On optimal initial value conditions for local strong solutions of the Navier-Stokes equations

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Abstract

Consider a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$, and the Navier-Stokes system in $[0,\infty) \times \Omega$ with initial value $u_0 \in L^2_\sigma(\Omega)$ and external force $f = \operatorname{div} F$, $F \in L^2(0,\infty; L^2(\Omega)) \cap L^{s/2}(0,\infty; L^{q/2}(\Omega))$ where $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$, are so-called Serrin exponents. It is an important question what is the optimal (weakest possible) initial value condition in order to obtain a unique strong solution $u \in L^s(0,T; L^q(\Omega))$ in some initial interval $[0,T)$, $0 < T \leq \infty$. Up to now several sufficient conditions on u_0 are known which need not be necessary. Our main result, see Theorem 1.2, shows that the condition $\int_0^\infty \|e^{-tA}u_0\|_q^s \,dt < \infty$, A denotes the Stokes operator, is sufficient and necessary for the existence of such a strong solution u. In particular, if $\int_0^\infty \|e^{-tA}u_0\|_q^s \,dt = \infty$, $u_0 \in L^2_\sigma(\Omega)$, then any weak solution u in the usual sense does not satisfy Serrin's condition $u \in L^s(0,T; L^q(\Omega))$ for each $0 < T \leq \infty$.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, and let $0 < T \leq \infty$. Then we consider in $[0,T) \times \Omega$ the Stokes and the Navier-Stokes system. Mainly we are interested in the notions of weak and strong solutions as follows.

Definition 1.1 Let $u_0 \in L^2_\sigma(\Omega)$ be the initial value and let $f = \operatorname{div} F$ with $F = (F_{ij})_{i,j=1}^3 \in L^2(0,T; L^2(\Omega))$ be the external force.

(1) A vector field

$$u \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; W_0^{1,2}(\Omega)) \tag{1.1}$$

is called a *weak solution* (*in the sense of Leray-Hopf*) of the Navier-Stokes system

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p = f, \quad \operatorname{div} u = 0, \\ u|_{\partial\Omega} = 0, \quad u(0) = u_0, \end{aligned} \tag{1.2}$$

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with data u_0, f, if the relation

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_{\Omega} - \langle F, \nabla w\rangle_{\Omega,T} \tag{1.3}$$

holds for each test function $w \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$, and if the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 \,\mathrm{d}\tau \le \frac{1}{2}\|u_0\|_2^2 - \int_0^t \langle F, \nabla u\rangle_\Omega \,\mathrm{d}\tau \tag{1.4}$$

is satisfied for $0 \le t < T$. A weak solution u of (1.2) is called a *strong solution* if there are exponents $2 < s < \infty$, $3 < q < \infty$ with $\frac{2}{s} + \frac{3}{q} = 1$ such that additionally *Serrin's condition*

$$u \in L^s(0,T; L^q(\Omega)) \tag{1.5}$$

is satisfied.

(2) A vector field u satisfying (1.1) is called a *weak solution* of the (linear) Stokes system

$$\begin{aligned} u_t - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0,\\ u|_{\partial\Omega} = 0, \quad u(0) = u_0, \end{aligned} \tag{1.6}$$

with data u_0, f, if

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_{\Omega} - \langle F, \nabla w\rangle_{\Omega,T} \tag{1.7}$$

holds for each $w \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$ and the energy equality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 \,\mathrm{d}\tau = \frac{1}{2}\|u_0\|_2^2 - \int_0^t \langle F, \nabla u\rangle_\Omega \,\mathrm{d}\tau, \quad 0 \le t < T, \tag{1.8}$$

is satisfied.

We may assume in the following, without loss of generality, that each weak solution

$$u : [0,T) \to L^2_\sigma(\Omega) \text{ is weakly continuous} \tag{1.9}$$

in both cases (1.2) and (1.6), see [S, V. Theorem 1.3.1]. Therefore $u(0) = u_0$ is well-defined.

Let u be a weak solution of (1.2) or of (1.6). Then there exists a distribution p in $(0,T) \times \Omega$, the associated pressure, such that $u_t - \Delta u + u \cdot \nabla u + \nabla p = f$ or $u_t - \Delta u + \nabla p = f$ holds, respectively, in the sense of distributions [S, V. 1.7].

Let u be a strong solution of (1.2) or a weak solution of (1.6), and assume that $\partial\Omega$ is of class C^∞ and that $F \in C^\infty((0,T) \times \overline{\Omega})$. Then, using (1.5) in the case (1.2), we obtain the regularity properties

$$u \in C^\infty((0,T) \times \overline{\Omega}), \quad p \in C^\infty((0,T) \times \overline{\Omega}), \tag{1.10}$$

see [S, V. Theorem 1.8.2].

The existence of at least one weak solution u of (1.2) or (1.6) is well-known, see [Le], [Ho], [S, V. 1.3]. The existence of a strong solution u of the nonlinear system (1.2) could be shown up to now at least in a sufficiently small interval $[0,T)$, $0 < T \le \infty$, and under additional smoothness conditions on the data u_0, f. The first sufficient existence condition in this context seems to be due to [KL], yielding a solution class of so-called local strong solutions. Since then there have been developed several sufficient initial value conditions for the existence of local strong solutions, getting weaker step by step and enlarging the corresponding solution class, see [FK], [G1], [He], [Ka],

[KY], [Mi], [S], [Sol], and the recent results in [A2]. Here we are mainly interested in conditions on initial values.

Our result in this context yields a condition on u_0 which is sufficient and also necessary, and therefore determines the largest possible class of strong solutions defined by Serrin's condition (1.5).

In this paper, $A = A_2$ means the Stokes operator in $L^2_\sigma(\Omega)$. More general, A_q, $1 < q < \infty$, means the Stokes operator in $L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$ where $C^\infty_{0,\sigma}(\Omega) = \{v \in C^\infty_0(\Omega) : \operatorname{div} v = 0\}$, and e^{-tA_q}, $t \geq 0$, is the semigroup generated by A_q in $L^q_\sigma(\Omega)$. Further, with $x = (x_1, x_2, x_3) \in \Omega$, $D_i = \partial/\partial x_i$, $i = 1, 2, 3$, $\nabla = (D_1, D_2, D_3)$, and for $F = (F_{ij})^3_{i,j=1}$, $u = (u_1, u_2, u_3)$ let $\operatorname{div} F = \left(\sum_{i=1}^3 D_i F_{ij}\right)^3_{j=1}$, $u \cdot \nabla\ u = (u \cdot \nabla)u = (u_1 D_1 + u_2 D_2 + u_3 D_3)u$, so that $u \cdot \nabla u = \operatorname{div}(uu)$, $uu = (u_i u_j)^3_{i,j=1}$ if $\operatorname{div} u = \nabla \cdot u = 0$; finally $u_t = \frac{\partial}{\partial t} u$.

Our main results read as follows, see [FS2] concerning results for $f = 0$.

Theorem 1.2 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, let $0 < T \leq \infty$, $2 < s < \infty$, $3 < q < \infty$ with $\frac{2}{s} + \frac{3}{q} = 1$, and let $u_0 \in L^2_\sigma(\Omega)$, $f = \operatorname{div} F$, $F \in L^2(0, T; L^2(\Omega)) \cap L^{s/2}(0, T; L^{q/2}(\Omega))$. Then there exists a constant $\varepsilon_* = \varepsilon_*(\Omega, q) > 0$ with the following property: If*

$$\left(\int_0^T \left\| e^{-tA} u_0 \right\|_q^s dt \right)^{1/s} + \left(\int_0^T \|F(t)\|_{\frac{q}{2}}^{\frac{s}{2}} dt \right)^{2/s} \leq \varepsilon_*, \tag{1.11}$$

then the Navier-Stokes system (1.2) has a unique strong solution $u \in L^s(0, T; L^q(\Omega))$ with data u_0, f.

Theorem 1.3 *Let Ω be as in Theorem 1.2, let $2 < s < \infty$, $3 < q < \infty$ with $\frac{2}{s} + \frac{3}{q} = 1$, and let $u_0 \in L^2_\sigma(\Omega)$, $f = \operatorname{div} F$ with $F \in L^2(0, \infty; L^2(\Omega)) \cap L^{s/2}(0, \infty; L^{q/2}(\Omega))$.*

(1) The condition

$$\int_0^\infty \left\| e^{-tA} u_0 \right\|_q^s dt < \infty \tag{1.12}$$

is sufficient and necessary for the existence of a unique strong solution $u \in L^s(0, T; L^q(\Omega))$ of the Navier-Stokes system (1.2), with data u_0, f, in some interval $[0, T)$, $0 < T \leq \infty$.

(2) Let u be a weak solution of the system (1.2) in $[0, \infty) \times \Omega$ with data u_0, f, and let

$$\int_0^\infty \left\| e^{-tA} u_0 \right\|_q^s dt = \infty. \tag{1.13}$$

Then Serrin's condition $u \in L^s(0, T; L^q(\Omega))$ does not hold for each $0 < T \leq \infty$. Moreover, the system (1.2) does not have a strong solution with data u_0, f and Serrin exponents s, q in any interval $[0, T)$, $0 < T \leq \infty$.

By Theorem 1.3, (2), (1.13) is a sufficient condition for the non-existence of a strong solution $u \in L^s(0, T; L^q(\Omega))$ of (1.2) with data u_0, f in each interval $[0, T)$, $0 < T \leq \infty$. Moreover, if $u_0 \in L^2_\sigma(\Omega)$ and $\int_0^\infty \left\| e^{-tA} u_0 \right\|_q^s \mathrm{dt} = \infty$ holds for all Serrin exponents $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$, and if $f = \operatorname{div} F$ with $F \in C^\infty_0([0, \infty); C^\infty(\overline{\Omega}))$, then the system has no strong solution $u \in L^s(0, T; L^q(\Omega))$ with data u_0, f for each q, s and all T.

Since $q > 2$ we have to explain that $\left\|e^{-tA}u_0\right\|_q$ with $u_0 \in L^2_\sigma(\Omega)$ is well-defined at least for $t > 0$. For this purpose we use the estimates (2.1), (2.2), see Section 2, with $0 < \alpha < \frac{3}{4}$, $2\alpha + \frac{3}{q} = \frac{3}{2}$ and constants $c, \delta > 0$ not depending on t, and obtain that $A^{-\alpha}u_0 \in L^q_\sigma(\Omega)$ and that

$$\begin{array}{rcccl} \left\|e^{-tA}u_0\right\|_q & = & \left\|A^{\alpha}e^{-tA}A^{-\alpha}u_0\right\|_q & = & \left\|A^{\alpha}_q e^{-tA_q}A^{-\alpha}u_0\right\|_q \\ & \leq & ct^{-\alpha}e^{-\delta t}\left\|A^{-\alpha}u_0\right\|_q & \leq & ct^{-\alpha}e^{-\delta t}\left\|u_0\right\|_2 \end{array} \tag{1.14}$$

for $t > 0$. Therefore, the condition (1.12) simply means that the (even continuous) function $t \mapsto \left\|e^{-tA}u_0\right\|_q^s$ is integrable in $(0, \infty)$. Further we see,

$$\text{(1.12) holds if and only if } \int_0^{T_0} \left\|e^{-tA}u_0\right\|_q^s \,\mathrm{d}t < \infty \tag{1.15}$$

for each given $T_0 > 0$.

We explain some further notations. The expression $\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\rangle$ denotes the pairing of functions on Ω, and $\langle\cdot,\cdot\rangle_{\Omega,T}$ means the corresponding pairing on $(0,T)\times\Omega$. For $1 < q < \infty$ and $k \in \mathbb{N}$ we need the usual Lebesgue and Sobolev spaces $L^q(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and $W^{k,q}(\Omega)$ with norm $\|\cdot\|_{W^{k,q}(\Omega)} = \|\cdot\|_{k;q}$, respectively. Let $L^s(0,T;L^q(\Omega))$, $1 < q, s < \infty$, with norm $\|\cdot\|_{L^s(0,T;L^q(\Omega))} = \|\cdot\|_{q,s;T} = \left(\int_0^T \|\cdot\|_q^s \,\mathrm{d}t\right)^{1/s}$ denote the classical Bochner spaces. Finally, we use the smooth function spaces $C_0^\infty(\Omega)$ and $C_{0,\sigma}^\infty(\Omega) = \{v \in C_0^\infty(\Omega) : \operatorname{div} v = 0\}$, and the spaces $L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$, $W^{1,q}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{W^{1,q}(\Omega)}}$, $W^{1,q}_{0,\sigma}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,q}(\Omega)}}$.

2 Proof of Theorems 1.2 and 1.3

To prepare the proof we first explain some well-known properties of Stokes operators and Stokes equations. Let Ω be as in these theorems, let $[0,T)$, $0 < T \leq \infty$, be a time interval and let $1 < q < \infty$.

Then $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$ denotes the Helmholtz projection, and the Stokes operator $A_q = -P_q\Delta : D(A_q) \to L^q_\sigma(\Omega)$ is defined with domain $D(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$ and range $R(A_q) = L^q_\sigma(\Omega)$. Note that $P_q v = P_\gamma v$ for $v \in L^q(\Omega) \cap L^\gamma(\Omega)$ and $A_q v = A_\gamma v$ for $v \in D(A_q) \cap D(A_\gamma)$, $1 < \gamma < \infty$. Therefore, we sometimes write $A_q = A$ if there is no misunderstanding, simplifying the notation. However, we always write $P = P_2$ and $A = A_2$ if $q = 2$. Let $A^\alpha_q : D(A^\alpha_q) \to L^q_\sigma(\Omega)$, $-1 \leq \alpha \leq 1$, denote the fractional powers of A_q. It holds $D(A_q) \subseteq D(A^\alpha_q) \subseteq L^q_\sigma(\Omega)$, $R(A^\alpha_q) = L^q_\sigma(\Omega)$ if $0 \leq \alpha \leq 1$. Note that $(A^\alpha_q)^{-1} = A^{-\alpha}_q$ and $(A_q)' = A_{q'}$ where $\frac{1}{q} + \frac{1}{q'} = 1$.

Important properties are the embedding estimate

$$\|v\|_q \leq c\left\|A^\alpha_\gamma v\right\|_\gamma, \quad v \in D(A^\alpha_\gamma),\ 1 < \gamma \leq q,\ 2\alpha + \frac{3}{q} = \frac{3}{\gamma},\ 0 \leq \alpha \leq 1, \tag{2.1}$$

and the estimate

$$\left\|A^\alpha_q e^{-tA_q}v\right\|_q \leq ct^{-\alpha}e^{-\delta t}\|v\|_q, \quad v \in L^q_\sigma(\Omega),\ 0 \leq \alpha \leq 1,\ t > 0 \tag{2.2}$$

with constants $c = c(\Omega, q) > 0$, $\delta = \delta(\Omega, q) > 0$, see [A1], [FS1], [Ga], [GiS], [Sol], [Va]. Further note that $D(A_q^{\frac{1}{2}}) = W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$ and that the norms

$$\left\|A_q^{\frac{1}{2}}v\right\|_q \approx \|\nabla v\|_q, \quad v \in D(A_q^{\frac{1}{2}}), \tag{2.3}$$

are equivalent. In particular, if $q = 2$, then

$$\left\|A^{\frac{1}{2}}v\right\|_2 = \|\nabla v\|_2, \quad v \in D(A^{\frac{1}{2}}). \tag{2.4}$$

Let $g = \operatorname{div} G$ with $G = (G_{ij})_{i,j=1}^3 \in L^q(\Omega)$. Then an approximation argument, see [S, III. Lemma 2.6.1], [FGS, p. 431], shows that $A_q^{-\frac{1}{2}} P_q \operatorname{div} G \in L^q_\sigma(\Omega)$ is well-defined by the identity

$$\langle A_q^{-\frac{1}{2}} P_q \operatorname{div} G, \varphi \rangle = \langle G, \nabla A_{q'}^{-\frac{1}{2}} \varphi \rangle, \quad \varphi \in L^{q'}_\sigma,$$

$\frac{1}{q} + \frac{1}{q'} = 1$, and that

$$\left\| A_q^{-\frac{1}{2}} P_q \operatorname{div} G \right\|_q \le c \, \|G\|_q \tag{2.5}$$

holds with $c = c(\Omega, q) > 0$.

Let $1 < q < \infty$, $1 < s < \infty$. Then for given $f \in L^s(0,T; L^q_\sigma(\Omega))$, there exists a unique solution $v \in C^0([0,T); L^q_\sigma(\Omega))$ of the instationary Stokes equation

$$v_t + A_q v = f, \quad v(0) = 0, \tag{2.6}$$

satisfying $v_t \in L^s(0,T; L^q_\sigma(\Omega))$, $A_q v \in L^s(0,T; L^q_\sigma(\Omega))$, and the maximal regularity estimate

$$\|v_t\|_{q,s;T} + \|A_q v\|_{q,s;T} \le c \, \|f\|_{q,s;T} \tag{2.7}$$

with constant $c = c(\Omega, q, s) > 0$, see [GiS, Theorem 2.7]. This solution has the representation

$$v(t) = \int_0^t e^{-(t-\tau)A_q} f(\tau) \, \mathrm{d}\tau, \quad 0 \le t < T. \tag{2.8}$$

Using (2.2) we obtain with $0 < \alpha < 1$ that

$$\left\| A_q^\alpha v(t) \right\|_q \le c \int_0^t (t-\tau)^{-\alpha} \, \|f(\tau)\|_q \, \mathrm{d}\tau, \quad 0 \le t < T, \tag{2.9}$$

with $c = c(\Omega, q) > 0$, and the Hardy-Littlewood estimate implies with $1 < \gamma < s < \infty$, $1 - \alpha + \frac{1}{s} = \frac{1}{\gamma}$, that

$$\left\| A_q^\alpha v \right\|_{q,s;T} \le c \, \|f\|_{q,\gamma;T} \tag{2.10}$$

where $c = c(\Omega, \alpha, q, s) > 0$ is independent of T.

Next we consider $f = \operatorname{div} F$ with $F \in L^r(0,T; L^2(\Omega))$, $1 < r < \infty$. Then a standard argument, see [S, IV. Theorem 2.4.1], shows that

$$E(t) = e^{-tA} u_0 + \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A} A^{-\frac{1}{2}} P \operatorname{div} F \, \mathrm{d}\tau, \quad 0 \le t < T \tag{2.11}$$

is well-defined with $u_0 \in L^2_\sigma(\Omega)$, it holds

$$E \in L^1_{loc}([0,T); W^{1,2}_{0,\sigma}(\Omega)), \tag{2.12}$$

and $u = E$ satisfies the relation (1.7). Using (2.5) we obtain that

$$A^{-\frac{1}{2}} P \operatorname{div} F \in L^r(0,T; L^2(\Omega)). \tag{2.13}$$

If $r = 2$ we obtain, see [S, IV. Lemma 2.4.2], that

E defined by (2.11) with $r = 2$ satisfies the energy equality (1.8) and the condition (1.1). (2.14)

Therefore, E in (2.14) is a weak solution of the Stokes system (1.6) in the sense of Definition 1.1. In particular, setting $E_0(t) = e^{-tA}u_0$, $u_0 \in L^2_\sigma(\Omega)$, (2.11) with $F = 0$ and (2.14) imply that

$$\frac{1}{2}\|E_0(t)\|_2^2 + \int_0^t \|\nabla E_0\|_2^2 \,\mathrm{d}\tau \le \frac{1}{2}\|u_0\|_2^2, \quad 0 \le t < T. \tag{2.15}$$

More general, if $u_0 \in L^2_\sigma(\Omega)$ and $F \in L^2(0,T;L^2(\Omega))$, E in (2.14) satisfies the inequality

$$\frac{1}{2}\|E(t)\|_2^2 + \int_0^t \|\nabla E\|_2^2 \,\mathrm{d}\tau \le c\left(\|u_0\|_2^2 + \int_0^t \|F\|_2^2 \,\mathrm{d}\tau\right), \quad 0 < t < T, \tag{2.16}$$

with some constant $c > 0$ not depending on t.

Next we use (2.11), with $F \in L^2(0,T;L^2(\Omega)) \cap L^{s/2}(0,T;L^{q/2}(\Omega))$, $\frac{2}{s} + \frac{3}{q} = 1$ as in Theorem 1.2, and set

$$E_1(t) = \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A} A^{-\frac{1}{2}} P \operatorname{div} F \,\mathrm{d}\tau, \quad 0 \le t < T. \tag{2.17}$$

Then we get

$$A^{-\frac{1}{2}} E_1(t) = \int_0^t e^{-(t-\tau)A} A^{-\frac{1}{2}} P \operatorname{div} F \,\mathrm{d}\tau, \quad 0 \le t < T. \tag{2.18}$$

Since (2.18) can be estimated in the same way as (2.6) with $v = A^{-\frac{1}{2}}E_1$, we obtain with (2.7) and (2.5) that

$$\left\|\left(A^{-\frac{1}{2}}E_1\right)_t\right\|_{2,2;T} + \left\|A^{\frac{1}{2}}E_1\right\|_{2,2;T} \le c\,\|F\|_{2,2;T} \tag{2.19}$$

and that

$$\left\|\left(A^{-\frac{1}{2}}E_1\right)_t\right\|_{\frac{q}{2},\frac{s}{2};T} + \left\|A^{\frac{1}{2}}E_1\right\|_{\frac{q}{2},\frac{s}{2};T} \le c\,\|F\|_{\frac{q}{2},\frac{s}{2};T} \tag{2.20}$$

with $c = c(\Omega, q) > 0$.

Looking at (2.17), we use the estimate (2.1) with $2\alpha + \frac{3}{q} = \frac{3}{q/2}$, i.e., $\alpha = \frac{3}{2q}$, then (2.9) with $\alpha = \frac{1}{2} + \frac{3}{2q}$, i.e., $1 - \alpha + \frac{1}{s} = \frac{1}{s/2}$, and finally (2.10), to get the inequality

$$\|E_1\|_{q,s;T} \le c\,\|F\|_{\frac{q}{2},\frac{s}{2};T} \tag{2.21}$$

with $c = c(\Omega, q) > 0$. Using (2.6), (2.8) we obtain the identities

$$A^{-\frac{1}{2}} E_1(t) = \int_0^t e^{-(t-\tau)A}\left((A^{-\frac{1}{2}}E_1)_\tau + A^{\frac{1}{2}}E_1\right)\mathrm{d}\tau, \tag{2.22}$$

$$E_1(t) = \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A}\left((A^{-\frac{1}{2}}E_1)_\tau + A^{\frac{1}{2}}E_1\right)\mathrm{d}\tau. \tag{2.23}$$

In the same way as for (2.21) there exists a constant $c = c(\Omega, q) > 0$ such that

$$\|E_1\|_{q,s;T} \le c\left(\left\|\left(A_{\frac{q}{2}}^{-\frac{1}{2}}E_1\right)_t\right\|_{\frac{q}{2},\frac{s}{2};T} + \left\|A_{\frac{q}{2}}^{\frac{1}{2}}E_1\right\|_{\frac{q}{2},\frac{s}{2};T}\right). \tag{2.24}$$

Proof of Theorem 1.2 Let Ω, T, s, q, u_0 and F be as in this theorem. First we assume that

$$\left(\int_0^T \left\|e^{-tA}u_0\right\|_q^s \,\mathrm{d}t\right)^{1/s} + \left(\int_0^T \|F\|_{\frac{q}{2}}^{\frac{s}{2}} \,\mathrm{d}t\right)^{2/s} \le C \tag{2.25}$$

in (1.11) holds with any constant $C > 0$. Later on we will choose $C = \varepsilon_* > 0$ sufficiently small. Further we assume in the first part of the proof that $u \in L^s(0,T;L^q(\Omega))$ is a given solution as desired in Theorem 1.2. Then we write the system (1.2) as a linear system in the form

$$\begin{aligned} u_t - \Delta u + \nabla p &= \operatorname{div}(-uu + F), \quad \operatorname{div} u = 0,\\ u|_{\partial\Omega} &= 0, \qquad\qquad\qquad\quad u(0) = u_0. \end{aligned} \tag{2.26}$$

By Hölder's inequality we see that $\|uu\|_{\frac{q}{2},\frac{s}{2};T} \le c\,\|u\|^2_{q,s;T}$ with some constant $c > 0$ not depending on T, so that

$$-uu + F \in L^{s/2}(0,T;L^{q/2}(\Omega)). \tag{2.27}$$

Since u is a weak solution, it holds $u \in L^2(0,T;W_0^{1,2}(\Omega)) \subset L^2(0,T;L^6(\Omega))$, and with Hölder's inequality and $1 < r < s$ defined by $\frac{1}{r} = \frac{1}{2} + \frac{1}{s}$, we obtain that

$$\|uu\|_{2,r;T} \le c\,\|u\|_{3,s;T}\,\|u\|_{6,2;T} \le c\,\|u\|_{q,s;T}\,\|u\|_{6,2;T} < \infty \tag{2.28}$$

holds with a constant $c > 0$ not depending on T. Using $F \in L^2(0,T;L^2(\Omega))$ and $r < 2$ we conclude that

$$-uu + F \in L^r_{\mathrm{loc}}([0,T);L^2(\Omega)). \tag{2.29}$$

Moreover, by (2.5),

$$\begin{aligned} A^{-\frac{1}{2}} P \operatorname{div}(uu) &\in L^r(0,T;L^2(\Omega)),\\ A^{-\frac{1}{2}} P \operatorname{div} F &\in L^2(0,T;L^2(\Omega)) \cap L^{s/2}(0,T;L^{q/2}(\Omega)). \end{aligned} \tag{2.30}$$

Now we can use (2.11) with $E(t)$, F replaced by $u(t)$, $-uu+F$, and obtain for (2.26) the representation

$$u(t) = e^{-tA}u_0 + \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A} A^{-\frac{1}{2}} P \operatorname{div}(-uu + F)\,\mathrm{d}\tau, \quad 0 \le t < T. \tag{2.31}$$

Let

$$\tilde{u}(t) = u(t) - E(t), \quad E(t) = E_0(t) + E_1(t), \tag{2.32}$$

with $E_0(t) = e^{-tA}u_0$, E_1 as in (2.17); then we obtain from (2.31) that

$$\tilde{u}(t) = -\int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A} A^{-\frac{1}{2}} P \operatorname{div}\left((\tilde{u}+E)(\tilde{u}+E)\right)\mathrm{d}\tau, \quad 0 \le t < T. \tag{2.33}$$

Setting

$$(\mathcal{F}(\tilde{u}))(t) = -\int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A} A^{-\frac{1}{2}} P \operatorname{div}\left((\tilde{u}+E)(\tilde{u}+E)\right)\mathrm{d}\tau \tag{2.34}$$

we write (2.33) as a fixed point equation

$$\tilde{u} = \mathcal{F}(\tilde{u}). \tag{2.35}$$

Let X be the Banach space defined by

$$X = \left\{ v : [0,T) \to L^{q/2}_\sigma(\Omega) : \left(A_{\frac{q}{2}}^{-\frac{1}{2}} v \right)_t, \ A_{\frac{q}{2}}^{\frac{1}{2}} v \in L^{s/2}(0,T; L^{q/2}_\sigma(\Omega)), \ A_{\frac{q}{2}}^{-\frac{1}{2}} v(0) = 0 \right\} \tag{2.36}$$

with norm

$$\|v\|_X = \left\| \left(A_{\frac{q}{2}}^{-\frac{1}{2}} v \right)_t \right\|_{\frac{q}{2}, \frac{s}{2}; T} + \left\| A_{\frac{q}{2}}^{\frac{1}{2}} v \right\|_{\frac{q}{2}, \frac{s}{2}; T}. \tag{2.37}$$

Since $\left(A_{\frac{q}{2}}^{-\frac{1}{2}} v \right)_t \in L^{s/2}(0,T; L^{q/2}_\sigma(\Omega))$ we see that $A_{\frac{q}{2}}^{-\frac{1}{2}} v \in C^0([0,T); L^{q/2}_\sigma(\Omega))$ and that $A_{\frac{q}{2}}^{-\frac{1}{2}} v(0) = 0$ is well defined.

Applying (2.20) with E_1, F replaced by $\mathcal{F}(\tilde{u})$, $(\tilde{u}+E)(\tilde{u}+E)$, respectively, to (2.34) and using Hölder's inequality we obtain for (2.34) the estimate

$$\|\mathcal{F}(\tilde{u})\|_X \le a \, \|(\tilde{u}+E)(\tilde{u}+E)\|_{\frac{q}{2}, \frac{s}{2}; T} \le a \left(\|\tilde{u}\|_{q,s;T} + \|E\|_{q,s;T} \right)^2 \tag{2.38}$$

with a constant $a = a(\Omega, q) > 0$. Thus (2.24), (2.38) yield the estimate

$$\|\mathcal{F}(\tilde{u})\|_X \le a \left(\|\tilde{u}\|_X + \|E\|_{q,s;T} \right)^2. \tag{2.39}$$

By (2.21) and the assumption (2.25) we obtain that

$$\|E\|_{q,s;T} \le b := \left(\int_0^T \left\| e^{-tA} u_0 \right\|_q^s \, \mathrm{d}t \right)^{1/s} + c \left(\int_0^T \|F\|_{\frac{q}{2}}^{\frac{s}{2}} \, \mathrm{d}t \right)^{2/s} \tag{2.40}$$

with c from (2.21). We thus obtain from (2.39) that

$$\|\mathcal{F}(\tilde{u})\|_X + b \le a \left(\|\tilde{u}\|_X + b \right)^2 + b. \tag{2.41}$$

Up to now u was a given strong solution as in Theorem 1.2 and $\tilde{u} = u - E$. In the next step we consider (2.35) as a fixed point equation in X yielding a solution $\tilde{u}$ in X. Then $u = \tilde{u} + E$ will be the desired solution for Theorem 1.2.

Thus let $\tilde{u} \in X$. Then the calculations (2.38), (2.39), (2.40) lead to the inequality (2.41). We choose $\varepsilon_* = \varepsilon_*(\Omega, q)$ in (1.11) sufficiently small in such a way that

$$4ab < 1. \tag{2.42}$$

Then the quadratic equation $y = ay^2 + b$ has a minimal positive root given by

$$0 < y_1 = 2b \left(1 + \sqrt{1 - 4ab} \right)^{-1} < 2b.$$

Since $y_1 = ay_1^2 + b > b$, the closed ball $B = \left\{ v \in X : \|v\|_X \le y_1 - b \right\}$ is not empty. Moreover, $\mathcal{F}(B) \in B$ by (2.41). Finally, for given $\tilde{u}, \hat{u} \in B$ the representation

$$\begin{aligned} &(\mathcal{F}(\tilde{u}) - \mathcal{F}(\hat{u}))(t) \\ &\qquad = -\int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A} A^{-\frac{1}{2}} P \operatorname{div}((\tilde{u}+E)(\tilde{u}-\hat{u}) + (\tilde{u}-\hat{u})(\hat{u}+E)) \, \mathrm{d}\tau, \end{aligned}$$

and the same arguments as used for (2.41) yield the estimate

$$\begin{aligned}\|\mathcal{F}(\tilde{u})-\mathcal{F}(\hat{u})\|_X &\le a\left(\|\tilde{u}\|_X+b+\|\hat{u}\|_X+b\right)\|\tilde{u}-\hat{u}\|_X\\ &\le 2ay_1\|\tilde{u}-\hat{u}\|_X \le 4ab\|\tilde{u}-\hat{u}\|_X .\end{aligned} \tag{2.43}$$

This shows that the map $\mathcal{F}: B \to B$ is a strict contraction, and Banach's fixed point principle yields the existence of some $\tilde{u} \in B$ satisfying $\tilde{u} = \mathcal{F}(\tilde{u})$. Using (2.24) with E_1 replaced by $\tilde{u}$ we conclude that

$$\|\tilde{u}\|_{q,s;T} \le c\|\tilde{u}\|_X \le c(y_1-b) \le cy_1 \le 2bc \tag{2.44}$$

with c from (2.24).

Next we define $u = \tilde{u} + E$ and prove that u is the desired solution in Theorem 1.2. Using (2.40), (2.44) we conclude that

$$\|u\|_{q,s;T} \le \|\tilde{u}\|_{q,s;T} + \|E\|_{q,s;T} \le 2bc + b \tag{2.45}$$

with c from (2.24). Thus $u \in L^s(0,T;L^q(\Omega))$, and it remains to show that u is a weak solution of the system (1.2). For this purpose we need some regularity properties of $\tilde{u}$ which we obtain writing $\tilde{u} = \mathcal{F}(\tilde{u})$ in the form

$$\tilde{u} = -\int_0^t A^{\frac12} e^{-(t-\tau)A} A^{-\frac12} P\,\mathrm{div}(u\tilde{u} + uE)\,\mathrm{d}\tau, \quad 0 \le t < T, \tag{2.46}$$

and applying Yosida's smoothing procedure in the following way:

We define the Yosida approximation of $\tilde{u}$ by $\tilde{u}_n = J_n\tilde{u}$ with $J_n = (I + \frac1n A^{\frac12})^{-1}$, $n \in \mathbb{N}$, where I means the identity, so that obviously $\tilde{u} = \tilde{u}_n + \frac1n A^{\frac12}\tilde{u}_n$. Note that J_n and $\frac1n A^{\frac12} J_n$ are uniformly bounded operators in $L^2_\sigma(\Omega)$ with respect to $n \in \mathbb{N}$, see [S, II. 3.4]. To smooth (2.46) we apply J_n to both sides and see that

$$\begin{aligned}&J_n P\,\mathrm{div}(u\tilde{u}+uE)\\ &\qquad = J_n P(u\cdot\nabla\tilde{u}_n) + \frac1n A^{\frac12} J_n A^{-\frac12} P\,\mathrm{div}(uA^{\frac12}\tilde{u}_n) + J_n P(u\cdot\nabla E).\end{aligned} \tag{2.47}$$

Further we use (2.4), (2.5) with exponent $\gamma = \left(\frac12 + \frac1q\right)^{-1}$, and Hölder's inequality to obtain the estimate

$$\begin{aligned}\|J_nP\,\mathrm{div}(u\tilde{u}+uE)\|_\gamma &\le c\left(\|u\cdot\nabla\tilde{u}_n\|_\gamma + \left\|A_\gamma^{-\frac12}P_\gamma\,\mathrm{div}(uA^{\frac12}\tilde{u}_n)\right\|_\gamma + \|u\cdot\nabla E\|_\gamma\right)\\ &\le c\|u\|_q\left(\|A^{\frac12}\tilde{u}_n\|_2 + \|\nabla E\|_2\right)\end{aligned}$$

with $c = c(\Omega, q) > 0$. Then we write (2.46) in the form

$$A^{\frac12}\tilde{u}_n(t) = -\int_0^t A^{\frac12} e^{-(t-\tau)A} J_n P\,\mathrm{div}(u\tilde{u}+uE)\,\mathrm{d}\tau,$$

and using (2.1), (2.2) with $2\alpha + \frac32 = \frac3\gamma$, $\alpha = \frac{3}{2q} < \frac12$, and the last inequality we obtain the estimate

$$\begin{aligned}\left\|A^{\frac12}\tilde{u}_n(t)\right\|_2 &= \left\|A^{\frac12}J_n\tilde{u}(t)\right\|_2\\ &\le c\int_0^t \left\|A^\alpha A^{\frac12} e^{-(t-\tau)A} J_nP\,\mathrm{div}(u\tilde{u}+uE)\right\|_\gamma \mathrm{d}\tau\\ &\le c\int_0^t (t-\tau)^{-\alpha-\frac12}\|u\|_q\left(\left\|A^{\frac12}\tilde{u}_n\right\|_2 + \|\nabla E\|_2\right)\mathrm{d}\tau\end{aligned}$$

with $c = c(\Omega, q) > 0$ not depending on $n \in \mathbb{N}$. Applying the Hardy-Littlewood estimate (2.10) with $1 - \left(\alpha + \frac{1}{2}\right) + \frac{1}{2} = \frac{1}{2} + \frac{1}{s}$, and Hölder's inequality, we obtain that

$$\left\| A^{\frac{1}{2}} \tilde{u}_n \right\|_{2,2;T} \leq c_1 \, \|u\|_{q,s;T} \left(\left\| A^{\frac{1}{2}} \tilde{u}_n \right\|_{2,2;T} + \|\nabla E\|_{2,2;T} \right) \tag{2.48}$$

with $c_1 = c_1(\Omega, q) > 0$.

The constant $\varepsilon_* = \varepsilon_*(\Omega, q)$ in (1.11) has been chosen up to now such that (2.42) is satisfied. Using (2.40) and (2.45) we see that ε_* can be chosen additionally in such a way that $c_1 \, \|u\|_{q,s;T} \leq \frac{1}{2}$ is satisfied. Then the absorption argument easily leads from (2.48) to the estimate

$$\left\| A^{\frac{1}{2}} \tilde{u}_n \right\|_{2,2;T} \leq 2c_1 \, \|u\|_{q,s;T} \, \|\nabla E\|_{2,2;T} < \infty. \tag{2.49}$$

Letting $n \to \infty$ we conclude that $\left\| A^{\frac{1}{2}} \tilde{u} \right\|_{2,2;T} \leq 2c_1 \, \|u\|_{q,s;T} \, \|\nabla E\|_{2,2;T}$, and consequently that even

$$\tilde{u}, u = \tilde{u} + E \in L^2(0, T; W_0^{1,2}(\Omega)). \tag{2.50}$$

In the next step we show that

$$uu \in L^2(0, T; L^2(\Omega)). \tag{2.51}$$

For this purpose we write (2.46) in the form

$$\tilde{u}(t) = -\int_0^t e^{-(t-\tau)A} P(u \cdot \nabla u) \, \mathrm{d}\tau, \quad 0 \leq t < T,$$

choose $\alpha = \frac{3}{q}$, $q_1 = \left(\frac{1}{2} - \frac{1}{q}\right)^{-1}$, $q_2 = \left(\frac{1}{2} + \frac{1}{q}\right)^{-1}$, use (2.1), (2.2) with $2\alpha + \frac{3}{q_1} = \frac{3}{q_2}$, and obtain that

$$\begin{aligned} \|\tilde{u}(t)\|_{q_1} &\leq c \int_0^t \left\| A_{q_2}^{\alpha} e^{-(t-\tau)A_{q_2}} P_{q_2}(u \cdot \nabla u) \right\|_{q_2} \mathrm{d}\tau \\ &\leq c \int_0^t (t-\tau)^{-\alpha} \, \|u \cdot \nabla u\|_{q_2} \, \mathrm{d}\tau \end{aligned}$$

with $c = c(\Omega, q) > 0$. Then we apply the Hardy-Littlewood estimate, see (2.9), (2.10) with $s_1 = \left(\frac{1}{2} - \frac{1}{s}\right)^{-1}$, $s_2 = \left(\frac{1}{2} + \frac{1}{s}\right)^{-1}$, $\alpha = 1 - \frac{2}{s} = \frac{3}{q}$, $1 - \alpha + \frac{1}{s_1} = \frac{1}{s_2}$, and obtain with Hölder's inequality that

$$\|\tilde{u}\|_{q_1,s_1;T} \leq c \, \|u \cdot \nabla u\|_{q_2,s_2;T} \leq c \, \|u\|_{q,s;T} \, \|\nabla u\|_{2,2;T} < \infty \tag{2.52}$$

with $c = c(\Omega, q) > 0$.

Further the standard embedding estimate

$$\|E\|_{q_1} \leq c \, \|\nabla E\|_2^{\beta} \, \|E\|_2^{1-\beta}$$

with $c = c(q) > 0$, $\beta\left(\frac{1}{2} - \frac{1}{3}\right) + (1-\beta)\frac{1}{2} = \frac{1}{q_1}$, i.e., $\beta = 3\left(\frac{1}{2} - \frac{1}{q_1}\right) = \frac{3}{q}$, see e. g. [S, II. 1.3.1], and (2.16) imply that

$$\|E\|_{q_1,s_1;T}^{s_1} \leq c \, \|\nabla E\|_{2,2;T}^{2} \, \|E\|_{2,\infty;T}^{s_1-2} < \infty \tag{2.53}$$

since $s_1\beta = 2$, $\frac{2}{s_1} + \frac{3}{q_1} = \frac{3}{2}$. Using (2.52), (2.53) we conclude with $u = \tilde{u} + E$ that $u \in L^{s_1}(0,T;L^{q_1}(\Omega))$, and Hölder's inequality yields

$$\|uu\|_{2,2;T} \le c\,\|u\|_{q,s;T}\,\|u\|_{q_1,s_1;T} < \infty$$

with $c > 0$.

Finally we write (2.46) in the form

$$u(t) = e^{-tA}u_0 + \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A} A^{-\frac{1}{2}} P \operatorname{div}(-uu + F)\,\mathrm{d}\tau, \quad 0 \le t < T. \tag{2.54}$$

Since $-uu + F \in L^2(0,T;L^2(\Omega))$, u satisfies the corresponding conditions as E in (2.11)-(2.14) with F replaced by $-uu + F$. Therefore, u satisfies the relation (1.3), it holds (1.1), the energy equality (1.8) with F replaced by $-uu + F$, and u is the weak solution of the linear system (1.6) with f replaced by $\operatorname{div}(-uu + F)$. Using the (well-defined) relation

$$\langle uu, \nabla u\rangle_\Omega = \frac{1}{2}\left\langle u, \nabla|u|^2\right\rangle_\Omega = -\frac{1}{2}\left\langle \operatorname{div} u, |u|^2\right\rangle_\Omega = 0,$$

we conclude that even (1.8) is satisfied. Thus the energy inequality (1.4) holds, and u is a strong solution of (1.2) in the sense of Definition 1.1. This proves Theorem 1.2. Note that the uniqueness of u is a consequence of the Serrin condition, see, e. g., [S, V. Theorem 1.5.1]. □

Proof of Theorem 1.3 (1) Using (1.12) and the assumption on F we can choose $0 < T \le \infty$ in such a way that (1.11) is satisfied. Then Theorem 1.2 yields the existence of a unique strong solution $u \in L^s(0,T;L^q(\Omega))$ of (1.2).

If conversely $u \in L^s(0,T;L^q(\Omega))$, $0 < T \le \infty$, is a strong solution of (1.2), then we may use (2.27)-(2.41) in the proof of Theorem 1.2. In particular the representation (2.33) yields

$$u(t) - E(t) = \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A} A^{-\frac{1}{2}} P \operatorname{div}(-uu)\,\mathrm{d}\tau, \quad 0 \le t < T. \tag{2.55}$$

Since $uu \in L^{s/2}(0,T;L^{q/2}(\Omega))$, we can apply to (2.55) the estimate (2.40) for the equation (2.11) with $u_0 = 0$, $E(t)$ replaced by $u(t) - E(t)$, and F replaced by $-uu$. This yields

$$\|u - E\|_{q,s;T} \le c\,\|uu\|_{\frac{q}{2},\frac{s}{2};T} \le c\,\|u\|^2_{q,s;T} < \infty \tag{2.56}$$

with some constant $c > 0$. Further (2.40) with $u_0 = 0$ implies that

$$\|E_1\|_{q,s;T} = \|E - E_0\|_{q,s;T} \le c\,\|F\|_{\frac{q}{2},\frac{s}{2};T} < \infty$$

which leads to

$$\|E_0\|_{q,s;T} \le \|u - E\|_{q,s;T} + \|u\|_{q,s;T} + \|E_1\|_{q,s;T} < \infty.$$

Therefore, $\|E_0\|_{q,s;T} < \infty$, and due to (1.15) also (1.12) is satisfied. This proves part (1) of Theorem 1.3.

(2) Let u be as in Theorem 1.3, (2), let (1.13) be satisfied, and suppose that $u \in L^s(0,T;L^q(\Omega))$ holds for some $T > 0$. Then we conclude from (1) that $\int_0^\infty \left\|e^{-tA}u_0\right\|_q^s \,\mathrm{d}t < \infty$ which is a contradiction to (1.13). □

3 Consequences

Let Ω, A, q, s, and u_0 be as in Theorem 1.3. We will use in this section some standard arguments on Besov spaces and interpolation theory, see [Tr], [BB], in order to give an equivalent formulation of Theorem 1.3, see Theorem 3.1 below. In particular we need the Besov space $\mathbb{B}^{-2/s}_{q,s}(\Omega)$, explained below, and will prove the equivalence

$$\left(\int_0^\infty \left\| e^{-tA} u_0 \right\|_q^s \, \mathrm{d}t\right)^{1/s} + \|u_0\|_2 \;\approx\; \|u_0\|_{\mathbb{B}^{-2/s}_{q,s}(\Omega)} + \|u_0\|_2 \tag{3.1}$$

of these norms. Let $B^{2/s}_{q',s'}(\Omega)$ be the usual Besov space, see [Tr, 4.2.1, (1)] for the definition, with the dual exponents q', s' defined by $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{s} + \frac{1}{s'} = 1$. Then the Besov space of solenoidal vector fields of $B^{2/s}_{q',s'}(\Omega)$ is defined by

$$\mathbb{B}^{2/s}_{q',s'}(\Omega) = B^{2/s}_{q',s'}(\Omega) \cap L^{q'}_\sigma(\Omega) = \left\{ v \in B^{2/s}_{q',s'}(\Omega) : \operatorname{div} v = 0,\; N \cdot v|_{\partial\Omega} = 0 \right\}$$

where $N \cdot v|_{\partial\Omega}$ means the (well-defined) normal component of v at $\partial\Omega$; see [A3, (0.5), (0.6)] concerning this space. The space

$$\mathbb{B}^{-2/s}_{q,s}(\Omega) = \left(\mathbb{B}^{2/s}_{q',s'}(\Omega)\right)' \tag{3.2}$$

is the dual space of $\mathbb{B}^{2/s}_{q',s'}(\Omega)$. Further we use the interpolation space $(L^q_\sigma(\Omega), D(A_q))_{1-1/s,s}$ and some similar spaces, see [Tr, 1.14.5] and [BB, Theorem 3.4.2].

For the next equivalences we use step by step the following arguments: [Tr, 1.14.5, (2)] for the first equivalence $\approx$, then the identity $\langle A^{-1}u_0, A\varphi\rangle_\Omega = \langle u_0, \varphi\rangle_\Omega$, $\varphi \in D(A)$, and then [Tr, 1.11.2]; for the last three equivalences we use [Tr, 1.3.3, (1)], then [A3, Prop. 3.4, (3.18)], and finally the definition in [A3, (0.6)]:

$$\begin{aligned}
\left(\int_0^\infty \left\| e^{-tA} u_0 \right\|_q^s \, \mathrm{d}t\right)^{1/s} + \left\|A^{-1}u_0\right\|_q \;&=\; \left(\int_0^\infty \left\| A e^{-tA} A^{-1} u_0 \right\|_q^s \, \mathrm{d}t\right)^{1/s} + \left\|A^{-1}u_0\right\|_q \\
\approx\; \left\|A^{-1}u_0\right\|_{(L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{s},s}} \;&\approx\; \|u_0\|_{(D(A_{q'})', L^q_\sigma(\Omega))_{1-\frac{1}{s},s}} \\
\approx\; \|u_0\|_{(D(A_{q'}), L^{q'}_\sigma(\Omega))'_{1-\frac{1}{s},s'}} \;&\approx\; \|u_0\|_{(L^{q'}_\sigma(\Omega), D(A_{q'}))'_{\frac{1}{s},s'}} \\
\approx\; \|u_0\|_{\left(\mathbb{B}^{2/s}_{q',s'}(\Omega)\right)'} \;&\approx\; \|u_0\|_{\mathbb{B}^{-2/s}_{q,s}(\Omega)}
\end{aligned}$$

Next use (2.1) with $\alpha = \frac{1}{2}(\frac{3}{2} - \frac{3}{q}) < 1$ to get that $\left\|A^{-1}u_0\right\|_q \le c \left\|A^\alpha A^{-1} u_0\right\|_2 \le c\,\|u_0\|_2$ with $c = c(\Omega, q) > 0$. Hence we obtain that

$$\begin{aligned}
\left(\int_0^\infty \left\| e^{-tA} u_0 \right\|_q^s \, \mathrm{d}t\right)^{1/s} + \|u_0\|_2 &\le \left(\int_0^\infty \left\| e^{-tA} u_0 \right\|_q^s \, \mathrm{d}t\right)^{1/s} + \left\|A^{-1}u_0\right\|_q + \|u_0\|_2 \\
&\le c\left(\|u_0\|_{\mathbb{B}^{-2/s}_{q,s}(\Omega)} + \|u_0\|_2\right)
\end{aligned}$$

as well as

$$\begin{aligned}
\|u_0\|_{\mathbb{B}^{-2/s}_{q,s}(\Omega)} + \|u_0\|_2 &\le \left(\int_0^\infty \left\| e^{-tA} u_0 \right\|_q^s \, \mathrm{d}t\right)^{1/s} + \left\|A^{-1}u_0\right\|_q + \|u_0\|_2 \\
&\le c\left(\|u_0\|_{\mathbb{B}^{-2/s}_{q,s}(\Omega)} + \|u_0\|_2\right)
\end{aligned}$$

with $c = c(\Omega, q) > 0$. This proves the equivalence (3.1).

Using (3.1) we get the following equivalent formulation of Theorem 1.3.

Theorem 3.1 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, let $u_0 \in L^2_\sigma(\Omega)$ and $f = \operatorname{div} F$ with $F \in L^2(0,\infty; L^2(\Omega)) \cap L^{s/2}(0,\infty; L^{q/2}(\Omega))$ where $2 < s < \infty$, $3 < q < \infty$ such that $\frac{2}{s} + \frac{3}{q} = 1$.*

(1) The condition

$$u_0 \in \mathbb{B}^{-2/s}_{q,s}(\Omega) \tag{3.3}$$

is sufficient and necessary for the existence of a unique strong solution $u \in L^s(0,T; L^q(\Omega))$ of the Navier-Stokes system (1.2), with data u_0, f in some interval $[0,T)$ with $0 < T \le \infty$.

(2) Let u be a weak solution of the system (1.2) in $[0,\infty) \times \Omega$, with data u_0, f, and assume that

$$u_0 \notin \mathbb{B}^{-2/s}_{q,s}(\Omega). \tag{3.4}$$

Then Serrin's condition $u \in L^s(0,T; L^q(\Omega))$ does not hold for each $0 < T \le \infty$. Moreover, the system (1.2) does not have a strong solution with data u_0, f and Serrin's exponents s, q in any interval $[0,T)$, $0 < T \le \infty$.

As a consequence of Theorem 3.1 we mention some sufficient conditions on the data u_0, f for the existence of a local strong solution. The optimal condition (3.5) yields the largest possible solution class in this context. Concerning the conditions (3.6) and (3.7) below, there are several known similar results with more regular external forces or with $f = 0$.

Theorem 3.2 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, let $u_0 \in L^2_\sigma(\Omega)$ and $f = \operatorname{div} F$ with $F \in L^2(0,\infty; L^2(\Omega)) \cap L^{s/2}(0,\infty; L^{q/2}(\Omega))$ where $2 < s < \infty$, $3 < q < \infty$ such that $\frac{2}{s} + \frac{3}{q} = 1$.*

Then each of the following conditions is sufficient for the existence of a unique strong solution $u \in L^s(0,T; L^q(\Omega))$ of the Navier-Stokes system (1.2) with data u_0, f in some interval $[0,T)$ with $0 < T \le \infty$.

(1) $$u_0 \in \mathbb{B}^{-2/s}_{q,s}(\Omega) \tag{3.5}$$

(2) $$u_0 \in L^3_\sigma(\Omega), \quad s \ge q \tag{3.6}$$

(3) $$u_0 \in D(A^{\frac{1}{4}}) \tag{3.7}$$

Proof (2) The result follows from Theorem 3.1 and the well-known embedding property $L^3_\sigma(\Omega) \subset \mathbb{B}^{-2/s}_{q,s}(\Omega)$, when $s \ge q$, see [A3, (0.16)] or [Tr, 4.6.1, (d)].

(3) We use (2.1) with $\alpha = \frac{3}{2}(\frac{1}{2} - \frac{1}{q})$ and obtain that

$$\begin{aligned}
\left(\int_0^\infty \left\| e^{-tA} u_0 \right\|_q^s \, \mathrm{d}t \right)^{1/s} &\le c \left(\int_0^\infty \left\| A^\alpha e^{-tA} u_0 \right\|_2^s \, \mathrm{d}t \right)^{1/s} \\
&= c \left(\int_0^\infty \left\| A^{\frac{1}{s}} e^{-tA} A^{\frac{1}{4}} u_0 \right\|_2^s \, \mathrm{d}t \right)^{1/s} \\
&\le c \left\| A^{\frac{1}{4}} u_0 \right\|_2
\end{aligned}$$

with $c = c(\Omega, q)$; see [S, IV. Lemma 1.5.3] concerning the last inequality. □

Remark 3.3 (1) The condition (3.5) is optimal and yields the largest possible class of local strong solutions $u \in L^s(0,T;L^q(\Omega))$, see Theorem 3.1, (1). This result extends the solution class considered in [A2, Theorem 11.1] where instead of $\mathbb{B}_{q,s}^{-2/s}(\Omega)$ a certain Bessel potential space is used.

(2) Of course, (3.6) can be replaced by the stronger condition $u_0 \in L^r_\sigma(\Omega)$, $r > 3$. The first result with $u_0 \in L^r_\sigma(\Omega)$, $r > 3$, has been given in [FJR], further results for $r > 3$ are given in [Mi]. The first result with $r = 3$ is contained in [Ka], further results in this case are developed in [G1], [Mi].

(3) The condition (3.6) can be replaced by the strictly weaker condition

$$u_0 \in L^{3,s}_\sigma(\Omega), \quad q \le s < \infty, \tag{3.8}$$

see [S2], which follows from the embedding $L^{3,s}_\sigma(\Omega) \subseteq \mathbb{B}_{q,s}^{-2/s}(\Omega)$ with $s \ge q$, see [A3, (0.16)]. Here $L^{3,s}_\sigma(\Omega)$ means the Lorentz space defined as the closure

$$L^{3,s}_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{L^{3,s}}},$$

where $L^{3,s} = L^{3,s}(\Omega)$ means the usual Lorentz space, see [Tr], [BB]. Since $s > 3$, it holds $L^{3,s}_\sigma \supset L^3_\sigma(\Omega)$. A result with (3.8) for $s = \infty$ has been given in [KY] with an additional smallness condition on the norm $L^{3,\infty}_\sigma(\Omega)$.

(4) The first result with initial conditions in $D(A) \subseteq L^2_\sigma(\Omega)$ has been given in [KL]; this seems to be the first result on the existence of local strong solutions. The first result with $u_0 \in D(A^{\frac{1}{4}})$ for smooth bounded domains is contained in [FK] and can be extended to general domains (i.e. open connected subsets of $\mathbb{R}^3$), see [S, V. Theorem 4.2.2].

4 Extension to completely general domains for the special exponents $s = 8$, $q = 4$

In this section $\Omega \subseteq \mathbb{R}^3$ means a general domain, i.e. an open and connected subset of $\mathbb{R}^3$ with boundary $\partial\Omega$. Note that Ω can be bounded or unbounded and may have edges and corners.

It turns out that in this case Theorems 1.2 and 1.3 remain true with the special Serrin exponents $s = 8$, $q = 4$. The reason is that each step of the proof can be carried out for general domains using only the L^2-approach to the Stokes operator. It is interesting to note that the smallness constant $\varepsilon_* = \varepsilon_*(\Omega, q) > 0$ in Theorem 1.2 for $s = 8$, $q = 4$ does not depend on Ω and is therefore an absolute constant. Indeed this follows since the L^2-approach is much simpler than the general L^q-approach; see [S, III. 2.1] for the Stokes operator $A = A_2$, and [S, II. 2.5] for the Helmholtz projection $P = P_2$ for general domains.

Further we note that Definition 1.1 remains valid for the general domain Ω with the only exception that the condition (1.1) has to be replaced by

$$u \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2_{\text{loc}}([0,T);W^{1,2}_0(\Omega)). \tag{4.1}$$

Thus in the next two theorems u is a strong solution in the sense of Definition 1.1 where (1.1) is replaced by (4.1).

Theorem 4.1 *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain with boundary $\partial\Omega$.*

(1) There exists an absolute constant $\varepsilon_* > 0$ with the following property: If $0 < T \le \infty$ and if $u_0 \in L^2_\sigma(\Omega)$, $f = \operatorname{div} F$, $F \in L^2(0,T;L^2(\Omega)) \cap L^4(0,T;L^2(\Omega))$ satisfy

$$\left(\int_0^T \left\| e^{-tA} u_0 \right\|_4^8 \, dt \right)^{1/8} + \left(\int_0^T \|F\|_2^4 \, dt \right)^{1/4} \le \varepsilon_*, \tag{4.2}$$

then the Navier-Stokes system (1.2) *has a unique strong solution* $u \in L^8(0,T;L^4(\Omega))$ *with data* u_0*,* f*.*

(2) Suppose $u_0 \in L^2_\sigma(\Omega)$*,* $f = \operatorname{div} F$*,* $F \in L^2(0,\infty;L^2(\Omega)) \cap L^4(0,\infty;L^2(\Omega))$*. Then the condition*

$$\int_0^\infty \left\| e^{-tA} u_0 \right\|_4^8 \, dt < \infty \tag{4.3}$$

is sufficient and necessary for the existence of a unique strong solution $u \in L^8(0,T;L^4(\Omega))$ *of the system* (1.2) *with data* u_0*,* f *in some interval* $[0,T)$*,* $0 < T \leq \infty$*.*

Proof (1) We only have to explain the modifications of the proof of Theorem 1.2 needed for the given general domain Ω.

The estimate (2.1) only holds for $0 \leq \alpha \leq \frac{1}{2}$, $2 \leq q < \infty$, $\gamma = 2$ but now with some constant $c = c(\alpha) > 0$ only depending on α, see [S, III. Lemma 2.4.2]. Concerning the semigroup e^{-tA}, $t \geq 0$, see [S, IV. 1.5]. Then the estimate (2.2) holds for $0 \leq \alpha \leq 1$, $q = 2$, and with constant $c = 1$. Applying (2.1) with $\alpha = \frac{3}{8}$, $q = 4$, and (2.2) with $q = 2$, $\alpha = \frac{3}{8}$ we obtain that

$$\left\| e^{-tA} u_0 \right\|_4^8 \leq c t^{-3} \left\| u_0 \right\|_2^8, \quad t > 0, \tag{4.4}$$

with some absolute constant $c > 0$. This shows that (4.3) is well-defined and means that the (continuous) function $t \mapsto \left\| e^{-tA} u_0 \right\|_4^8$, $t > 8$, is integrable in $(0,\infty)$. Further we note that $\left\| A^{\frac{1}{2}} v \right\|_2 = \left\| \nabla v \right\|_2$ for all $v \in D(A^{\frac{1}{2}})$, and that

$$\left\| A^\alpha v \right\|_2 \leq \left\| A v \right\|_2^\alpha \left\| v \right\|_2^{1-\alpha}, \quad v \in D(A), \quad 0 \leq \alpha \leq 1, \tag{4.5}$$

see [S, III. Lemma 2.2.1 and III.2, (2.2.8)].

Next we note that (2.7) holds for the solution of (2.6) with the exponents $q = 2$, $1 < s < \infty$, and with constant $c = c(s) > 0$, see [S, IV. (2.5.13)]. The estimate (2.9) holds for $q = 2$, $0 < \alpha < 1$, $c = 1$, (2.10) holds for $q = 2$, $1 < \gamma < s < \infty$, $1 - \alpha + \frac{1}{s} = \frac{1}{\gamma}$, $c = c(s,\gamma)$, and (2.5) holds with $q = 2$ and $c = 1$.

Starting with a given strong solution $u \in L^8(0,T;L^4(\Omega))$ of (2.26) we obtain (2.29) with $r = 4$, yielding the representation (2.33) with $\tilde{u}$ as in (2.32). This leads to (2.38)-(2.41) with $q = 4$, $s = 8$, and with absolute constants a, $c > 0$.

Next we solve the fixed point equation (2.35) in X, see (2.36), where $\frac{q}{2} = 2$, $\frac{s}{2} = 4$. Using (2.41)-(2.45) with $q = 4$, $s = 8$ and the smallness condition (4.2) we obtain a solution $u \in L^8(0,T;L^4(\Omega))$ of (2.33) where $\tilde{u}$ is defined as in (2.32).

Using (2.27) with $\frac{s}{2} = 4$, $\frac{q}{2} = 2$, we see that

$$-uu + F \in L^2_{loc}([0,T);L^2(\Omega)).$$

Therefore, we do not need the Yosida approximation as in (2.47)-(2.49) and directly apply the results in [S, IV. Theorem 2.4.1, and Theorem 2.4.2, d)] to obtain from the representation (2.33) that u is a strong solution of the system (1.2). Here we argue as for (2.54). This proves Theorem 4.1, (1).

(2) The argument to prove (2) is the same as in the proof of Theorem 1.3, (1). □

Using (2.1) with $\alpha = \frac{3}{8}$, $\gamma = 2$, (2.2) with $\alpha = \frac{1}{8}$, $q = 2$, and the estimate

$$\left(\int_0^\infty \left\| A^{\frac{1}{8}} e^{-tA} u_0 \right\|_2^8 \mathrm{d}t \right)^{1/8} \leq \left\| u_0 \right\|_2, \tag{4.6}$$

see [S, IV. Lemma 1.5.3], we obtain with $u_0 \in D(A^{\frac{1}{4}})$ that

$$\left(\int_0^\infty \left\|e^{-tA}u_0\right\|_4^8 \,\mathrm{d}t\right)^{1/8} \leq c\left(\int_0^\infty \left\|A^{\frac{3}{8}}e^{-tA}u_0\right\|_2^8 \,\mathrm{d}t\right)^{1/8}$$

$$= \left(\int_0^\infty \left\|A^{\frac{1}{8}}e^{-tA}A^{\frac{1}{4}}u_0\right\|_2^8 \,\mathrm{d}t\right)^{1/8} \leq c\left\|A^{\frac{1}{4}}u_0\right\|_2$$

with some absolute constant $c > 0$. This yields the following corollary which extends Fujita-Kato's result [FK] to general domains.

Corollary 4.2 *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, and let $u_0 \in D(A^{\frac{1}{4}})$, $f = \operatorname{div} F$, $F \in L^2(0,\infty;L^2(\Omega)) \cap L^4(0,\infty;L^2(\Omega))$. Then there exists a unique strong solution $u \in L^8(0,T;L^4(\Omega))$ of the system (1.2) with data u_0, f in some interval $[0,T)$, $0 < T \leq \infty$.*

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