Stochastic Taylor Expansions: Weight functions of B-series expressed as multiple integrals

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Abstract

The exact solution of stochastic differential equations can be expressed as stochastic B-series. In this paper, we present an algorithm using rooted trees for expanding the weight functions occurring in this representation in terms of multiple integrals using multi-indices.

Keywords: Stochastic differential equations, stochastic Taylor expansions, Wagner-Platen series, B-series
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1 Introduction

Given the stochastic differential equation (SDE)

$$X(t)=x_0+\int_0^t g_0(X(s))ds+\sum_{l=1}^{m}\int_0^t g_l(X(s))\star dW_l(s). \tag{1}$$

Here $W_l(s)$, $l=1,\ldots,m$, are m independent Wiener processes defined on a probability space $(\Omega,\mathcal{F},P)$, and $g_l(X)$ are assumed to be measurable $\mathbb{R}^d$-valued functions on $\mathbb{R}^d$. The integral w. r. t. the Wiener process can be interpreted e. g. as Itô integral with $\star dW_l(s)=dW_l(s)$ or as Stratonovich integral with $\star dW_l(s)=\circ dW_l(s)$. The solution $X(t)$ and $f(X(t))$ can formally be written in terms of stochastic Taylor expansions. These can take two forms: Either as Wagner-Platen series [4, 5] or as B-series [1, 2, 6, 7, 3]. The aim of this paper is to express the weight functions of the B-series in terms of multiple integrals using multi-indices, as this is in practice the more common representation and e. g. especially useful for calculating expectation values in the Itô case. To achieve this, we use a representation of the coefficient functions of the Wagner-Platen series in terms of rooted trees.

To simplify the presentation, we define $W_0(s)=s$, so that (1) can be written as

$$X(t)=x_0+\sum_{l=0}^{m}\int_0^t g_l(X(s))\star dW_l(s). \tag{2}$$

The Wagner-Platen series is derived by repeated use of the Itô formula

$$f(X(t))=f(x_0)+\sum_{l=0}^{m}\int_0^t L^l f(X(s))\star dW_l(s), \tag{3}$$

in which

$$L^0 f=f'g_0+\gamma^\star\sum_{l=1}^{m}f''(g_l,g_l),\qquad L^l f=f'g_l,\quad l=1,2,\ldots,m$$

and $\gamma^\star=1/2$ in the Itô case and 0 in the Stratonovich case. Our exposition is independent of the choice of $\gamma^\star$. The following fundamental theorem is taken from [5, chap. 5]:

Theorem 1 (Wagner-Platen series). *If $X(t)$ is a solution of SDE* (1)*, then $f(X(t))$ can be written as a formal series of the form*

$$f(X(t))=\sum_{\alpha\in\mathcal{M}}I_\alpha^\star(t)f_\alpha(x_0).$$

Here, the set of multi-indices $\mathcal{M}$ is

$$\mathcal{M}=\{\alpha=(j_1,j_2,\ldots,j_r):j_i\in\{0,1,2,\ldots,m\},\ i\in\{1,2,\ldots,r\}\ \text{for}\ r=1,2,3,\ldots\}\cup\{\emptyset\}.$$

The coefficient functions f_α and the multiple stochastic integrals $I_\alpha^\star$ are given by

$$f_\emptyset(x_0)=f(x_0),\qquad f_\alpha(x_0)=\left(L^{j_r}L^{j_{r-1}}\cdots L^{j_1}f\right)(x_0),$$

$$I_\emptyset^\star(t)=1,\qquad I_\alpha^\star(t)=\int_0^t\int_0^{s_{s_r}}\cdots\int_0^{s_2}dW_{j_1}(s_1)\star\cdots\star dW_{j_{r-1}}(s_{r-1})\star dW_{j_r}(s_r).$$

The series for the solution $X(t)$ is given by using $f(x) = x$. While the Wagner-Platen series uses multi-indices, the B-series denotes the terms by trees:

Definition 1 (Trees). *The set of $m+1$-colored, rooted trees*

$$T = \{\emptyset\} \cup T_0 \cup T_1 \cup \cdots \cup T_m$$

is recursively defined as follows:

a) *The graph $\textcircled{l} = [\emptyset]_l$ with only one vertex of color l belongs to T_l.*

Let $\tau = [\tau_1, \tau_2, \ldots, \tau_\kappa]_l$ be the tree formed by joining the subtrees $\tau_1, \tau_2, \ldots, \tau_\kappa$ each by a single branch to a common root of color l.

b) *If $\tau_1, \tau_2, \ldots, \tau_\kappa \in T$, then $\tau = [\tau_1, \tau_2, \ldots, \tau_\kappa]_l \in T_l$ for $\kappa = 1, 2, \ldots$.*

Thus, T_l is the set of trees with an l-colored root, and T is the union of these sets. We will also need the following set:

$$U_f = \{u = [\tau_1, \tau_2, \ldots, \tau_\kappa]_f \ : \ \tau_j \in T,\ j = 1, 2, \ldots, \kappa, \text{ for } \kappa = 1, 2, \ldots\}. \tag{4}$$

The following theorem has been proved in [3]:

Theorem 2 (B-series). *The exact solution $X(t)$ of (1) can be written in terms of formal series as*

$$X(t) = \sum_{\tau \in T} \alpha_\tau \cdot \varphi_\tau(t) \cdot F_\tau(x_0) \qquad \textit{and} \qquad f(X(t)) = \sum_{u \in U_f} \beta_u \cdot \psi_u(t) \cdot G_u(x_0). \tag{5}$$

The elementary differentials $F_\tau(x_0)$, the weight function $\varphi_\tau(t)$, and α_τ are given by

$$\begin{aligned}
&F_\emptyset(x_0) = x_0, && F_{\textcircled{l}}(x_0) = g_l(x_0), && F_\tau(x_0) = g_l^{(\kappa)}(x_0)\big(F_{\tau_1}(x_0), F_{\tau_2}(x_0), \ldots, F_{\tau_\kappa}(x_0)\big),\\
&\varphi_\emptyset(t) \equiv 1, && \varphi_{\textcircled{l}}(t) = W_l(t), && \varphi_\tau(t) = \int_0^t \prod_{j=1}^{\kappa} \varphi_{\tau_j}(s) \star dW_l(s),\\
&\alpha_\emptyset = 1, && \alpha_{\textcircled{l}} = 1, && \alpha_\tau = \frac{1}{r_1! r_2! \cdots r_q!} \prod_{j=1}^{\kappa} \alpha_{\tau_j},
\end{aligned}$$

where $\tau = [\tau_1, \tau_2, \ldots, \tau_\kappa]_l \in T_l$ and $r_1, r_2, \ldots, r_q$ count equal trees among $\tau_1, \tau_2, \ldots, \tau_\kappa$. For $u \in U_f$ we get

$$\begin{aligned}
&G_\emptyset(x_0) = f(x_0), && G_\tau(x_0) = f^{(\kappa)}(x_0)\big(F_{\tau_1}(x_0), F_{\tau_2}(x_0), \ldots, F_{\tau_\kappa}(x_0)\big),\\
&\psi_{[\emptyset]_f}(t) \equiv 1, && \psi_u(t) = \prod_{j=1}^{\kappa} \varphi_{\tau_j}(t)
\end{aligned}$$

for $u = [\tau_1, \tau_2, \ldots, \tau_\kappa]_f \in U_f$. The expression for β_u is the same as for α_τ.

In this paper, we prove that for each $\alpha \in \mathcal{M}$ and for each $u \in U_f$ there exist a corresponding set $V(\alpha) \in U_f$ and $\mathcal{A}(u) \in \mathcal{M}$ respectively such that

$$f_\alpha(x_0) = \sum_{u \in V(\alpha)} \mu_u^\alpha \cdot G_u(x_0), \tag{6}$$

$$\beta_u \psi_u(t) = \sum_{\alpha \in \mathcal{A}(u)} \mu_u^\alpha \cdot I_\alpha^\star(t). \tag{7}$$

This will be done by the use of monotonically labelled trees. A tree $u \in U_f$ of $r(u)$ vertices (the root excluded) is *monotonically labelled* if each node is assigned a unique integer in the set $(1, 2, \ldots, r(u))$, so that these numbers increase monotonically along each branch. We will use $\omega(u)$ for the set of all possible monotonically labellings of a tree u and LU_f for the set of all possible monotonically labellings of all trees in U_f. If $\hat{w} \in LU_f$, then $LW^j(\hat{w}) \subset LU_f$ is constructed by all possible combinations of

- adding one vertex of color j to a vertex of $\hat{w}$ (including the root) and label it $r(\hat{w}) + 1$,

- and, if $j=0$, adding two nodes of the same color l, $l=1,\ldots,m$, with labels $r(\hat{w})+1, r(\hat{w})+2$ directly to vertices of $\hat{w}$.

We will use $\hat{w}-$ for the tree obtained from $\hat{w}$ by deleting $r(\hat{w})$. A vertex of a monotonically labelled tree is denoted $(j)_k$ where the subscript j denotes the color and k the label. Following [5], if $\alpha=(j_1,j_2,\ldots,j_r)\in\mathcal{M}$, then we use $\alpha-=(j_1,\ldots,j_{r-1})\in\mathcal{M}$ and $(\alpha_1,\ldots,\alpha_l)\circ j=(\alpha_1,\ldots,\alpha_l,j)\in\mathcal{M}$.

2 The coefficient functions expressed by elementary differentials

We can now state the following result:

Theorem 3. *For an $\alpha\in\mathcal{M}$ the coefficient function $f_\alpha(x_0)$ can be written as*

$$f_\alpha(x_0)=\sum_{\hat{u}\in LV(\alpha)}\hat{\mu}^\alpha_{\hat{u}}\cdot G_{\hat{u}}(x_0)=\sum_{u\in V(\alpha)}\mu^\alpha_u\cdot G_u(x_0), \tag{8}$$

where $LV(\alpha)$ and $\hat{\mu}^\alpha_{\hat{u}}$ are given by the iterative process

- *For $\alpha=(j)$ we get*

$$\begin{aligned} LV((0)) &= \{[(0)_1]_f, [(l)_1,(l)_2]_f,\ l=1,\ldots,m\},\\ LV((j)) &= \{[(j)_1]_f\}, \qquad\qquad\qquad \textit{for } j=1,\ldots,m,\\ \hat{\mu}^\alpha_{\hat{u}} &= \begin{cases} 1 & \textit{for } \hat{u}=[(j)_1]_f,\\ \gamma^\star & \textit{for } \hat{u}=[(l)_1,(l)_2]_f, \end{cases}. \end{aligned}$$

- *For $\alpha=(j_1,j_2,\ldots,j_r)$ we get*

$$\begin{aligned} LV(\alpha) &= \bigcup_{\hat{w}\in LV(\alpha-)} LW^{j_r}(\hat{w}),\\ \hat{\mu}^\alpha_{\hat{u}} &= \nu_{\hat{u}}\cdot\hat{\mu}^\alpha_{\hat{w}}, \qquad \textit{for} \quad \hat{u}\in LW^{(j_r)}(\hat{w}), \end{aligned}$$

  where $\nu_{\hat{u}}=\gamma^\star$ if $r(\hat{u})=r(\hat{w})+2$, otherwise 1.

The set $V(\alpha)$ contains the trees of $LV(\alpha)$ with the labellings removed, and $\mu^\alpha_u=\sum_{\hat{u}\in\omega(u)}\hat{\mu}^\alpha_{\hat{u}}$.

Proof. We have

$$f_{(j)}(x_0)=(L^jf)(x_0)=\begin{cases}(f'g_0)(x_0)+\gamma^\star\sum_{l=1}^m(f''(g_l,g_l))(x_0)=G_{[(0)_1]_f}(x_0)+\gamma^\star\sum_{l=1}^m G_{[(l)_1,(l)_2]}(x_0) & \text{for } j=0,\\ (f'g_j)(x_0)=G_{[(j)_1]_f}(x_0) & \text{for } j=1,2,\ldots,m.\end{cases}$$

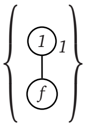
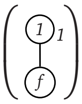
proving the theorem for $\alpha=(j)$. For each of those trees, there is only one possible labelling. Now, let $\alpha=(j_1,j_2,\ldots,j_r)$ and assume the theorem to be true for $\alpha-$. Notice that

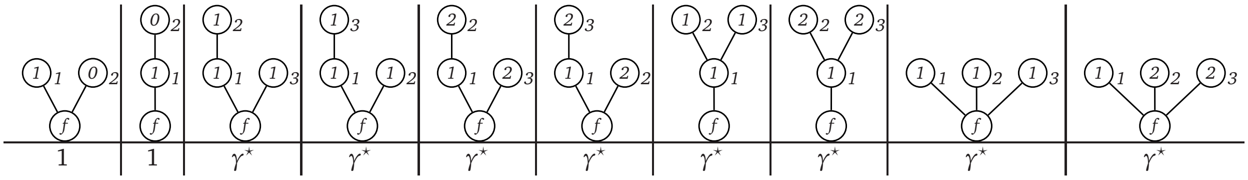
$$L^{j_r}G_{\hat{w}}=\begin{cases}(G_{\hat{w}})'g_0+\gamma^\star\sum_{l=1}^m(G_{\hat{w}})''(g_l,g_l) & \text{if } j_r=0,\\ (G_{\hat{w}})'g_{j_r} & \text{if } j_r=1,\ldots,m.\end{cases}$$

Using the definition of the set $LW^j(\hat{w})$, the definition of $G_{\hat{u}}(x_0)$, and the induction hypothesis, we get

$$f_\alpha(x_0)=(L^{j_r}f_{\alpha-})(x_0)=\sum_{\hat{w}\in LV(\alpha-)}\hat{\mu}^\alpha_{\hat{w}}\cdot(L^{j_r}G_{\hat{w}})(x_0)=\sum_{\hat{w}\in LV(\alpha-)}\hat{\mu}^\alpha_{\hat{w}}\sum_{\hat{v}\in LW^{j_r}(\hat{w})}\nu_{\hat{v}}\cdot G_{\hat{v}}(x_0),$$

where $\nu_{\hat{v}}$ is given as in the theorem. As the sets $LW^{j_r}(\hat{w})$ for $\hat{w}\in LV(\alpha-)$ are disjoint, this concludes the proof. ☐

Example 1. *Consider $\alpha = (1,0)$ and $m = 2$ Wiener processes. As $LV((1)) =$*  *we have $LV((1,0)) = LW^0$*  *. We obtain*



where the elements $\hat{u} \in LV((1,0))$ are given in the first line and $\hat{\mu}_{\hat{u}}^{(1,0)}$ are given in the second line. Therefore finally

$$\begin{aligned} f_{(1,0)}(x_0) = &\left(f''(g_1, g_0) + f' g_1' g_0 + 2\gamma^\star f''(g_1' g_1, g_1) + 2\gamma^\star f''(g_1' g_2, g_2)\right)(x_0) \\ &+ \left(\gamma^\star f' g_1''(g_1, g_1) + \gamma^\star f' g_1''(g_2, g_2) + \gamma^\star f'''(g_1, g_1, g_1) + \gamma^\star f'''(g_1, g_2, g_2)\right)(x_0). \end{aligned}$$

3 Weight functions expressed as multiple integrals

We will now prove (7). The set $\mathcal{A}(u)$ has to be given by

$$\mathcal{A}(u) = \{\alpha \in \mathcal{M} \text{ such that } u \in V(\alpha)\}.$$

We observe that $\bigcup_{u \in U_f} \mathcal{A}(u) = \mathcal{M}$, as well as $\bigcup_{\alpha \in \mathcal{M}} V(\alpha) = U_f$. From Theorems 1 and 3 we formally get

$$f(X(t)) = \sum_{\alpha \in \mathcal{M}} \left(\sum_{u \in V(\alpha)} \mu_u^\alpha G_u(x_0) \right) I_\alpha^\star(t) = \sum_{u \in U_f} \left(\sum_{\alpha \in \mathcal{A}(u)} \mu_u^\alpha I_\alpha^\star(t) \right) G_u(x_0).$$

Let j_s be the color of vertex s. The set $\mathcal{A}(u)$ can be found as follows: For each $\hat{u} \in \omega(u)$ we want to find $L\mathcal{A}(\hat{u}) = \{\alpha \in \mathcal{M}$ such that $\hat{u} \in LV(\alpha)\}$. By construction of $LV(\alpha)$, for $r(\hat{u}) = 1$ we have $L\mathcal{A}(\hat{u}) = \{(j_1)\}$ and $\mu_{\hat{u}}^{(j_1)} = 1$. For $r(\hat{u}) \geq 2$ we distinguish two cases. If $j_{r(\hat{u})} = j_{r(\hat{u})-1} \neq 0$ and vertex $r(\hat{u})$ is not above vertex $r(\hat{u}) - 1$ on the same branch, then

$$L\mathcal{A}(\hat{u}) = \{\alpha \circ j_{r(\hat{u})} : \ \alpha \in L\mathcal{A}(\hat{u}-)\} \cup \{\beta \circ 0 : \ \beta \in L\mathcal{A}((\hat{u}-)-)\},$$

with

$$\mu_{\hat{u}}^{\alpha \circ j_{r(\hat{u})}} = \mu_{\hat{u}}^\alpha, \qquad \mu_{\hat{u}}^{\beta \circ 0} = \gamma^\star \mu_{\hat{u}}^\beta.$$

Else, $L\mathcal{A}(\hat{u}) = \{\alpha \circ j_{r(\hat{u})} : \ \alpha \in L\mathcal{A}(\hat{u})-\}$, and $\mu_{\hat{u}}^{\alpha \circ j_{r(\hat{u})}} = \mu_{\hat{u}}^\alpha$. Here, $L\mathcal{A}(\emptyset) = \emptyset$, $\mu_{\hat{u}}^\emptyset = 1$. Finally,

$$\mathcal{A}(u) = \bigcup_{\omega(u)} L\mathcal{A}(\hat{u}), \qquad \nu_u^\alpha = \sum_{\omega(u)} \nu_{\hat{u}}^\alpha.$$

By induction on α, we find the intended result of this article.

Theorem 4. *The sets $L\mathcal{A}(\hat{u})$ and coefficients $\mu_{\hat{u}}^\alpha$ can be found as follows:*

For each $\hat{u} \in \omega(u)$

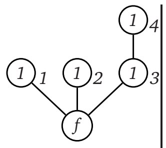
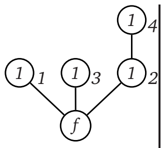
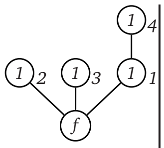
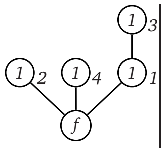
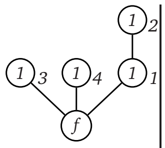
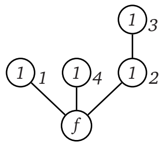
1. *$\alpha = (j_1, j_2, \ldots, j_{r(u)}) \in L\mathcal{A}(\hat{u})$ and $\mu_{\hat{u}}^\alpha = 1$.*

2. *If $j_s = j_{s-1} \neq 0$ and vertex s is not above vertex $s-1$ on the same branch, then $(j_1, j_2, \ldots, j_{s-2}, 0, j_{s+1}, \ldots, j_{r(u)}) \in L\mathcal{A}(\hat{u})$ and $\mu_{\hat{u}}^\alpha = \gamma^\star$.*

3. *If there are k such pairs of indices, then each pair is replaced by 0, and the resulting multi-index is an element of $L\mathcal{A}(\hat{u})$. In this case $\mu_{\hat{u}}^\alpha = (\gamma^\star)^k$.*

We conclude this article with an example.

Example 2. *For* $u = [①, ①, [①]_1]_f$ *we get*

$\hat{u} \in \omega(u)$						
$L\mathcal{A}(\hat{u})$	$\{(1,1,1,1),$ $(0,1,1)$ $(1,0,1)\}$	$\{(1,1,1,1)$ $(0,1,1)$ $(1,0,1)$ $(1,1,0)$ $(0,0)\}$	$\{(1,1,1,1),$ $(0,1,1)$ $(1,0,1)$ $(1,1,0)$ $(0,0)\}$	$\{(1,1,1,1),$ $(0,1,1)$ $(1,0,1)$ $(1,1,0)$ $(0,0)\}$	$\{(1,1,1,1),$ $(1,0,1)$ $(1,1,0)\}$	$\{(1,1,1,1),$ $(0,1,1)$ $(1,1,0)\}$ $(0,0)\}$.

Taking into account $\beta_u = \frac{1}{2}$ *we obtain finally*

$$\frac{1}{2}\psi_u(t) = 6I_{(1111)} + \gamma^\star(5I_{(110)} + 5I_{(101)} + 5I_{(011)}) + 4(\gamma^\star)^2 I_{(00)}.$$

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