Omega\times Y,\eta}$. Note that A_η is obtained from $A_{[\eta]}$ by replacing $\sigma_{[\eta]}$ in the argument of G_η by σ_η.

Step three: estimation of $z_{[\eta]} - z_\eta$. To estimate the difference $z_{[\eta]}(t) - z_\eta(t)$ we use a perturbation estimate for solutions of evolution equations given in [25]. This estimate, which is stated in the following lemma, goes back to the theory of the operator distance introduced in [31].

Lemma 3.7. *Let $z_{[\eta]}, z_\eta \in W^{1,\infty}(0, T_e; L^2(\Omega \times Y, \mathbb{R}^N))$ be solutions of* (65), (66) *and* (67), (68), *respectively. Then there is a constant C independent of η, such that for all $0 \le t \le T_e$*

$$\|z_{[\eta]}(t) - z_\eta(t)\|^2_{\Omega\times Y} \le C\Big(\int_0^t \|\sigma_{[\eta]}(s) - \sigma_\eta(s)\|_{\Omega\times Y} ds + \|z^{(0)}_{[\eta]} - z^{(0)}_\eta\|^2_{\Omega\times Y}\Big). \tag{69}$$

For completeness we give the *proof of Lemma 3.7.* Since $z_{[\eta]}(t)$ and $z_\eta(t)$ satisfy (65) and (67), respectively, and since G_η is monotone, we obtain

$$\begin{aligned}
&\|z_{[\eta]}(t) - z_\eta(t)\|^2_{\Omega\times Y,\eta} - \|z_{[\eta]}(0) - z_\eta(0)\|^2_{\Omega\times Y,\eta} \\
&\quad = 2\int_0^t \langle \partial_s z_{[\eta]}(s) - \partial_s z_\eta(s), z_{[\eta]}(s) - z_\eta(s)\rangle_{\Omega\times Y,\eta}\, ds \\
&\quad = -2\int_0^t \Big(\partial_s z_{[\eta]}(s) - \partial_s z_\eta(s), (-M_\eta z_{[\eta]}(s) + B^T\sigma_{[\eta]}(s)) \\
&\qquad\qquad - (-M_\eta z_\eta(s) + B^T\sigma_\eta(s))\Big)_{\Omega\times Y} \\
&\qquad\quad + 2\Big(\partial_s z_{[\eta]}(s) - \partial_s z_\eta(s), B^T\sigma_{[\eta]}(s) - B^T\sigma_\eta(s)\Big)_{\Omega\times Y} ds \qquad (70) \\
&\quad \le 2\|B^T\| \int_0^t \big(\|\partial_s z_{[\eta]}(s)\|_{\Omega\times Y} + \|\partial_s z_\eta(s)\|_{\Omega\times Y}\big)\|\sigma_{[\eta]}(s) - \sigma_\eta(s)\|_{\Omega\times Y}\, ds.
\end{aligned}$$

From the assumption (27) and from the inequalities (10) and (39) we conclude that there are constants C_1, C_2 such that for all $\eta > 0$

$$\|z_\eta\|_{W^{1,\infty}(0,T_e;L^2(\Omega\times Y))} \le C_1\,, \tag{71}$$

$$\|\tilde{z}_\eta\|_{W^{1,\infty}(0,T_e;L^2(\Omega\times Y))} \le C_2\,. \tag{72}$$

Since $z_{[\eta]} = \mathcal{T}_\eta \tilde{z}_\eta$, we infer from (72) and from (42) that

$$\|z_{[\eta]}\|_{W^{1,\infty}(0,T_e;L^2(\Omega\times Y))} \le C_2\,. \tag{73}$$

To obtain (69) we combine (70), (71), (73) and note (63). The proof of Lemma 3.7 is complete. $\square$

Now we are in a position to verify the limit relation (43) for the z–component. This relation follows from Lemma 3.7 if we show that $\|\sigma_{[\eta]}(s) - \sigma_\eta(s)\|_{\Omega\times Y} \to 0$ for $\eta \to 0$ and

$$\lim_{\eta\to 0} \|z^{(0)}_{[\eta]} - z^{(0)}_\eta\|^2_{\Omega\times Y} = 0. \tag{74}$$

To verify the last relation note that $z^{(0)} \in L^2(\Omega, C(Y,\mathbb{R}^N))$, by assumption, which implies

$$\begin{aligned}
&\|z^{(0)}_{[\eta]} - z^{(0)}_\eta\|^2_{\Omega\times Y}\\
&= \int_Y \Big(\int_{\Omega_{\eta y}} |z^{(0)}(x-\eta y, \tfrac{x}{\eta}) - z^{(0)}(x, \tfrac{x}{\eta})|^2 dx + \int_{\Omega'_{\eta y}} |z^{(0)}(x, \tfrac{x}{\eta})|^2 dx \Big) dy\\
&\le \int_Y \int_{\Omega_{\eta y}} \|z^{(0)}(x-\eta y, \cdot) - z^{(0)}(x,\cdot)\|^2_{C(Y)} dx dy + \int_{\Omega'_{\eta y}} \|z^{(0)}(x,\cdot)\|^2_{C(Y)} dx,
\end{aligned}$$

with $\Omega'_{\eta y} = \Omega \setminus (\Omega + \eta y)$. Here we use that by Pettis' theorem for $z^{(0)} \in L^2(\Omega, C(Y,\mathbb{R}^N))$ the function $x \to \|z^{(0)}(x,\cdot)\|_{C(Y)}$ is measurable, see [7, Lemma 1.3]. The right hand side of this inequality converges to zero for $\eta \to 0$, whence (74) follows.

In order to estimate $\|\sigma_{[\eta]}(t) - \sigma_\eta(t)\|_{\Omega\times Y}$ we first observe that the functions $(v_\eta(t), \sigma_\eta(t))$ and $(v_{[\eta]}(t), \sigma_{[\eta]}(t))$ both are solutions of the linear boundary value problem (60) – (62), yet to different data. The difference $(v_\eta(t) - v_{[\eta]}(t), \sigma_\eta(t) - \sigma_{[\eta]}(t))$ thus satisfies this boundary value problem to the difference of these data, that is to $(\hat{b}, \hat{\gamma}, \hat{\varepsilon}_p) = (b(t) - b_{[\eta]}(t), -\gamma_{[\eta]}(t), 0)$. The standard theory of this elliptic boundary value problem thus yields

$$\begin{aligned}
\|\sigma_{[\eta]}(t) - \sigma_\eta(t)\|_{\Omega\times Y} \le C\big(&\|b(t) - b_{[\eta]}(t)\|_{L^2(Y,H^{-1}(\Omega))}\\
&+ \|\gamma_{[\eta]}(t)\|_{L^2(Y,H^{1/2}(\partial\Omega))}\big),
\end{aligned} \tag{75}$$

with a constant C independent of η and t.

In order to prove that

$$\lim_{\eta\to 0}\|\gamma_{[\eta]}(t)\|_{L^2(Y,H^{1/2}(\partial\Omega))}=0, \tag{76}$$

remember first that in the proof of (51) we extended the function $u_{[\eta]}$ to a function in the space $L^\infty(0,T_e;L^2(Y,H^1(\mathbb{R}^3)))$. We denote the extended function again by $u_{[\eta]}$. By (58) we have $\gamma_{[\eta]}(x,y,t)=u_{[\eta]}(x,y,t)$. The trace theorem thus yields that

$$\|\gamma_{[\eta]}(\cdot,y,t)\|_{H^{1/2}(\partial\Omega)}\le C\|u_{[\eta]}(\cdot,y,t)\|_{H^1(\mathbb{R}^3\setminus\Omega)},$$

with a constant C, which only depends on the domain $\mathbb{R}^3\setminus\Omega$, and which is therefore independent of η. However, $u_{[\eta]}(x,y,t)$ is different from zero outside of Ω only on the set $(\Omega+\eta y)\setminus\Omega$. This implies that

$$\begin{aligned}\|\gamma_{[\eta]}(\cdot,y,t)\|_{H^{1/2}(\partial\Omega)}&\le C\|u_{[\eta]}(\cdot,y,t)\|_{H^1((\Omega+\eta y)\setminus\Omega)}\\ &=C\|\tilde u_\eta(\cdot,y,t)\|_{H^1(\Omega\setminus(\Omega-\eta y))}\le C\|\tilde u_\eta(\cdot,y,t)\|_{H^1(\Gamma_\eta)}\,,\end{aligned}$$

with the set

$$\Gamma_\eta=\{x\in\Omega\mid \operatorname{dist}(x,\partial\Omega)<\eta d\},$$

where $d=\sup_{y\in Y}|y|$. Thus,

$$\|\gamma_{[\eta]}(t)\|_{L^2(Y,H^{1/2}(\partial\Omega))}\le C_1\big(\|\tilde u_\eta(t)\|_{\Gamma_\eta\times Y}+\|\nabla_x\tilde u_\eta(t)\|_{\Gamma_\eta\times Y}\big). \tag{77}$$

The relation (40) yields that

$$\lim_{\eta\to 0}\|\tilde u_\eta(t)\|_{\Gamma_\eta\times Y}\le\lim_{\eta\to 0}\big(\|\tilde u_\eta(t)-u_0(t)\|_{\Omega\times Y}+\|u_0(t)\|_{\Gamma_\eta\times Y}\big)=0, \tag{78}$$

since meas $(\Gamma_\eta\times Y)\to 0$ as $\eta\to 0$ and u_0 is independent of η. Similarly, (41) implies that

$$\begin{aligned}\lim_{\eta\to 0}\|\nabla_x\tilde u_\eta(t)\|_{\Gamma_\eta\times Y}\;&\le\;\lim_{\eta\to 0}\|\nabla_x\tilde u_\eta(t)-(\nabla_x u_0(t)+\nabla_y\hat u_{1\eta}(t))\|_{\Omega\times Y}\\ &+\;\lim_{\eta\to 0}\|\nabla_x u_0(t)\|_{\Gamma_\eta\times Y}+\lim_{\eta\to 0}\|\nabla_y\hat u_{1\eta}(t)\|_{\Gamma_\eta\times Y}\\ &=\;0,\end{aligned} \tag{79}$$

since meas $(\Gamma_\eta\times Y)\to 0$, and since the periodicity of $u_1(x,y,t)$ with respect to y yields

$$\begin{aligned}\lim_{\eta\to 0}\|\nabla_y\hat u_{1\eta}(t)\|^2_{\Gamma_\eta\times Y}\;&=\;\lim_{\eta\to 0}\int_{\Gamma_\eta}\int_Y\big|\nabla_y u_1\big(x,\frac{x}{\eta}+y,t\big)\big|^2dydx\\ &=\;\lim_{\eta\to 0}\int_{\Gamma_\eta}\int_Y|\nabla_y u_1(x,y,t)|^2dydx=0.\end{aligned} \tag{80}$$

Combination of (77) – (79) yields (76).

In order to prove that

$$\lim_{\eta\to 0} \|b(t) - b_{[\eta]}(t)\|_{L^2(Y,H^{-1}(\Omega))} = 0, \tag{81}$$

we consider a family of cut-off functions $\theta_\eta \in C_0^\infty(\Omega)$ such that $0 \le \theta_\eta \le 1$ and

$$\theta_\eta(x) = \begin{cases} 0, & x \in \Gamma_\eta, \\ 1, & x \in \Omega \setminus \Gamma_{2\eta}, \end{cases} \qquad \max|\nabla_x \theta_\eta| \le \frac{C}{\eta}.$$

Remembering the definition of $b_{[\eta]}$ in (57), we have for $\phi \in L^2(Y, H_0^1(\Omega, \mathbb{R}^3))$ that

$$\begin{aligned}
&[b(t) - b_{[\eta]}(t), \phi]_{\Omega\times Y} \\
&= (b(t),\phi)_{\Omega\times Y} + (T_{[\eta]}(t), \nabla_x\phi)_{\Omega\times Y} \\
&= (b(t),\phi)_{\Omega\times Y} + \big((1-\theta_\eta)T_{[\eta]}(t) + \theta_\eta T_{[\eta]}(t), \nabla_x\phi\big)_{\Omega\times Y} \\
&= (b(t) - \theta_\eta \operatorname{div}_x T_{[\eta]}(t), \phi)_{\Omega\times Y} + \big((1-\theta_\eta)T_{[\eta]}(t), \nabla_x\phi\big)_{\Omega\times Y} \\
&\quad -\big(T_{[\eta]}(t), \phi\,\nabla_x\theta_\eta\big)_{\Omega\times Y} \\
&=: I_1 + I_2 + I_3\,.
\end{aligned} \tag{82}$$

In this computation we also use that for $y \in Y$ and $x \in (\Omega\setminus\Gamma_\eta) \subseteq (\Omega \cap (\Omega + \eta y))$ we have $T_{[\eta]}(x,y,t) = \tilde{T}_\eta(x-\eta y, y, t)$. For such x and y we thus obtain

$$-\operatorname{div}_x T_{[\eta]}(x,y,t) = -\operatorname{div}_x \tilde{T}_\eta(x-\eta y,y,t) = b(x-\eta y, t).$$

Hence, with the notation $b_\eta(x,y,t) = b(x-\eta y,t)$ for $(x,y) \in (\Omega\setminus\Gamma_\eta)\times Y$,

$$\begin{aligned}
|I_1| &\le |(b(t) - \theta_\eta b_\eta(t), \phi)_{\Omega\times Y}| \\
&\le \|b(t) - \theta_\eta b_\eta(t)\|_{\Omega\times Y}\|\phi\|_{\Omega\times Y} \le C_b(\eta)\|\phi\|_{\Omega\times Y},
\end{aligned} \tag{83}$$

where

$$C_b(\eta)^2 = \int_Y\int_\Omega |b(x,t) - \theta_\eta(x) b(x-\eta y,t)|^2 dxdy \to 0, \quad \text{for } \eta \to 0\,. \tag{84}$$

I_2 satisfies the inequality

$$|I_2| \le \|(1-\theta_\eta)T_{[\eta]}(t)\|_{\Omega\times Y}\,\|\phi\|_{L^2(Y,H^1(\Omega))} \le C_T(\eta)\,\|\phi\|_{L^2(Y,H^1(\Omega))}\,, \tag{85}$$

with

$$\begin{aligned}
C_T^2(\eta) &= \int_Y\int_{\Gamma_{2\eta}} |T_{[\eta]}(x,y,t)|^2 dxdy \\
&= \int_Y\int_{\Gamma_{2\eta}\cap(\Omega+\eta y)} |\tilde{T}_\eta(x-\eta y,y,t)|^2 dxdy \\
&= \int_Y\int_{(\Gamma_{2\eta}-\eta y)\cap\Omega} |\tilde{T}_\eta(x,y,t)|^2 dxdy \\
&\le \int_Y\int_{\Gamma_{3\eta}} |\tilde{T}_\eta(x,y,t)|^2 dxdy \\
&\le \big(\|\tilde{T}_\eta(t) - \hat{T}_\eta(t)\|_{\Omega\times Y} + \|\hat{T}_\eta(t)\|_{\Gamma_{3\eta}\times Y}\big)^2 \to 0,
\end{aligned} \tag{86}$$

for $\eta \to 0$. The last convergence relation follows from (40) and from

$$\lim_{\eta\to 0} \|\hat{T}_\eta(t)\|_{\Gamma_{3\eta}\times Y} = \lim_{\eta\to 0} \|T_0(t)\|_{\Gamma_{3\eta}\times Y} = 0,$$

which is obtained by the same computation as in (80).

It remains to estimate I_3. Since $\nabla\theta_\eta(x)$ differs from zero only for $x \in \Gamma_{2\eta} \setminus \Gamma_\eta$, we have

$$\begin{aligned} |I_3| &= |\,(T_{[\eta]}(t), \phi\nabla_x\theta_\eta)_{\Gamma_{2\eta}\times Y}\,| \\ &\le (\max|\nabla_x\theta_\eta|)\,\|T_{[\eta]}(t)\|_{\Gamma_{2\eta}\times Y}\,\|\phi\|_{\Gamma_{2\eta}\times Y} \\ &\le \frac{C}{\eta} C_T(\eta) C_1 \eta \|\phi\|_{L^2(Y,H^1(\Omega))} = C_2\, C_T(\eta)\,\|\phi\|_{L^2(Y,H^1(\Omega))}\,, \end{aligned} \tag{87}$$

with $C_T(\eta)$ from (86). In the second last step we used Poincare's inequality; since the width of the domain $\Gamma_{2\eta}$ is $2d\eta$, this inequality yields

$$\|\phi(\cdot,y)\|_{\Gamma_{2\eta}} \le C_1\eta\|\nabla_x\phi(\cdot,y)\|_{\Gamma_{2\eta}}\,,$$

whence $\|\phi\|_{\Gamma_{2\eta}\times Y} \le C_1\eta\|\phi\|_{L^2(Y,H^1(\Omega))}$. If we combine now the estimates and limit relations (82) – (87), we obtain (81).

Insertion of (76) and (81) into (75) yields that for all $0 \le s \le T_e$

$$\lim_{\eta\to 0} \|\sigma_{[\eta]}(s) - \sigma_\eta(s)\|_{\Omega\times Y} = 0. \tag{88}$$

To conclude that the integral in (69) tends to zero, we finally show that the integrand is uniformly bounded with respect to s and η. By construction, $(v_\eta(t), \sigma_\eta(t))$ is the solution of the boundary value problem (60) – (62) to the data $\hat{b} = b(t)$, $\hat{\gamma} = 0$, $\hat{\varepsilon}_p = 0$. The elliptic regularity theory for this boundary value problem thus yields that there is a constant, independent of η, such that

$$\|\sigma_\eta\|_{L^\infty(0,T_e;L^2(\Omega\times Y))} \le C\|b\|_{L^\infty(0,T_e;L^2(\Omega))}\,, \tag{89}$$

where the right hand side is finite since by assumption

$$b \in W^{2,1}(0,T_e;L^2(\Omega)) \subseteq L^\infty(0,T_e;L^2(\Omega)).$$

The definition of $b_{[\eta]}$ in (57) and the inequality (42) together imply

$$\|b_{[\eta]}(t)\|_{L^2(Y,H^{-1}(\Omega))} \le C_1\|T_{[\eta]}(t)\|_{\Omega\times Y} \le C_1\|\tilde{T}_\eta(t)\|_{\Omega\times Y}\,,$$

with a constant C_1 independent of t and η. Since $(v_{[\eta]}(t), \sigma_{[\eta]}(t))$ solves the boundary value problem (60) – (62) to the data $\hat{b} = b_{[\eta]}(t)$, $\hat{\gamma} = \gamma_{[\eta]}(t)$,

$\hat{\varepsilon}_p = 0$, we obtain from elliptic regularity theory, from the last estimate and from (77) with constants $C_2, \ldots, C_4$ independent of t and η that

$$\begin{aligned} \|\sigma_{[\eta]}(t)\|_{\Omega\times Y} &\leq C_2\Big(\|b_{[\eta]}(t)\|_{L^2(Y,H^{-1}(\Omega))} + \|\gamma_{[\eta]}(t)\|_{L^2(Y,H^{1/2}(\partial\Omega))}\Big) \\ &\leq C_3\Big(\|\tilde{T}_\eta(t)\|_{\Omega\times Y} + \|\tilde{u}_\eta(t)\|_{L^2(Y,H^1(\Omega))}\Big) \leq C_4\,, \end{aligned} \tag{90}$$

where the last inequality is obtained by again using the regularity theory to the boundary value problem (32), (33), (36) to estimate the solution $(\tilde{u}_\eta, \tilde{T}_\eta)$, noting the estimate (39). From (90), (89) and (88) we see by the Lebesque convergence theorem that the integral in (69) tends to zero for $\eta \to 0$. Together with (74) we thus infer from (69) that for $0 \leq t \leq T_e$

$$\lim_{\eta\to 0} \|z_{[\eta]}(t) - z_\eta(t)\|_{\Omega\times Y} = 0. \tag{91}$$

Last step: estimation of $u_\eta - u_{[\eta]}$ and $T_\eta - T_{[\eta]}$. Since the functions $(u_\eta(t), T_\eta(t))$ and $(u_{[\eta]}(t), T_{[\eta]}(t))$ are solutions of the boundary value problems (1), (2), (5) and (52), (53), (56), respectively, and since the data of these boundary value problems satisfy (76), (81) and (91), we obtain by the standard theory of these linear elliptic boundary value problems that

$$\begin{aligned} &\lim_{\eta\to 0}(\|u_{[\eta]}(t) - u_\eta(t)\|_{\Omega\times Y} + \|T_{[\eta]}(t) - T_\eta(t)\|_{\Omega\times Y}) \\ &\leq C \lim_{\eta\to 0}\Big(\|\gamma_{[\eta]}(t)\|_{L^2(Y,H^{1/2}(\partial\Omega))} + \|b(t) - b_{[\eta]}(t)\|_{L^2(Y,H^{-1}(\Omega))} \\ &\qquad + \|z_{[\eta]}(t) - z_\eta(t)\|_{\Omega\times Y}\Big) = 0\,. \end{aligned}$$

This completes the proof of Proposition 3.2. □

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