Mean field modeling of isotropic random Cauchy elasticity versus microstretch elasticity.

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Abstract

We show that the averaged response of random isotropic Cauchy elastic material can be described analytically. It leads to a higher gradient model with explicit expressions for the dependence on the second derivatives of the mean field. A subsequent penalty formulation coincides with a linear elastic micro-stretch model with specific choice of constitutive parameters, depending only on the average cut-off length (the internal length scale $L_c > 0$). Thus the microstretch displacement field can be viewed as an approximated mean field response for these parameter ranges. The mean field free energy in this micro-stretch formulation is not uniformly positive, nevertheless, the model is well posed.

Key words: polar-materials, microstructure, couple-stress model
structured continua, solid mechanics, variational methods.

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1 Introduction

Higher gradient or extended continuum models involve an increasing number of constitutive parameters which cannot easily be interpreted and determined. In this contribution we assume we are dealing with micro-heterogeneous random isotropic Cauchy material for which we would like to determine a variational principle for the averaged response over a given length scale $L_c > 0$. Averaging at scale L_c introduces naturally a scale-dependence into the problem. It is clear that the L_c-averaged (henceforth called mean field) displacement field $\overline{u}$ will acquire additional smoothness compared to linear elasticity. Thus we are prepared to allow for second gradients of the mean field $D^2\overline{u}$ to appear in the variational formulation for the mean field problem. How should this additional dependence in the energy on the second gradient look like? Infinitesimal invariance principles like objectivity, meaning that the response is invariant under superposition of constant infinitesimal rotations $u \mapsto u + \widehat{A}.x + \widehat{b},\ \widehat{A} \in \mathfrak{so}(3), \widehat{b} \in \mathbb{R}^3$ gives no restriction for the second gradient. Descending from a finite strain development we know [25, eq.(10.25)] that the strain energy of a non-simple material of grade two is expressible as $W = W(F^TF, \nabla_x(F^TF))$ which, after linearization, reduces to $W = W(\operatorname{sym}\nabla u, \nabla \operatorname{sym}\nabla u)$ but gives us not much further insight.[1]

In our approach, we try to directly expand isotropic linear elasticity and to read off the appearing higher gradient terms acting on the averaged response. To do this, we need to invoke an orthogonality assumption (3.6), motivated by assumed **statistical micro-randomness**. Moreover we assume that the third gradient $D^3\overline{u}$ of the mean field is much smaller then its second gradient $D^2\overline{u}$. It was surprising for us to be able to determine the explicit formulas for the higher gradient terms using established formulas for spherical averaging. Equally surprising was the fact that the higher gradient contribution is not controlling the full second gradient! The finally obtained second gradient model for the mean field response will be modified by considering a better manageable second gradient expression. For this modification then we offer a penalized model which can be interpreted in a natural way as a microstretch model. The microstretch model is a slight generalization of the micropolar model in that it involves, apart from the **mirorotations** $\overline{A} \in \mathfrak{so}(3)$ also a scalar **microdilation** $\overline{p} \in \mathbb{R}$ as additional field variable. Thus, the microstretch model is "nearly" a penalty formulation for our mean-field response.

The mathematical analysis establishing well-posedness for the infinitesimal strain, microstretch elastic model is given in [11, 9, 10]. This analysis has always been based on the **uniform positivity** of the free quadratic energy of the microstretch solid. The first author has extended the existence results for the more general micromorphic models [23, 18, 5] to the geometrically exact, finite-strain case, see e.g. [21, 20, 22]. It is interesting in this respect to note that our penalty-micro-stretch formulation for the mean field is not uniformly positive. Nevertheless one can show that the model is well-posed along the lines presented in [12] but we abstain from presenting further details here.

This paper is organized as follows. First we recall the microstretch model in variational form together with its subvariants, the Cosserat or micropolar model and the indeterminate couple stress model. Then we consider the averaged displacement field for linear isotropic elasticity and derive a second order correction term. Subsequently we modify slightly the second gradient contribution, taking care not to impart additional control which is not originally present. Finally, we present the penalty formulation and relate it to the microstretch model. The notation and some detailed calculations concerning spherical averages and second derivatives are found in the appendix.

2 The linear isotropic microstretch model

The investigation of microstretch and micromorphic continua (which are prominent examples of so-called extended continua) dates back to Eringens pioneering works in the mid 1960. The linear isotropic microstretch model of Eringen [2, p.254] and [3] features one additional degree of freedom as compared to a Cosserat or micropolar model: a scalar variable $\overline{p}$ for the "pressure", also called microdilation. In a variational format, the model can be shortly stated

[1] Note that all second derivatives D^2u are expressible as linear combinations of strain gradient derivatives $\nabla \operatorname{sym}\nabla u$, since $u^i_{,jk} = \varepsilon_{ik,j} + \varepsilon_{ij,k} - \varepsilon_{jk,i}$. This identity implies noteworthy the existence of a constant $C^+ > 0$ such that $\|D^2u\|^2 \le C^+\,\|\nabla \operatorname{sym}\nabla u\|^2$; a fact sometimes used to prove Korn's inequality. Moreover, we see that $W = W(\operatorname{sym}\nabla u, \nabla \operatorname{sym}\nabla u) = \tilde{W}(\operatorname{sym}\nabla u, D^2u)$ places in reality no condition on the higher gradient term.

as follows: find the **displacement** $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$, the **skew-symmetric infinitesimal microrotation** $\overline{A} : \Omega \subset \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ and the **scalar pressure** $\overline{p} : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ minimizing the **three-field** problem

$$I(u, \overline{A}, \overline{p}) = \int_\Omega W_{\text{stretch}}(\overline{\varepsilon}, \overline{p}) + W_{\text{curv}}(\nabla \operatorname{axl} \overline{A}, \nabla \overline{p}) - \langle f, u\rangle \, \mathrm{dx} \mapsto \quad \min . \text{ w.r.t. } (u, \overline{A}, \overline{p}), \tag{2.1}$$

under the following constitutive requirements and boundary conditions

$$\begin{aligned}
\overline{\varepsilon} &= \nabla u - (\overline{A} + \overline{p}\, 1\!\!1), \qquad \textbf{generalized stretch tensor} \\
u_{|\Gamma} &= u_{\rm d}\,, \qquad \textbf{essential displacement boundary conditions} \\
W_{\text{stretch}}(\overline{\varepsilon}, \overline{p}) &= \mu_e \,\|\operatorname{dev}\operatorname{sym}\overline{\varepsilon}\|^2 + \mu_c \,\|\operatorname{skew}\overline{\varepsilon}\|^2 + \frac{K_c}{2}\operatorname{tr}[\overline{\varepsilon}]^2 + \frac{1}{2}\left(\sqrt{K_0}\operatorname{tr}[\overline{\varepsilon}] + \sqrt{\lambda_c}\operatorname{tr}[\overline{p}\, 1\!\!1]\right)^2 \\
\phi &:= \operatorname{axl}\overline{A} \in \mathbb{R}^3, \quad \|\operatorname{curl}\phi\|^2_{\mathbb{R}^3} = 4\|\operatorname{axl}\operatorname{skew}\nabla\phi\|^2_{\mathbb{R}^3} = 2\|\operatorname{skew}\nabla\phi\|^2_{\mathbb{M}^{3\times3}}\,, \\
W_{\text{curv}}(\nabla\phi, \nabla\overline{p}) &= \frac{\gamma+\beta}{2}\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2 + \frac{\gamma-\beta}{2}\|\operatorname{skew}\nabla\phi\|^2 \\
&\quad + \frac{3\alpha + (\beta+\gamma)}{6}\operatorname{tr}[\nabla\phi]^2 + \frac{a_0}{2}\|\nabla\overline{p}\|^2 .
\end{aligned} \tag{2.2}$$

Here, f are given volume forces while $u_{\rm d}$ are Dirichlet boundary conditions[2] for the displacement at $\Gamma \subset \partial\Omega$, where $\Omega \subset \mathbb{R}^3$ is the referential domain. Surface tractions, volume couples and surface couples can be included in the standard way. The strain energy W_{stretch} and the curvature energy W_{curv} are isotropic quadratic forms in the **infinitesimal non-symmetric generalized strain tensor** $\overline{\varepsilon} = \nabla u - (\overline{A} + \overline{p} 1\!\!1)$, the **micropolar curvature tensor** $\overline{\mathfrak{k}} = \nabla \operatorname{axl}\overline{A} = \nabla\phi$ (curvature-twist tensor) and the **microdilation gradient** $\nabla\overline{p}$. The parameters $\mu_e, \mu_c, K_c, K_0, \lambda_c$[MPa] are constitutive parameters and $\alpha, \beta, \gamma, a_0$ are additional microstretch curvature moduli with dimension $[\mathrm{Pa}\cdot\mathrm{m}^2] = [\mathrm{N}]$ of a force. Experimentally, the determination of the curvature moduli is extremely difficult compared to the first set of parameters [14].

2.1 Non-negativity of the micro-stretch energy

From the representation of the energy in (2.2) we can read off immediately the necessary and sufficient conditions for the non-negativity of this free energy. We must have

$$\begin{gathered}
\mu \ge 0\,, \quad \mu_c \ge 0\,, \quad K_c \ge 0\,, \quad K_0 \ge 0\,, \quad \lambda_c \ge 0\,, \\
\gamma+\beta \ge 0\,, \quad \gamma-\beta \ge 0\,, \quad 3\alpha + (\beta+\gamma) \ge 0\,, \quad a_0 \ge 0\,.
\end{gathered} \tag{2.3}$$

Certain of these inequalities need to be strict in order for the well-posedness of the model. However, the uniform pointwise positivity (strict inequalities everywhere) is not necessary, although it is assumed most often in treatments of linear microstretch elasticity.

2.2 The micro-stretch balance equations: strong form

For our choice of energy we collect the induced balance equations. Taking variations of the energy in (2.5) w.r.t. displacement $u \in \mathbb{R}^3$, infinitesimal microrotation $\overline{A} \in \mathfrak{so}(3)$ and microdilation $\overline{p} \in \mathbb{R}$, one arrives at the equilibrium system (the Euler-Lagrange equations of (2.5))

$$\begin{aligned}
\operatorname{Div}\sigma &= f\,, \qquad \textbf{balance of linear momentum} \\
-\operatorname{Div} m &= 4\,\mu_c \cdot \operatorname{axl}\operatorname{skew}\overline{\varepsilon}\,, \quad \textbf{balance of angular momentum} \\
a_0\,\Delta\overline{p} &= \left(\sqrt{K_0}\operatorname{tr}[\overline{\varepsilon}] + \sqrt{\lambda_c}\operatorname{tr}[\overline{p}\,1\!\!1]\right) 3\,(\sqrt{\lambda_c} - \sqrt{K_0}) - 3\,K_c\operatorname{tr}[\overline{\varepsilon}] \\
&\qquad \textbf{balance of micro-dilational momentum} \\
\sigma &= 2\mu\cdot\operatorname{dev}\operatorname{sym}\overline{\varepsilon} + 2\mu_c\cdot\operatorname{skew}\overline{\varepsilon} + \left[K_c\operatorname{tr}[\overline{\varepsilon}] + \sqrt{K_0}\,\left(\sqrt{K_0}\operatorname{tr}[\overline{\varepsilon}] + \sqrt{\lambda_c}\operatorname{tr}[\overline{p}\,1\!\!1]\right)\right]\cdot 1\!\!1 \\
m &= \gamma\,\nabla\phi + \beta\,\nabla\phi^T + \alpha\operatorname{tr}[\nabla\phi]\cdot 1\!\!1 \\
&= (\gamma+\beta)\,\operatorname{dev}\operatorname{sym}\nabla\phi + (\gamma-\beta)\,\operatorname{skew}\nabla\phi + \frac{3\alpha+(\gamma+\beta)}{2}\operatorname{tr}[\nabla\phi]\,1\!\!1\,, \\
\phi &= \operatorname{axl}\overline{A}\,, \quad u_{|\Gamma} = u_{\rm d}\,.
\end{aligned} \tag{2.4}$$

[2]Note that it is always possible to prescribe essential boundary values for the microrotations $\overline{A}$ and the microdilation $\overline{p}$ but we abstain from such a prescription because the physical meaning of this is doubtful. Choosing $\lambda_c = K_0$ we are able to generate the term $\frac{K_0}{2}\operatorname{tr}[\nabla u]^2$.

Here, m is the (second order) **moment stress tensor** which is given as a linear function of the curvature $\nabla\phi = \nabla \operatorname{axl} \overline{A}$. For more on the microstretch model, its applications and parameter determination we refer to [13, 17, 14].

The Cosserat or micropolar model can be obtained formally by letting $\lambda_c \to \infty$, in which case the bulk modulus is $K = K_c + K_0$.

2.3 The linear elastic isotropic Cosserat model in variational form

For the **displacement** $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ and the **skew-symmetric infinitesimal microrotation** $\overline{A} : \Omega \subset \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ we consider the **two-field** minimization problem

$$I(u, \overline{A}) = \int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon}) + W_{\mathrm{curv}}(\nabla \operatorname{axl} \overline{A}) - \langle f, u\rangle \, \mathrm{dx} \mapsto \quad \min.\ \text{w.r.t.}\ (u, \overline{A}), \tag{2.5}$$

under the following constitutive requirements and boundary conditions

$$\begin{aligned} \overline{\varepsilon} &= \nabla u - \overline{A}, \quad \textbf{first Cosserat stretch tensor} \\ W_{\mathrm{mp}}(\overline{\varepsilon}) &= \mu\, \|\operatorname{dev}\operatorname{sym} \nabla u\|^2 + \mu_c\, \|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{K}{2} \operatorname{tr}[\nabla u]^2 \\ W_{\mathrm{curv}}(\nabla\phi) &= \frac{\gamma+\beta}{2} \|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2 + \frac{\gamma-\beta}{2}\|\operatorname{skew}\nabla\phi\|^2 + \frac{3\alpha + (\beta+\gamma)}{6} \operatorname{tr}[\nabla\phi]^2 . \end{aligned} \tag{2.6}$$

Here, the strain energy W_{mp} and the curvature energy W_{curv} are the most general isotropic quadratic forms in the **infinitesimal non-symmetric first Cosserat strain tensor** $\overline{\varepsilon} = \nabla u - \overline{A}$ and the **micropolar curvature tensor** $\overline{\mathfrak{k}} = \nabla \operatorname{axl} \overline{A} = \nabla\phi$ (curvature-twist tensor). The parameters μ, K[MPa] are the classical shear and bulk modulus, respectively. The parameter $\mu_c \geq 0$[MPa] in the strain energy is the **Cosserat couple modulus**. For $\mu_c = 0$ the two fields of displacement and microrotations decouple and one is left formally with classical linear elasticity for the displacement u. Next, we obtain the indeterminate couple stress model by letting $\mu_c \to \infty$.

2.4 The indeterminate couple stress problem

The indeterminate couple stress problem [19, 24, 15, 26] is characterized by the identification $\frac{1}{2}\operatorname{curl} u = \operatorname{axl}\overline{A} = \phi$ which can be formally obtained from the genuine Cosserat model by setting $\mu_c = \infty$. Since here the infinitesimal microrotations $\overline{A} \in \mathfrak{so}(3)$ cease to be an independent field the model has the advantage of conceptional simplicity and improved physical transparency.

For the **displacement** $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ we consider therefore the **one-field** second gradient minimization problem

$$I(u) = \int_\Omega W_{\mathrm{mp}}(\nabla u) + W_{\mathrm{curv}}(\nabla \operatorname{curl} u)\, \mathrm{dV} \mapsto \quad \min.\ \text{w.r.t.}\ u,$$

under the constitutive requirements and boundary conditions

$$\begin{aligned} W_{\mathrm{mp}}(\nabla u) &= \mu\, \|\operatorname{sym}\nabla u\|^2 + \frac{\lambda}{2}\operatorname{tr}[\operatorname{sym}\nabla u]^2 , \quad u_{|\Gamma} = u_{\mathrm{d}} , \\ W_{\mathrm{curv}}(\nabla \operatorname{curl} u) &= \frac{\gamma+\beta}{8}\, \|\operatorname{dev}\operatorname{sym}\nabla\operatorname{curl} u\|^2 + \frac{\gamma-\beta}{8}\, \|\operatorname{skew}\nabla\operatorname{curl} u\|^2 \\ &\quad + \frac{3\alpha+(\beta+\gamma)}{24} \operatorname{tr}[\nabla\operatorname{curl} u]^2 \\ &= \frac{\gamma+\beta}{8}\, \|\operatorname{sym}\nabla\operatorname{curl} u\|^2 + \frac{\gamma-\beta}{8}\, \|\operatorname{skew}\nabla\operatorname{curl} u\|^2 . \end{aligned} \tag{2.7}$$

In this limit model, the curvature parameter α, related to the spherical part of the (higher order) couple stress tensor m remains **indeterminate**, since $\operatorname{tr}[\nabla\phi] = \operatorname{Div}\operatorname{axl}\overline{A} = \operatorname{Div}\frac{1}{2}\operatorname{curl} u = 0$. Following [15], it is usually assumed that $-1 < \eta := \frac{\beta}{\gamma} < 1$ in order to guarantee uniform positive definiteness.

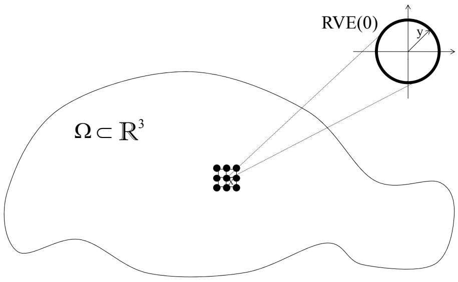

Figure 1: The basic situation of our multiscale approach. Here, the black points symbolize the mesoscale.

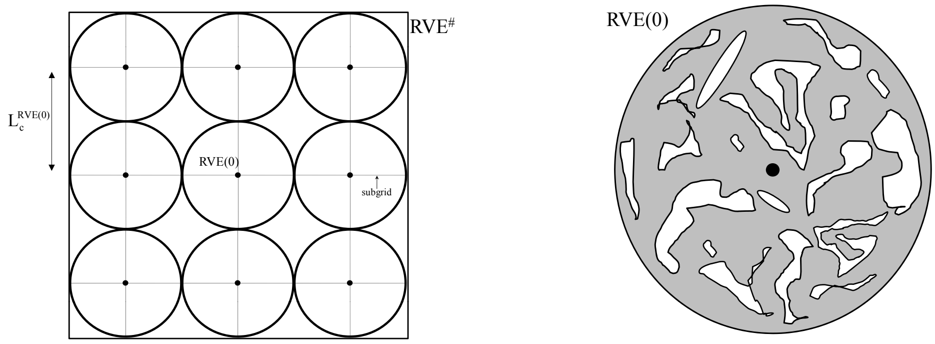

Figure 2: **Left**: The $RVE^\sharp$ fill Ω. Each $RVE^\sharp$ represents a cluster of smaller $RVE(0)$ which themselves define the cutoff length (averaging scale). The mean field will be determined on a coarser grid. **Right**: We assume that each $RVE(0)$ consists of random Cauchy elastic material.

3 Formal homogenization through averaging

Let us now switch to our homogenization procedure. In order to approach the question of homogenization we consider a rudimentary, formal, two-scale homogenization method in which we use the word "homogenization" in a loose sense. The underlying assumption is that there exist two distinct levels in the body of interest: a discontinuous, heterogeneous microscopic one, consisting of matrix material, voids and other inhomogeneities, and a continuous macroscopic one. The representative volume element $RVE^\sharp$ [4] defines the order of the scale of resolution of the envisaged continuum model, effects below this scale do not appear explicitly in the final model, cf. Figure 1. The classical continuum limit arises then as a doubly asymptotic, namely

- The $RVE^\sharp$ is big enough to be representative of the microstructure in a statistical sense.
- The $RVE^\sharp$ is small enough compared to the actual sample size for it to be considered to be infinitesimal.

In our case it is not even clear whether a unique homogenized medium exists. We assume to deal with statistically random Cauchy material in general. Nevertheless, there is a certain scale below which we are not interested in the displacement details. Thus we cover the body of interest with $RVE^\sharp$ containing a representative microstructure. The precise form of the $RVE^\sharp$ is irrelevant.

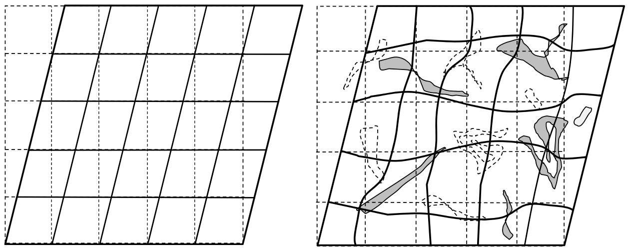

Figure 3: **Left**: Homogeneous deformation of the subgrid cluster $RVE^\sharp$ which is homogeneous inside due to homogeneous boundary conditions $y \mapsto \widehat{B}.y$. **Right**: Inhomogeneous response (micro-fluctuations) for same homogeneous boundary conditions due to random Cauchy material inside the $RVE^\sharp$.

3.1 Formal homogenization procedure: expansion to second order for the mean field

I order to get rid of the fine scale details we average the inhomogeneous, random response over a scale which is smaller then the scale induced by $RVE^\sharp$. This scale is given by a ball $RVE(0)$ with diameter L_c. The mean field on the averaging scale $RVE(0)$ is defined as box-filtered quantity [8]

$$\overline{u}(x) := \frac{1}{|B(0,L_c)|} \int\limits_{\xi\in B(0,L_c)} u(x+\xi)\, d\xi\,, \tag{3.1}$$

where $B(x_0,L_c) := \{x \in \mathbb{R}^3 \,|\, \|x-x_0\| \le L_c\}$. Let us consider random linear elastic Cauchy material inside $RVE^\sharp$ and consider the original linear elastic solution (including all fine scale features)

$$\begin{aligned}
&\int_{x\in RVE^\sharp} \mu(x)\, \|\operatorname{dev}\operatorname{sym}\nabla u(x)\|^2 + \frac{K(x)}{2}\operatorname{tr}\left[\nabla u(x)\right]^2 \mathrm{dx} \;\mapsto\; \min.\, u\\
&\frac{1}{|RVE^\sharp|}\int_{x\in RVE^\sharp} \nabla_x u(x)\,\mathrm{dx} = \widehat{B}\,, \qquad \text{average constraint over } RVE^\sharp \text{ or}\\
&u_{|_{\partial RVE^\sharp}}(x) = \widehat{B}.x\,, \quad \widehat{B}\in\mathfrak{gl}(3)\,, \quad \text{Dirichlet constraint}\,.
\end{aligned} \tag{3.2}$$

Here, $\mu(x), K(x)$ represent the random subgrid variation of the elastic moduli due to the inhomogeneities on this scale. This problem is well-posed for $\mu(x),\, K(x) \ge c_0 > 0$, but the solution u will in general develop irregular fluctuations with high gradients inside $RVE^\sharp$, see Fig. 3. Nevertheless, the solution u is in $H^1(RVE^\sharp)$ and the box-filtered $\overline{u}$ is an $H^2(RVE^\sharp)$-function.

Next, the distance between the mean field gradient $\nabla\overline{u}$ and the total field gradient ∇u can be expressed to highest order as

$$\begin{aligned}
\|\nabla\overline{u}(x) - \nabla u(x)\|^2 &= \|\frac{1}{|B(0,L_c)|}\int\limits_{\xi\in B(0,L_c)} \nabla u(x+\xi)\,d\xi - \nabla u(x)\|^2\\
&\approx \|\frac{1}{|B(0,L_c)|}\int\limits_{\xi\in B(0,L_c)} D^2u(x).\xi + \frac{1}{2}D^3u(x).(\xi,\xi) + \ldots\, d\xi\|^2\\
&\approx \|\frac{1}{|B(0,L_c)|}\int\limits_{\xi\in B(0,L_c)} \frac{1}{2}D^3u(x).(\xi,\xi) + \ldots\, d\xi\|^2 \sim C^+\,\|D^3u(x)\|^2\,L_c^4\,,
\end{aligned} \tag{3.3}$$

since the second derivative $D^2 u$ here is linear in ξ and does therefore not contribute to the average.

Further, we want to relate the mean field energy and the averaged energy through

$$\|\operatorname{dev}\operatorname{sym}\nabla\overline{u}(x)\|^2 + \text{"Correction"} = \frac{1}{|B(0,L_c)|}\int\limits_{\xi\in B(0,L_c)} \|\operatorname{dev}\operatorname{sym}\nabla u(x+\xi)\|^2\, d\xi\,. \tag{3.4}$$

Thus we start from the right hand side and write

$$\begin{aligned}
&\frac{1}{|B(0,L_c)|}\int\limits_{\xi\in B(0,L_c)} \|\operatorname{dev}\operatorname{sym}\nabla u(x+\xi)\|^2\, d\xi\\
&= \frac{1}{|B(0,L_c)|}\int\limits_{\xi\in B(0,L_c)} \|\operatorname{dev}\operatorname{sym}[\nabla u(x+\xi)-\nabla\overline{u}(x+\xi)+\nabla\overline{u}(x+\xi)]\|^2\, d\xi\\
&= \frac{1}{|B(0,L_c)|}\int\limits_{\xi\in B(0,L_c)} \|\operatorname{dev}\operatorname{sym}[\nabla u(x+\xi)-\nabla\overline{u}(x+\xi)]\|^2\\
&\qquad\qquad + 2\langle\operatorname{dev}\operatorname{sym}[\nabla u(x+\xi)-\nabla\overline{u}(x+\xi)],\nabla\overline{u}(x+\xi)\rangle\\
&\qquad\qquad\qquad + \|\operatorname{dev}\operatorname{sym}\nabla\overline{u}(x+\xi)\|^2\, d\xi\,.
\end{aligned} \tag{3.5}$$

The last term in the integrand will be much simplified through expansion and analytic integration in (3.8). For the middle term (which is in principle of order L_c^2) we invoke an orthogonality assumption on the difference between average and total displacement gradient. This is the assumption that these functions are **statistically uncorrelated**. More precisely, we **assume the orthogonality** on the L_c-averaging scale[3]

$$\begin{aligned}
&\int\limits_{\xi\in B(0,L_c)} \langle\operatorname{dev}\operatorname{sym}[\nabla u(x+\xi)-\nabla\overline{u}(x+\xi)],\ \nabla\overline{u}(x+\xi)\rangle\, d\xi = 0\,,\\
&\int\limits_{\xi\in B(0,L_c)} \operatorname{tr}[\nabla u(x+\xi)-\nabla\overline{u}(x+\xi)]]\cdot\operatorname{tr}[\nabla\overline{u}(x+\xi)]\, d\xi = 0\,.
\end{aligned} \tag{3.6}$$

Similar assumptions in a one dimensional setting are made in [6]. It remains to estimate the first term in (3.5). Here, we use for the integrand the highest order expansion in (3.3) which shows that the term is of order L_c^4 (before and after integration).

For the mean field $\overline{u}$ itself it seems also appropriate to assume that

$$\forall\,\xi\in\mathbb{S}^2: \qquad \|D^3\overline{u}.(\xi,\xi)\|^2_{\mathbb{M}^{3\times3}} \ll \|D^2\overline{u}.\xi\|^2_{\mathbb{M}^{3\times3}}\,, \tag{3.7}$$

since averaging will in general damp oscillations. This motivates to skip third gradients as compared to second gradients in the subsequent development. Finally, then

$$\begin{aligned}
&\frac{1}{|B(0,L_c)|}\int\limits_{\xi\in B(0,L_c)} \|\nabla\overline{u}(x+\xi)\|^2\, d\xi\\
&\approx \frac{1}{|B(0,L_c)|}\int\limits_{\xi\in B(0,L_c)} \|\nabla\overline{u}(x)+D^2\overline{u}(x).\xi+\dots\|^2\, d\xi\\
&= \frac{1}{|B(0,L_c)|}\int\limits_{\xi\in B(0,L_c)} \|\nabla\overline{u}(x)\|^2 + 2\langle\nabla\overline{u}(x),D^2\overline{u}(x).\xi\rangle + \|D^2\overline{u}(x).\xi\|^2+\dots\, d\xi\\
&= \|\nabla\overline{u}(x)\|^2 + 0 + \frac{1}{|B(0,L_c)|}\int\limits_{\xi\in B(0,L_c)} \|D^2\overline{u}(x).\xi\|^2\, d\xi + \dots
\end{aligned}$$

[3] It would suffice to have an estimate $\leq C^+\, L_c^3$ for these two terms. The condition is, effectively, the requirement $\overline{(g-\overline{g})\,\overline{g}} = 0$ or $\overline{g\,\overline{g}} = \overline{g}\,\overline{g}$ for a generic field g, where the overline denotes averaging. In this setting, the well known micro-macro homogeneity condition [7] would read $\overline{g\,g} = \overline{g}\,\overline{g}$.

$$
\begin{aligned}
&= \|\nabla\overline{u}(x)\|^2 + \frac{1}{|B(0,L_c)|}\int_{t=0}^{L_c}\int_{\xi\in\partial B(0,t)} \|D^2\overline{u}(x).\xi\|^2\, d\xi\,\mathrm{dt} + \dots \\
&= \|\nabla\overline{u}(x)\|^2 + \frac{1}{|B(0,L_c)|}\int_{t=0}^{L_c}\int_{\xi\in t\,\mathbb{S}^2} \|D^2\overline{u}(x).\xi\|^2\, d\mathbb{S}^2_t\,\mathrm{dt} + \dots \\
&= \|\nabla\overline{u}(x)\|^2 + \frac{1}{|B(0,L_c)|}\int_{t=0}^{L_c}\int_{\tilde{h}\in\mathbb{S}^2} \|D^2\overline{u}(x).(t\,\tilde{h})\|^2\, t^2\, d\mathbb{S}^2\,\mathrm{dt} + \dots \\
&= \|\nabla\overline{u}(x)\|^2 + \frac{1}{|B(0,L_c)|}\int_{t=0}^{L_c} t^4 \int_{\tilde{h}\in\mathbb{S}^2} \|D^2\overline{u}(x).\tilde{h}\|^2\, d\mathbb{S}^2\,\mathrm{dt} + \dots \\
&= \|\nabla\overline{u}(x)\|^2 + \frac{1}{\frac{4\,\pi\,L_c^3}{3}}\,\frac{L_c^5}{5}\int_{\tilde{h}\in\mathbb{S}^2} \|D^2\overline{u}(x).\tilde{h}\|^2\, d\mathbb{S}^2 + \dots \\
&= \|\nabla\overline{u}(x)\|^2 + \frac{3\,L_c^2}{20\,\pi}\int_{\tilde{h}\in\mathbb{S}^2} \|D^2\overline{u}(x).\tilde{h}\|^2\, d\mathbb{S}^2 + \dots \quad .
\end{aligned}
\tag{3.8}
$$

Using the same procedure for the full linear elastic energy expression is now possible, together with a complete analytic result of the integration. We obtain to highest order under the assumption (3.6)

$$
\begin{aligned}
&\frac{1}{|B(0,L_c)|}\int_{\xi\in B(0,L_c)} \widehat{\mu}\,\|\,\mathrm{dev}\,\mathrm{sym}\,\nabla u(x+\xi)\|^2 + \frac{\widehat{K}}{2}\,\mathrm{tr}\left[\nabla u(x+\xi)\right]^2 d\xi \\
&= \widehat{\mu}\left(\|\,\mathrm{dev}\,\mathrm{sym}\,\nabla\overline{u}(x)\|^2 + \frac{3\,L_c^2}{20\,\pi}\int_{\tilde{h}\in\mathbb{S}^2}\|\,\mathrm{dev}\,\mathrm{sym}\,D^2\overline{u}(x).\tilde{h}\|^2\, d\mathbb{S}^2\right) \\
&\quad + \frac{\widehat{K}}{2}\left(\mathrm{tr}\left[\nabla\overline{u}(x)\right]^2 + \frac{3\,L_c^2}{20\,\pi}\int_{\tilde{h}\in\mathbb{S}^2}\mathrm{tr}\left[D^2\overline{u}(x).\tilde{h}\right]^2 d\mathbb{S}^2\right) \\
&= \widehat{\mu}\|\,\mathrm{dev}\,\mathrm{sym}\,\nabla\overline{u}(x)\|^2 + \frac{\widehat{K}}{2}\,\mathrm{tr}\left[\nabla\overline{u}(x)\right]^2 \\
&\quad + \frac{L_c^2}{5}\left(\widehat{\mu}\left(\|D^2\overline{u}(x)\|^2_{\mathbb{R}^{27}} - \frac{1}{2}\,\|\nabla\,\mathrm{curl}\,\overline{u}(x)\|^2\right) + \frac{\widehat{\lambda}}{2}\|\nabla\,\mathrm{Div}\,\overline{u}(x)\|^2\right),
\end{aligned}
\tag{3.9}
$$

where we have used the calculation in the appendix equation (5.16). As a preliminary result we see that the such determined mean field[4] $\overline{u}$ will be smoother than the original field u but full pointwise control of all second derivatives of the mean field is not implied.[5]

In the two-dimensional, planar case, one can show (5.17) that for smooth functions with compact support $u\in C_0^\infty(\Omega\subset\mathbb{R}^2,\mathbb{R}^2)$ it holds after integration

$$
\int_\Omega \|D^2\overline{u}(x)\|^2_{\mathbb{R}^{27}} - \frac{1}{2}\,\|\nabla\,\mathrm{curl}\,\overline{u}(x)\|^2\,\mathrm{dV} \geq 1\int_\Omega \|\,\mathrm{sym}\,\nabla\,\mathrm{curl}\,u(x)\|^2\,\mathrm{dV}\,, \tag{3.10}
$$

where the constant 1 is sharp. Thus, for $\widehat{\lambda}=0$ in the higher order term and using the last result (formally also in three-dimensions) as a simplification, we obtain the classical indeterminate couple stress integrand, see (2.7)

$$
\widehat{\mu}\,\|\,\mathrm{dev}\,\mathrm{sym}\,\nabla\overline{u}(x)\|^2 + \frac{\widehat{K}}{2}\,\mathrm{tr}\left[\nabla\overline{u}(x)\right]^2 + \widehat{\mu}\,\frac{L_c^2}{5}\,\|\,\mathrm{sym}\,\nabla\,\mathrm{curl}\,u\|^2\,. \tag{3.11}
$$

[4] Of course, once the original integrand is replaced, $\overline{u}$ is not any longer the true averaged field but only related to it in the obvious way.

[5] Since $\Delta\overline{u} = \nabla\,\mathrm{Div}\,\overline{u} - \mathrm{curl}\,\mathrm{curl}\,\overline{u}$ we obtain from the modification (3.10) at least that $\Delta\overline{u}\in L^2(\Omega)$ if the energy is bounded for zero boundary conditions. This reflects that the "truly" averaged field $\overline{u}$ would satisfy $\overline{u}\in H^2_{loc}(\Omega,\mathbb{R}^3)$.

Note, however, that this representation implies the **symmetry of the moment stress tensor**! Based on different considerations of point mechanics in [26] the authors have also arrived at the conclusion that the moment stress tensor in the couple stress theory should be symmetric. It violates the condition $-1 < \frac{\beta}{\gamma} < 1$.

3.2 Relaxed penalty formulation of the mean field energy

Using the obtained second order corrected integrand for the mean field we would be bound to implement a fourth order problem on the equilibrium level. To avoid this, we formulate a penalty approach. We add a scalar pressure variable $\overline{p}(x)$ which is supposed to take on the role of $\frac{1}{3}\operatorname{Div}\overline{u}$ and we define a microrotation vector $\operatorname{axl}(\overline{A}) \in \mathbb{R}^3$ for $\overline{A} \in \mathfrak{so}(3)$ taking on the role of $\frac{1}{2}\operatorname{curl}\overline{u}$. Then we **define the relaxed and penalized integrand**

$$\begin{aligned}&\widehat{\mu}\,\|\operatorname{dev}\operatorname{sym}\nabla\overline{u}(x)\|^2 + \frac{\widehat{K}}{2}\operatorname{tr}\left[\nabla\overline{u}(x)\right]^2 + \frac{\mu_c^\infty}{2}\,\|\operatorname{curl}\overline{u} - 2\operatorname{axl}(\overline{A})\|^2 + \frac{K_c^\infty}{2}\operatorname{tr}\left[\nabla\overline{u}(x) - \overline{p}\mathbb{1}\right]^2 \\ &\quad + \frac{L_c^2}{5}\left(\widehat{\mu}\,\|\operatorname{dev}\operatorname{sym}\nabla(2\operatorname{axl}(\overline{A})\|^2 + \frac{\widehat{\lambda}}{2}\|\nabla 3\overline{p}(x)\|^2\right). \end{aligned} \tag{3.12}$$

Here, $L_c > 0$ is the physical cut-off length and μ_c^∞, $K_c^\infty \gg \mu$ are numerical penalty factors (not physical constants). Clearly, this is now a microstretch model, having the advantage that we can precisely interpret the appearing terms. For μ_c^∞, $K_c^\infty \to \infty$ we recover, formally, the higher-order mean field integrand. In terms of the micro-stretch formulation presented in (2.1) we can write

$$\begin{aligned}&\widehat{\mu}\,\|\operatorname{dev}\operatorname{sym}\nabla\overline{u}(x)\|^2 + \frac{\widehat{K}}{2}\operatorname{tr}\left[\nabla\overline{u}(x)\right]^2 + \frac{\mu_c^\infty}{2}\,\|\operatorname{curl}\overline{u} - 2\operatorname{axl}(\overline{A})\|^2 + \frac{K_c^\infty}{2}\operatorname{tr}\left[\nabla\overline{u}(x) - \overline{p}\mathbb{1}\right]^2 \\ &\quad + \frac{L_c^2}{5}\left(\widehat{\mu}\,\|\operatorname{dev}\operatorname{sym}\nabla(2\operatorname{axl}(\overline{A})\|^2 + \frac{\widehat{\lambda}}{2}\|\nabla 3\overline{p}(x)\|^2\right) \\ &\quad = \mu\,\|\operatorname{dev}\operatorname{sym}\overline{\varepsilon}\|^2 + \mu_c\,\|\operatorname{skew}\overline{\varepsilon}\|^2 + \frac{K_c}{2}\operatorname{tr}\left[\overline{\varepsilon}\right]^2 + \frac{1}{2}\left(\sqrt{K_0}\operatorname{tr}\left[\overline{\varepsilon}\right] + \sqrt{\lambda_c}\operatorname{tr}\left[\overline{p}\,\mathbb{1}\right]\right)^2 \\ &\quad + \frac{\gamma+\beta}{2}\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2 + \frac{\gamma-\beta}{2}\|\operatorname{skew}\nabla\phi\|^2 + \frac{3\alpha+(\beta+\gamma)}{6}\operatorname{tr}\left[\nabla\phi\right]^2 + \frac{a_0}{2}\,\|\nabla\overline{p}\|^2\,, \end{aligned} \tag{3.13}$$

where $\mu_c = \mu_c^\infty$, $K_c = K_c^\infty$, $\widehat{K} = K_0$, $\lambda_c = K_0$ and $\gamma = \beta = \frac{4\widehat{\mu}L_c^2}{5}$ and $3\alpha + (\beta+\gamma) = 0$ and $a_0 = \frac{9\widehat{\lambda}\,L_c^2}{5}$. The well-posedness of this relaxed formulation, although not pointwise positive definite, can be shown along the lines in [12].

4 Conclusion

We have shown that mean field modeling of random isotropic Cauchy material leads in a straight forward way to the well established micro stretch model. Thereby, the usually difficult to determine coefficients in the curvature part can be uniquely given and are related primarily to the one cut-off length L_c used for the averaging scale. Thus we have provided an additional argument for the usefulness of the micro stretch model in general, independent of more detailed micro-structure arguments. It is interesting to note the consequences for a linear isotropic Cosserat model which is meant to be a homogeneous substitute medium of the random Cauchy material. In this case, in terms of the Cosserat parameters, μ_c should be large and the moment stress tensor m should be symmetric and trace free. This conclusion is consistent with experimental observations and identifications by Lakes [16] for foams. In addition we see that the mean field, obtained by spherical averaging, gives rise to a more "symmetric" response.

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5 Appendix

5.1 Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters and Sym denotes symmetric second orders tensors. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times3}$. The identity tensor on $\mathbb{M}^{3\times3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\,1\!\!1 \in \mathfrak{sl}(3)$ where $\mathfrak{sl}(3)$ is the Lie-algebra of traceless matrices. The set $\operatorname{Sym}(n)$ denotes all symmetric $n \times n$-matrices. The Lie-algebra of $\operatorname{SO}(3) := \{X \in \operatorname{GL}(3)\ |X^T X = 1\!\!1,\ \det[X] = 1\}$ is given by the set $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times3}\ |\ X^T = -X\}$ of all skew symmetric tensors. The canonical identification of $\mathfrak{so}(3)$ and $\mathbb{R}^3$ is denoted by $\operatorname{axl}\overline{A} \in \mathbb{R}^3$ for $\overline{A} \in \mathfrak{so}(3)$. The Curl operator on the three by three matrices acts row-wise, i.e.

$$\operatorname{Curl}\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} = \begin{pmatrix} \operatorname{curl}(X_{11}, X_{12}, X_{13})^T \\ \operatorname{curl}(X_{21}, X_{22}, X_{23})^T \\ \operatorname{curl}(X_{31}, X_{32}, X_{33})^T \end{pmatrix} . \tag{5.1}$$

Moreover, we have

$$\forall\, A \in \mathrm{C}^1(\mathbb{R}^3, \mathfrak{so}(3)) : \quad \operatorname{Div} A(x) = -\operatorname{curl}\operatorname{axl}(A(x)) . \tag{5.2}$$

Note that $(\operatorname{axl}\overline{A}) \times \xi = \overline{A}.\xi$ for all $\xi \in \mathbb{R}^3$, such that

$$\begin{aligned} &\operatorname{axl}\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix} , \quad \overline{A}_{ij} = \sum_{k=1}^{3} -\varepsilon_{ijk} \cdot \operatorname{axl}\overline{A}_k , \\ &\|\overline{A}\|^2_{\mathbb{M}^{3\times3}} = 2\,\|\operatorname{axl}\overline{A}\|^2_{\mathbb{R}^3} , \qquad \langle \overline{A}, \overline{B}\rangle_{\mathbb{M}^{3\times3}} = 2\langle \operatorname{axl}\overline{A}, \operatorname{axl}\overline{B}\rangle_{\mathbb{R}^3} , \end{aligned} \tag{5.3}$$

where ε_{ijk} is the totally antisymmetric permutation tensor. Here, $\overline{A}.\xi$ denotes the application of the matrix $\overline{A}$ to the vector ξ and $a \times b$ is the usual cross-product. Moreover, the inverse of axl is denoted by anti and defined by

$$\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \operatorname{anti}\begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix} , \quad \operatorname{axl}(\operatorname{skew}(a \otimes b)) = -\frac{1}{2}\, a \times b , \tag{5.4}$$

and

$$2\operatorname{skew}(b \otimes a) = \operatorname{anti}(a \times b) = \operatorname{anti}(\operatorname{anti}(a).b) . \tag{5.5}$$

Moreover,

$$\operatorname{curl} u = 2\operatorname{axl}(\operatorname{skew}\nabla u) . \tag{5.6}$$

5.2 Second order expansions

Let us gather some expansions and developments which we need in the following. Note first that D^2u is interpreted as $D^2u(x) \in Lin(\mathbb{R}^3, \mathbb{M}^{3\times3})$ and therefore $[D^2u(x)]^T[D^2u(x)] \in Lin(\mathbb{R}^3, \mathbb{R}^3) = \mathbb{M}^{3\times3}$.

$$\begin{aligned} [\mathbb{1} + \nabla u(x+h)].y - [\mathbb{1} + \nabla u(x)].y &= [D^2u(x).h].y + \dots , \\ \operatorname{tr}\left[[\mathbb{1} + \nabla u(x+h)] - [\mathbb{1} + \nabla u(x)]\right] &= \operatorname{tr}\left[D^2u(x).h\right] + \dots , \\ \operatorname{tr}\left[[\mathbb{1} + \nabla u(x+h)] - [\mathbb{1} + \nabla u(x)]\right] &= \operatorname{Div} u(x+h) - \operatorname{Div} u(x) = \langle \nabla \operatorname{Div} u(x), h\rangle + \dots \\ &= \operatorname{tr}\left[D^2u(x).h\right] + \dots , \\ \operatorname{sym}[[\mathbb{1} + \nabla u(x+h)] - [\mathbb{1} + \nabla u(x)]] &= \operatorname{sym}[D^2u(x).h] + \dots , \\ \operatorname{dev}\operatorname{sym}[[\mathbb{1} + \nabla u(x+h)] - [\mathbb{1} + \nabla u(x)]] &= \operatorname{dev}\operatorname{sym}[D^2u(x).h] + \dots , \\ \operatorname{skew}([\mathbb{1} + \nabla u(x+h)] - [\mathbb{1} + \nabla u(x)]) &= \operatorname{skew}(D^2u(x).h) + \dots , \\ 2\operatorname{axl}[\operatorname{skew}([\mathbb{1} + \nabla u(x+h)] - [\mathbb{1} + \nabla u(x)])] &= 2\operatorname{axl}[\operatorname{skew}(D^2u(x).h)] + \dots , \\ \operatorname{curl} u(x+h) - \operatorname{curl} u(x) &= \nabla \operatorname{curl} u(x).h + \dots = 2\operatorname{axl}[\operatorname{skew}(D^2u(x).h)] + \dots , \\ \nabla \operatorname{curl} u(x).h &= 2\operatorname{axl}[\operatorname{skew}(D^2u(x).h)] . \end{aligned} \tag{5.7}$$

5.3 Spherical integration

We make constantly use of the following simple closed form expressions for integrals over the unit sphere which ca be found, e.g., in [1]

$$\begin{aligned} &\int_{h\in\mathbb{S}^2} \langle X.h, h\rangle^2 d\mathbb{S}^2 = \frac{4\,\pi}{15}\left(2\,\|\operatorname{sym} X\|^2 + \operatorname{tr}[X]^2\right) , \\ &\int_{h\in\mathbb{S}^2} \langle X.h, h\rangle d\mathbb{S}^2 = \frac{4\,\pi}{3}\operatorname{tr}[X] , \qquad \int_{h\in\mathbb{S}^2} \langle h, h\rangle^2 d\mathbb{S}^2 = \int_{h\in\mathbb{S}^2} 1\, d\mathbb{S}^2 = 4\,\pi , \\ &\int_{h\in\mathbb{S}^2} \langle v, h\rangle^2 d\mathbb{S}^2 = \int_{h\in\mathbb{S}^2} \langle (v\otimes v).h, h\rangle d\mathbb{S}^2 = \frac{4\,\pi}{3}\operatorname{tr}[v\otimes v] = \frac{4\,\pi}{3}\,\|v\|^2 . \end{aligned} \tag{5.8}$$

On this basis,

$$\begin{aligned} \int_{\tilde{h}\in\mathbb{S}^2} \langle \nabla \operatorname{curl} u.\tilde{h}, \tilde{h}\rangle^2\, d\mathbb{S}^2 &= \frac{4\,\pi}{15}\,\left(2\,\|\operatorname{sym}\nabla\operatorname{curl} u\|^2 + \operatorname{tr}[\operatorname{sym}\nabla\operatorname{curl} u]^2\right) \\ &= \frac{4\,\pi}{15}\,\left(2\,\|\operatorname{sym}\nabla\operatorname{curl} u\|^2 + (\operatorname{Div}\operatorname{curl} u)^2\right) = \frac{8\,\pi}{15}\,\|\operatorname{sym}\nabla\operatorname{curl} u\|^2 . \end{aligned} \tag{5.9}$$

On the other hand,

$$\begin{aligned} \int_{\tilde{h}\in\mathbb{S}^2} \langle \nabla \operatorname{curl} u.\tilde{h}, \tilde{h}\rangle^2\, d\mathbb{S}^2 &= \int_{\tilde{h}\in\mathbb{S}^2} \langle 2\operatorname{axl}[\operatorname{skew}(D^2u(x).\tilde{h})], \tilde{h}\rangle^2\, d\mathbb{S}^2 = 2\int_{\tilde{h}\in\mathbb{S}^2} \frac{1}{4}\langle \operatorname{skew}(D^2u(x).\tilde{h}), \operatorname{anti}(\tilde{h})\rangle^2\, d\mathbb{S}^2 \\ &= \frac{1}{2}\int_{\tilde{h}\in\mathbb{S}^2} \langle D^2u(x).\tilde{h}, \operatorname{anti}(\tilde{h})\rangle^2\, d\mathbb{S}^2 . \end{aligned} \tag{5.10}$$

Using $(5.8)_3$ we get as well

$$\int_{h\in\mathbb{S}^2} \operatorname{tr}\left[D^2u(x).h\right]^2 d\mathbb{S}^2 = \int_{h\in\mathbb{S}^2} \langle \nabla \operatorname{Div} u(x), h\rangle^2 d\mathbb{S}^2 = \frac{4\,\pi}{3}\, \|\nabla \operatorname{Div} u(x)\|^2\,, \tag{5.11}$$

and

$$\begin{aligned}
&\int_{h\in\mathbb{S}^2} \|\operatorname{skew}[D^2u(x).h]\|^2_{\mathbb{M}^{3\times3}}\, d\mathbb{S}^2 = \int_{h\in\mathbb{S}^2} 2\|\operatorname{axl}[\operatorname{skew}[D^2u(x).h]]\|^2_{\mathbb{R}^3}\, d\mathbb{S}^2 = \int_{h\in\mathbb{S}^2} \frac{1}{2}\|2\operatorname{axl}[\operatorname{skew}[D^2u(x).h]]\|^2_{\mathbb{R}^3}\, d\mathbb{S}^2\\
&= \int_{h\in\mathbb{S}^2} \frac{1}{2}\|\nabla \operatorname{curl} u(x).h\|^2_{\mathbb{R}^3}\, d\mathbb{S}^2 = \int_{h\in\mathbb{S}^2} \frac{1}{2}\langle [\nabla \operatorname{curl} u(x)]^T[\nabla \operatorname{curl} u(x)].h, h\rangle\, d\mathbb{S}^2\\
&= \frac{4\pi}{6}\, \|\nabla \operatorname{curl} u(x)\|^2\,.
\end{aligned} \tag{5.12}$$

Moreover,

$$\int_{h\in\mathbb{S}^2} \|D^2u(x).h\|^2_{\mathbb{M}^{3\times3}}\, d\mathbb{S}^2 = \int_{h\in\mathbb{S}^2} \langle [D^2u(x)]^T[D^2u(x)].h, h\rangle\, d\mathbb{S}^2 = \frac{4\pi}{3}\, \|D^2u(x)\|^2_{\mathbb{R}^{27}}\,. \tag{5.13}$$

Thus, observing that

$$\|D^2u(x).h\|^2_{\mathbb{M}^{3\times3}} = \|\operatorname{dev}\operatorname{sym}[D^2u(x).h]\|^2_{\mathbb{M}^{3\times3}} + \|\operatorname{skew}[D^2u(x).h]\|^2_{\mathbb{M}^{3\times3}} + \frac{1}{3}\operatorname{tr}\left[D^2u(x).h\right]^2, \tag{5.14}$$

we obtain

$$\begin{aligned}
&\int_{h\in\mathbb{S}^2} \|\operatorname{dev}\operatorname{sym}[D^2u(x).h]\|^2_{\mathbb{M}^{3\times3}}\, d\mathbb{S}^2 = \int_{h\in\mathbb{S}^2} \|[D^2u(x).h]\|^2 - \|\operatorname{skew}[D^2u(x).h]\|^2 - \frac{1}{3}\operatorname{tr}\left[D^2u(x).h\right]^2 d\mathbb{S}^2\\
&= \frac{4\pi}{3}\, \|D^2u(x)\|^2_{\mathbb{R}^{27}} - \frac{4\pi}{6}\, \|\nabla \operatorname{curl} u(x)\|^2_{\mathbb{M}^{3\times3}} - \frac{4\,\pi}{9}\, \|\nabla \operatorname{Div} u(x)\|^2\,.
\end{aligned} \tag{5.15}$$

Therefore

$$\begin{aligned}
&\int_{h\in\mathbb{S}^2} \widehat{\mu}\, \|\operatorname{dev}\operatorname{sym}[D^2u(x).h]\|^2_{\mathbb{M}^{3\times3}} + \frac{\widehat{K}}{2}\operatorname{tr}\left[D^2u(x).h\right]^2 d\mathbb{S}^2\\
&= 4\pi\,\widehat{\mu}\, \left(\frac{1}{3}\,\|D^2u(x)\|^2_{\mathbb{R}^{27}} - \frac{1}{6}\,\|\nabla \operatorname{curl} u(x)\|^2\right) + 4\pi\, \left(\frac{\widehat{K}}{2} - \frac{\widehat{\mu}}{3}\right)\, \|\nabla \operatorname{Div} u(x)\|^2\\
&= \frac{4\pi}{3}\,\widehat{\mu}\, \left(\|D^2u(x)\|^2_{\mathbb{R}^{27}} - \frac{1}{2}\,\|\nabla \operatorname{curl} u(x)\|^2\right) + \frac{2\pi\widehat{\lambda}}{3}\|\nabla \operatorname{Div} u(x)\|^2\\
&= \frac{4\pi}{3}\, \left(\widehat{\mu}\, \left(\|D^2u(x)\|^2_{\mathbb{R}^{27}} - \frac{1}{2}\,\|\nabla \operatorname{curl} u(x)\|^2\right) + \frac{\widehat{\lambda}}{2}\|\nabla \operatorname{Div} u(x)\|^2\right)\,.
\end{aligned} \tag{5.16}$$

5.4 Estimate for two-dimensional displacement fields

Here we show that for $u(x,y,z) = (u_1(x,y), u_2(x,y), 0)^T \in C_0^\infty(\Omega, \mathbb{R}^3)$ it holds that

$$\int_\Omega \|D^2u\|^2_{\mathbb{R}^{27}} - \frac{1}{2}\|\nabla \operatorname{curl} u\|^2_{\mathbb{M}^{3\times3}}\, \mathrm{dx} \geq \int_\Omega \|\operatorname{sym}\nabla \operatorname{curl} u\|^2_{\mathbb{M}^{3\times3}}\, \mathrm{dx}\,. \tag{5.17}$$

Proof. . We first observe that $\operatorname{curl} u = (0, 0, u_{2,x} - u_{1,y})^T$ from which follows

$$\begin{aligned}
\nabla \operatorname{curl} u &= \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ (u_{2,x} - u_{1,y})_x & (u_{2,x} - u_{1,y})_y & 0\end{pmatrix}\,,\\
\operatorname{sym}\nabla \operatorname{curl} u &= \frac{1}{2}\begin{pmatrix} 0 & 0 & (u_{2,x} - u_{1,y})_x\\ 0 & 0 & (u_{2,x} - u_{1,y})_y\\ (u_{2,x} - u_{1,y})_x & (u_{2,x} - u_{1,y})_y & 0\end{pmatrix}\,,
\end{aligned} \tag{5.18}$$

and thus pointwise $\|\operatorname{sym}\nabla \operatorname{curl} u\|^2_{\mathbb{M}^{3\times3}} = \frac{1}{2}\|\nabla \operatorname{curl} u\|^2_{\mathbb{M}^{3\times3}}$. The inequality (5.17) is therefore equivalent to

$$\int_\Omega \|D^2u\|^2_{\mathbb{R}^{27}}\, \mathrm{dx} \geq \int_\Omega \|\nabla \operatorname{curl} u\|^2_{\mathbb{M}^{3\times3}}\, \mathrm{dx}\,. \tag{5.19}$$

The second derivatives are

$$\begin{aligned}
D^2u &= \begin{pmatrix} u_{1,xx} & u_{1,xy} & u_{2,xx} & u_{2,xy}\\ u_{1,yx} & u_{1,yy} & u_{2,yx} & u_{2,xy}\end{pmatrix}\,,\\
\|D^2u\|^2 &= u_{1,xx}^2 + u_{1,xy}^2 + u_{2,xx}^2 + u_{2,xy}^2 + u_{1,yx}^2 + u_{1,yy}^2 + u_{2,yx}^2 + u_{2,xy}^2\,,\\
\|\nabla \operatorname{curl} u\|^2 &= u_{2,xx}^2 - 2u_{2,xx}\, u_{1,yx} + u_{1,yx}^2 + u_{2,xy}^2 - 2u_{2,xy}\, u_{1,yy} + u_{1,yy}^2\,,\\
\|D^2u\|^2 - \|\nabla \operatorname{curl} u\|^2 &= u_{1,xx}^2 + u_{1,xy}^2 + u_{2,xy}^2 + u_{2,yy}^2 + 2u_{2,xx}\, u_{1,yx} + 2u_{2,xy}\, u_{1,yy}\,.
\end{aligned} \tag{5.20}$$

Partial integration for functions with compact support and the Theorem of Schwarz for the mixed products give

$$\begin{aligned}
\int_\Omega \|D^2u\|^2 - \|\nabla \operatorname{curl} u\|^2\, dx &= \int_\Omega u_{1,xx}^2 + u_{1,xy}^2 + u_{2,xy}^2 + u_{2,yy}^2 + 2u_{2,xx}\, u_{1,yx} + 2u_{2,xy}\, u_{1,yy}\, \mathrm{dx}\\
&= \int_\Omega u_{1,xx}^2 + u_{1,xy}^2 + u_{2,xy}^2 + u_{2,yy}^2 + 2u_{2,xy}\, u_{1,xx} + 2u_{2,yy}\, u_{1,yx}\, \mathrm{dx} \qquad (5.21)\\
&= \int_\Omega (u_{2,xy}^2 + 2u_{2,xy}\, u_{1,xx} + u_{1,xx}^2) + (u_{2,yy}^2 + 2u_{2,yy}\, u_{1,yx} + u_{1,xy}^2)\, \mathrm{dx}\\
&= \int_\Omega (u_{2,xy} + u_{1,xx})^2 + (u_{2,yy} + u_{1,xy})^2\, \mathrm{dx} = \int_\Omega \|\nabla \operatorname{Div} u\|^2\, \mathrm{dx} \geq 0\,.
\end{aligned}$$

Summarizing we have in the two-dimensional situation

$$\int_\Omega \|D^2u\|^2_{\mathbb{R}^{27}} - \frac{1}{2}\,\|\nabla \operatorname{curl} u\|^2\, \mathrm{dx} = \int_\Omega \|\operatorname{sym}\nabla \operatorname{curl} u\|^2_{\mathbb{M}^{3\times3}} + \|\nabla \operatorname{Div} u\|^2\, \mathrm{dx} + \text{Null-Lagrangian}\,. \tag{5.22}$$