

# $H_{loc}^1$ -stress and strain regularity in Cosserat plasticity

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## Abstract

We show  $H_{loc}^1$ -regularity of the Cauchy stress tensor and  $H_{loc}^1$ -regularity of the infinitesimal strain tensor and the plastic strain tensor in infinitesimal Cosserat plasticity with monotone flow rule. We use energy estimates for difference quotients.

**Key words:** plasticity, visco-plasticity, polar-materials, non-simple materials, solid mechanics, elliptic systems, regularity theory, Cosserat model

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## 1 Introduction: Plasticity and Cosserat models

The regularity question for small strain models of elasto-plastic behavior has recently found renewed interest [17, 18, 16, 7, 8, 4, 5], in part motivated by the need for qualitative statements on the rate of convergence of finite element methods in elasto-plasticity. There it is necessary to know precisely the regularity of the function to be approximated, see [15]. This article addresses the regularity question for time-continuous formulations of *geometrically linear* elasto-plasticity. As a representative model problem we consider generalized continua of *Cosserat-micropolar* type. The basic difference of a Cosserat model as compared with classical continuum models is the appearance of a nonsymmetric stress tensor which is augmented by a generalized balance of angular momentum equation allowing to model interaction of particles not only by surface forces (classical Cauchy continuum) but also through surface couples (Cosserat continuum). For an introduction to the theory of Cosserat and micropolar models we refer to [19, 15, 10, 11, 14]. The first author has also proposed an elasto-plastic Cosserat model [11] in a finite strain framework.

A geometrical linearization of this model has been investigated in [12, 14, 13, 3] and is shown to be well-posed also in the rate-independent limit for both quasistatic and dynamic processes.

Time-incremental formulations for this and other models have already been shown to possess smooth updates, see [16] and the references therein. However, the employed method did not allow to pass to the continuous time limit and it was not clear what kind of regularity to expect for the time-continuous setting. An early statement can be found in [6]. A first major breakthrough regarding global spatial regularity was obtained recently by Alber and Nesenenko [1] where  $L^\infty(0, T; H^{1/3-\delta}(\Omega))$ -regularity is shown for stresses and plastic strains for classical rate-dependent viscoplasticity and rate-independent models with linear kinematic hardening. This is followed by Knees [9] where viscosity is replaced by the linear hardening assumption together with the subdifferential structure of the flow rule. She obtains the improved  $L^\infty(0, T; H^{1/2-\delta}(\Omega))$ -regularity.

Local regularity results for elasto-plasticity with linear hardening and variants thereof have been derived by several authors [2, 4, 5, 21, 20]. Typically, one gets  $L^\infty(0, T; H_{loc}^1(\Omega))$ . This is also what we will obtain for the Cosserat model, however, without any hardening and for both the quasistatic and dynamic case and without using a subdifferential structure.

Our focus on Cosserat models is justified by the fact that the Cosserat type models are today increasingly advocated as a means to regularize the pathological mesh size dependence of localization computations where shear failure mechanisms play a dominant role.

This contribution is now organized as follows: first, we recall the time-continuous geometrically linear elasto-plastic Cosserat model as introduced in [11] and investigated mathematically in [12, 14, 13] together with the major statements obtained for this model. Then we prove that for initial plastic strain  $\varepsilon_p^0 \in H_{loc}^1(\Omega)$  and body force  $f \in L^2(0, T; H_{loc}^1(\Omega))$  the solution obtained in the existence theorem is more regular. In sections 4 and 5 we repeat the regularity procedure defined in section 3 in the dynamical setting of the problem and for general flow rules of monotone type.

## 2 The infinitesimal Cosserat elasto-plastic model

We consider the infinitesimal Cosserat elasto-plastic model as introduced in [12, 14, 13]. The goal is to prove a higher regularity of the stress tensor and the strain tensors assuming that the external force and the initial plastic strain are more regular. In the quasistatic setting of the problem the system of equations is in the form

$$\begin{aligned}
\operatorname{Div} \sigma &= -f, \\
\sigma &= 2\mu(\varepsilon - \varepsilon_p) + 2\mu_c(\operatorname{skew}(\nabla u) - A) + \lambda \operatorname{tr}[\varepsilon] \cdot \mathbb{1}, \\
-l_c \Delta \operatorname{axl}(A) &= \mu_c \operatorname{axl}(\operatorname{skew}(\nabla u) - A), \\
\dot{\varepsilon}_p &\in \mathbf{f}(T_E), \quad T_E = 2\mu(\varepsilon - \varepsilon_p), \\
u|_{\partial\Omega} &= u_d, \quad A|_{\partial\Omega} = A_d, \quad \varepsilon_p(0) = \varepsilon_p^0.
\end{aligned} \tag{2.1}$$

Here  $u$  is the displacement vector,  $\sigma$  is the Cauchy stress tensor, which for  $\mu_c > 0$  is not necessary symmetric,  $\varepsilon = \text{sym } \nabla u$  is the infinitesimal strain tensor,  $\varepsilon_p$  is the symmetric plastic strain tensor,  $A \in \mathfrak{so}(3)$  is the infinitesimal skew-symmetric microrotation matrix,  $\text{axl} : \mathfrak{so}(3) \mapsto \mathbb{R}^3$  is the canonical identification of the Lie-algebra of skew-symmetric real  $3 \times 3$ -matrices  $\mathfrak{so}(3)$  and vectors in  $\mathbb{R}^3$ ,  $T_E$  is the reduced Eshelby tensor,  $f$  is the density of the external force acting on the material,  $\mathbf{f} : D(\mathbf{f}) \subset \text{Sym}(3) \rightarrow \mathfrak{sl}(3) \cap \text{Sym}(3)$  is supposed to be a maximal monotone mapping with trace free, symmetric image and  $\{0\} \in \mathbf{f}(0)$ ,  $\varepsilon_p^0$  is the initial plastic strain and  $u_d, A_d$  are given boundary data. In this article, in extension of [12], we assume that the coefficients  $\mu, \lambda, \mu_c, l_c$  depend also on  $x \in \Omega$ . We require that these functions are positive and continuous on  $\overline{\Omega}$  and locally Lipschitz. The parameter  $l_c$  abbreviates  $l_c = \mu L_c^2$  with an internal length scale  $L_c$ . In all of the following we assume  $\Omega \subset \mathbb{R}^3$  is a bounded, open domain with smooth boundary  $\partial\Omega$ .

In [12] the following existence and uniqueness theorem for system (2.1) with constant coefficients  $\mu, \lambda, \mu_c, l_c \geq c > 0$  is proved:<sup>1</sup>

**Theorem 2.1 (Existence for the infinitesimal elasto-plastic Cosserat model)**

Suppose that the given data  $f, u_d, A_d$  satisfy: for all times  $T > 0$

$$\begin{aligned} f &\in C^1([0, T], L^2(\Omega, \mathbb{R}^3)), & \dot{f} &\in L^2((0, T) \times \Omega, \mathbb{R}^3), \\ u_d &\in C^2([0, T], H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^3)), & \ddot{u}_d &\in L^2((0, T) \times \partial\Omega, \mathbb{R}^3), \\ A_d &\in C^2([0, T], H^{\frac{3}{2}}(\partial\Omega, \mathfrak{so}(3))), & \ddot{B}_d &\in L^2((0, T) \times \partial\Omega, \mathfrak{so}(3)), \end{aligned}$$

where  $v_d = \dot{u}_d$  and  $B_d = \dot{A}_d$ . Moreover, assume that the initial data  $\varepsilon_p^0 \in L^2(\Omega, \text{Sym}(3))$  is chosen such that the initial value of the reduced Eshelby tensor  $T_E(0) = 2\mu(\varepsilon(0) - \varepsilon_p^0)$  defined by the initial data  $\varepsilon_p^0$  belongs to the domain of the maximal monotone operator  $\mathbf{f}$ . Then system (2.1) possesses a global in time, unique solution  $(u, \varepsilon, \varepsilon_p, A)$  with the regularity: for all times  $T > 0$

$$\begin{aligned} u &\in H^{1,\infty}((0, T), H^1(\Omega, \mathbb{R}^3)), & \varepsilon, \varepsilon_p &\in H^{1,\infty}((0, T), L^2(\Omega, \text{Sym}(3))), \\ A &\in H^{1,\infty}((0, T), H^2(\Omega, \mathfrak{so}(3))). \end{aligned}$$

In Theorem 2.1  $\text{Sym}(3)$  denotes the set  $3 \times 3$  real-valued symmetric matrices. If the coefficients of the model are locally Lipschitz and positive Theorem 2.1 can be proved using the same technics as in [12]. ■

### 3 $H_{loc}^1$ -regularity in the quasistatic case

The goal of this section is to prove that for  $\varepsilon_p^0 \in H_{loc}^1(\Omega)$  and  $f \in L^2(0, T; H_{loc}^1(\Omega))$  with  $f(0) \in H_{loc}^1(\Omega)$  the solution of system (2.1) is more regular. We are using the difference quotient method. Let  $V, U \subset \Omega$  be open sets such that  $V \Subset U \Subset \Omega$ . Let  $\eta \in C_0^\infty(\Omega)$  be a cutoff function such that  $\eta(x) \in [0, 1]$  for each  $x \in \Omega$ ,  $\eta \equiv 1$  on  $V$  and  $\text{supp } \eta \subset \overline{U}$ . Let us observe that using the standard regularity theory of linear elliptic

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<sup>1</sup>Note that in contrast to linear elasticity,  $\lambda > 0$  is mandatory, but is verified for metals.

systems if  $\varepsilon_p^0 \in H_{loc}^1(\Omega)$  and  $f(0) \in H_{loc}^1(\Omega)$  then the initial stress, the initial strain tensors and the initial microrotation are more regular as obtained in [12]. Namely, these initial functions are solutions to the system

$$\begin{aligned} \operatorname{Div} \sigma(0) &= -f, \\ \sigma(0) &= 2\mu(\varepsilon(0) - \varepsilon_p(0)) + 2\mu_c(\operatorname{skew}(\nabla u(0)) - A(0)) + \lambda \operatorname{tr}[\varepsilon(0)] \cdot \mathbb{1}, \\ -l_c \Delta \operatorname{axl}(A(0)) &= -\mu_c \operatorname{axl}(A(0)) + \mu_c \operatorname{axl}(\operatorname{skew}(\nabla u(0))), \\ u(0)|_{\partial\Omega} &= u_d, \quad A(0)|_{\partial\Omega} = A_d, \end{aligned} \tag{3.1}$$

where  $\varepsilon(0) = \operatorname{sym} \nabla u(0)$ . Hence, we have that  $\sigma(0), \varepsilon(0) \in H_{loc}^1(\Omega; \operatorname{Sym}(3))$ . Moreover, if additionally  $\nabla l_c$  and  $\nabla \mu_c$  are locally Lipschitz then the initial microrotation  $A(0) \in H_{loc}^3(\Omega; \mathfrak{so}(3))$ . Let us recall the energy function associated with system (2.1)

$$\mathcal{E}(u, \varepsilon, \varepsilon_p, A)(t) = \int_{\Omega} \left( \mu \|\varepsilon - \varepsilon_p\|^2 + \frac{\lambda}{2} \operatorname{tr}[\varepsilon]^2 + \mu_c \|\operatorname{skew}(\nabla u) - A\|^2 + 2l_c \|\nabla \operatorname{axl}(A)\|^2 \right) dx.$$

The following coerciveness property of the energy function is proved in [12]

**Theorem 3.1 (Coerciveness of the energy)**

*The energy function is elastically coercive with respect to  $\nabla u$ . This means that*

$$\exists C_E > 0, \forall u \in H_0^1(\Omega), \forall A \in H_0^1(\Omega), \forall \varepsilon_p \in L^2(\Omega)$$

$$\mathcal{E}(u, \varepsilon, \varepsilon_p, A) \geq C_E (\|u\|_{H^1(\Omega)}^2 + \|A\|_{H^1(\Omega)}^2).$$

Moreover,

$\exists C_E > 0, \forall u_d, A_d \in H^{\frac{1}{2}}(\partial\Omega), \exists C_d > 0, \forall \varepsilon_p \in L^2(\Omega), \forall u \in H^1(\Omega), \forall A \in H^1(\Omega)$  with  $u|_{\partial\Omega} = u_d$  and  $A|_{\partial\Omega} = A_d$  it holds that

$$\mathcal{E}(u, \varepsilon, \varepsilon_p, A) + C_d \geq C_E (\|u\|_{H^1(\Omega)}^2 + \|A\|_{H^1(\Omega)}^2).$$

This theorem was proved for constant coefficients only. Nevertheless, a simple modification of the proof from [12] allows to conclude the same result for locally Lipschitz and positive coefficients. ■

Let us denote by  $\mathcal{E}_V(u, \varepsilon, \varepsilon_p, A)$  the energy calculated on the set  $V$  only. Let us also choose a basis  $e_k, k = 1, \dots, n$  of  $\mathbb{R}^n$ . For  $h \in \mathbb{R}$  and a fixed  $k \in \{1, \dots, n\}$  we denote by  $D_k^h$  the difference quotient in the direction  $e_k$  with the step  $h$ . This means that for a function  $w$  defined on  $\Omega$

$$D_k^h w(x) := \frac{w(x + he_k) - w(x)}{h} \quad \text{defined for } x + he_k \in \Omega.$$

**Theorem 3.2 (Main estimate)**

*Let us assume that  $\varepsilon_p^0 \in H_{loc}^1(\Omega; \mathcal{P}(\operatorname{Sym}(3)))$ ,  $f \in L^2(0, T; H_{loc}^1(\Omega; \mathbb{R}^3))$  with  $f(0) \in H_{loc}^1(\Omega; \mathbb{R}^3)$  and  $\mu, \lambda, \mu_c, l_c$  are positive and continuous on  $\overline{\Omega}$  and locally Lipschitz. Then*

for all  $k \in \{1, \dots, n\}$ ,  $t \in [0, T]$  and sufficiently small  $h \in \mathbb{R}$  the solution of system (2.1) satisfies

$$\begin{aligned} \mathcal{E}_V(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)(t) &\leq C \left( \mathcal{E}_{U^*}(u_{x_k}, \varepsilon_{x_k}, \varepsilon_{p, x_k}, A_{x_k})(0) \right. \\ &\quad + \|f\|_{L^2(H^1(\Omega))}^2 + \int_0^t (\mathcal{E}_{U^*}(u, \varepsilon, \varepsilon_p, A) + \mathcal{E}_{U^*}(\dot{u}, \dot{\varepsilon}, \dot{\varepsilon}_p, \dot{A})) d\tau \\ &\quad \left. + \|\dot{u}\|_{L^2(H^1(\Omega))}^2 + \|\dot{A}\|_{L^2(H^1(\Omega))}^2 + 1 \right), \end{aligned} \quad (3.2)$$

where  $U^* = U + B(0, r)$  with  $r = 1/2 \text{ dist}(U, \partial\Omega)$  and the constant  $C > 0$  does not depend on  $h$ .

**Proof.** Let us fix  $k$  and assume that  $h \neq 0$  and  $|h| \leq 1/2 \text{ dist}(U, \partial\Omega)$ . For  $x \in U$  the difference operator  $D_k^h$  is well defined. We are going to estimate the time derivative of  $\mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)(t)$ . Note that for  $x \notin U$  the products  $\eta D_k^h(\cdot)$  are equal to zero.

$$\begin{aligned} \dot{\mathcal{E}}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)(t) &= \int_{\Omega} \eta^2 \left( 2\mu \langle D_k^h \varepsilon - D_k^h \varepsilon_p, D_k^h \dot{\varepsilon} - D_k^h \dot{\varepsilon}_p \rangle + \lambda \text{tr} [D_k^h \dot{\varepsilon}] \text{tr} [D_k^h \dot{\varepsilon}] \right. \\ &\quad \left. + 2\mu_c \langle \text{skew}(\nabla D_k^h u) - D_k^h A, \text{skew}(\nabla D_k^h \dot{u}) - D_k^h \dot{A} \rangle + 4l_c \langle \nabla \text{axl}(D_k^h A), \nabla \text{axl}(D_k^h \dot{A}) \rangle \right) dx = \\ &\quad \int_{\Omega} \eta^2 \left( 2\mu \langle D_k^h \varepsilon - D_k^h \varepsilon_p + \lambda \text{tr} [D_k^h \varepsilon] + 2\mu_c (\text{skew}(\nabla D_k^h u) - D_k^h A), \nabla D_k^h \dot{u} \rangle \right) dx \\ &\quad - \int_{\Omega} \eta^2 \langle 2\mu (D_k^h \varepsilon - D_k^h \varepsilon_p), D_k^h \dot{\varepsilon}_p \rangle dx - 2 \int_{\Omega} \mu_c \eta^2 \langle \text{skew}(\nabla D_k^h u) - D_k^h A, D_k^h \dot{A} \rangle dx \\ &\quad + 4 \int_{\Omega} \eta^2 l_c \langle \nabla \text{axl}(D_k^h A), \nabla \text{axl}(D_k^h \dot{A}) \rangle dx. \end{aligned} \quad (3.3)$$

In the first integral on the right hand side of (3.3) we are using the balance of forces. Unfortunately, the term  $2\mu D_k^h \varepsilon - D_k^h \varepsilon_p + \lambda \text{tr} [D_k^h \varepsilon] + 2\mu_c (\text{skew}(\nabla D_k^h u) - D_k^h A)$  is not equal to the difference quotient  $D_k^h \sigma$  because the coefficients are not constant. By the property of the operator  $D_k^h$  similar to the product rule we have

$$\begin{aligned} D_k^h \sigma &= 2\mu D_k^h \varepsilon - D_k^h \varepsilon_p + \lambda \text{tr} [D_k^h \varepsilon] + 2\mu_c (\text{skew}(\nabla D_k^h u) - D_k^h A) \\ &\quad + 2D_k^h \mu (\varepsilon^h - \varepsilon_p^h) + D_k^h \lambda \text{tr} [\varepsilon^h] + 2D_k^h \mu_c (\text{skew}(\nabla u^h) - A^h), \end{aligned} \quad (3.4)$$

where the **superscript**  $(\cdot)^h$  **denotes the shifted function**  $(\cdot)(x + he_k)$ . In a similar manner we transform the integrand from the second integral on the right hand side of (3.3)

$$D_k^h T_E = D_k^h (2\mu(\varepsilon - \varepsilon_p)) = 2\mu D_k^h (\varepsilon - \varepsilon_p) + 2D_k^h \mu (\varepsilon^h - \varepsilon_p^h). \quad (3.5)$$

In the fourth integral on the right hand side of (3.3) we integrate by parts to obtain

$$\begin{aligned} \int_{\Omega} \eta^2 l_c \langle \nabla \text{axl}(D_k^h A), \nabla \text{axl}(D_k^h \dot{A}) \rangle dx &= - \int_{\Omega} \eta^2 l_c \langle \Delta \text{axl}(D_k^h A), \text{axl}(D_k^h \dot{A}) \rangle dx \\ &\quad - \int_{\Omega} \eta \langle \nabla \text{axl}(D_k^h A), \text{axl}(D_k^h \dot{A} \otimes (2l_c \nabla \eta + \eta \nabla l_c)) \rangle dx. \end{aligned} \quad (3.6)$$

Inserting (3.4)-(3.6) into (3.3) and using the balance of forces we obtain

$$\begin{aligned}
\dot{\mathcal{E}}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)(t) &= - \int_{\Omega} \eta^2 \langle D_k^h f, D_k^h \dot{u} \rangle dx - \int_{\Omega} 2\eta \langle D_k^h \sigma, D_k^h \dot{u} \otimes \nabla \eta \rangle dx \\
&- \int_{\Omega} \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h) + D_k^h \lambda \text{tr} [\varepsilon^h] + 2D_k^h \mu_c(\text{skew}(\nabla u^h) - A^h), \nabla D_k^h \dot{u} \rangle dx \\
&- \int_{\Omega} \eta^2 \underbrace{\langle D_k^h T_E, D_k^h \dot{\varepsilon}_p \rangle}_{\text{flow-rule} \geq 0} dx - \int_{\Omega} \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h), D_k^h \dot{\varepsilon}_p \rangle dx \\
&- 4 \int_{\Omega} \mu_c \eta^2 \langle \text{axl}(\text{skew}(\nabla D_k^h u) - D_k^h A), \text{axl}(D_k^h \dot{A}) \rangle dx \\
&- 4 \int_{\Omega} \eta^2 l_c \langle \Delta \text{axl}(D_k^h A), \text{axl}(D_k^h \dot{A}) \rangle dx \\
&+ 4 \int_{\Omega} \eta \langle \nabla \text{axl}(D_k^h A), \text{axl}(D_k^h \dot{A} \otimes (2l_c \nabla \eta + \eta \nabla l_c)) \rangle dx. \tag{3.7}
\end{aligned}$$

Next, using the balance of angular momentum equation for the microrotation and the monotonicity of the flow rule after integration over the time interval  $(0, t)$  we arrive at the inequality

$$\begin{aligned}
\mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)(t) &\leq \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)(0) \\
&- \int_0^t \int_{\Omega} \eta^2 \langle D_k^h f, D_k^h \dot{u} \rangle dx d\tau - \int_0^t \int_{\Omega} 2\eta \langle D_k^h \sigma, D_k^h \dot{u} \otimes \nabla \eta \rangle dx d\tau \\
&- \int_0^t \int_{\Omega} \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h) + D_k^h \lambda \text{tr} [\varepsilon^h] + 2D_k^h \mu_c(\text{skew}(\nabla u^h) - A^h), \nabla D_k^h \dot{u} \rangle dx d\tau \\
&- \int_0^t \int_{\Omega} \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h), D_k^h \dot{\varepsilon}_p \rangle dx d\tau \\
&+ 4 \int_0^t \int_{\Omega} \eta \langle \nabla \text{axl}(D_k^h A), \text{axl}(D_k^h \dot{A} \otimes (2l_c \nabla \eta + \eta \nabla l_c)) \rangle dx d\tau. \tag{3.8}
\end{aligned}$$

By the regularity of  $\varepsilon_p^0$  and  $f(0)$  we conclude that  $\mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)(0)$  is bounded by  $C \mathcal{E}_{U^*}(u_{x_k}, \varepsilon_{x_k}, \varepsilon_{p, x_k}, A_{x_k})(0)$ , where  $C > 0$  does not depend on  $h$ . Next, we are going to estimate all integrals in the right hand side on (3.8). To estimate the first integral we use the regularity of  $f$  and  $L^2(H^1)$  regularity of the velocity  $\dot{u}$ . Hence, we obtain

$$- \int_0^t \int_{\Omega} \eta^2 \langle D_k^h f, D_k^h \dot{u} \rangle dx d\tau \leq C \|f_{x_k}\|_{L^2((0,t) \times \Omega)} \|\dot{u}_{x_k}\|_{L^2((0,t) \times \Omega)}. \tag{3.9}$$

The second integral can be estimated as follows

$$\begin{aligned}
- \int_0^t \int_{\Omega} 2\eta \langle D_k^h \sigma, D_k^h \dot{u} \otimes \nabla \eta \rangle dx d\tau &\leq C (\|\eta D_k^h \sigma\|_{L^2((0,t) \times U)}^2 + \|\dot{u}_{x_k}\|_{L^2((0,t) \times \Omega)}^2) \leq \\
C \left( \int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A) d\tau + \int_0^t \mathcal{E}_{U^*}(u, \varepsilon, \varepsilon_p, A) d\tau + \|\dot{u}_{x_k}\|_{L^2((0,t) \times \Omega)}^2 \right), \tag{3.10}
\end{aligned}$$

where  $C > 0$  does not depend on  $h$ . In the last estimate we have used (3.4) and the regularity of  $\mu, \lambda, \mu_c$  and  $l_c$ . To estimate the third integral we integrate by parts with respect to  $\tau$  and get

$$\begin{aligned}
& - \int_0^t \int_{\Omega} \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h) + D_k^h \lambda \operatorname{tr} [\varepsilon^h] + 2D_k^h \mu_c (\operatorname{skew}(\nabla u^h) - A^h), \nabla D_k^h \dot{u} \rangle \, dx \, d\tau \\
& = \int_0^t \int_{\Omega} \eta^2 \langle 2D_k^h \mu(\dot{\varepsilon}^h - \dot{\varepsilon}_p^h) + D_k^h \lambda \operatorname{tr} [\dot{\varepsilon}^h] + 2D_k^h \mu_c (\operatorname{skew}(\nabla \dot{u}^h) - \dot{A}^h), \nabla D_k^h u \rangle \, dx \, d\tau \\
& \quad - \int_{\Omega} \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h) + D_k^h \lambda \operatorname{tr} [\varepsilon^h] + 2D_k^h \mu_c (\operatorname{skew}(\nabla u^h) - A^h), \nabla D_k^h u \rangle \, dx \\
& \quad + \int_{\Omega} \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h) + D_k^h \lambda \operatorname{tr} [\varepsilon^h] + 2D_k^h \mu_c (\operatorname{skew}(\nabla u^h) - A^h), \nabla D_k^h u \rangle|_{t=0} \, dx \\
& \leq C \left( \int_0^t \mathcal{E}_{U^*}(\dot{u}, \dot{\varepsilon}, \dot{\varepsilon}_p, \dot{A}) \, d\tau + \|\eta D_k^h \nabla u\|_{L^2((0,t) \times U)}^2 + C(\alpha) \mathcal{E}_{U^*}(\dot{u}, \dot{\varepsilon}, \dot{\varepsilon}_p, \dot{A})(t) \right. \\
& \quad \left. + \alpha \|\eta D_k^h \nabla u\|_{L^2(U)}^2 + \mathcal{E}_{U^*}(\dot{u}, \dot{\varepsilon}, \dot{\varepsilon}_p, \dot{A})(0) + \|\eta D_k^h \nabla u(0)\|_{L^2(U)}^2 \right), \tag{3.11}
\end{aligned}$$

where  $\alpha > 0$  is an arbitrary constant and  $C(\alpha)$  depends on  $\alpha$  only. By the regularity of the initial data the last two terms on the right hand side of (3.11) are bounded by a constant which is independent of  $h$ . Using the coerciveness of the energy function we have

$$\|\eta D_k^h \nabla u\|_{L^2(U)}^2 \leq \|\eta D_k^h \varepsilon\|_{L^2(U)}^2 + \|\eta D_k^h \operatorname{skew} \nabla u\|_{L^2(U)}^2 \leq C \mathcal{E}(\eta D_k^h u, \eta D_k^h, \eta D_k^h \varepsilon_p, \eta D_k^h A). \tag{3.12}$$

Inserting (3.12) into (3.11) we arrive at the inequality

$$\begin{aligned}
& - \int_0^t \int_{\Omega} \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h) + D_k^h \lambda \operatorname{tr} [\varepsilon^h] + 2D_k^h \mu_c (\operatorname{skew}(\nabla u^h) - A^h), \nabla D_k^h \dot{u} \rangle \, dx \, d\tau \\
& \leq C \left( \int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h, \eta D_k^h \varepsilon_p, \eta D_k^h A) \, d\tau + \alpha \mathcal{E}(\eta D_k^h u, \eta D_k^h, \eta D_k^h \varepsilon_p, \eta D_k^h A) \right. \\
& \quad \left. + \int_0^t \mathcal{E}_{U^*}(\dot{u}, \dot{\varepsilon}, \dot{\varepsilon}_p, \dot{A}) \, d\tau + C(\alpha) \mathcal{E}_{U^*}(\dot{u}, \dot{\varepsilon}, \dot{\varepsilon}_p, \dot{A}) + 1 \right). \tag{3.13}
\end{aligned}$$

In the same manner we estimate the fourth integral from the right hand side of (3.8)

$$\begin{aligned}
& - \int_0^t \int_{\Omega} \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h), D_k^h \dot{\varepsilon}_p \rangle \, dx \, d\tau = \int_0^t \int_{\Omega} \eta^2 \langle 2D_k^h \mu(\dot{\varepsilon}^h - \dot{\varepsilon}_p^h), D_k^h \varepsilon_p \rangle \, dx \, d\tau \\
& \quad - \int_{\Omega} \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h), D_k^h \varepsilon_p \rangle \, dx + \int_{\Omega} \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h), D_k^h \dot{\varepsilon}_p \rangle|_{t=0} \, dx \\
& \leq \hat{C} \left( \int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h, \eta D_k^h \varepsilon_p, \eta D_k^h A) \, d\tau + \beta \mathcal{E}(\eta D_k^h u, \eta D_k^h, \eta D_k^h \varepsilon_p, \eta D_k^h A) \right. \\
& \quad \left. + \int_0^t \mathcal{E}_{U^*}(\dot{u}, \dot{\varepsilon}, \dot{\varepsilon}_p, \dot{A}) \, d\tau + C(\beta) \mathcal{E}_{U^*}(\dot{u}, \dot{\varepsilon}, \dot{\varepsilon}_p, \dot{A}) + 1 \right), \tag{3.14}
\end{aligned}$$

where  $\beta > 0$  is an arbitrary constant,  $C(\beta) > 0$  depends on  $\beta$  only and  $\hat{C}$  does not depend on  $h$ . Finally, the last integral from the right hand side of (3.8) can be estimated

as follows

$$\begin{aligned} & \int_0^t \int_{\Omega} \eta \langle \nabla \operatorname{axl}(D_k^h A), \operatorname{axl}(D_k^h \dot{A} \otimes (2l_c \nabla \eta + \eta \nabla l_c)) \rangle dx d\tau \\ & C \left( \int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A) d\tau + \|\dot{A}_{x_k}\|_{L^2((0,t) \times \Omega)}^2 \right), \end{aligned} \quad (3.15)$$

where again the constant  $C > 0$  does not depend on  $h$ . Let us choose now  $\alpha$  and  $\beta$  such that  $C\alpha + \hat{C}\beta < 1$  where the constants  $C$  and  $\hat{C}$  are from inequality (3.13) and (3.14) respectively. On inserting (3.9)-(3.10) and (3.13)-(3.15) into (3.8) we obtain

$$\begin{aligned} \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)(t) & \leq C \left( \mathcal{E}_{U^*}(u_{x_k}, \varepsilon_{x_k}, \varepsilon_{p,x_k}, A_{x_k})(0) \right. \\ & \int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A) d\tau + \int_0^t (\mathcal{E}_{U^*}(u, \varepsilon, \varepsilon_p, A) + \mathcal{E}_{U^*}(\dot{u}, \dot{\varepsilon}, \dot{\varepsilon}_p, \dot{A})) d\tau \\ & \left. \mathcal{E}_{U^*}(\dot{u}, \dot{\varepsilon}, \dot{\varepsilon}_p, \dot{A}) + \|\dot{u}_{x_k}\|_{L^2((0,t) \times \Omega)}^2 + \|\dot{A}_{x_k}\|_{L^2((0,t) \times \Omega)}^2 + 1 \right). \end{aligned} \quad (3.16)$$

Finally, using the Gronwall Lemma and the inequality

$$\mathcal{E}_V(D_k^h u, D_k^h \varepsilon, D_k^h \varepsilon_p, D_k^h A) \leq \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A),$$

we easily complete the proof. ■

### Corollary 3.3

*Assuming that the initial plastic strain  $\varepsilon_p^0$  and the external force  $f$  satisfy all requirements of Theorem 3.2 the solution to system (2.1) is more regular:  $u \in L^\infty(0, T; H_{loc}^2(\Omega; \mathbb{R}^3))$ ,  $\sigma, \varepsilon_p \in L^\infty(0, T; H_{loc}^1(\Omega; \operatorname{Sym}(3)))$ . If additionally  $\nabla l_c, \nabla \mu_c$  are locally Lipschitz then  $A \in L^\infty(0, T; H_{loc}^3(\Omega; \mathfrak{so}(3)))$ .*

**Proof.** By Theorem 3.2 we immediately have that for all subsets  $V \Subset \Omega$  all functions appearing in the energy function have the regularity  $\varepsilon - \varepsilon_p \in L^\infty(0, T; H^1(V; \operatorname{Sym}(3)))$ ,  $\operatorname{tr}[\varepsilon] \in L^\infty(0, T; H^1(V; \mathbb{R}))$  and  $\operatorname{skew}(\nabla u - A) \in L^\infty(0, T; H^1(V; \mathfrak{so}(3)))$ . From the  $H^2$ -regularity we obtain that  $\sigma \in L^\infty(0, T; H^1(V; \operatorname{Sym}(3)))$ . Using the coerciveness of the energy function we conclude that  $\varepsilon \in L^\infty(0, T; H^1(V; \operatorname{Sym}(3)))$ , which implies that  $u \in L^\infty(0, T; H_{loc}^2(\Omega; \mathbb{R}^3))$ . The  $H_{loc}^3$ -regularity of  $A$  follows by  $H_{loc}^2$ -regularity of  $u$  and the standard regularity theory of elliptic equations. ■

## 4 $H_{loc}^1$ -regularity in the dynamic case

In the dynamical setting the system of equations which we are going to study is in the form

$$\begin{aligned} \ddot{u} - \operatorname{Div} \sigma &= f, \\ \sigma &= 2\mu(\varepsilon - \varepsilon_p) + 2\mu_c(\operatorname{skew}(\nabla u) - A) + \lambda \operatorname{tr}[\varepsilon] \cdot \mathbb{1}, \end{aligned}$$

$$\begin{aligned}
\text{axl}(\dot{A}) - l_c \Delta \text{axl}(A) &= \mu_c \text{axl}(\text{skew}(\nabla u) - A), \\
\dot{\varepsilon}_p &\in \mathbf{f}(T_E), \quad T_E = 2\mu(\varepsilon - \varepsilon_p), \\
u|_{\partial\Omega} &= u_d, \quad A|_{\partial\Omega} = A_d, \\
u(0) = u^0, \quad \dot{u}(0) = u^1, \quad A(0) = A^0, \quad \dot{A}(0) = A^1, \quad \varepsilon_p(0) = \varepsilon_p^0,
\end{aligned} \tag{4.1}$$

where  $f$  is a given volume force  $u_d, A_d$  are given boundary data and  $u^0, u^1, A^0, A^1, \varepsilon_p^0$  are given initial data. This initial boundary-value problem was studied in [14] and the Main Theorem from [14] yields an existence and uniqueness result similar to Theorem 2.1. The energy function associated with system (4.1) is in the form

$$\begin{aligned}
\mathcal{E}(u, \varepsilon, \varepsilon_p, A)(t) &:= \int_{\Omega} \left( \frac{1}{2} \|\dot{u}\|^2 + 2 \|\text{axl}(\dot{A})\|^2 \right. \\
&\quad \left. + \mu \|\varepsilon - \varepsilon_p\|^2 + \frac{\lambda}{2} \text{tr}[\varepsilon]^2 + \mu_c \|\text{skew}(\nabla u) - A\|^2 + 2l_c \|\nabla \text{axl}(A)\|^2 \right) dx.
\end{aligned}$$

This function is also coercive which means that  $\mathcal{E}$  satisfies the statements of Theorem 3.1. Using the same methods as in Section 2 we can conclude the following regularity result for the initial boundary-value problem (4.1).

**Theorem 4.1**

*Let us assume that  $u^0 \in H_{loc}^2(\Omega; \mathbb{R}^3)$ ,  $u^1 \in H_{loc}^1(\Omega; \mathbb{R}^3)$ ,  $\varepsilon_p^0 \in H_{loc}^1(\Omega; \text{Sym}(3))$ ,  $A^0 \in H^2(\Omega; \mathfrak{so}(3))$ ,  $A^1 \in H^1(\Omega; \mathfrak{so}(3))$  and  $f \in L^2(0, T; H_{loc}^1(\Omega; \mathbb{R}^3))$ . If  $\mu, \lambda, \mu_c, l_c$  are continuous and positive on  $\bar{\Omega}$  and locally Lipschitz then the solution to (4.1) is more regular:  $u \in L^\infty(0, T; H_{loc}^2(\Omega; \mathbb{R}^3))$ ,  $\sigma, \varepsilon_p \in L^\infty(0, T; H_{loc}^1(\Omega; \text{Sym}(3)))$ .*

## 5 Quasistatic case with a general flow rule

In [3] the quasistatic problem was studied with the following general flow rule of monotone type

$$z_t \in \mathbf{f}(-\nabla_z \psi(\varepsilon, z, A)),$$

where  $z = (\varepsilon_p, \tilde{z})$  is the vector of internal variables,  $Bz = \varepsilon_p$  is the projector on the direction of the plastic strain,  $\mathbf{f}$  is a maximal monotone mapping satisfying  $\{0\} \in \mathbf{f}(0)$  and  $\psi$  is the free energy function. The free energy considered in this article is in the form

$$\psi(\varepsilon, z, A) = \mu \|\varepsilon - \varepsilon_p\|^2 + \frac{\lambda}{2} \text{tr}[\varepsilon]^2 + \mu_c \|\text{skew}(\nabla u) - A\|^2 + 2l_c \|\nabla \text{axl}(A)\|^2 + \langle Lz, z \rangle,$$

where  $L$  is a symmetric and semi-positive matrix. Using the same procedure as in Section 3 the following regularity result can be obtained

**Theorem 5.1**

*If the initial value  $z^0 \in H_{loc}^1(\Omega; \mathbb{R}^N)$  and  $f \in L^2(0, T; H_{loc}^1(\Omega; \mathbb{R}^3))$  then the solution to the quasistatic problem in the Cosserat plasticity with a general flow rule of monotone type is more regular:  $\sigma \in L^\infty(0, T; H_{loc}^1(\Omega; \text{Sym}(3)))$  and  $Lz \in L^\infty(0, T; H_{loc}^1(\Omega; \mathbb{R}^N))$ . Moreover, the coerciveness of the energy function yields additionally that  $u \in L^\infty(0, T; H_{loc}^2(\Omega; \mathbb{R}^3))$  and  $\varepsilon_p \in L^\infty(0, T; H_{loc}^1(\Omega; \text{Sym}(3)))$ . ■*

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