Hierarchical modelling and model adaptivity for gas flow on networks

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Abstract

We are interested in simulation and optimization of gas transport in networks. Different regions of the network may be modelled by different equations. There are three models based on the Euler equations that describe the gas flow in pipelines qualitatively different: a nonlinear model, a semilinear model and a stationary also called algebraic model. For the whole network, adequate initial and boundary values as well as coupling conditions at the junctions are needed. Using adjoint techniques one can specify model error estimators for the simplified models. A strategy to adaptively apply the different models in different regions of the network while maintaining the accuracy of the solution is presented.

1 Introduction

During the last years, there has been intense research in the field of simulation and optimization of gas transport in networks [2–4, 8, 9]. The equations describing the transport of gas in pipelines are based on the Euler equations, a hyperbolic system of nonlinear partial differential equations, mainly consisting of the conservation of mass, momentum and energy. The transient flow of gas may be described appropriately by equations in one space dimension. For the whole network, adequate initial and boundary values as well as coupling conditions at the junctions are needed.

Although solving one-dimensional equations does not pose a challenge, the complexity increases with the size of the network. Thus, we present a hierarchy of models that describes the flow of gas in pipelines qualitatively different: The most detailed model we use consists of the isothermal Euler equations (continuity equation and momentum equation). A common simplification of the momentum equation leads to a semilinear model, which is only valid if the velocity of the gas is much less than the speed of sound, that is, $|v| \ll c$. Further simplifications lead to the steady state model. The different models are introduced in section 2. The modelling of the network as well as the boundary and coupling conditions are presented in section 3.

In order to estimate the model error of the simplified models, that is, of the semilinear and the steady state model with respect to some quantity of interest, one has to solve adjoint systems on the network. For the adjoint equations appropriate coupling conditions are required which are introduced in section 4. There, we also present a strategy, how to decide in which regions of the network which model has to be used to reduce the complexity of the whole problem, whereas the accuracy of the solution is maintained. We give numerical examples of this algorithm that switches adaptively between the models on one pipe in section 5.

2 Model hierarchy

In this section, we introduce a hierarchy of models consisting of three different models. Each model results from the previous one by making further simplifying assumptions [1].

2.1 Nonlinear model

The isothermal Euler equations, which describe the flow of gas, consist of the continuity and the momentum equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0. \tag{1a}$$

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2)}{\partial x} + \frac{\partial p}{\partial x} = -g\rho h' - \frac{\lambda}{2d}\rho |v| v. \tag{1b}$$

Additionally the equation of state for real gases holds:

$$\rho = \frac{p}{z(p,T)RT}. \tag{2}$$

Here, ρ denotes the density, v the velocity of the gas, p the pressure, g the gravity constant, h' the slope of the pipe, λ the friction coefficient, d the diameter of the pipe, R the (special) gas constant, T the temperature of the gas (assumed to be constant) and $z = z(p,T)$ the compressibility factor.

For the sake of simplicity, we assume the pipe to be horizontal and the compressibility factor to be constant. This results in a simplified equation of state with constant speed of sound c:

$$\rho = \frac{p}{c^2}, \qquad c = \sqrt{RT}.$$

Since the mass flow M can be traced back to the flux under standard conditions ($M = \rho v A = \rho_0 q$) the system can be rewritten in the following way:

$$p_t + \frac{\rho_0 c^2}{A} q_x = 0 \tag{3a}$$

$$q_t + \frac{A}{\rho_0} p_x + \frac{\rho_0 c^2}{A} \left(\frac{q^2}{p} \right)_x = -\frac{\lambda \rho_0 c^2 |q| q}{2dAp} \tag{3b}$$

or in a more compact form

$$u_t + f(u)_x = \psi(u) \tag{4}$$

with

$$u = \begin{pmatrix} p \\ q \end{pmatrix}, \qquad f(u) = \begin{pmatrix} \frac{\rho_0 c^2}{A} q \\ \frac{A}{\rho_0} p + \frac{\rho_0 c^2}{A} \frac{q^2}{p} \end{pmatrix}, \qquad \psi(u) = \begin{pmatrix} 0 \\ -\frac{\lambda \rho_0 c^2 |q| q}{2dAp} \end{pmatrix}.$$

Here, ρ_0 denotes the density under standard conditions (1 atm air pressure, temperature of 0 °C), A the cross-sectional area of the pipe and q the flux under standard conditions.

For a scalar conservation law, information always travels with speed $f'(u)$ (also called characteristic speed). For a system of conservation laws, one has to compute the eigenvalues of the Jacobian matrix of f. Thus we get

$$\mathbf{A}(u) = f'(u) = \begin{pmatrix} 0 & \frac{\rho_0 c^2}{A} \\ \frac{A}{\rho_0} - \frac{\rho_0 c^2}{A}\frac{q^2}{p^2} & \frac{2\rho_0 c^2}{A}\frac{q}{p} \end{pmatrix}$$

with eigenvalues

$$\lambda_{1/2}(u) = \frac{\rho_0 c^2}{A}\frac{q}{p} \mp c = v \mp c.$$

Therefore, for subsonic flow, the characteristics travel in opposite directions with the characteristic speeds $\lambda_{1/2}$ depending on the velocity of the gas.

2.2 Semilinear model

We can rewrite the term of the spatial derivative in the momentum equation (1b) as follows:

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial}{\partial x}\left(p\left(1 + \frac{v^2}{c^2}\right)\right) = -\frac{\lambda\rho_0 c^2 |q| q}{2dAp}.$$

If the velocity v of the gas is much less than the speed of sound, we can neglect $\frac{v^2}{c^2}$. Together with the equation of state as above, this yields a semilinear model

$$u_t + \mathbf{A}u_x = \psi(u) \tag{5}$$

with

$$u = \begin{pmatrix} p \\ q \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & \frac{c^2\rho_0}{A} \\ \frac{A}{\rho_0} & 0 \end{pmatrix} \text{ and } \psi(u) = \begin{pmatrix} 0 \\ -\frac{\lambda\rho_0 c^2 |q| q}{2dAp} \end{pmatrix}.$$

For the semilinear model we can also specify characteristic speeds. The eigenvalues of the matrix $\mathbf{A}$ are $\lambda_{1/2} = \mp c$. Thus, information always travels in both directions with sonic speed.

2.3 Algebraic model

A further simplification leads to the stationary model: Setting the time derivatives in (5) to zero results in

$$\begin{aligned} q_x &= 0 \\ \frac{A}{\rho_0} p_x &= -\frac{\lambda\rho_0 c^2 |q| q}{2dAp}. \end{aligned}$$

Thus, q is constant in space and the exact solution for p is

$$p(x) = \sqrt{p(x_0)^2 + \frac{\lambda\rho_0^2 c^2 |q| q}{dA^2}(x_0 - x)}\,.$$

Here, $p(x_0)$ denotes the pressure at an arbitrary point $x_0 \in [0, L]$. Setting $x_0 = 0$, that is, $p(x_0) = p(0) = p_{in}$ at the inbound of the pipe, and $x = L$, that is, $p(x) = p(L) = p_{out}$ at the end of the pipe, yields the algebraic model [10]

$$p_{out} = \sqrt{p_{in}^2 - \frac{\lambda \rho_0^2 c^2 |q| q}{dA^2} L}. \tag{6}$$

For the other two models, we computed characteristic speeds at which information propagates in different directions. Since this model is stationary, information given at any place instantaneously influences all other points.

3 Modelling of the network

We now want to describe the gas flow on networked pipelines. For this purpose, we model the network as a directed graph $\mathcal{G} = (\mathcal{J}, \mathcal{V})$ with edges $\mathcal{J}$ (pipes) and vertices $\mathcal{V}$ (nodes, branching points). Each edge $j \in \mathcal{J}$ is defined as an interval (x_j^a, x_j^b) with a direction from x_j^a to x_j^b. Of course, all intervals are disjoint. Then, for any inner node v, we can define two sets of edges. Let the set of ingoing pipes be denoted by δ_v^-, that is, the set of any edge $j \in \mathcal{J}$ with endpoint x_j^b being adjacent to v. Then, analogously, δ_v^+ denotes the set of outgoing pipes (see figure 1).

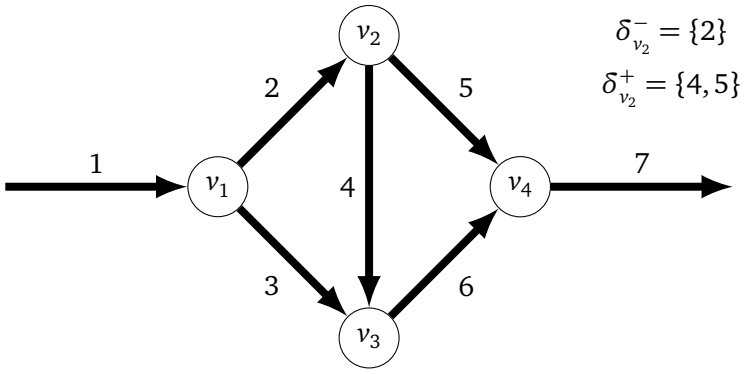

Figure 1: A small network; the ingoing pipes of node v_2 are $\delta_{v_2}^- = \{2\}$ and the set of outgoing pipes is $\delta_{v_2}^+ = \{4,5\}$

Inside each pipe one of the models described above holds. In order to obtain a unique solution, we have to pose coupling conditions at the inner nodes of the network as well as boundary conditions at the sources and sinks.

3.1 Coupling conditions

A first coupling condition is the conservation of mass at each inner node. Let $v \in \mathcal{V}$ be a node with ingoing pipes $j \in \delta_v^-$ and outgoing pipes $i \in \delta_v^+$. Then, Kirchhoff's law (conservation of mass) yields

$$\sum_{j \in \delta_v^-} q(x_j^b, t) = \sum_{i \in \delta_v^+} q(x_i^a, t) \quad \forall t \geq 0. \tag{7}$$

This law is also called Rankine-Hugoniot condition at the node.

Next, we need further coupling conditions and there are several possibilities. The most common condition used is the equality of pressure at the node v as pointed out in [3]:

$$p(x_i^b,t) = p(x_j^a,t), \quad \forall i \in \delta_v^+, j \in \delta_v^-. \tag{8}$$

For the nonlinear model one could also use equality of the sum of pressure and momentum at the node v:

$$\frac{A_i}{\rho_0} p(x_i^b,t) + \frac{\rho_0 c^2}{A_i} \frac{\left(q(x_i^b,t)\right)^2}{p(x_i^b,t)} = \frac{A_j}{\rho_0} p(x_j^a,t) + \frac{\rho_0 c^2}{A_j} \frac{\left(q(x_j^a,t)\right)^2}{p(x_j^a,t)} \quad \forall i \in \delta_v^+, j \in \delta_v^-. \tag{9}$$

Here, A_i and A_j denote the cross-sectional area of pipe i and j respectively.

In case of a bend of two pipes i and j, a so-called "minor loss" of pressure can be observed:

$$\frac{A_i}{\rho_0} p(x_i^b,t) + \frac{\rho_0 c^2}{A_i} \frac{\left(q(x_i^b,t)\right)^2}{p(x_i^b,t)} - k \cdot q(x_i^b,t) = \frac{A_j}{\rho_0} p(x_j^a,t) + \frac{\rho_0 c^2}{A_j} \frac{\left(q(x_j^a,t)\right)^2}{p(x_j^a,t)}. \tag{10}$$

The factor k is empirical and may depend on the flow as well as the angle of the bend.

Since in practice the velocity of the gas is rather small, the momentum is negligible compared to the pressure. Thus, we will use conditions (7) and (8) as coupling conditions.

3.2 Boundary conditions

Let $\mathcal{J}_{in}$ denote the set of ingoing pipes of the network, i.e. the pipes connecting the sources with the network and let $\mathcal{J}_{out}$ denote the set of outgoing pipelines connected with sinks.

Since for subsonic flow the characteristics of the nonlinear model propagate in different directions and for the semilinear model the characteristics always propagate in reverse directions, one can prescribe the characteristic variables only on opposing sides of a pipe. Thus, there are some limitations on the boundary conditions of the edges.

One possibility is to specify the pressure p at one end of the pipe and the flux q at the other. So, we usually prescribe the pressure at $x_j^a, j \in \mathcal{J}_{in}$ and the flux at $x_j^b, j \in \mathcal{J}_{out}$, i. e. the pressure at the sources and the flux at the sinks.

3.3 Gas flow on the network

We can now describe the flow of gas on the network. With the notations $\Omega = \bigcup_{j \in \mathcal{J}} [x_j^a, x_j^b]$ and $Q := \Omega \times (0,T)$, the equations for the nonlinear model read as follows:

$$\begin{aligned}
&u_t + f(u)_x = \psi(u) && \text{in } Q \\
&p(x,0) = p_0(x) && \text{in } \Omega \\
&q(x,0) = q_0(x) && \text{in } \Omega \\
&p(x_i^a,t) = w_i(t) && i \in \mathcal{J}_{in}, t \in (0,T) \\
&q(x_i^b,t) = v_i(t) && i \in \mathcal{J}_{out}, t \in (0,T) \\
&p(x_i^b,t) = p(x_j^a,t) && \forall v \in \mathcal{V}, i \in \delta_v^-, j \in \delta_v^+, t \in (0,T) \\
&\sum_{i \in \delta_v^-} q(x_i^b,t) = \sum_{i \in \delta_v^+} q(x_i^a,t) && \forall v \in \mathcal{V}, t \in (0,T) \\
&w_i(t) > 0 && i \in \mathcal{J}_{in}, t \in (0,T).
\end{aligned} \tag{11}$$

For the semilinear model, the equations are analogous to (11) with the corresponding PDE in the first line. The boundary conditions $p(x_i^a,t), i \in \mathcal{J}_{in}, t \in (0,T)$ and $q(x_i^b,t), i \in \mathcal{J}_{out}, t \in (0,T)$ are determined by control variables/functions $w_i(t)$ and $v_i(t)$. Since the flux at the sinks is given by the consumers, the variable that can be controlled by us will only be the pressure at the sources.

For the algebraic model, one has a similar structure. At each time step the algebraic equation (6) holds at every pipe as well as the coupling and boundary conditions at the nodes. For arbitrary $t \in (0,T)$, the system to be solved is

$$\begin{aligned}
&p(x,t) = \sqrt{\left(p(x_i^a,t)\right)^2 - \tfrac{\lambda \rho_0^2 c^2 |q|q}{d_i A_i^2}(x - x_i^a)} && i \in \mathcal{J}, x \in (x_i^a, x_i^b] \\
&p(x_i^a,t) = w_i(t) && i \in \mathcal{J}_{in} \\
&q(x_i^b,t) = v_i(t) && i \in \mathcal{J}_{out} \\
&p(x_i^b,t) = p(x_j^a,t) && \forall v \in \mathcal{V}, i \in \delta_v^-, j \in \delta_v^+ \\
&\sum_{i \in \delta_v^-} q(x_i^b,t) = \sum_{i \in \delta_v^+} q(x_i^a,t) && \forall v \in \mathcal{V} \\
&w_i(t) > 0 && i \in \mathcal{J}_{in}.
\end{aligned} \tag{12}$$

Again, the boundary conditions $p(x_i^a), i \in \mathcal{J}_{in}$ and $q(x_i^b), i \in \mathcal{J}_{out}$ are determined by control variables/ functions $w_i(t_n)$ and $v_i(t_n)$ of which we only control the pressure at the sources.

4 Adjoint equations on the network

A possibility to achieve a compromise between the accuracy of the model and the computational costs is to use the more complex model only when necessary. Using the solution of adjoint equations as done in [5, 6] we deduce a model error estimator to measure the influence of the model on a user-defined output functional.

Let the functional M be of the form

$$M(u) = \int_Q N(u)\,\mathrm{d}t\,\mathrm{d}x + \sum_{i \in \mathcal{J}_{in}} \int_0^T N_{x_i^a}(q)\,\mathrm{d}t + \sum_{i \in \mathcal{J}_{out}} \int_0^T N_{x_i^b}(p)\,\mathrm{d}t + \int_\Omega N_T(u)\,\mathrm{d}x. \tag{13}$$

As pointed out in [6], we only need to solve the dual problem of the simplified models in order to obtain a first order error estimator. Let $\xi = \binom{\xi_1}{\xi_2}$ be the solution of the dual problem of the semilinear model (5) or the algebraic model (6) with respect to the functional M.

For a given solution of the semilinear equations $u^* = \binom{p^*}{q^*}$, the adjoint system on the network reads as follows:

$$\begin{aligned}
&\xi_t + \mathbf{A}^T \xi_x = -\partial_u \psi(u^*)^T \xi - \partial_u N(u^*)^T && \text{in } Q \\
&\xi(\cdot,T) = \partial_u N_T(u^*(\cdot,T))^T && \text{in } \Omega \\
&\xi_1(x_i^a,t) = -\tfrac{A_i}{\rho_0 c^2} \partial_q N_{x_i^a}(q^*(x_i^a,t)) && i \in \mathcal{J}_{in}, t \in (0,T) \\
&\xi_2(x_i^b,t) = \tfrac{\rho_0}{A_i} \partial_p N_{x_i^b}(p^*(x_i^b,t)) && i \in \mathcal{J}_{out}, t \in (0,T).
\end{aligned} \tag{14}$$

The adjoint system for the algebraic equations is again similar to that of the semilinear model. For arbitrary $t \in (0,t)$ the system to be solved is

$$\begin{aligned} \mathbf{A}^T \xi_x &= -\partial_u \psi(u^*)^T \xi - \partial_u N(u^*)^T && \text{in } \Omega \\ \xi_1(x_i^a, t) &= -\tfrac{A_i}{\rho_0 c^2} \partial_q N_{x_i^a}(q^*(x_i^a, t)) && i \in \mathcal{J}_{in} \\ \xi_2(x_i^b, t) &= \tfrac{\rho_0}{A_i} \partial_p N_{x_i^b}(p^*(x_i^b, t)) && i \in \mathcal{J}_{out}. \end{aligned} \tag{15}$$

With these equations, the functional M can only be of the form

$$M(u) = \int_Q N(u)\,\mathrm{d}t\,\mathrm{d}x + \sum_{i \in \mathcal{J}_{in}} \int_0^T N_{x_i^a}(q)\,\mathrm{d}t + \sum_{i \in \mathcal{J}_{out}} \int_0^T N_{x_i^b}(p)\,\mathrm{d}t,$$

that is, one cannot measure the influence of the algebraic model at the final time T.

For the adjoint systems, one also has to specify coupling conditions. Conservation of mass and equality of pressure at the node v yield for the adjoint variables:

$$\begin{aligned} \frac{1}{A_i} \xi_1(x_i^b, t) &= \frac{1}{A_j} \xi_1(x_j^a, t) && i \in \delta_v^-, j \in \delta_v^+, t \in (0,T) \\ \sum_{i \in \delta_v^-} A_i \xi_2(x_i^b, t) &= \sum_{j \in \delta_v^+} A_j \xi_2(x_j^a, t) && t \in (0,T) \end{aligned}$$

4.1 Error estimators

We now use the adjoint equations to assess the simplified models with respect to the quantity of interest. Let $u = \binom{p}{q}$ be the solution of the nonlinear model (4) and $u^h = \binom{p^h}{q^h}$ the discretized solution of the semilinear model (5). Then the difference between the output functional of u, $M(u)$, and $M(u^h)$ is

$$\begin{aligned} M(u) - M(u^h) = &\int_Q N(u) - N(u^h)\,\mathrm{d}t\,\mathrm{d}x + \sum_{i \in \mathcal{J}_{in}} \int_0^T N_{x_i^a}(q) - N_{x_i^a}(q^h)\,\mathrm{d}t \\ &+ \sum_{i \in \mathcal{J}_{out}} \int_0^T N_{x_i^b}(p) - N_{x_i^b}(p^h)\,\mathrm{d}t + \int_\Omega N_T(u) - N_T(u^h)\,\mathrm{d}x. \end{aligned}$$

Taylor expansion of first order yields

$$\begin{aligned} = &\int_Q \partial_u N(u^h)(u - u^h)\,\mathrm{d}t\,\mathrm{d}x + \sum_{i \in \mathcal{J}_{in}} \int_0^T \partial_q N_{x_i^a}(q^h)(q - q^h)\,\mathrm{d}t \\ &+ \sum_{i \in \mathcal{J}_{out}} \int_0^T \partial_p N_{x_i^b}(p^h)(p - p^h)\,\mathrm{d}t + \int_\Omega \partial_u N_T(u^h)(u - u^h)\,\mathrm{d}x + H.O.T. \end{aligned}$$

with $H.O.T.$ being higher order terms. Inserting the solution ξ of the adjoint system (14) we get a first order error estimator for the model and the discretization error respectively as in [6]:

$$M(u) - M(u^h) \approx \eta_m + \eta_h \tag{16}$$

with the estimators η_m and η_h as follows:

$$\eta_m^{nl-sl} = \int_Q -\xi^T \begin{pmatrix} 0 \\ \frac{\rho_0 c^2 (q^h)^2}{A p^h} \end{pmatrix}_x \mathrm{d}x\,\mathrm{d}t \tag{17}$$

$$\eta_h^{nl-sl} = \int_Q \xi^T \left(-u_t^h - \mathbf{A} u_x^h + \psi(u^h) \right) \mathrm{d}x\,\mathrm{d}t. \tag{18}$$

Since the algebraic model can be solved exactly, the discretization error disappears and one only gets an estimator for the model error

$$\eta_m^{sl-alg} = \int_Q -\xi^T \begin{pmatrix} p \\ q \end{pmatrix}_t \mathrm{d}x\,\mathrm{d}t \tag{19}$$

with ξ being the solution of the adjoint equations either of the semilinear model (14) or of the algebraic model (15). Here, $u = \binom{p}{q}$ denotes the solution of the stationary model (6).

Assuming that the discretization error estimator is much smaller than the model error estimator, that is, $\eta_h \ll \eta_m$, one can use η_m to compute the time intervals in which the model error is above or below a given tolerance. If it is above, one has to use a more complex model in the hierarchy.

Here, for the stationary model, a problem arises. The model error estimator of the algebraic model vanishes from that time on, when the boundary values become stationary, although dynamics may remain in the pipe. Thus, the point of "shifting down" cannot be computed this way. Therefore, a strategy has been developed to get around this problem.

4.2 Adaptive switching strategy

The hierarchy consists of three models. The most complex model is the nonlinear model followed by the linear model. The most simple model used is the algebraic/stationary model (see figure 2).

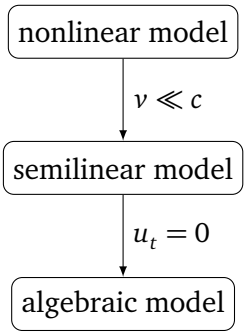

Figure 2: model hierarchy

In order to avoid the mentioned problem of the error estimator for the algebraic model, we divide the time interval $(0, T)$ into equal subintervals (T_{k-1}, T_k), $k = 1, \ldots, N_B$, with $T_0 = 0$ and $T_{N_B} = T$. Thus, we can split up the computational domain $Q = \Omega \times (0, T)$ into N_B blocks $Q_k = \Omega \times (T_{k-1}, T_k)$, $k = 1, \ldots, N_B$, of equal size (see figure 3).

We start with simulating the first block Q_1. Each pipe is assigned to one of the three models. Then, we solve the corresponding adjoint system in order to estimate the model error using (17) and (19)

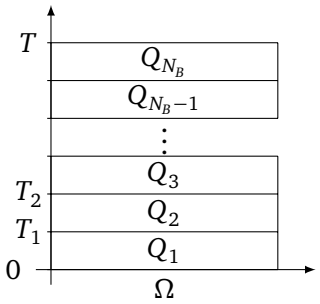

Figure 3: Partition of the computational domain

respectively. The model error estimator on Q_1 can now be computed for each pipe separately. For the semilinear case (17) this reads as follows.

$$\eta_m = \sum_{k=1}^{N_B} \int_{Q_k} -\xi^T \begin{pmatrix} 0 \\ \frac{\rho_0 c^2 (q^h)^2}{A p^h} \end{pmatrix}_x \mathrm{d}x\,\mathrm{d}t = \sum_{k=1}^{N_B} \sum_{j\in\mathcal{J}} \int_{T_{k-1}}^{T_k} \int_{x_j^a}^{x_j^b} -\xi^T \begin{pmatrix} 0 \\ \frac{\rho_0 c^2 (q^h)^2}{A p^h} \end{pmatrix}_x \mathrm{d}x\,\mathrm{d}t =: \sum_{k=1}^{N_B} \sum_{j\in\mathcal{J}} \eta_m(k,j)$$

with the "local" estimators $\eta_m(k,j)$.

Given a tolerance `TOL` one can decide in which pipe the model used is appropriate and in which it is not. We want to accept the model if the relative deviation of the simpler model u^h from the exact solution of the more complex model u is below `TOL`:

$$\frac{\left|M(u) - M(u^h)\right|}{\left|M(u^h)\right|} \leq \texttt{TOL}.$$

Provided that the discretization error is nonsignificant compared to the model error we can approximate $\left|M(u) - M(u^h)\right|$ by $\left|\eta_m\right|$ which yields

$$\left|\eta_m\right| \leq \texttt{TOL} \left|M(u^h)\right|. \tag{20}$$

Just like the error estimator η_m we can evaluate the target functional M at every pipe $j \in \mathcal{J}$ and every time interval (T_{k-1}, T_k), $k = 1, \ldots, N_B$ individually, giving $M_{k,j}$:

$$M(u^h) = \sum_{k=1}^{N_B} \sum_{j\in\mathcal{J}} M_{k,j}(u^h).$$

Thus, for inequality (20) to hold, it suffices to claim

$$\left|\eta_m(k,j)\right| \leq \texttt{TOL} \left|M_{k,j}(u^h)\right|, \quad \forall k \in \{1,\ldots,N_B\}, j \in \mathcal{J}. \tag{21}$$

If any of the estimators $\eta_m(k,j)$ violates the inequality (21), the computation of this time interval has to be repeated and the models used in these pipes have to be exchanged by a more complex model.

The pipes of which the estimators fulfil the inequality (21) may keep the model until the next time interval. If (21) is satisfied for all pipes in the current time interval, the block Q_1 is accepted and we may continue with the next block. Potentially, a more simple model may be used in some pipes.

5 Numerical results

We give an example of the algorithm for one pipe. The models are as above the non-linear, semilinear and algebraic model. The quantity of interest used is:

$$M(\binom{p}{q}) = \int_Q p \, \mathrm{d}x \, \mathrm{d}t.$$

The simulation time totals $T = 1000\text{s}$ with time step size T/N_t and $N_t = 8000$, the length of the pipe is $L = 17\text{km}$ with spatial step size L/N_x and $N_x = 200$.

In this setting, the pressure remains constant at the ingoing end of the pipe (69.5 bar). The flux at the outflow starts with $250\frac{\text{m}^3}{\text{s}}$ and increases in the time between 200s and 250s up to $350\frac{\text{m}^3}{\text{s}}$. Thus, if we start with the algebraic model, presumably we have to switch at that time to the semilinear model.

We computed a reference solution with the nonlinear model and a very fine discretization ($N_t = 32000$, $N_x = 800$) using Finite Volume Methods [7]. The reference solution is shown in figure 4.

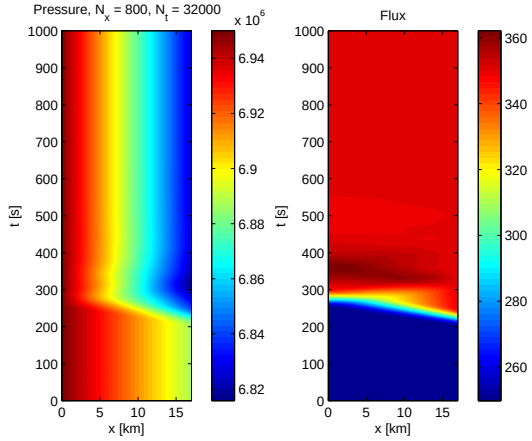

Figure 4: Reference solution

The left picture shows the pressure in the pipe, which corresponds to the x-axis, at every time t. One can see that the pressure remains constant at the left hand side of the pipe but varies inside the pipe as the flux (right picture) increases at the outflow ($x = L$) of the pipe.

We choose $\texttt{TOL} = 10^{-4}$ and to simplify matters, we use an approximation of the quantity of interest M

$$M_{k,1}(u^h) \approx \frac{T\,L}{N_B} p(L, 0)$$

which means that relative influence of the estimated model error on the quantity of interest restricted to one computation block is at most 0.01%.

Figure 5 shows the adaptive solution with the semilinear and the algebraic model. The blocks are visualized as black lines, the characters refer to the models: A for the algebraic model, L for the semilinear model. Starting with the algebraic model the algorithm switches to the semilinear model at that time when the boundary conditions stop being stationary. Not until the variables reach an almost steady state, the algorithm changes down to the algebraic model.

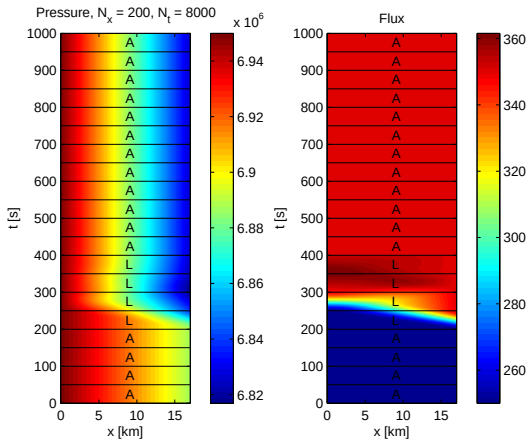

Figure 5: Adaptive solution

Now, we are interested in a comparison with the reference solution. For the results of the algebraic model, the semilinear model and the adaptive algorithm, we compute the maximum relative deviation in the pipe from the reference solution in time. This is shown in figure 6.

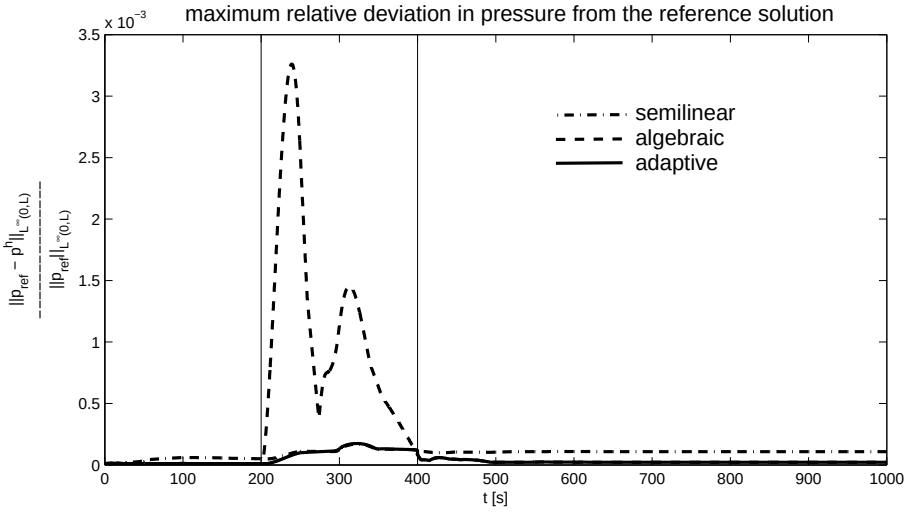

Figure 6: maximum relative deviation in the pipe from the reference solution per time step

6 Summary

We introduced a model hierarchy for the simulation of gas transport in networked pipelines. This hierarchy consists of a nonlinear and a semilinear system of hyperbolic partial differential equations and of an algebraic steady state model. We discussed coupling and boundary conditions for the wellposedness of the whole system. For the network, adjoint equations as well as adjoint coupling conditions were given that allow us to valuate the different models with respect to a quantity of interest. An algorithm was developed that switches adaptively between the three models using model error estimators deduced from the adjoint systems. A numerical example was given for one pipe. The results presented show that an adaptive coupling of the different models can achieve a certain accuracy while the computational time decreases. In the future we will implement the algorithm for arbitrary networks.

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Bibliography

[1] Pia Bales. Hierarchische Modellierung der Eulerschen Flussgleichungen in der Gasdynamik. Diplomarbeit, Technische Universität Darmstadt, Fachbereich Mathematik, Arbeitsgruppe „Numerik partieller Differentialgleichungen", Darmstadt, July 2005.

[2] Mapundi Banda and Michael Herty. Multiscale modeling for gas flow in pipe networks, August 2006.

[3] Mapundi Banda, Michael Herty, and Axel Klar. Coupling conditions for gas networks governed by the isothermal euler equations. *Networks and Heterogeneous Media*, 1(2):295–314, June 2006.

[4] Mapundi Banda, Michael Herty, and Axel Klar. Gas flow in pipeline networks. *Networks and Heterogeneous Media*, 1(1):41–56, March 2006.

[5] Roland Becker and Rolf Rannacher. An optimal control approach to a posteriori error estimation in finite element methods. *Acta numerica*, 10:1–102, 2001.

[6] M. Braack and A. Ern. A posteriori control of modeling errors and discretization errors. *SIAM J. Multiscale Modeling and Simulation*, 2003.

[7] Randall J. Le Veque. *Finite Volume Methods for Hyperbolic Problems*. Cambridge University Press, 2002.

[8] A. Martin, M. Möller, and S. Moritz. Mixed integer models for the stationary case of gas network optimization. *Mathematical Programming*, 105:563–582, 2006.

[9] S. Moritz. *A Mixed Integer Approach for the Transient Case of Gas Network Optimization*. PhD thesis, TU Darmstadt, 2006.

[10] Erwin Sekirnjak. Transiente Technische Optimierung. Entwurf, PSI AG, November 2000.