A Note on Existence Result for Viscoplastic Models with Nonlinear Hardening

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Abstract

In the recent work of H.-D. Alber and K. Chelminski [3] the existence of the solutions to a model of inelastic (viscoplastic) behavior of materials at small strain is derived. In this work we show that the conditions of the existence theorem in [3] can be relaxed and the same result can be proved under less restrictive assumptions. The relaxation of the conditions of the existence theorem in [3] allows to give the answer on the question raised by Alber and Chelminski in [3] concerning the solvability of the model of nonlinear kinematic hardening without assuming a higher exponent in the constitutive law for one of the internal variables than the exponent in the constitutive law for the other one.

Key words: existence, plasticity, viscoplasticity, maximal monotone operator, general duality principle, degenerate equation.

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1 Setting of the problem

The authors of [3] studied the existence of the solutions to the initial boundary value problem modeled the behaviour of viscoplastic materials at small strains. The problem is formulated as follows:
Let $\Omega \subset \mathbb{R}^3$ denote the set of material points of the body. $\mathcal{S}^3$ denotes the space of symmetric 3×3-matrices. One searches the displacement

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$u(x,t) \in \mathbb{R}^3$, the Cauchy stress tensor $T(x,t) \in \mathcal{S}^3$ and the vector of internal variables $z(x,t) = (\varepsilon_p(x,t), \tilde{z}(x,t)) \in \mathcal{S}^3 \times \mathbb{R}^{N-6}$ of the following model equations

$$-\mathrm{div}_x T(x,t) = b(x,t), \tag{1}$$

$$T(x,t) = \mathcal{D}\big(\varepsilon(\nabla_x u(x,t)) - \varepsilon_p(x,t)\big), \tag{2}$$

$$\partial_t \varepsilon_p(x,t) = g_1\Big(T(x,t), -\tilde{z}(x,t)\Big), \tag{3}$$

$$\partial_t \tilde{z}(x,t) = g_2\Big(T(x,t), -\tilde{z}(x,t)\Big), \tag{4}$$

with the initial condition

$$\varepsilon_p(x,0) = 0, \qquad \tilde{z}(x,0) = \tilde{z}^{(0)}(x) \tag{5}$$

and with the Dirichlet boundary condition

$$u(x,t) = \gamma(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty). \tag{6}$$

The term $\varepsilon(\nabla_x u(x,t))$ in these equations denotes the symmetric 3×3–matrix

$$\varepsilon(\nabla_x u(x,t)) = \frac{1}{2}\Big(\nabla_x u(x,t) + (\nabla_x u(x,t))^T\Big) \in \mathcal{S}^3,$$

the strain tensor. We denote by $\mathcal{D} : \mathcal{S}^3 \to \mathcal{S}^3$ a linear, symmetric, positive definite mapping, the elasticity tensor. The functions $b : \Omega \times [0,\infty) \to \mathbb{R}^3$ is the volume force and $\gamma : \partial\Omega \times [0,\infty) \to \mathrm{I\!R}^3$ is the boundary data.
The functions $g_1 : \mathcal{S}^3 \times \mathbb{R}^{N-6} \to \mathcal{S}^3$ and $g_1 : \mathcal{S}^3 \times \mathbb{R}^{N-6} \to \mathbb{R}^{N-6}$ are given such that

$$(T,y) \to (g_1(T,y), g_2(T,y)) : \mathbb{R}^N \to \mathbb{R}^N$$

is a monotone mapping.

Functional spaces. Let Ω be an open bounded set with C^1-boundary $\partial\Omega$. T_e denotes a positive number (time of existence) and for $0 < t \le T_e$

$$\Omega_t = \Omega \times (0,t), \quad Q = \Omega \times (0, T_e).$$

We denote the Banach space of Lebesgue integrable with the power p together with their weak derivatives up to the order m functions by $W^{m,p}(\Omega, \mathbb{R}^N)$. The norm in $W^{m,p}(\Omega, \mathbb{R}^N)$ is $\|\cdot\|_{m,p,\Omega}$. We choose the numbers p, q satisfying

$$1 < p, q < \infty \ \text{ and } \ 1/p + 1/q = 1.$$

For such p and q one can define the bilinear form on the product space $L^p(\Omega, \mathbb{R}^N) \times L^q(\Omega, \mathbb{R}^N)$ by

$$(\xi, \zeta)_\Omega = \int_\Omega \xi(x) \cdot \zeta(x) dx.$$

We define another bilinear form on $L^p(\Omega, \mathbb{R}^N) \times L^q(\Omega, \mathbb{R}^N)$ by

$$[\xi, \zeta]_\Omega = (\mathcal{D}\xi, \zeta)_\Omega.$$

Spaces of Bochner-measurable functions. If (X, H, X^*) is an evolution triple (known as "Gelfand triple") and $1 < p, q < \infty$, $1/p + 1/q = 1$, then

$$W_{p,q}(0, T_e; X) := \big\{ u \in L^p(0, T_e; X) \mid \dot{u} \in L^q(0, T_e; X^*) \big\}$$

are separable reflexive Banach spaces when furnished with the norm

$$\|u\|^2_{W_{p,q}} = \|u\|^2_{L^p(0,T_e;X)} + \|\dot{u}\|^2_{L^q(0,T_e;X^*)},$$

where the time derivative of $u(\cdot)$ is understood in the sense of vector-valued distributions. The space $L^p(0, T_e; X)$ in the definition of $W_{p,q}(0, T_e; X)$ denotes the Banach space of all Bochner-measurable functions $u : [0, T_e) \to X$ such that $t \mapsto \|u(t)\|^p_X$ is integrable on $[0, T_e)$. We recall that the embedding $W_{p,q}(0, T_e; X) \subset C([0, T_e], H)$ is continuous ([10, p. 4], for instance).

Finally, we frequently use the spaces $W^{k,p}(0, T_e; X)$, which consist of Bochner measurable functions having a p-integrable weak derivatives up to order k.

Main result. Following [3] we define the operator $\mathcal{H} : F(Q, \mathcal{S}^3) \to F(Q, \mathcal{S}^3)$, where $F(Q, \mathcal{S}^3)$ denotes the set of all function from Q to $\mathcal{S}^3$, by the following rule:
Let $(h, \tilde{z})$ be a sulution of the problem

$$h(x,t) = g_1\Big(T(x,t), -\tilde{z}(x,t)\Big), \tag{7}$$

$$\partial_t \tilde{z}(x,t) = g_2\Big(T(x,t), -\tilde{z}(x,t)\Big), \tag{8}$$

$$\tilde{z}(x,0) = \tilde{z}^{(0)}(x), \tag{9}$$

for given $\tilde{z}^{(0)}$ and T and $(x,t) \in Q$. Then the operator $\mathcal{H}$ on $F(Q, \mathcal{S}^3)$ is given by

$$\mathcal{H}(T) = h.$$

In terms of the operator $\mathcal{H}$ the problem (1) - (6) can be written as follows:

$$-\mathrm{div}_x T(x,t) = b(x,t), \tag{10}$$

$$T(x,t) = \mathcal{D}\big(\varepsilon(\nabla_x u(x,t)) - \varepsilon_p(x,t)\big), \tag{11}$$

$$\partial_t \varepsilon_p(x,t) = \mathcal{H}(T), \tag{12}$$

$$\varepsilon_p(x,0) = 0, \tag{13}$$

$$u(x,t) = \gamma(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty). \tag{14}$$

Now we state the existence result for the problem (10) - (14).

Theorem 1.1. *Let $2 \le p < \infty$ and $1 < q \le 2$ be numbers with $1/p + 1/q = 1$. Assume that $\mathcal{H} : L^p(Q, \mathcal{S}^3) \to L^q(Q, \mathcal{S}^3)$ satisfies* *(a) $\mathcal{H}$ is demicontinuous and monotone;*

(b) $\mathcal{H}$ enjoys the growth condition

$$\|\mathcal{H}^{-1}(v)\|_{p,Q} \leq C(1 + \|v\|_{p,Q}^{q/p});$$

(c) the inverse $\mathcal{H}^{-1}$ is strongly coercive, i.e.

$$\frac{\langle v^*, v\rangle}{\|v\|} \to +\infty \quad \textit{as } \|v\| \to \infty, \quad v^* \in \mathcal{H}^{-1}(v).$$

Suppose that $b \in L^p(0, T_e; W^{-1,p}(\Omega, \mathbb{R}^3))$ and $\gamma \in L^p(0, T_e; W^{1,p}(\Omega, \mathbb{R}^3))$. Then there exists a solution

$$u \in L^q(0, T_e; W^{1,q}(\Omega, \mathbb{R}^3)), \quad T \in L^p(0, T_e; L^p(\Omega, \mathcal{S}^3)),$$

$$\varepsilon_p \in W^{1,q}(0, T_e, L^q(\Omega, \mathcal{S}^3))$$

of the problem (10) - (14).

We note that monotonicity of $\mathcal{H}$ is implied by monotonicity of the mapping $(T, y) \to (g_1(T, y), g_2(T, y))$ (see Lemma 4.1, [3]).
In [3] Theorem 1.1 is proved under the additional assumption that the operator $\mathcal{H}$ is coercive. This assumption caused difficulties in the derivation of the existence of the solutions to the model of nonlinear kinematic hardening. Since to show that the operator $\mathcal{H}$ defined by the constitutive relations (specific choice of the functions g_1 and g_2) used for modeling of nonlinear kinematic hardening is coercive the authors of [3] had to impose the restriction on the exponents in the constitutive relations for the different internal variables (see Section 5). Our approach is actually based on the constructions from [3] and repeats main steps of that work in the reverse direction with only one difference that we use the general duality principle for the sum of two operators from [4] to derive the existence of the solutions to the problem (10) - (14). The application of this duality principle gives us the possibility to avoid the coercivity assumption on $\mathcal{H}$.

2 The Helmholtz projection on tensor fields

The material for this section we borrow from [3]. Therefore we state only main results presented there without going into details and for the further reading we refer the reader to that work.
In this work we need projection operators to spaces of tensor fields, which are symmetric gradients and to spaces of tensor fields with vanishing divergence.

We recall ([15]) that a Dirichlet boundary value problem from the linear elasticity theory formed by equations

$$-\mathrm{div}_x T(x) = \hat{b}(x), \quad x \in \Omega, \tag{15}$$

$$T(x) = \mathcal{D}(\varepsilon(\nabla_x u(x)) - \hat{\varepsilon}_p(x)), \quad x \in \Omega, \tag{16}$$

$$u(x) = \hat{\gamma}(x), \quad x \in \partial\Omega, \tag{17}$$

to given $\hat{b} \in W^{-1,p}(\Omega, \mathbb{R}^3)$, $\hat{\varepsilon}_p \in L^p(\Omega, \mathcal{S}^3)$ and $\hat{\gamma} \in W^{1,p}(\Omega, \mathbb{R}^3)$ has a unique weak solution $(u, T) \in W^{1,p}(\Omega, \mathbb{R}^3) \times L^p(\Omega, \mathcal{S}^3)$ with $1 < p < \infty$ and $1/p + 1/q = 1$. For $\hat{b} = \hat{\gamma} = 0$ the solution of (15) - (17) satisfies the inequality

$$\|\varepsilon(\nabla_x u)\|_{p,\Omega} \leq C \|\hat{\varepsilon}_p\|_{p,\Omega}$$

with some positive constant C.

Definition 2.1. *For every $\hat{\varepsilon}_p \in L^p(\Omega, \mathcal{S}^3)$ we define a linear operator $P_p : L^p(\Omega, \mathcal{S}^3) \to L^p(\Omega, \mathcal{S}^3)$ by*

$$P_p \hat{\varepsilon}_p = \varepsilon(\nabla_x u),$$

where $u \in W_0^{1,p}(\Omega, \mathrm{I\!R}^3)$ is a unique weak solution of (15) - (17) to the given function $\hat{\varepsilon}_p$ and $\hat{b} = \hat{\gamma} = 0$.

A subset $\mathcal{G}^p$ of $L^p(\Omega, \mathcal{S}^3)$ is defined by

$$\mathcal{G}^p = \{\varepsilon(\nabla_x u) \mid u \in W_0^{1,p}(\Omega, \mathrm{I\!R}^3)\}.$$

The following lemma gives the main properties of P_p.

Lemma 2.1. *For every $1 < p < \infty$ the operator P_p is a bounded projector onto the subset $\mathcal{G}^p$ of $L^p(\Omega, \mathcal{S}^3)$. The projector $(P_p)^*$ adjoint with respect to the bilinear form $[\xi, \zeta]_\Omega$ on $L^p(\Omega, \mathcal{S}^3) \times L^q(\Omega, \mathcal{S}^3)$ satisfy*

$$(P_p)^* = P_q, \quad \textit{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

This implies $\ker(P_p) = H^p_{sol}$ *with*

$$H^p_{sol} = \{\xi \in L^p(\Omega, \mathcal{S}^3) \mid [\xi, \zeta]_\Omega = 0 \quad for\ all\ \zeta \in \mathcal{G}^q\}.$$

Since $\mathcal{D}$ is symmetric, the relation $[\xi, \zeta]_\Omega = 0$ holds for all $\zeta \in \mathcal{G}^q$ if and only if

$$(\mathcal{D}\xi, \nabla_x v)_\Omega = (\mathcal{D}\xi, \varepsilon(\nabla_x v))_\Omega = [\xi, \varepsilon(\nabla_x v)]_\Omega = 0$$

for all $v \in W_0^{1,q}(\Omega, \mathrm{I\!R}^3)$. Consequently

$$H^p_{sol} = \{\xi \in L^p(\Omega, \mathcal{S}^3) \mid \mathrm{div}(\mathcal{D}\xi) = 0\}.$$

Therefore the projection operator

$$Q_p = (I - P_p) : L^p(\Omega, \mathcal{S}^3) \to L^p(\Omega, \mathcal{S}^3)$$

with $Q_p(L^p(\Omega, \mathcal{S}^3)) = H^p_{sol}$ is a generalization of the classical Helmholtz projection.

Corollary 2.0.1. *Let $(\mathcal{D}P_p)^T$ be the operator adjoint to $\mathcal{D}P_p : L^p(\Omega, \mathrm{I\!R}^{\mathrm{N}}) \to L^p(\Omega, \mathrm{I\!R}^{\mathrm{N}})$ with respect to the bilinear form $(\xi, \zeta)_\Omega$ on the product space $L^p(\Omega, \mathrm{I\!R}^{\mathrm{N}}) \times L^q(\Omega, \mathrm{I\!R}^{\mathrm{N}})$. Then*

$$(\mathcal{D}P_p)^T = \mathcal{D}P_q : L^q(\Omega, \mathrm{I\!R}^{\mathrm{N}}) \to L^q(\Omega, \mathrm{I\!R}^{\mathrm{N}}).$$

Moreover, the operator $\mathcal{D}Q_2$ is non-negative and self-adjoint.

The last result in this corollary is proved in [2].

3 Maximal monotone operators

In this section we present our necessary tools for the construction of the existence theory for the problem (10) - (14), which will be used in the next section.
Let V be a reflexive Banach space with the norm $\|\cdot\|$, V^* be its dual space with the norm $\|\cdot\|_*$. The brackets $\langle\cdot,\cdot\rangle$ denotes the dual pairing between V and V^*. Under V we shall always mean a reflexive Banach space throughout this section.
For a multivalued mapping $A : V \to 2^{V^*}$ the sets

$$D(A) = \{v \in V \mid Av \neq \emptyset\}$$

and

$$GrA = \{[v, v^*] \in V \times V^* \mid v \in D(A),\ v^* \in Av\}$$

are called the *effective domain* and the *graph* of A, respectively.

Definition 3.1. *The mapping $A : V \to 2^{V^*}$ is called* monotone *if the inequality*

$$\langle v^* - u^*, v - u\rangle \geq 0$$

holds for all $[u, u^*], [v, v^*] \in GrA$.
The mapping $A : V \to 2^{V^*}$ is called maximal monotone *iff the inequality*

$$\langle v^* - u^*, v - u\rangle \geq 0 \quad (\forall)\ [u, u^*] \in GrA$$

implies $[v, v^*] \in GrA$.
The mapping $A : V \to 2^{V^*}$ is called strongly coercive *iff either $D(A)$ is bounded or $D(A)$ is unbounded and the condition*

$$\frac{\langle v^*, v - w\rangle}{\|v\|} \to +\infty \quad \text{as } \|v\| \to \infty, \quad [v, v^*] \in GrA,$$

is satisfied for each $w \in D(A)$.

It is well known ([11, p. 105]) that if A is a maximal monotone operator, then, for any $v \in D(A)$, the image Av is closed convex subset of V^* and the graph GrA is demiclosed[1].

Remark 3.1. We recall that the subdifferential of a lower semi-continuous and convex function is maximal monotone (Theorem 2.25 [12, p. 27]).

The next theorem gives a criteria for a linear monotone operator to be maximal.

Theorem 3.1. *The following assertions are equivalent:*

(a) $A : V \to V^*$ *is maximal monotone;*

(b) A *is a densely defined closed operator such that its adjoint* A^* *is monotone;*

(c) A *is a densely defined closed operator such that* A^* *is maximal monotone.*

Proof. See Theorem 1 in [6]. □

We also need the following result on the maximality of the sum of two maximal monotone operators.

Theorem 3.2. *Let* V *be a reflexive Banach space, and let* A *and* B *be maximal monotone. Suppose that the condition*

$$\operatorname{int} D(A) \cap D(B) \neq \emptyset$$

is fulfilled. The the sum $A + B$ *is a maximal monotone operator.*

Proof of Theorem 3.2. See Theorem III.3.6 in [11] or Theorem II.1.7 in [5]. □

For deeper results on the maximality of the sum of two maximal monotone operators we refer the reader to [14], see also [7].
The next surjectivity result on maximal monotone operators is the key tool in the proof of our main existence result.

Theorem 3.3. *If* V *is a (strictly convex) reflexive Banach space and* $A : V \to 2^{V^*}$ *is maximal monotone and coercive, then* A *is surjective.*

Proof of Theorem 3.3. See Theorem III.2.10 in [11]. □

For further reading on maximal monotone operators we refer the reader to [5, 8, 9, 11, 12] or [16].

[1]A set $A \in V \times V^*$ is demiclosed if v_n converges strongly to v_0 in V and v_n^* converges weakly to v_0^* in V^* (or v_n converges weakly to v_0 in V and v_n^* converges strongly to v_0^* in V^*) and $[v_n, v_n^*] \in GrA$, then $[v, v^*] \in GrA$

4 Proof of the existence result

This section is devoted to the study of the existence of the solutions for the initial boundary value problem (10) - (14).

Proof of Theorem 1.1. Before we start the proof of Theorem 1.1, let us introduce the following notations

$$X = L^p(\Omega, \mathcal{S}^3), \quad \mathcal{X} = L^p(0, T_e; W), \quad M_p = \mathcal{D}Q_p : X \to X.$$

We note that the operator M_2 is non-negative by Corollary 2.0.1. Next, we define a linear maximal monotone operator $\mathcal{L} : \mathcal{X} \to \mathcal{X}^*$ by

$$\mathcal{L}z = \partial_t z \ \text{ with } D(\mathcal{L}) = \{z \in W_{p,q}(0, T_e; \mathcal{X}) \mid z(0) = 0\}.$$

The idea of the proof of Theorem 1.1 is to show the solvability of the abstract equation (18) in a reflexive Banach space $\mathcal{X}^*$ applying Theorem 3.3 and then, based on this result, to construct solutions for the initial boundary value problem (1) - (5). We note that the idea of the proof is strongly connected to the general duality principle for the sum of two operators obtained in [4].

Let us consider now the following inclusion in $\mathcal{X}^*$

$$\mathcal{L}^{-1} M_q w + \mathcal{H}^{-1} w \ni \sigma, \quad w \in \mathcal{X}^*, \tag{18}$$

where $(v(t), \sigma(t))$ is a solution of the Dirichlet boundary value problem (15) - (17) to the data $\hat{b} = b(t)$, $\hat{\gamma} = \gamma(t)$, $\hat{\varepsilon}_p = 0$. The next lemma proves that the operator $\mathcal{L}^{-1} M_q$ in (18) is maximal monotone.

Lemma 4.1. *The operator $\mathcal{L}^{-1} M_q : D(\mathcal{L}^{-1} M_q) \subset \mathcal{X}^* \to \mathcal{X}$ is linear maximal monotone.*

Proof of Lemma 4.1. According to Theorem 3.1, the operator $\mathcal{L}^{-1} M_q$ with $D(\mathcal{L}^{-1} M_q) = \{v \in \mathcal{X}^* \mid M_q v \in D(\mathcal{L}^{-1})\}$ is maximal monotone, if it is a densely defined closed monotone operator such that its adjoint $(\mathcal{L}^{-1} M_q)^*$ is monotone.

We note that the operator $\mathcal{L}^{-1} M_q$ is the closure in $\mathcal{X}^* \times \mathcal{X}$ of the operator $\mathcal{L}_0$ given by

$$\mathcal{L}_0 v := \mathcal{L}^{-1} M_q v, \quad v \in D(\mathcal{L}_0) = \{v \in \mathcal{X}^* \mid \mathcal{L}^{-1} v \in \mathcal{X}\}.$$

The last operator is monotone, what can be shown using the generalized integration by parts formula and the following identity

$$\mathcal{L}^{-1} M_q v = M_p \mathcal{L}^{-1} v, \quad v \in D(\mathcal{L}_0). \tag{19}$$

The identity (19) is proved in the end of this work. Therefore, the operator $\mathcal{L}^{-1} M_q$ is monotone as the closure in $\mathcal{X}^* \times \mathcal{X}$ of the monotone operator

$\mathcal{L}_0$. Since the operator $(\mathcal{L}^{-1}M_q)$ is the closure of $\mathcal{L}_0$, thier adjoint operators coincide. The adjoint of $\mathcal{L}_0$ is easy to compute and is equal to $(\mathcal{L}^{-1})^*M_q$, by a well-known result from the functional analysis. Therefore, by arguing in the same way as above, we obtain that the adjoint $(\mathcal{L}^{-1}M_q)^*$ (we recall that $(\mathcal{L}^{-1}M_q)^* = (\mathcal{L}^{-1})^*M_q$) is monotone. Thus, since $\mathcal{L}^{-1}M_q$ verifies all assumptions of Theorem 3.1, it is a maximal monotone operator. The proof of Lemma 4.1 is complete. □

In order to apply Theorem 3.3 we note the operator $\mathcal{H}^{-1}$ is maximal monotone as the inverse of a maximal monotone operator. By the assumption (b) of Theorem 1.1 the operators $\mathcal{L}^{-1}M_q$ and $\mathcal{H}^{-1}$ satisfy the condition

$$D(\mathcal{L}^{-1}M_q) \cap \operatorname{int} D(\mathcal{H}^{-1}) \neq \emptyset.$$

Therefore, by Theorem 3.2, the sum $\mathcal{L}^{-1}M_q + \mathcal{H}^{-1}$ is maximal monotone. Moreover, the coecivity of $\mathcal{H}^{-1}$ implies that

$$\frac{\big\langle \mathcal{L}^{-1}M_q v + v^*, v\big\rangle}{\|v\|} \geq \frac{\langle v^*, v\rangle}{\|v\|} \to +\infty \quad \text{as } \|v\| \to \infty,$$

for $v^* \in \mathcal{H}^{-1}(v)$. Thus, in virtue of Theorem 3.3, the maximal monotone and coercive operator $\mathcal{L}^{-1}M_q + \mathcal{H}^{-1}$ is surjective. Therefore, the equition (18) has a solution. Denoting by $\tau = \mathcal{L}^{-1}M_q w$ we obtain from (18) that τ solves the problem

$$\mathcal{L}\tau = M_q G(-\tau + \sigma), \quad \tau \in L^p(\Omega_{T_e}, \mathbb{R}^N). \tag{20}$$

Using the last result, the construction of the solution of the problem (10) - (14) can be now performed as in [3]:

Let $\tau \in \mathcal{X}$ be the unique solution of (20). With the function τ let $\varepsilon_p \in W^{1,q}(0, T_e, L^q(\Omega, \mathcal{S}^3))$ be the solution of

$$\partial_t \varepsilon_p(t) = \mathcal{H}\big(-\tau(t) + \sigma(t)\big), \quad \text{for a.e. } t \in (0, T_e) \tag{21}$$

$$\varepsilon_p(0) = 0. \tag{22}$$

Moreover, by the linear elliptic theory, there is a unique solution $(\tilde{u}(t), \tilde{T}(t))$ of problem (15) - (17) to the data $\hat{b} = \hat{\gamma} = 0$, $\hat{\varepsilon}_p = \varepsilon_p(t)$. The solution of (10) - (14) is now given as follows

$$(u, T, \varepsilon_p) = (\tilde{u} + v, \tilde{T} + \sigma, \varepsilon_p) \in$$
$$L^q(0, T_e; W^{1,q}(\Omega, \mathbb{R}^3)) \times L^p(\Omega_{T_e}\mathcal{S}^3) \times W^{1,q}(0, T_e, L^q(\Omega, \mathcal{S}^3)).$$

To see that (u, T, ε_p) satisfies (12), we apply the operator Q_q to (21) - (22) from the left and obtain that

$$\partial_t(Q_q\varepsilon_p) = Q_q\mathcal{H}\big(-\tau(t) + \sigma(t)\big) = \partial_t\tau, \quad Q_q\varepsilon_p(0) = \tau(0) = 0.$$

The last line implies that $Q_q\varepsilon_p = \tau$. Thus

$$T = \tilde{T} + \sigma = -Q_q\varepsilon_p + \sigma = -\tau + \sigma \in L^p(\Omega_{T_e}\mathcal{S}^3).$$

The last observation completes the proof of Theorem 1.1. □

5 Nonlinear kinematic hardening

Now we apply Theorem 1.1 to the model of of nonlinear hardening. The model of nonlinear kinematic hardening consists of the equations ([1, 3])

$$
\begin{aligned}
-\mathrm{div}_x T &= b, &(23)\\
T &= \mathcal{D}\big(\varepsilon(\nabla_x u) - \varepsilon_p\big), &(24)\\
\partial_t \varepsilon_p &= c_1 |T - k(\varepsilon_p - \varepsilon_n)|^r \frac{T - k(\varepsilon_p - \varepsilon_n)}{|T - k(\varepsilon_p - \varepsilon_n)|}, &(25)\\
\partial_t \varepsilon_n &= c_2 |k(\varepsilon_p - \varepsilon_n)|^m \frac{k(\varepsilon_p - \varepsilon_n)}{|k(\varepsilon_p - \varepsilon_n)|}, &(26)\\
\varepsilon_n(0) &= \varepsilon_n^0, \quad \varepsilon_p(0) = 0, &(27)\\
u &= \gamma, \qquad x \in \partial\Omega. &(28)
\end{aligned}
$$

The equations (23) - (27) can be written in the general form (1) - (6) with $g = (g_1, g_2) : \mathcal{S}^3 \times \mathcal{S}^3 \to \mathcal{S}^3 \times \mathcal{S}^3$ defined by

$$
\begin{aligned}
g_1(T, \tilde{z}) &= c_1 |T + k^{1/2}\tilde{z}|^r \frac{T + k^{1/2}\tilde{z}}{|T + k^{1/2}\tilde{z}|}, &(29)\\
g_2(T, \tilde{z}) &= c_1 k^{1/2} |T + k^{1/2}\tilde{z}|^r \frac{T + k^{1/2}\tilde{z}}{|T + k^{1/2}\tilde{z}|} + c_2 k^{1/2} |k^{1/2}\tilde{z}|^m \frac{\tilde{z}}{|\tilde{z}|}, &(30)
\end{aligned}
$$

where $\tilde{z} = k^{1/2}(\varepsilon_p - \varepsilon_n)$.
Monotonicity of the mapping $(T, \tilde{z}) \to (g_1(T, \tilde{z}), g_2(T, \tilde{z}))$ follows from the fact that $g = (g_1, g_2)$ is the gradient of the convex function

$$
\phi(T, \tilde{z}) = \frac{c_1}{r+1} |T + k^{1/2}\tilde{z}|^{r+1} + \frac{c_2}{m+1} |k^{1/2}\tilde{z}|^{m+1}.
$$

Now we can prove the main result of this section.

Theorem 5.1. *Let c_1, c_2, k be positive constants and let r and γ satisfy $r, m > 1$. Let us define $p = 1 + r$, $q = 1 + 1/r$, $\hat{p} = \max\{p, 1 + m\}$ and $\hat{q} = \min\{q, 1 + 1/m\}$.*
Suppose that $b \in L^p(0, T_e; W^{-1,p}(\Omega, \mathbb{R}^3))$, $\gamma \in L^p(0, T_e, W^{1,p}(\Omega, \mathbb{R}^3))$ and $\varepsilon_n^{(0)} \in L^2(\Omega, \mathcal{S}^3)$. Then there exists a solution

$$
u \in L^q(0, T_e; W^{1,q}(\Omega, \mathbb{R}^3)), \quad T \in L^p(0, T_e; L^p(\Omega, \mathcal{S}^3)),
$$

$$
\varepsilon_p \in W^{1,q}(0, T_e, L^q(\Omega, \mathcal{S}^3)), \quad \varepsilon_n \in W^{1,\hat{q}}(0, T_e, L^{\hat{q}}(\Omega, \mathcal{S}^3))
$$

of the problem (23) - (27). Moreover, $\varepsilon_p - \varepsilon_n \in W_{\hat{p},\hat{q}}(0, T_e, L^{\hat{p}}(\Omega, \mathcal{S}^3))$.

Remark 5.1. *In [3] Theorem 5.1 is proved provided m and r satisfy the inequality $m > r$. This condition the authors of [3] use to show that the operator $\mathcal{H}$ defined by the equations (25) - (27) according to the rule given above is coercive.*

Remark 5.2. *Using the theory of Orlic spaces and the monotone operator method similar results are obtained in [13] with the same restrictions on m and r as in Theorem 5.1.*

Proof of Theorem 5.1. The maximal monotonicity of $\mathcal{H}$ is shown in [3]. Therefore, we refer the reader for the prove of the condition (a) of Theorem 1.1 to that work and show here only that the operator $\mathcal{H}$ defined by (25) - (27) satisfies the assumptions (b) and (c) of Theorem 1.1. First, we prove that $\mathcal{H}^{-1}$ is coercive.
We note first that

$$\|\mathcal{H}(T)\|_{q,Q}^q = \int_Q \left(c_1|T - k(\varepsilon_p - \varepsilon_n)|^r\right)^{1+1/r} d(x,t) = c_1^q \|T - k(\varepsilon_p - \varepsilon_n)\|_{p,Q}^p.$$

Equations (25) - (26) yield

$$\mathcal{H}(T)\cdot T = \partial_t \frac{k}{2}|\varepsilon_p - \varepsilon_n|^2 + (T - k(\varepsilon_p - \varepsilon_n))\cdot c_1|T - k(\varepsilon_p - \varepsilon_n)|^r \frac{T - k(\varepsilon_p - \varepsilon_n)}{|T - k(\varepsilon_p - \varepsilon_n)|}$$

$$+k(\varepsilon_p - \varepsilon_n)\cdot c_2|k(\varepsilon_p - \varepsilon_n)|^m \frac{k(\varepsilon_p - \varepsilon_n)}{|k(\varepsilon_p - \varepsilon_n)|} = \partial_t \frac{k}{2}|\varepsilon_p - \varepsilon_n|^2$$

$$+c_1|T - k(\varepsilon_p - \varepsilon_n)|^p + c_2|k(\varepsilon_p - \varepsilon_n)|^{m+1}.$$

Then the integration and previous computations show that

$$(\mathcal{H}(T), T)_Q = \frac{k}{2}\|(\varepsilon_p - \varepsilon_n)(T_e)\|_\Omega^2 - \frac{k}{2}\|(\varepsilon_p - \varepsilon_n)(0)\|_\Omega^2 + c_1\|T - k(\varepsilon_p - \varepsilon_n)\|_{p,Q}^p$$

$$+c_2\|k(\varepsilon_p - \varepsilon_n)\|_{m+1,Q}^{m+1} \geq c_1\|T - k(\varepsilon_p - \varepsilon_n)\|_{p,Q}^p - \frac{k}{2}\|(\varepsilon_p - \varepsilon_n)(0)\|_\Omega^2$$

$$= c_1^{1-q}\|\mathcal{H}(T)\|_{q,Q}^q - \frac{k}{2}\|(\varepsilon_p - \varepsilon_n)(0)\|_\Omega^2.$$

The last inequality implies the coercivity of the inverse $\mathcal{H}^{-1}$. Therefore, it remains to verify that $\mathcal{H}^{-1}$ has a polynomical growth.

For the function $y = \varepsilon_p - \varepsilon_n$ we have

$$\partial_t \frac{k}{2}|y(x,t)|^2 = ky\cdot c_1|T - ky|^r \frac{T - ky}{|T - ky|} - ky\cdot c_2|ky|^m \frac{ky}{|ky|}$$

$$\leq c_1\Big(\frac{1}{p\alpha^p}|ky|^p + \frac{\alpha^q}{q}|T - ky|^{qr}\Big) - c_2|ky|^{m+1}.$$

Here we used Young's inequality with $\alpha > 0$. Therefore,

$$\frac{k}{2}\|y(T_e)\|_{2,\Omega}^2 + c_2\|ky\|_{m+1,\Omega_{T_e}}^{m+1} \leq c_1\Big(\frac{1}{p\alpha^p}\|ky\|_{p,\Omega_{T_e}}^p + \frac{\alpha^q}{q}\|T - ky\|_{p,\Omega_{T_e}}^p\Big) + \frac{k}{2}\|y(0)\|_{2,\Omega}^2$$

and consequently

$$c_2\|ky\|_{m+1,\Omega_{T_e}}^{m+1} \le c_1\Big(\frac{1}{p\alpha^p}\|ky\|_{p,\Omega_{T_e}}^p + \frac{\alpha^q}{q}\|T-ky\|_{p,\Omega_{T_e}}^p\Big) + \frac{k}{2}\|y(0)\|_{2,\Omega}^2. \quad (31)$$

On the other hand we have

$$\|T\|_{p,\Omega_{T_e}}^p \le \|ky\|_{p,\Omega_{T_e}}^p + \|T-ky\|_{p,\Omega_{T_e}}^p. \quad (32)$$

Multiplying (32) by $\frac{1}{p\alpha^p}$ and then subtracting (31) we get the estimate

$$\begin{aligned}\frac{1}{p\alpha^p}\|T\|_{p,\Omega_{T_e}}^p - \frac{c_2}{c_1}\|ky\|_{m+1,\Omega_{T_e}}^{m+1} &\le \Big(\frac{1}{p\alpha^p} - \frac{\alpha^q}{q}\Big)\|T-ky\|_{p,\Omega_{T_e}}^p - \frac{k}{2c_1}\|y(0)\|_{2,\Omega}^2 \\ &\le \Big(\frac{1}{p\alpha^p} - \frac{\alpha^q}{q}\Big)\|T-ky\|_{p,\Omega_{T_e}}^p. \quad (33)\end{aligned}$$

For sufficiently small α the constant $\Big(\frac{1}{p\alpha^p} - \frac{\alpha^q}{q}\Big)$ is positive. More precisely, $\alpha \in (0,\alpha_0)$ with $\alpha_0 := (q/p)^{1/(p+q)}$. Later we give more precisely the upper bound for α. Now we derive the estimate for $\|ky\|_{m+1,\Omega_{T_e}}$ in terms of $\|T\|_{p,\Omega_{T_e}}$:

$$\partial_t\frac{k}{2}|y(x,t)|^2 = -(T-ky)\cdot c_1|T-ky|^r\frac{T-ky}{|T-ky|} - ky\cdot c_2|ky|^m\frac{ky}{|ky|}$$

$$+T\cdot c_1|T-ky|^r\frac{T-ky}{|T-ky|} \le -c_1|T-ky|^p - c_2|ky|^{m+1} + c_1|T||T-ky|^r$$

$$\le -c_1|T-ky|^p - c_2|ky|^{m+1} + c_1\Big(\frac{1}{p\delta^p}|T|^p + \frac{\delta^q}{q}|T-ky|^{qr}\Big).$$

Here we used Young's inequality with δ. Choosing $\delta = (q/2)^{1/q}$ we arrive at the estimate

$$\frac{k}{2}\|y(T_e)\|_{2,\Omega}^2 + \frac{c_1}{2}\|T-ky\|_{p,\Omega_{T_e}}^p + c_2\|ky\|_{m+1,\Omega_{T_e}}^{m+1} \le \frac{k}{2}\|y(0)\|_{2,\Omega}^2 + c_1\frac{1}{p\delta^p}\|T\|_{p,\Omega_{T_e}}^p$$

and consequently

$$c_2\|ky\|_{m+1,\Omega_{T_e}}^{m+1} \le \frac{k}{2}\|y(0)\|_{2,\Omega}^2 + c_1\frac{1}{p\delta^p}\|T\|_{p,\Omega_{T_e}}^p. \quad (34)$$

Thus from (33) and (34) we obtain

$$\Big(\frac{1}{p\alpha^p} - \frac{1}{p\delta^p}\Big)\|T\|_{p,\Omega_{T_e}}^p - \frac{k}{2c_1}\|y(0)\|_{2,\Omega}^2 \le \Big(\frac{1}{p\alpha^p} - \frac{\alpha^q}{q}\Big)\|T-ky\|_{p,\Omega_{T_e}}^p. \quad (35)$$

Choosing $\alpha = \min\{\delta/2, \alpha_0/2\}$ in (35) we obtain

$$C_1\|T\|_{p,\Omega_{T_e}}^p - C_2 \le C_3\|T-ky\|_{p,\Omega_{T_e}}^p \quad (36)$$

with some positive constants C_1, C_2 and C_3. Recalling that $\|\mathcal{H}(T)\|^q_{q,\Omega_{T_e}} = c_1^q \|T - ky\|^p_{p,\Omega_{T_e}}$, the inequality (36) implies

$$C_1 \|T\|^p_{p,\Omega_{T_e}} - C_2 \le C_3 c_1^q \|\mathcal{H}(T)\|^q_{q,\Omega_{T_e}},$$

which yields the polynomial growth for the inverse of $\mathcal{H}(T)$, i.e.

$$\|\mathcal{H}^{-1}(v)\|_{p,\Omega_{T_e}} \le C_4 (1 + \|v\|^{q/p}_{q,\Omega_{T_e}}) \tag{37}$$

with some positive constant C_4. Thus $\mathcal{H}^{-1}$ is coercive and bounded. Hence, Theorem 1.1 yields the existence of u, T and ε_p.
To show the existence of ε_n we proceed as in [3]: The definition of $\mathcal{H}$ (see Section 1) implies that the function $y = k^{-1/2}\tilde{z} = \varepsilon_p - \varepsilon_n$ solves the problem

$$y_t = c_1 |T - ky|^r \frac{T - ky}{|T - ky|} - c_2 |ky|^m \frac{y}{|y|}, \tag{38}$$

$$y(0) = y^{(0)}, \tag{39}$$

where $y^{(0)} = \varepsilon_p^{(0)} - \varepsilon_n^{(0)}$. The solvability of (38) - (39) can be obtained, for example, in the same manner as for the equation (18). Namely, the direct computations together with Young's inequality with ϵ imply that the operator $B : L^{\hat{p}}(Q, \mathcal{S}^3) \to L^{\hat{q}}(Q, \mathcal{S}^3)$ defined by

$$B(T, y) = -c_1 |T - ky|^r \frac{T - ky}{|T - ky|} + c_2 |ky|^m \frac{y}{|y|}$$

enjoys the inequality

$$(B(T, y), y)_Q \ge C_1 \|y\|^{m+1}_{m+1} + C_2 \|T - ky\|^p_p - C_3 \|T\|^p_p$$

or

$$(B(T, y), y)_Q \ge C_1 \|y\|^{m+1}_{m+1} + C_2 k \|y\|^p_p - (1 + C_3) \|T\|^p_p,$$

where C_1, C_2 and C_3 are some constants. The last inequality yields the coercivity of B in $L^{\hat{p}}(Q, \mathcal{S}^3)$. Repeating the arguments of the proof of the existence for (18) we get the solvability of (38) - (39). Uniqueness of the solution is obvious. From the existence of ε_p and y follows the existence and the required regularity of ε_n. The last computations also show that the operator $\mathcal{H}$ defined by the equations (25) - (27) is well-defined.
The proof of Theorem 5.1 is complete. □

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