Existence, uniqueness and stability in linear Cosserat elasticity for weakest curvature conditions

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Abstract

We investigate the weakest possible constitutive assumptions on the curvature energy in linear Cosserat models still providing for existence, uniqueness and stability. The assumed curvature energy is $\mu L_c^2 \, \|\operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \overline{A}\|^2$ where $\operatorname{axl} \overline{A}$ is the axial vector of the skewsymmetric microrotation $A \in \mathfrak{so}(3)$ and dev is the orthogonal projection on the Lie-algebra $\mathfrak{sl}(3)$ of trace free matrices. The proposed Cosserat parameter values coincide with values adopted in the experimental literature by Lakes [25, 27]. It is observed that unphysical stiffening for small samples is avoided in torsion and bending while size effects are still present. The number of Cosserat parameters is reduced from six to four. One Cosserat coupling parameter $\mu_c > 0$ and only one length scale parameter $L_c > 0$. Use is made of a new coercive inequality for conformal Killing vectorfields. An interesting point is that no (controversial) essential boundary conditions on the microrotations need to be specified; thus avoiding boundary layer effects. Since the curvature energy is the weakest possible consistent with non-negativity of the energy, it seems that the Cosserat couple modulus $\mu_c > 0$ remains a material parameter independent of the sample size which is impossible for stronger curvature expressions.

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1 Introduction

We establish well-posedness of the linear elastic Cosserat model in parameter ranges hitherto not considered, extending and precising previous work of the second author [31].

General continuum models involving **independent rotations** have been introduced by the Cosserat brothers [4] at the beginning of the last century. Their originally nonlinear, geometrically exact development has been largely forgotten for decades only to be rediscovered in a restricted linearized setting in the early sixties [12, 10, 36, 37, 17, 29, 35, 38]. Since then, the original Cosserat concept has been generalized in various directions, notably by Eringen and his coworkers who extended the Cosserat concept to include also microinertia effects and to rename it subsequently into **micropolar theory**. For an overview of these so called **microcontinuum** theories we refer to [11, 9, 3, 2, 28, 34].

The Cosserat model includes in a natural way **size effects**, i.e. small samples behave comparatively stiffer than large samples. These effects have recently received new attention in conjunction with nano-devices. The mathematical analysis establishing well-posedness for the infinitesimal strain, Cosserat elastic solid is presented in [20, 7, 18, 15, 16] and in [23, 21, 22] for so called linear microstretch models. This analysis has always been based on the **uniform positivity** of the free energy of the linear elastic Cosserat solid.

The second author has extended the existence results for both the Cosserat model and the more general micromorphic models to the geometrically exact, finite-strain case, see e.g. [33, 30, 32].

The important problem of the determination of Cosserat material parameters for **continuous solids** with random microstructure is still an open problem. Usually, a series of experiments with specimens of different slenderness is performed in order to determine the Cosserat parameters [13, 27]. By using the traditional curvature energy complying with pointwise definiteness, one observes, however, an **unphysical unbounded stiffening** behavior for slender specimens which seems to make it impossible to arrive at consistent values for the Cosserat parameters: the value for the parameters will depend strongly on the smallest investigated specimen size. This inconsistency may be in part responsible for the fact that 1. (linear) Cosserat parameters for continuous solids have never gained general acceptance even in the "Cosserat community" and 2. that the linear elastic Cosserat model has never been really accepted by a majority of applied scientists as a useful model to describe size effects in continuous solids.

As a possible way out of this problem we propose to use a weaker curvature energy of the type

$$W_{\mathrm{curv}}(\nabla \operatorname{axl} \overline{A}) = \mu\, L_c^2\, \| \operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \overline{A} \|^2 \tag{1.1}$$

which is not pointwise positive. But now it is not at all clear that a suitable mathematical model can be built with this energy. This is the question we address here.

This contribution is then organized as follows: first, we recall the linear elastic static isotropic Cosserat model in variational form and re-derive the necessary conditions for non-negativity of the energy. Then we focus on weaker conditions than the uniform positivity of the strain and curvature energy which still lead to a well-posed boundary value problem. In the proof, which is based on the direct methods of the calculus of variations, we also re-derive a new coercive inequality for vector fields in $\mathbb{R}^3$. The decisive new observation is that pointwise positivity is not really necessary for existence and stability. Rather, only after integration over the domain Ω uniform convexity holds true. This mirrors the case of linear elasticity, which itself is not pointwise positive in the displacement gradient but gets so only after integration and use of Korn's inequality. We believe that the usually assumed pointwise positivity of the Cosserat curvature energy is responsible for the fact that material parameters for the Cosserat solid have not been successfully determined. Thus, relaxing the curvature energy might allow for a new chance of parameter determination, notably of the Cosserat couple modulus μ_c. Our notation is found in the appendix.

2 The linear elastic isotropic Cosserat model revisited

This section does not contain any new results, rather it serves to accommodate the widespread notations used in Cosserat elasticity and to introduce the problem.

2.1 The linear elastic Cosserat model in variational form

For the **displacement** $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ and the **skew-symmetric infinitesimal microrotation** $\overline{A} : \Omega \subset \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ we consider the **two-field** minimization problem

$$I(u, \overline{A}) = \int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon}) + W_{\mathrm{curv}}(\nabla \operatorname{axl} \overline{A}) - \langle f, u\rangle - \langle \overline{M}, \overline{A}\rangle \,\mathrm{dx} \qquad (2.1)$$
$$- \int_{\Gamma_S} \langle f_S, u\rangle - \langle \overline{M}_S, \overline{A}\rangle \,\mathrm{dS} \mapsto \quad \min.\ \text{w.r.t. } (u, \overline{A}),$$

under the following constitutive requirements and boundary conditions[1]

$$\begin{aligned}
\overline{\varepsilon} &= \nabla u - \overline{A}, \quad u_{|\Gamma} = u_{\mathrm{d}}\,, \\
W_{\mathrm{mp}}(\overline{\varepsilon}) &= \mu \,\|\operatorname{sym}\overline{\varepsilon}\|^2 + \mu_c \,\|\operatorname{skew}\overline{\varepsilon}\|^2 + \frac{\lambda}{2}\operatorname{tr}[\operatorname{sym}\overline{\varepsilon}]^2 \qquad \textbf{strain energy} \\
&= \mu \,\|\operatorname{sym}\nabla u\|^2 + \mu_c \,\|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda}{2}\operatorname{tr}[\operatorname{sym}\nabla u]^2 \qquad (2.3)\\
&= \mu \,\|\operatorname{dev}\operatorname{sym}\nabla u\|^2 + \mu_c \,\|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{2\mu + 3\lambda}{6}\operatorname{tr}[\operatorname{sym}\nabla u]^2 \\
&= \mu \,\|\operatorname{sym}\nabla u\|^2 + \frac{\mu_c}{2}\,\|\operatorname{curl} u - 2\operatorname{axl}\overline{A}\|^2_{\mathbb{R}^3} + \frac{\lambda}{2}\,(\operatorname{Div} u)^2\ , \\
\phi &:= \operatorname{axl}\overline{A} \in \mathbb{R}^3, \quad \overline{\mathfrak{k}} = \nabla\phi, \quad \|\operatorname{curl}\phi\|^2_{\mathbb{R}^3} = 4\|\operatorname{axl}\operatorname{skew}\nabla\phi\|^2_{\mathbb{R}^3} = 2\|\operatorname{skew}\nabla\phi\|^2_{\mathbb{M}^{3\times 3}}\,, \\
W_{\mathrm{curv}}(\nabla\phi) &= \frac{\gamma+\beta}{2}\|\operatorname{sym}\nabla\phi\|^2 + \frac{\gamma-\beta}{2}\|\operatorname{skew}\nabla\phi\|^2 + \frac{\alpha}{2}\operatorname{tr}[\nabla\phi]^2 \qquad \textbf{curvature energy} \\
&= \frac{\gamma+\beta}{2}\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2 + \frac{\gamma-\beta}{2}\|\operatorname{skew}\nabla\phi\|^2 + \frac{3\alpha + (\beta+\gamma)}{6}\operatorname{tr}[\nabla\phi]^2 \\
&= \frac{\gamma}{2}\|\nabla\phi\|^2 + \frac{\beta}{2}\langle\nabla\phi, \nabla\phi^T\rangle + \frac{\alpha}{2}\operatorname{tr}[\nabla\phi]^2 \\
&= \frac{\gamma+\beta}{2}\|\operatorname{sym}\nabla\phi\|^2 + \frac{\gamma-\beta}{4}\|\operatorname{curl}\phi\|^2_{\mathbb{R}^3} + \frac{\alpha}{2}\,(\operatorname{Div}\phi)^2\ .
\end{aligned}$$

Here, $f, \overline{M}$ are volume force and volume couples, respectively; $f_s, \overline{M}_S$ are surface tractions and surface couples at $\Gamma_S \subset \partial\Omega$, respectively, while u_{d} are Dirichlet boundary conditions for displacement at $\Gamma \subset \partial\Omega$.[2] The strain energy W_{mp} and the curvature energy W_{curv} are the most general isotropic, centro-symmetric quadratic forms in the **non-symmetric strain tensor** $\overline{\varepsilon} = \nabla u - \overline{A}$ and the **micropolar curvature tensor** $\overline{\mathfrak{k}} = \nabla \operatorname{axl}\overline{A}$ (curvature-twist tensor). The parameters μ, λ[MPa] are the classical Lamé moduli and α, β, γ are additional micropolar moduli with dimension $[\mathrm{Pa}\cdot\mathrm{m}^2] = [\mathrm{N}]$ of a force. It is usually clearer to write $\alpha, \beta, \gamma \sim \mu\, L_c^2\alpha', \mu\, L_c^2\beta', \mu\, L_c^2\gamma'$ with corresponding non-dimensional parameters α', β', γ' and a material length scale $L_c > 0\,$[m].

The additional parameter $\mu_c \geq 0$[MPa] in the strain energy is the **Cosserat couple modulus**. For $\mu_c = 0$ the two fields of displacement and microrotations decouple and one is left formally with classical linear elasticity for the displacement u.

2.2 The linear elastic Cosserat balance equations: hyperelasticity

Taking free variations of the energy in (2.1) w.r.t. both displacement $u \in \mathbb{R}^3$ and infinitesimal microrotation $\overline{A} \in \mathfrak{so}(3)$, one arrives at the equilibrium system (the Euler-Lagrange equations

[1] More detailed than strictly necessary in order to accommodate the different representations in the literature. Note that $\operatorname{axl}\overline{A} \times \xi = \overline{A}.\xi$ for all $\xi \in \mathbb{R}^3$, such that

$$\operatorname{axl}\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad \overline{A}_{ij} = \sum_{k=1}^{3} -\varepsilon_{ijk}\cdot \operatorname{axl}\overline{A}_k\,, \qquad (2.2)$$

where ε_{ijk} is the totally antisymmetric permutation tensor. Here, $\overline{A}.\xi$ denotes the application of the matrix $\overline{A}$ to the vector ξ and $a \times b$ is the usual cross-product. Note that it is always possible to prescribe essential boundary values for $\overline{A}$ but we abstain from such a prescription throughout.

[2] For simplicity only we assume that $\Gamma \cap \Gamma_S = \emptyset$ and that surface tractions and surface couples are prescribed at the same portion of the boundary. Much more general combinations could be considered.

of (2.1))

$$\begin{aligned}
&\operatorname{Div}\sigma = f\,, \qquad -\operatorname{Div} m = 2\mu_c\cdot\operatorname{axl}\operatorname{skew}\overline{\varepsilon} + \operatorname{axl}\operatorname{skew}(\overline{M})\,, \quad \overline{\varepsilon} = \nabla u - \overline{A},\\
&\quad \sigma = 2\mu\cdot\operatorname{sym}\overline{\varepsilon} + 2\mu_c\cdot\operatorname{skew}\overline{\varepsilon} + \lambda\cdot\operatorname{tr}[\overline{\varepsilon}]\cdot 1\!\!1 = (\mu+\mu_c)\cdot\overline{\varepsilon} + (\mu-\mu_c)\cdot\overline{\varepsilon}^T + \lambda\cdot\operatorname{tr}[\overline{\varepsilon}]\cdot 1\!\!1\,,\\
&\quad m = \gamma\,\nabla\phi + \beta\,\nabla\phi^T + \alpha\operatorname{tr}[\nabla\phi]\cdot 1\!\!1, \qquad \phi = \operatorname{axl}\overline{A}\,, \qquad (2.4)\\
&\quad u_{|\Gamma} = u_{\mathrm{d}}\,, \quad \sigma.\vec{n}_{|\Gamma_S} = f_S\,, \quad m.\vec{n}_{|\Gamma_S} = \frac{1}{2}\operatorname{axl}(\operatorname{skew}(\overline{M}_S))\,,\\
&\quad \sigma.\vec{n}_{|\partial\Omega\setminus(\Gamma_S\cup\Gamma)} = 0\,, \quad m.\vec{n}_{|\partial\Omega\setminus(\Gamma_S\cup\Gamma)} = 0\,.
\end{aligned}$$

Here, m is the **couple stress tensor**. For comparison, in [9, p.111] or [1, 27, 14] the elastic moduli in our notation are defined to be $\mu = \mu^* + \frac{\kappa}{2}$, $\mu_c = \frac{\kappa}{2}$.[3] But in this last definition (see [5]), μ^* cannot be regarded as one of the classical Lamé constants.[4] [5] We note that under the usual positivity requirements on the curvature energy, the couple stress/ curvature relation can be pointwise inverted. The case which we have in mind is characterized by $\gamma = \beta > 0$ and $\alpha = -\frac{2\gamma}{3}$. Thus, the constant α cannot be considered to to be a "spring constant"-like quantity!

3 Constitutive restrictions for Cosserat hyperelasticity

3.1 Pointwise positivity of the micropolar energy

For a mathematical treatment in the hyperelastic case it is often assumed that for arbitrary nonzero strain and curvature $\overline{\varepsilon}, \overline{\mathfrak{k}} \in \mathbb{M}^{3\times 3}$ one has the **local positivity condition**

$$\forall\, \overline{\varepsilon}, \overline{\mathfrak{k}} \neq 0: \qquad W_{\mathrm{mp}}(\overline{\varepsilon}) > 0\,, \qquad W_{\mathrm{curv}}(\overline{\mathfrak{k}}) > 0\,. \tag{3.1}$$

This condition is most often invoked as the basis of uniqueness proofs in static micropolar elasticity, see e.g. [20, 18, 9, 8]. By splitting $\overline{\varepsilon}$ in its deviatoric and volumetric part, i.e. writing

$$\overline{\varepsilon} = \operatorname{dev}\operatorname{sym}\overline{\varepsilon} + \operatorname{skew}\overline{\varepsilon} + \frac{1}{3}\operatorname{tr}[\overline{\varepsilon}]\cdot 1\!\!1 \tag{3.2}$$

and inserting this into the energy W_{mp} one gets

$$W_{\mathrm{mp}}(\overline{\varepsilon}) = \mu\,\|\operatorname{dev}\operatorname{sym}\overline{\varepsilon}\|^2 + \mu_c\,\|\operatorname{skew}\overline{\varepsilon}\|^2 + \frac{2\mu+3\lambda}{6}\operatorname{tr}[\overline{\varepsilon}]^2\,. \tag{3.3}$$

Since all three contributions in (3.2) can be chosen independent of each other, one obtains from (3.1) the **pointwise positive-definiteness condition**

$$\begin{aligned}
&\mu > 0\,, \quad 2\mu+3\lambda > 0\,, \quad \mu_c > 0\,,\\
&\gamma+\beta > 0\,, \quad (\gamma+\beta)+3\alpha > 0\,, \quad \gamma-\beta > 0\,, \quad (\Rightarrow\ \gamma > 0)\,,
\end{aligned} \tag{3.4}$$

where the argument pertaining to the curvature energy W_{curv} is exactly similar, cf. [21, (2.9)]. In effect, one ensures **uniform convexity** of the integrand w.r.t $\overline{\varepsilon}, \overline{\mathfrak{k}}$. In this case, then, the stress/strain and couple-stress/curvature relation can be inverted, simplifying the mathematical treatment considerably.

By a **thermodynamical stability argument** [9] one may similarly infer the **non-negativity** of the energy (material stability), leading only to

$$\begin{aligned}
&\mu \geq 0\,, \quad 2\mu+3\lambda \geq 0\,, \quad \mu_c \geq 0\,,\\
&\gamma+\beta \geq 0\,, \quad (\gamma+\beta)+3\alpha \geq 0\,, \quad \gamma-\beta \geq 0\,, \quad (\Rightarrow\ \gamma \geq 0)\,,
\end{aligned} \tag{3.5}$$

which allows for classical linear elasticity but which condition alone is not strong enough to guarantee existence and uniqueness of the corresponding boundary value problem. Nevertheless, all constitutive restrictions on a linear Cosserat solid must at least be consistent with (3.5) from a purely physical point of view.

[3] In [21, 9] the Cauchy stress tensor σ is defined as $\sigma = (\mu^*+\kappa)\,\overline{\varepsilon} + \mu^*\,\overline{\varepsilon}^T + \lambda\operatorname{tr}[\overline{\varepsilon}]\cdot 1\!\!1$ with given constants μ^*, κ, λ and one must identify $\mu^*+\kappa = \mu+\mu_c$, $\quad \mu^* = \mu-\mu_c$.

[4] A simple definition of the Lamé constants in micropolar elasticity is that they should coincide with the classical Lamé constants for symmetric situations. Equivalently, they are obtained by the classical formula $\mu = \frac{E}{2(1+\nu)}$, $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, where E and ν are uniquely determined from uniform traction where Cosserat effects are absent.

[5] Unfortunately, while authors are consistent in their usage of material parameters, one should be careful when identifying the actually used parameters with his own usage. The different representations in (2.3) might be useful for this purpose.

3.2 Legendre-Hadamard ellipticity conditions for the Cosserat model

For a dynamic problem, another condition, implying **real wave speeds** in wave propagation problems, is useful. This is the Legendre-Hadamard ellipticity condition. Let us investigate the restrictions which it imposes on the constitutive parameters of the Cosserat model. In the following we treat the generic case of a quadratic form which can then be applied to the balance of linear and angular momentum system. Our quadratic form is

$$W(\nabla\phi) := a_1 \, \|\operatorname{sym}\nabla\phi\|^2 + a_2 \, \|\operatorname{skew}\nabla\phi\|^2 + a_3 \operatorname{tr}[\nabla\phi]^2 \,. \tag{3.6}$$

Replacing $\nabla\phi$ by $\xi\otimes\eta$ we obtain

$$\begin{aligned}
&a_1 \, \|\operatorname{sym}\xi\otimes\eta\|^2 + a_2 \, \|\operatorname{skew}\xi\otimes\eta\|^2 + a_3 \operatorname{tr}[\xi\otimes\eta]^2 \\
&\quad= \frac{a_1}{4}\,\|\xi\otimes\eta+\eta\otimes\xi\|^2 + \frac{a_2}{4}\,\|\xi\otimes\eta-\eta\otimes\xi\|^2 + a_3\,\langle\xi,\eta\rangle^2 \qquad (3.7)\\
&\quad= \frac{a_1}{4}\,\left(2\,\|\xi\otimes\eta\|^2 + 2\,\langle\xi\otimes\eta,\eta\otimes\xi\rangle\right) + \frac{a_2}{4}\,\left(2\,\|\xi\otimes\eta\|^2 - 2\,\langle\xi\otimes\eta,\eta\otimes\xi\rangle\right) + a_3\,\langle\xi,\eta\rangle^2 \\
&\quad= \frac{a_1}{4}\,\left(2\,\|\xi\|^2\|\eta\|^2 + 2\,\langle\xi,\eta\rangle^2\right) + \frac{a_2}{4}\,\left(2\,\|\xi\|^2\|\eta\|^2 - 2\,\langle\xi,\eta\rangle^2\right) + a_3\,\langle\xi,\eta\rangle^2 \\
&\quad= \frac{a_1+a_2}{2}\,\|\xi\|^2\|\eta\|^2 + \frac{a_1-a_2+2\,a_3}{2}\langle\xi,\eta\rangle^2 \\
&\quad= \frac{a_1+a_2}{2}\,\|\xi\|^2\|\eta\|^2 + \frac{a_1-a_2+2\,a_3}{2}\|\xi\|^2\|\eta\|^2\cos^2\vartheta \\
&\quad= \frac{a_1+a_2}{2}\,\|\xi\|^2\|\eta\|^2\,(\sin^2\vartheta+\cos^2\vartheta) + \frac{a_1-a_2+2\,a_3}{2}\|\xi\|^2\|\eta\|^2\cos^2\vartheta \\
&\quad= \frac{a_1+a_2}{2}\,\|\xi\|^2\|\eta\|^2\,\sin^2\vartheta + (\frac{a_1+a_2}{2} + \frac{a_1-a_2+2\,a_3}{2})\|\xi\|^2\|\eta\|^2\cos^2\vartheta \\
&\quad= \frac{a_1+a_2}{2}\,\|\xi\|^2\|\eta\|^2\,\sin^2\vartheta + \frac{a_1+2\,a_3}{2}\|\xi\|^2\|\eta\|^2\cos^2\vartheta \,.
\end{aligned}$$

Thus

$$D^2W(\nabla\phi).(\xi\otimes\eta,\xi\otimes\eta) = (a_1+a_2)\,\|\xi\|^2\|\eta\|^2\,\sin^2\vartheta + (a_1+2\,a_3)\,\|\xi\|^2\|\eta\|^2\cos^2\vartheta\,, \tag{3.8}$$

and Legendre-Hadamard ellipticity demands that the acoustic tensor $Q(\xi):\mathbb{R}^3\mapsto\mathbb{R}^3$ defined through $D^2W(\nabla\phi).(\xi\otimes\eta,\xi\otimes\eta) = \langle\eta,Q(\xi).\eta\rangle$ is strictly positive definite for any nonzero wave direction $\xi\in\mathbb{R}^3$. We infer the necessary and sufficient conditions for strict Legendre-Hadamard ellipticity of the quadratic form (3.6)

$$a_1+a_2>0\,,\quad a_1+2\,a_3>0\,. \tag{3.9}$$

Applying this result to both the strain energy and curvature energy in (2.3) we obtain the **Legendre-Hadamard ellipticity condition for linear, isotropic Cosserat solids**

$$\begin{aligned}
&\mu+\mu_c>0\,,\quad \mu+\lambda>0\,,\\
&\gamma>0\,,\quad \gamma+\beta+\alpha>0\,.
\end{aligned} \tag{3.10}$$

In the case of $\mu_c=0$ we recover the well known ellipticity condition for linear elasticity. It is clear that (3.4) is sufficient for (3.10). But (3.10) alone is not sufficient for well-posedness of the Cosserat boundary value problem.

3.3 Coercivity of the micropolar energy

What one really needs for a mathematical treatment of the boundary value problem in the variational context, is, however, a **coercivity condition**, in the sense that a bounded energy I implies a bound on the displacement u and the infinitesimal microrotation $\overline{A}$ in appropriate Sobolev spaces. More precisely, for H^1-coercivity it must hold that

$$I(u,\overline{A})\le K_1<\infty \quad\Rightarrow\quad u\in H^{1,2}(\Omega,\mathbb{R}^3),\ \overline{A}\in H^{1,2}(\Omega,\mathfrak{so}(3))\,. \tag{3.11}$$

In this contribution we consider a limit case of non-negativity

$$\begin{aligned}
&\mu>0\,,\quad 2\mu+3\lambda>0\,,\quad \mu_c>0\,,\\
&\gamma+\beta>0\,,\quad (\gamma+\beta)+3\alpha=0\,,\quad \gamma-\beta=0\,,\quad (\gamma>0)\,,
\end{aligned} \tag{3.12}$$

in which the **curvature energy is not pointwise positive definite, but only non-negative**. In terms of the non-dimensional **polar ratio** Ψ defined as $\Psi = \frac{\beta+\gamma}{\alpha+\beta+\gamma}$ one has

$$\Psi := \frac{\beta+\gamma}{\alpha+\beta+\gamma} = \frac{3(\beta+\gamma)}{3(\alpha+\beta+\gamma)} = \frac{3(\beta+\gamma)}{3\alpha+(\beta+\gamma)+2(\beta+\gamma)}\,, \tag{3.13}$$

which leads with (3.12) to the restriction $\Psi = \frac{3}{2}$.[6] Our energy can be written as

$$\begin{aligned} W_{\rm mp}(\nabla u, \overline{A}) &= \mu\, \|\operatorname{sym}\nabla u\|^2 + \mu_c\, \|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}\nabla u\right]^2, \\ W_{\rm curv}(\nabla \operatorname{axl}\overline{A}) &= \mu\, L_c^2\, \|\operatorname{dev}\operatorname{sym}\nabla \operatorname{axl}\overline{A}\|^2\,. \end{aligned} \tag{3.14}$$

A first observation is in order. Considering ever smaller samples means, by a simple scaling argument, that $L_c \to \infty$. In this case, since the curvature energy must remain bounded, we must have $\operatorname{dev}\operatorname{sym}\nabla\operatorname{axl}\overline{A} = 0$. This does not imply, however, that $\overline{A}$ is constant[7] or affine linear. It is exactly this indeterminacy which is necessary for our purpose. For this curvature energy, the couple stress/curvature relation in (2.4) cannot be inverted.

Here, $\operatorname{dev}\operatorname{sym} X = 0$ implies that $X = p\,1\!\!1 + A$ with $p \in \mathbb{R}$ and $A \in \mathfrak{so}(3)$. The kernel of the linear operator $\operatorname{dev}\operatorname{sym} : \mathbb{M}^{3\times3} \mapsto \mathbb{M}^{3\times3}$ is thus the Lie-algebra of the **conformal group** which comprises all invertible deformations $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that $\nabla\varphi \in \mathbb{R}^+\,\mathrm{SO}(3)$.

3.4 The conformal indeterminate couple stress model

This model is formally obtained by setting $\mu_c = \infty$, which enforces the constraint $\operatorname{curl} u = 2\operatorname{axl}\overline{A}$ [29, 36, 24]. For the **displacement** $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ we consider therefore the **one-field** minimization problem

$$\begin{aligned} I(u) = \int_\Omega W_{\rm mp}(\nabla u) + W_{\rm curv}(\nabla \operatorname{curl} u) - \langle f, u\rangle - \frac{1}{2}\langle \operatorname{axl}(\overline{M}), \operatorname{curl} u\rangle\, \mathrm{dV} \\ - \int_{\Gamma_S} \langle f_S, u\rangle - \frac{1}{2}\langle \operatorname{axl}(\overline{M}_S), \operatorname{curl} u\rangle\, \mathrm{dS} \mapsto \quad \min.\ \text{w.r.t. } u, \end{aligned} \tag{3.15}$$

under the constitutive requirements and boundary conditions

$$\begin{aligned} W_{\rm mp}(\overline{\varepsilon}) &= \mu\, \|\operatorname{sym}\nabla u\|^2 + \frac{\lambda}{2}\operatorname{tr}\left[\operatorname{sym}\nabla u\right]^2, \quad u_{|\Gamma} = u_{\rm d}\,, \\ W_{\rm curv}(\nabla \operatorname{curl} u) &= \mu\, L_c^2\, \|\operatorname{dev}\operatorname{sym}\nabla\operatorname{curl} u\|^2 = \frac{\mu\, L_c^2}{4}\, \|\operatorname{sym}\nabla\operatorname{curl} u\|^2\,. \end{aligned} \tag{3.16}$$

In this limit model, the curvature parameter α, related to the spherical part of the couple stress tensor m remains **indeterminate**, since $\operatorname{tr}\left[\nabla\phi\right] = \operatorname{Div}\operatorname{axl}\overline{A} = \operatorname{Div}\frac{1}{2}\operatorname{curl} u = 0$. We remark the intricate relation between $\mu_c \to \infty$[8] and the indeterminacy of α. Since our curvature energy is much weaker than the one usually considered, the criticism which has been raised against this limit model might not apply here. This will be subject of future research.

4 A new coercive inequality

In order to show existence, uniqueness and stability we make use of a recent observation in [6]. Let $\Omega \subset \mathbb{R}^3$ be a three-dimensional domain with Lipschitz boundary. It holds

Theorem 4.1 (Weak coercivity for $n = 3$)

$$\exists\, C_D > 0 \quad \forall\, \phi \in H^1(\Omega, \mathbb{R}^3): \quad \int_\Omega \|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2 + \|\phi\|^2\, \mathrm{dx} \geq C_D\, \|\phi\|^2_{H^1(\Omega,\mathbb{R}^3)}\,. \tag{4.1}$$

The proof of (4.1) is given in [6]. In fact, however, the inequality follows already from a general result of Necas on the weak coercivity of formally positive quadratic forms. For the convenience of the reader we include here an independent proof, based on the following statement of Necas.

[6] If we require only $(3.10)_2$ then the polar ratio Ψ may become negative.
[7] $\overline{A}$ would be constant if the curvature energy was pointwise positive definite.
[8] This formal limit $\mu_c \to \infty$ excludes that $\operatorname{axl}\overline{A}$ is an independent field.

Theorem 4.2
Let $N : \mathbb{M}^{3\times 3} \mapsto \mathbb{M}^{3\times 3}$ be a constant coefficient linear operator and let $\Omega \subset \mathbb{R}^3$ be a $C^{1,1}$-domain. The formally positive quadratic form $\|N.\nabla u\|^2_{\mathbb{M}^{3\times 3}}$ is weakly coercive, i.e.

$$\exists\, C > 0 \quad \forall\, \phi \in H^1(\Omega, \mathbb{R}^3) : \quad \int_\Omega \|N.\nabla\phi\|^2_{\mathbb{M}^{3\times 3}} + \|\phi\|^2_{\mathbb{R}^3}\, \mathrm{dx} \geq \|\phi\|^2_{H^1(\Omega,\mathbb{R}^3)}\,, \tag{4.2}$$

if for all $\xi \in \mathbb{C}^3$, $\xi \neq 0$ the system $N.(\xi \otimes \hat{u}) = 0$ implies $\hat{u} = 0$.

Proof. This statement is a simple consequence of Theorem 3.2 in [19]. ■
Let us now use this result and re-proof (4.1).
Proof. (weak coercivity) The quadratic form $\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2$ is formally positive with constant coefficients. Using Theorem 4.2 we check first the necessary (but not sufficient) Legendre-Hadamard ellipticity condition (equivalent to allow only for $\xi, \hat{u} \in \mathbb{R}^3$) to the extent that for all $\xi, \hat{u} \in \mathbb{R}^3$ with $\hat{u} \neq 0$

$$\operatorname{dev}\operatorname{sym}\xi \otimes \hat{u} = 0 \;\Rightarrow\; \xi = 0\,. \tag{4.3}$$

To see this, it is enough to compute

$$\begin{aligned}\|\operatorname{dev}\operatorname{sym}\xi\otimes\hat{u}\|^2 &= \|\operatorname{sym}\xi\otimes\hat{u}\|^2 - \frac{1}{3}\operatorname{tr}[\xi\otimes\hat{u}]^2 \\ &\geq \frac{1}{2}\|\xi\|^2\,\|\hat{u}\|^2 - \frac{1}{3}\|\xi\|^2\,\|\hat{u}\|^2 = \frac{1}{6}\|\xi\|^2\,\|\hat{u}\|^2\,. \end{aligned} \tag{4.4}$$

Thus $\xi = 0$ if $\operatorname{dev}\operatorname{sym}\xi\otimes\hat{u} = 0$ and $\hat{u} \neq 0$. This suffices to obtain (4.1) in case of the space $H^1_0(\Omega)$, i.e., zero boundary values throughout which is also included in [19]. In order to strengthen the result we need to show the same for complex-valued vectors. Thus we have to show that for all complex-valued vectors $\xi, \hat{u} \in \mathbb{C}^3$ with $\xi \neq 0$

$$\operatorname{dev}\operatorname{sym}\xi \otimes \hat{u} = 0 \;\Rightarrow\; \hat{u} = 0\,. \tag{4.5}$$

The corresponding complex linear problem in matrix format reads

$$\begin{pmatrix} 2\xi_1 & -\xi_2 & -\xi_3 \\ \xi_2 & \xi_1 & 0 \\ -\xi_1 & 2\xi_2 & -\xi_3 \\ \xi_3 & 0 & \xi_1 \\ -\xi_1 & -\xi_2 & 2\xi_3 \\ 0 & \xi_3 & \xi_2 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.6}$$

Our remaining task is to evaluate whether the 6×3 matrix has full rank, for a nonzero $\xi \in \mathbb{C}^3$.

Elementary operations lead to the equivalent question whether the following 5×3 matrix has rank three for a nonzero $\xi \in \mathbb{C}^3$

$$\begin{pmatrix} \xi_2 & \xi_1 & 0 \\ -\xi_1 & \xi_2 & 0 \\ \xi_3 & 0 & \xi_1 \\ 0 & \xi_3 & \xi_2 \\ 0 & \xi_2 & -\xi_3 \end{pmatrix}. \tag{4.7}$$

We proceed by case distinction:

Let first $\xi_3 \neq 0$. If then $\xi_2^2 + \xi_3^2 \neq 0$ (the determinant of rows $3,4,5$), we are done. If $\xi_2^2 + \xi_3^2 = 0$, consider the determinant of rows $2,3,4$, which is $\xi_3\,(\xi_1^2 - \xi_2^2)$. If $\xi_1^2 - \xi_2^2 \neq 0$ we are done. Assume thus that $\xi_1^2 - \xi_2^2 = 0$ and consider finally the determinant of rows $1,3,4$ which is $-2\,\xi_1\xi_2\xi_3 \neq 0$ since we are in the case $\xi_1^2 - \xi_2^2 = 0$ and $\xi_2^2 + \xi_3^2 = 0$ and $\xi_3 \neq 0$. Thus, for $\xi_3 \neq 0$ the rank is three.

Assume now that $\xi_2 \neq 0$. If the determinant of rows $1,4,5 :\ -\xi_2\,(\xi_3^2 + \xi_2^2) \neq 0$ we are done. If $(\xi_3^2 + \xi_2^2) = 0$ consider the determinant of rows $1,2,4$. If $\xi_2\,(\xi_2^2 + \xi_1^2) \neq 0$ we are done. If not, then $\xi_2^2 + \xi_1^2 = 0$ and consider the determinant of rows $1,3,4 :\ -2\xi_1\,\xi_2\,\xi_3$. This determinant is now non-vanishing since we are in the case that both $(\xi_3^2 + \xi_2^2) = 0$ and $\xi_2^2 + \xi_1^2 = 0$ and $\xi_2 \neq 0$. Thus, for $\xi_2 \neq 0$ the rank is three.

Assume finally that $\xi_1 \neq 0$. If the determinant of rows $1,2,3 :\ \xi_1\,(\xi_2^2 + \xi_1^2) \neq 0$ we are done. If $(\xi_1^2 + \xi_2^2) = 0$ consider the determinant of rows $2,4,5 :\ \xi_1\,(\xi_2^2 + \xi_3^2)$. If $\xi_2^2 + \xi_3^2 \neq 0$ we are done. If $\xi_2^2 + \xi_3^2 = 0$ consider the determinant of rows $1,3,4 :\ -2\,\xi_1\xi_2\xi_3$ which must be nonzero now. Thus, for $\xi_1 \neq 0$ the rank is also three. This finishes the proof. ■

Remark 4.3
The statement is not true in case of space-dimension $n = 2$ *together with the definition of a two-dimensional deviator* $\operatorname{dev}_2 X := X - \frac{1}{2}\operatorname{tr}[X]\mathbb{1}_2$*. In this case, the quadratic form is not algebraically closed. Here, the corresponding system is*

$$\begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\det \begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{pmatrix} = \xi_1^2 + \xi_2^2 = 0 \quad \textit{for} \quad \xi = (1, i)^T, \tag{4.8}$$

showing that the operator is not algebraically complete.

4.1 Existence

Without loss of generality, we consider the case without body and surface couples but with given body forces $f \in L^2(\Omega, \mathbb{R}^3)$. Moreover, the infinitesimal microrotations $\overline{A}$ at the boundary are left free (no essential boundary conditions for the microrotation field). We fix the body at a part $\Gamma \subset \partial\Omega$ which translates into the essential boundary conditions for the displacement $u_{|_\Gamma} = 0$. In addition, we assume $\mu > 0$ and $\lambda \geq 0$ (this condition could be weakened as in linear elasticity: only the bulk modulus $K = \frac{2\mu+3\lambda}{3}$ needs to be strictly positive). The task is to show that

$$I(u, \overline{A}) := \int_\Omega \mu \, \|\operatorname{sym} \nabla u\|^2 + \mu_c \, \|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda}{2} \operatorname{tr}[\operatorname{sym} \nabla u]^2$$
$$+ \mu \, L_c^2 \, \|\operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \overline{A}\|^2 - \langle f, u \rangle \, \mathrm{dx} \tag{4.9}$$

admits minimizers in the set of admissible functions $\mathcal{A}$ with bounded energy I:

$$\mathcal{A} := \{u \in H^1(\Omega, \mathbb{R}^3), \quad \overline{A} \in H^1(\Omega, \mathfrak{so}(3)), \quad I(u, \overline{A}) < \infty\,, \quad u_{|_\Gamma} = 0\,\}\,. \tag{4.10}$$

Since I is quadratic, it is clear that I is convex. Using the classical Korn's first inequality of linear elasticity (usually, for stronger curvature expressions, Korn's inequality is not needed for existence in linear Cosserat models!) we obtain the estimate

$$\begin{aligned} I(u, \overline{A}) &\geq \mu \, C_K^+ \|u\|^2_{H^1(\Omega)} - \|f\|_{L^2(\Omega)} \, \|u\|_{L^2(\Omega)} \\ &\quad + \int_\Omega \mu_c \, \|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \mu \, L_c^2 \, \|\operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \overline{A}\|^2 \, \mathrm{dx} \\ &\geq \mu \, C_K^+ \|u\|^2_{H^1(\Omega)} - \|f\|_{L^2(\Omega)} \, \|u\|_{H^1(\Omega)} \\ &\quad + \int_\Omega \mu_c \, \|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \mu \, L_c^2 \, \|\operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \overline{A}\|^2 \, \mathrm{dx} \end{aligned} \tag{4.11}$$

The first line of the last estimate gives us a quadratic inequality for $\|u\|_{H^1(\Omega)}$. Since I is bounded on $\mathcal{A}$ we infer that $u \in \mathcal{A}$ is bounded in $H^1(\Omega, \mathbb{R}^3)$. Since $\|u\|_{H^1(\Omega)}$ is bounded and the remaining terms in I are positive we infer that I is bounded below. Thus the value

$$\inf_{\mathcal{A}} I(u, \overline{A}) = m \tag{4.12}$$

exists. We choose minimizing sequences, i.e. sequences $(u_k, \overline{A}_k) \in \mathcal{A}$ with the property that the lowest energy level is approximated:

$$\lim_{k \to \infty} I(u_k, \overline{A}_k) = m\,. \tag{4.13}$$

The previous estimate shows that u_k is bounded in $H^1(\Omega, \mathbb{R}^3)$. Since $\mu_c > 0$ we infer that $\overline{A}_k$ is also bounded in $L^2(\Omega)$. Thus, for the microrotations we have that

$$\int_\Omega \mu_c \, \|\overline{A}_k\|^2 + \mu \, L_c^2 \, \|\operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \overline{A}_k\|^2 \, \mathrm{dx} \tag{4.14}$$

is bounded. Using the new coercivity estimate (4.1) shows that $\|\overline{A}_k\|_{H^1(\Omega, \mathfrak{so}(3))}$ is bounded as well. Extracting weakly convergent subsequences of displacements $u_{k_j} \rightharpoonup u \in H^1(\Omega, \mathbb{R}^3)$, strongly converging in $L^2(\Omega)$ by Rellichs compact embedding, and similarly for microrotations $\overline{A}_{k_j} \to \overline{A} \in H^1(\Omega, \mathfrak{so}(3))$ together with weak lower semicontinuity of the energy

$$I(u, \overline{A}) \leq \liminf_{k_j} I(u_{k_j}, \overline{A}_{k_j}) = m \tag{4.15}$$

implies that $I(u, \overline{A}) \leq m$, which is only possible if $I(u, \overline{A}) = m$. Thus the pair $(u, \overline{A})$ is a minimizer.

4.2 Uniqueness

For uniqueness we have to show strict convexity of the energy. The energy is two-times differentiable. Thus we may consider the second derivative of I, which is given by

$$\begin{aligned} D^2 I(u, \overline{A}).(\delta u, \delta\overline{A})^2 = & \int_\Omega 2\mu \| \operatorname{sym} \nabla \delta u\|^2 + 2\mu_c \, \| \operatorname{skew}(\nabla \delta u - \delta\overline{A})\|^2 + \lambda \operatorname{tr} [\operatorname{sym} \nabla \delta u]^2 \\ & \quad + 2\mu \, L_c^2 \, \| \operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \delta\overline{A}\|^2 \, \mathrm{dx} \\ \geq & \int_\Omega 2\mu \| \operatorname{sym} \nabla \delta u\|^2 + 2\mu_c \, \| \operatorname{skew}(\nabla \delta u - \delta\overline{A})\|^2 \\ & \quad + 2\mu \, L_c^2 \, \| \operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \delta\overline{A}\|^2 \, \mathrm{dx} \, . \end{aligned} \tag{4.16}$$

Obviously, $D^2 I(u, \overline{A}).(\delta u, \delta\overline{A})^2 = 0$ implies that $(\delta u, \delta\overline{A}) = 0$ by using Korn's first inequality on δu and $\mu_c > 0$. This shows that the second derivative is strictly positive, implying strict convexity of I and uniqueness of the minimizer.

4.3 Stability for all $\mu_c > 0$

An interesting observation is that strict convexity of our energy does not immediately lead to uniform convexity or stability. For this the second derivative must satisfy the uniform inequality

$$D^2 I(u, \overline{A}).(\delta u, \delta\overline{A})^2 \geq C^+ \left(\|\delta u\|^2_{H^1(\Omega)} + \|\delta\overline{A}\|^2_{H^1(\Omega)} \right) \tag{4.17}$$

with a constant $C^+ > 0$ independent of $(u, \overline{A})$.

In order to see under what conditions we might be able to get such an inequality, we turn back to the second derivative

$$\begin{aligned} & D^2 I(u, \overline{A}).(\delta u, \delta\overline{A})^2 \\ & = \int_\Omega 2\mu \| \operatorname{sym} \nabla \delta u\|^2 + 2\mu_c \, \| \operatorname{skew}(\nabla \delta u - \delta\overline{A})\|^2 + \lambda \operatorname{tr} [\operatorname{sym} \nabla \delta u]^2 \\ & \quad + 2\mu \, L_c^2 \, \| \operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \delta\overline{A}\|^2 \, \mathrm{dx} \\ & \geq \int_\Omega 2\mu \| \operatorname{sym} \nabla \delta u\|^2 + 2\mu_c \, \| \operatorname{skew}(\nabla \delta u - \delta\overline{A})\|^2 \\ & \quad + 2\mu \, L_c^2 \, \| \operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \delta\overline{A}\|^2 \, \mathrm{dx} \\ & \geq \int_\Omega 2\mu C_K^+ \|\nabla \delta u\|^2 + 2\mu_c \, \| \operatorname{skew}(\nabla \delta u - \delta\overline{A})\|^2 \\ & \quad + 2\mu \, L_c^2 \, \| \operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \delta\overline{A}\|^2 \, \mathrm{dx} \\ & \geq \int_\Omega 2\mu C_K^+ \|\nabla \delta u\|^2 + 2\mu_c \left(\| \operatorname{skew}(\nabla \delta u)\|^2 - 2\langle \operatorname{skew} \nabla \delta u, \delta\overline{A}\rangle + \|\delta\overline{A}\|^2 \right) \\ & \quad + 2\mu \, L_c^2 \, \| \operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \delta\overline{A}\|^2 \, \mathrm{dx} \\ & \geq \int_\Omega 2\mu C_K^+ \left(\| \operatorname{sym} \nabla \delta u\|^2 + \| \operatorname{skew} \nabla \delta u\|^2 \right) + 2\mu_c \left(\| \operatorname{skew}(\nabla \delta u)\|^2 - 2\langle \operatorname{skew} \nabla \delta u, \delta\overline{A}\rangle + \|\delta\overline{A}\|^2 \right) \\ & \quad + 2\mu \, L_c^2 \, \| \operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \delta\overline{A}\|^2 \, \mathrm{dx} \\ & \geq \int_\Omega 2\mu C_K^+ \| \operatorname{sym} \nabla \delta u\|^2 + (2\mu C_K^+ + 2\mu_c) \, \| \operatorname{skew} \nabla \delta u\|^2 - 2\mu_c \, 2\langle \operatorname{skew} \nabla \delta u, \delta\overline{A}\rangle \\ & \quad + 2\mu_c \, \|\delta\overline{A}\|^2 + 2\mu \, L_c^2 \, \| \operatorname{dev} \operatorname{sym} \nabla \operatorname{axl} \delta\overline{A}\|^2 \, \mathrm{dx} \\ & \geq \int_\Omega 2\mu C_K^+ \| \operatorname{sym} \nabla \delta u\|^2 + (2\mu C_K^+ + 2\mu_c) \, \| \operatorname{skew} \nabla \delta u\|^2 - 2\mu_c \, 2\| \operatorname{skew} \nabla \delta u\| \, \|\delta\overline{A}\| \end{aligned}$$

$$
\begin{aligned}
&\quad + 2\mu_c\,\|\delta\overline{A}\|^2 + 2\mu\,L_c^2\,\|\operatorname{dev}\operatorname{sym}\nabla\operatorname{axl}\delta\overline{A}\|^2\,\mathrm{dx}\\
&\geq \int_\Omega 2\mu C_K^+\|\operatorname{sym}\nabla\delta u\|^2 + (2\mu C_K^+ + 2\mu_c)\,\|\operatorname{skew}\nabla\delta u\|^2 - 2\mu_c\,(\frac{1}{\varepsilon}\|\operatorname{skew}\nabla\delta u\|^2 + \varepsilon\|\delta\overline{A}\|^2)\\
&\quad + 2\mu_c\,\|\delta\overline{A}\|^2 + 2\mu\,L_c^2\,\|\operatorname{dev}\operatorname{sym}\nabla\operatorname{axl}\delta\overline{A}\|^2\,\mathrm{dx}\\
&\geq \int_\Omega 2\mu C_K^+\|\operatorname{sym}\nabla\delta u\|^2 + (2\mu C_K^+ + 2\mu_c - 2\mu_c\frac{1}{\varepsilon})\,\|\operatorname{skew}\nabla\delta u\|^2 + (2\mu_c - 2\mu_c\varepsilon)\,\|\delta\overline{A}\|^2\\
&\quad + 2\mu\,L_c^2\,\|\operatorname{dev}\operatorname{sym}\nabla\operatorname{axl}\delta\overline{A}\|^2\,\mathrm{dx}\\
&\geq \int_\Omega 2\mu C_K^+\|\operatorname{sym}\nabla\delta u\|^2 + 2(\mu C_K^+ + \mu_c - \mu_c\frac{1}{\varepsilon})\,\|\operatorname{skew}\nabla\delta u\|^2 + 2\mu_c(1-\varepsilon)\,\|\delta\overline{A}\|^2\\
&\quad + 2\mu\,L_c^2\,\|\operatorname{dev}\operatorname{sym}\nabla\operatorname{axl}\delta\overline{A}\|^2\,\mathrm{dx}\,. \qquad (4.18)
\end{aligned}
$$

We need to choose $\varepsilon > 0$ such that

$$
1-\varepsilon > 0 \quad \text{and} \quad (\mu C_K^+ + \mu_c - \mu_c\frac{1}{\varepsilon}) \geq 0\,. \qquad (4.19)
$$

This is satisfied whenever

$$
1 > \varepsilon \geq \frac{1}{1+\frac{\mu}{\mu_c}C_K^+}\,. \qquad (4.20)
$$

Since the constant in Korn's inequality C_K^+ is strictly positive, this can always be achieved. With such an $\varepsilon > 0$ we obtain therefore the estimate

$$
\begin{aligned}
D^2 I(u,\overline{A}).(\delta u,\delta\overline{A})^2 &\geq \int_\Omega 2\mu C_K^+\|\operatorname{sym}\nabla\delta u\|^2 + 2\mu_c(1-\varepsilon)\,\|\delta\overline{A}\|^2\\
&\quad + 2\mu\,L_c^2\,\|\operatorname{dev}\operatorname{sym}\nabla\operatorname{axl}\delta\overline{A}\|^2\,\mathrm{dx}\\
&\geq 2\mu(C_K^+)^2\|\delta u\|^2_{H^1(\Omega)} + C_D^+\,\|\delta\overline{A}\|^2_{H^1(\Omega)}\\
&\geq \min(2\mu(C_K^+)^2, C_D^+)\,\left(\|\delta u\|^2_{H^1(\Omega)} + \|\delta\overline{A}\|^2_{H^1(\Omega)}\right)\,. \qquad (4.21)
\end{aligned}
$$

4.4 Relations for micropolar constants

In the literature on Cosserat or micropolar solids the following abbreviations and definitions are frequently encountered. As a convenience for the reader, we collect these technical constants here.

$$
\begin{aligned}
\Psi &:= \frac{\beta+\gamma}{\alpha+\beta+\gamma}\,, \qquad \text{non-dimensional polar ratio}\,, \quad 0 \leq \Psi \leq \frac{3}{2}\,,\\
\ell_t^2 &:= \left(\frac{\beta+\gamma}{2\mu^*+\kappa}\right) = \frac{\beta+\gamma}{2}\frac{1}{\mu}\,, \qquad \text{"characteristic length for torsion"}\,, \qquad (4.22)\\
\ell_b^2 &= \frac{\gamma}{2\,(2\mu^*+\kappa)} = \frac{\gamma}{4\mu}\,, \qquad \text{"characteristic length for bending"}\,,\\
p^2 &:= \frac{2\,\kappa}{\alpha+\beta+\gamma}\,, \qquad \kappa := 2\,\mu_c\,,\\
N^2 &:= \frac{\mu_c}{\mu+\mu_c} = \frac{\kappa}{2\,(\mu^*+\kappa)}\,, \quad \text{Cosserat coupling number}, \quad 0 \leq N \leq 1\,.\\
\nu &= \frac{\lambda}{2\mu^*+2\lambda+\kappa} = \frac{\lambda}{2\,(\mu+\lambda)}\,, \qquad \text{classical Poisson ratio.}
\end{aligned}
$$

For every physical material, it is essential that small samples still have bounded rigidity. This may or may not be true for Cosserat models, depending on the values of Cosserat parameters. Based on analytic solution formulas for simple three-dimensional Cosserat boundary value problems it has been shown in [31] that for bounded stiffness for arbitrary small samples we must have

1. torsion of a cylinder: $\beta+\gamma = 0$ or $\Psi = \frac{\beta+\gamma}{\alpha+\beta+\gamma} = \frac{3}{2}$.

2. bending of a cylinder: $(\beta+\gamma)\,(\gamma-\beta) = 0$.

Our curvature energy satisfies both requirements through $\beta=\gamma$ and $\Psi=\frac{3}{2}$.[9]
Foams and bones have been identified by Lakes as prototype Cosserat solids. In order to identify the material parameters, however, Lakes had to leave the traditionally admitted parameter range motivated by strict pointwise positivity. In [25, 27] the value $\Psi=\frac{3}{2}$ has been chosen in order to accommodate bounded stiffness with experimental findings. For a syntactic foam [25] $\beta=\gamma$ has been taken for a best fit. In this case, the curvature energy looks like $W_{\mathrm{curv}}(\nabla\phi)=\gamma\,\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2$ with $\gamma>0$. With our new result it is shown that this is a well-posed fit. For a polyurethane foam [25] $\beta\neq\gamma$ and the curvature energy looks like $W_{\mathrm{curv}}(\nabla\phi)=\frac{\beta+\gamma}{2}\,\|\operatorname{dev}\operatorname{sym}\nabla\phi\|^2+\frac{\gamma-\beta}{4}\,\|\operatorname{curl}\phi\|^2$. Our result can be easily adapted to this case as well. The problem is well-posed.

4.5 Stress concentration along a cylindrical hole

In [9, p.222] or [8, p.238] the analytical solution for the stress distribution around a cylindrical hole with radius $r>0$ of an infinite plate is recalled. The stress concentration factor K_t, which classically is $K_t=3$ turns for the linear Cosserat model into

$$K_t=\frac{3+F_1}{1+F_1}\leq 3\,,\quad F_1=8\,(1-\nu)\,N^2\,\frac{1}{4+\frac{r^2}{c^2}+2\frac{r}{c}\frac{K_0(\frac{r}{c})}{K_1(\frac{r}{c})}}\,,\quad c^2:=\frac{\gamma\,(\mu^*+\kappa)}{\kappa\,(2\mu^*+\kappa)}=\frac{\ell_b^2}{N^2}\,,$$
$$K_0(\xi),\,K_1(\xi)\quad\text{modified }\textbf{Bessel functions}\text{ of the second kind}\,. \tag{4.23}$$

In the genuine micropolar case with pointwise positive curvature, the stress intensities are somewhat weakened: the Cosserat solid has the ability to distribute the stresses more smoothly which is one of the salient features of the Cosserat model. For our relaxed curvature energy, the same stress intensity formula still applies.[10] This shows that the new conformal linear Cosserat model still shares this classical feature of the linear Cosserat model.

5 Conclusion

That linear elastic Cosserat models may show singular stiffening behaviour has already been observed previously. In [26, p.17] we read "For some combinations of elastic constants, the apparent modulus tends to infinity as the bar or plate size goes to zero. Large stiffening effects might be seen in composite materials consisting of very stiff fibers or laminae in a compliant matrix. However, infinite stiffening effects are unphysical. For very slender specimens, it is likely that a continuum theory more general than Cosserat elasticity; or use of a discrete structural model, would be required to deal with the observed phenomena".

But Lakes himself still used the linear Cosserat model, albeit within uncommon parameter ranges. Adopting the same values as Lakes did, pointwise strict positivity of the energy is not true and has been criticized in [9]. However, the limit case $\gamma=\beta>0$, $\alpha=-\frac{2\gamma}{2}$, does still allow for existence and uniqueness and stability. Usually, it is also a problem to determine some sound boundary conditions for the microrotations $\overline{A}$. Even within the weaker parameter range we need not, however, specify Dirichlet boundary conditions for the microrotations! Thus boundary layer effects can be completely avoided. In a sequel paper, we will consider the FEM-implementation of the linear Cosserat model with the weakened curvature energy. It is hoped that within the new parameter range, the Cosserat couple modulus μ_c can be identified with a material parameter, independent of the size of the sample - which it cannot be in case of pointwise strict positivity of the curvature energy, as shown in [31]. We think that our relaxed curvature expression offers a fresh departure for the experimental determination of the linear isotropic Cosserat constants.

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[9] The additional conditions in [31] for bounded stiffness have been based on dimensionally reduced models and must therefore be taken with care.

[10] While the boundary conditions in this problem differ from those used in the existence part of this paper, the analysis can be extended easily.

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Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3\times3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times3}$ is given by $\langle X, Y\rangle_{\mathbb{M}^{3\times3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{M}^{3\times3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{M}^{3\times3}$. The identity tensor on $\mathbb{M}^{3\times3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\, 1\!\!1 \in \mathfrak{sl}(3)$ where $\mathfrak{sl}(3)$ is the Lie-algebra of traceless matrices. The Lie-algebra of $\mathrm{SO}(3) := \{X \in \mathrm{GL}(3)\,|X^TX = 1\!\!1,\ \det[X] = 1\}$ is given by the set $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times3}\,|X^T = -X\}$ of all skew symmetric tensors. The canonical identification of $\mathfrak{so}(3)$ and $\mathbb{R}^3$ is denoted by $\operatorname{axl}\overline{A} \in \mathbb{R}^3$ for $\overline{A} \in \mathfrak{so}(3)$. Finally, w.r.t. abbreviates with respect to.