Vorticity, Rotation and Symmetry – Stabilizing and Destabilizing Fluid Motion

Reinhard Farwig[*], Jiří Neustupa[†], Patrick Penel[‡]

The mathematical theory of the Navier–Stokes equations poses a famous open problem for instationary solutions in three dimensions: It is unknown whether the so–called weak solution, which does exist globally in time, is regular or smooth for all times if the initial value and/or the prescribed external force are large. Analogously, the question, whether a regular solution which may be constructed on a sufficiently small time interval does exist globally in time, is open. Both these problems are closely related to another open and very important question, i.e. the uniqueness of weak solutions of the instationary Navier–Stokes equations.

To be more precise, given a domain $\Omega \subset \mathbb{R}^3$ and a time interval $(0,T)$, an external force field f on $\Omega \times (0,T)$ and an initial value u_0, we are looking for a velocity field u and a pressure function p solving the Navier–Stokes system

$$\begin{array}{rcll} u_t - \nu\Delta u + u\cdot\nabla u + \nabla p &=& f & \text{in } \Omega\times(0,T) \\ \operatorname{div} u &=& 0 & \text{in } \Omega\times(0,T) \\ u(0) &=& u_0 & \text{at } t=0 \\ u &=& 0 & \text{on } \partial\Omega\times(0,T). \end{array} \tag{1}$$

Under relatively weak assumptions on u_0, f, say,

$$u_0 \in L^2_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2}, \quad C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega) : \operatorname{div} u = 0\},$$

and $f \in L^1\big(0,T; L^2(\Omega)\big)$, there exists a weak solution

$$u \in L^\infty\big(0,T; L^2(\Omega)\big) \cap L^2_{\text{loc}}\big(0,T; H^1_0(\Omega)\big)$$

[*] Technische Universität Darmstadt, Fachbereich Mathematik, 64283 Darmstadt, Germany, email address: farwig@mathematik.tu-darmstadt.de

[†] Czech Academy of Sciences, Mathematical Institute, 115 67 Praha 1, Czech Republic, email address: neustupa@math.cas.cz

[‡] Département de Mathématique, Université du Sud, Toulon-Var, 83957 La Garde Cedex, France, email address: penel@univ-tln.fr

to (1). It is an open problem since the pioneering work of J. Leray (1934) whether u is a strong solution, e.g. in the sense that $u \in L^\infty\big(0,T;H_0^1(\Omega)\big) \cap L^2\big(0,T;H^2(\Omega)\big)$ or even $u \in C^\infty\big(\overline{\Omega} \times (0,T)\big)$ under suitable assumptions on u_0, f and $\partial\Omega$. Up to now, this result can be proved only under additional assumptions on u (or p or on other quantities). The classical result of *conditional regularity* is due to J. Serrin (1962/63) and requires that

$$u \in L^s\big(0,T;L^q(\Omega)\big), \ \frac{2}{s} + \frac{3}{q} = 1, \ s > 2, \ q > 3.$$

Since then Serrin's condition has been generalized to the limit cases $s = 2$, $q = \infty$ and, more recently, $s = \infty$, $q = 3$, as well as to related conditions on ∇u, on specific components either of u or of ∇u. Other conditions on the given weak solution concern the signs of the eigenvalues of the symmetric matrix of deformation, $\frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$, see J. Neustupa, P. Penel (2001), or the behavior of the pressure which is unique only up to a time–dependent function.

Of special interest is the vorticity $\omega = \operatorname{curl} u$. On the one hand, the nonlinear term $u \cdot \nabla u$ may be written in the form

$$u \cdot \nabla u = \omega \times u + \nabla\big(\frac{1}{2}|u|^2\big) \tag{2}$$

so that $\frac{1}{2}|u|^2$ can be considered as part of the pressure p; then $\frac{1}{2}|u|^2 + p$ defines the so–called total head pressure. Actually, in most results on the Navier–Stokes system the term $u \cdot \nabla u$ is directly estimated ignoring the special decomposition (2). On the other hand, ω satisfies the *vorticity transport equation*

$$\omega_t - \nu\Delta\omega + u \cdot \nabla\omega - \omega \cdot \nabla u = \operatorname{curl} f. \tag{3}$$

In two dimensions, $\omega = (0,0,\omega_3)$ where ω_3 satisfies the scalar equation

$$\partial_t\omega_3 - \nu\Delta\omega_3 + u \cdot \nabla\omega_3 = \partial_1 f_2 - \partial_2 f_1,$$

the maximum principle holds for ω_3 provided data for $\omega_3|_{\partial\Omega}$ and $\omega_3|_{t=0}$ are available. In three dimensions, the term $\omega \cdot \nabla u$ in (3) prevents the application of a maximum principle and may lead to the phenomenon of *vortex stretching*, a local increase of $|\omega|$ when certain geometrical conditions on ω and u are satisfied. In the whole space case the identity $\operatorname{curl}\omega = \operatorname{curl}\ \operatorname{curl} u = -\Delta u$ may be used to get *Biot-Savart's law*

$$u = (-\Delta)^{-1}\operatorname{curl}\omega = \frac{1}{4\pi}\int_{\mathbb{R}^3} \omega(y) \times \frac{x-y}{|x-y|^3}\,dy, \tag{4}$$

i.e., u is defined by ω via a weakly singular integral operator. Moreover, we see that (3) is a nonlinear and nonlocal equation in ω. In view of Serrin-type results integrability conditions on two components of ω prove that weak solutions are regular, but it is still open whether a condition on only one component is sufficient for an analogous result.

Additional problems occur in the analysis of viscous fluid flow around rotating obstacles. Assume that a compact obstacle $K \subset \mathbb{R}^3$ is rotating around a fixed axis of rotation $w = (0, 0, w_3)$ with angular velocity $|w| = w_3 \neq 0$. If K is not axially symmetric with respect to w, the domain $\Omega(t)$ occupied by the fluid is changing in time. Then the change to a coordinate system attached to the rotating body yields the Navier–Stokes equation

$$u_t - \nu \Delta u + u \cdot \nabla u - (w \times x) \cdot \nabla u + w \times u + \nabla p = F. \tag{5}$$

The additional linear term $w \times u$ represents the Coriolis force, whereas the term $(w \times x) \cdot \nabla u$ is increasing as $|(x_1, x_2)| \to \infty$, not subordinate to the Laplacian and totally changes the mathematical analysis of this problem. Actually, this latter term adds an hyperbolic effect to the parabolic Navier–Stokes system. With regard to this hyperbolic influence, the semigroup generated by the operator

$$A_w u := P\big(- \nu \Delta u - (w \times x) \cdot \nabla u + w \times u \big)$$

is no longer analytic, see T. Hishida (1999), but only strongly continuous; here P denotes the Helmholtz projection. Moreover, the spectrum of $-A_w$ contains an infinite set of equidistant half lines in the left complex half plane, cf. J. Neustupa, R. Farwig (2007). However, it is an open problem whether the spectrum equals this set of half lines when the obstacle is not axially symmetric.

In geophysics and atmospheric flows the Navier–Stokes system with initial values of rotational type are considered. Then a coordinate transform yields the Navier–Stokes equation

$$u_t - \nu \Delta u + u \cdot \nabla u + 2w \times u + \nabla p = F \tag{6}$$

with Coriolic force $2w \times u$, cf. (5). A. Babin, A. Mahalov and B. Nicolaenko (1999) have shown that for sufficiently large $|w|$ solutions to (6) are regular. In other words, a large Coriolis force may help to stabilize and regularize fluid flow. Moreover, it is well known that symmetry also helps to prove regularity of weak solutions to the Navier–Stokes system. Besides the trivial example of planar flow this remark also applies e.g. to helical flow in a pipe, cf. A. Mahalov, E. S. Titi, S. Leibovich (1990).

Recent progress on the above–mentioned topics will be discussed during the conference "Vorticité, Rotation et Symétrie – Stabilité des Ecoulements“ to be held at the *Centre International de Rencontres Mathématiques* (CIRM) in Luminy (Marseille), May 19 to May 23, 2008. Several talks concern the open problem of regularity of weak solutions to the Navier–Stokes system in three dimensions and discuss e.g. criteria using the vorticity. Based on the integral representation (4) it suffices to assume that the direction $\frac{\omega}{|\omega|}$ of the vorticity does not change too fast. This result well–known for the whole space problem cannot easily be transferred to domains. Moreover, the set of initial values to guarantee the existence of regular solutions in general or with further specific properties is analyzed. Further talks will show that the influence of the boundary itself and of the choice of boundary conditions is crucial to prove local and/or global regularity of weak solutions.

The second main topic concerns the flow around a single or several rotating obstacles – even including translational motions. Special emphasis is put on the interaction of the fluid with rigid bodies which are allowed to move freely with the flow and may approach each other or the wall in finite or infinite time. These results depend on the rheology of the fluid and on the shape of the bodies.

Besides non–Newtonian fluids, compressible fluid flow poses special problems to be discussed in this conference. In particular, the incompressible limit is analyzed from different points of view. Other kinds of nonlinearities have to be taken into account for turbulence models involving the turbulent kinetic energy, or for reactive flows. Finally, the analysis of fluid flow in unbounded domains requires special tools such as weighted estimates or the careful choice of function spaces adapted to the problem at hand and – for numerical computations – the approximation of the domain by bounded domains.

Serrin type regularity criterion and time singularity for the non-Newtonian flow

Hyeong-Ohk Bae

Ajou University
Suwon, Republic of Korea
hobae@ajou.ac.kr

Kyungkeun Kang and Jihoon Lee

Sungkyunkwan University
Suwon, Republic of Korea
kkang@skku.edu, jihoonlee@skku.edu

We show that there exists a strong solution to a class of non-Newtonian flows if it satisfies Serrin type condition. We estimate the Hausdorff dimension of the set of singular times for the weak solutions.

Instabilities in the incompressible Euler equation

Claude Bardos

Université Denis Diderot Paris
Laboratoire JL Lions, 175 rue du Chevaleret, 75013 Paris, FRANCE
claude.bardos@gmail.com

Many (not all) instabilities for Navier-Stokes equations are related to the corresponding behavior at the level of the Euler equation. Therefore I intend to discuss old and recent results for the Euler equation and try to show how they may improve our mathematical approach of the subject.

Vorticity and regularity for the Navier-Stokes equations

Hugo Beirão da Veiga

University of Pisa
Department of Applied Mathematics "U.Dini"
via F. Buonarroti 1/C I 56127 Pisa ITALY
bveiga@dma.unipi.it

In our talk we consider Leray-Hopf weak solutions to initial boundary value problems for the 3D Navier-Stokes equations. The physical domain is a bounded open set Ω with a smooth boundary $\partial\Omega$, the half-space $\mathbb{R}^3_+$, or the whole of $\mathbb{R}^3$. We denote by u the velocity, by p the pressure and by $\omega = \nabla \times u$ the vorticity field. Furthermore, $\theta(x, y, t)$ denotes the angle between the vorticity ω at two distinct points x and y, at time t, i.e., $\theta(x, y, t) \overset{def}{=} \angle(\widehat{\omega}(x, t), \widehat{\omega}(y, t))$, where $\widehat{v} \overset{def}{=} v/|v|$. The initial data u_0 satisfies $u_0 \in H^1(\Omega)$ and $\nabla \cdot u_0 = 0$. We address here the problem of global existence of smooth solutions, under additional hypotheses on the vorticity-direction.

The study of conditions involving the direction of vorticity, and its physical-geometric interpretation, started with Constantin and Fefferman [5], who first derived some exact formulas and employed them in order to prove regularity in the whole of $\mathbb{R}^3$. In particular, in the fundamental paper [5], it is proved that if $\sin\theta(x, y, t) \leq c\,|x - y|$, for a.a. $x, y \in \mathbb{R}^3$ and $t \in]0, T[$, then the solution u is strong in $[0, T]$ and, consequently, is regular.

Later on, this result has been improved by Berselli and us in reference [3], by replacing the above Lipschitz condition by a 1/2-Hölder condition. More precisely, if $\sin\theta(x, y, t) \leq c\,|x - y|^{1/2}$, *for* a.a. $x, y \in \mathbb{R}^3$ and $t \in]0, T[$, then the solution u is necessarily regular.

More recently, see [1], we extended the 1/2-Hölder condition to solutions in the half-space $\Omega = \mathbb{R}^3_+$ of the Navier (or slip) boundary condition $u\cdot n = 0$, $n\cdot\nabla u - (n\cdot\nabla u\cdot n)\,n = 0$, where n denotes the exterior unit normal vector. Note that, on flat boundaries, the slip boundary condition is equivalent to the stress-free boundary condition $u \cdot n = 0$, $\omega \times n = 0$. In reference [1] the above result is proved by appealing, separately, to the classical Dirichlet and Neumann Green's functions, in the half space. This separation can be done since, for flat boundaries, the above boundary conditions are equivalent to $\omega_1 = \omega_2 = 0$, plus $\frac{\partial\,\omega_3}{\partial\,x_3} = 0$.

Finally, in reference [4], in collaboration with Berselli, we extend the above regularity result to arbitrary, regular, open sets Ω. Since the boundary is not flat, we have to localize the problem, a not trivial and quite technical matter. In the very fundamental contribution [7], Solonnikov constructs global Green's matrices for a large class of boundary value problems and systems of partial differential equations. Our problem falls within this class. The first step in [7] consists in constructing a local version of Green's matrices, in a neighborhood of each boundary point. With the help of these local kernels, Solonnikov constructs the global kernel. Unfortunately, it seems not possible to treat our problem by applying directly to the global Green's matrices. Hence, in [4] we appeal to the "local" Green's matrices.

It is of interest to compare the above situation with that faced in the presence of a Dirichlet boundary condition. In spite of the arbitrary boundary, in reference [1] the fundamental integral estimate of $(\omega(x) \cdot \nabla u(x)) \cdot \omega(x)$ is proved by appealing directly to the global Green's function for the Dirichlet problem, without the need of a localization argument. However, a new obstacle appears. Integration in Ω of the scalar product $-\Delta\,\omega \cdot \omega$ gives rise to the boundary integral of $\frac{\partial \omega}{\partial n} \cdot \omega$. Under the stress-free boundary condition we are able to estimate this term in a suitable way (if the boundary is flat, see [1], the above integral vanishes). On the contrary, under the Dirichlet boundary condition, a suitable additional assumption on the above boundary integral seems necessary.

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Some geometric constraints and the problem of global regularity for the Navier-Stokes equations

Luigi C. Berselli

Università di Pisa,
Dipartimento di Matematica Applicata "U. Dini," Via F. Buonarroti 1/c
56127 Pisa, ITALY.
berselli@dma.unipi.it

We consider the Cauchy problem for the 3D Navier-Stokes equations and study geometric criteria similar to that introduced by Constantin [1] Constantin and Fefferman [2] and later analyzed (among the others) by Beirão da Veiga and Berselli [3,4]. We show that weak solutions satisfying suitable geometric conditions are smooth. The first condition we consider requires the vorticity's direction at the point y (near to x) to be within a circular cone of given *small* amplitude, with vertex at x, and axis along the vorticity's directions at x. Hence, vorticity's direction need not to be (Hölder) continuous to ensure regularity, as in the previous results. Our new results show that a possible singularity deriving from non-trivial collision of strong vortex tubes is depleted if the angle formed by the tubes is *small* enough.

Other conditions regarding the direction of the curl of the vorticity are considered: it is shown that if the direction of curl of the vorticity at point x is either "nearly parallel" or "nearly orthogonal" to the same quantity at neighboring points, then weak solutions are smooth.

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Concentration-diffusion effects in viscous incompressible flows

Lorenzo Brandolese

Université de Lyon, Université Lyon 1, CNRS UMR 5208 Institut Camille Jordan,
43, bd. du 11 novembre 1918, 69622 Villeurbanne Cedex, FRANCE
brandolese@math.univ-lyon1.fr

One of the most important questions in mathematical Fluid Mechanics, which is still far from being understood, is to know whether a finite energy, and initially smooth, nonstationary Navier–Stokes flow will always remain regular during its evolution, or can become turbulent in finite time.

As a first step toward the understanding of possible blow-up mechanisms, it is interesting to exhibit examples of smooth and decaying initial data such that, even if the corresponding solutions remain regular for all time, "something strange" happens around a given point (x_0, t_0) in space-time. This is the goal of the present talk.

Given a finite sequence of times $0 < t_1 < \cdots < t_N$, we construct an example of a smooth solution of the nonstationary Navier–Stokes equations in $\mathbb{R}^3$ such that, in the absence of any external force: (i) The velocity field $u(x,t)$ is spatially poorly localized at the beginning of the evolution but tends to concentrate until, as the time t approaches t_1, it becomes well-localized. (ii) Then u spreads out again after t_1, and such concentration-diffusion phenomena are later reproduced near the instants t_2, t_3, ...

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The problem of global wellposedness for incompressible 3D viscous fluids

Jean-Yves Chemin

Université Pierre et Marie Curie
Laboratoire JL Lions, 175 rue du Chevaleret,
75013 Paris, FRANCE
chemin@ann.jussieu.fr

In this talk, we first present an overview of the series of works about global wellposedness for small data for the incompressible Navier-Stokes equations. This will be the opportunity to define the concept of large data. After that we construct a large class of examples of (very) large data which generate global smooth solutions. These data are slowly varying in one space variable. The proof is based of a method that involves analytic smoothing effect of the Laplacian.

On the Boltzmann equations for mixed molecules

Hi Jun Choe

Yonsei University
Seoul, Republic of Korea
choe@yonsei.ac.kr

In this talk, we consider a collision model for two different species fluid. Molecules are assumed hard spheres of two different radii and masses. For the simplicity, one species has a nice distribution like the Maxwellian and we consider only binary collision. These assumptions are well established for the dilute neutral gases. With these assumptions, the collision dynamics are governed by Boltzmann equations. We will also show there is an equilibrium under a mild assumption.

Asymptotic limits in the theory of viscous fluids

Eduard Feireisl

Institute of Mathematics AV ČR
Žitná 25, 115 67 Praha 1, Czech Republic
feireisl@math.cas.cz

We discuss several problems connected with singular limits of solutions to the complete Navier-Stokes-Fourier system describing the motion of a general compressible, viscous, and heat conducting fluid. The main topics include:

(i) The incompressible limit for small Mach and Froude numbers, where the limit system is represented by the Oberbeck-Boussinesq approximation.

(ii) The incompressible limit under strong stratification leading to the so-called anelastic approximation.

(iii) Attenuation and refined analysis of the acoustic waves.

A free boundary problem related to the spin-coating process

Matthias Geissert

TU Darmstadt
Schlossgartenstr. 7, 64289 Darmstadt, Germany
geissert@mathematik.tu-darmstadt.de

We consider the spin-coating process which is described by the Navier-Stokes equations in a layer-like domain in $\mathbb{R}^3$ in the rotating setting. Our model takes into account Coriolis forces, centrifugal forces as well as surface tension on the free boundary. On the fixed boundary we prescribe Robin boundary conditions.

Our aim is to show local existence and uniqueness of strong solutions. In order to do so, we transform this problem to a fixed layer by the Hanzawa transform and show maximal regularity estimates for a suitable linearized problem.

Identification problems for the Stokes equations

Horst Heck

TU Darmstadt
Schlossgartenstr. 7, 64289 Darmstadt, Germany
heck@mathematik.tu-darmstadt.de

In this talk we consider two types of identifiability results for the stationary Stokes equations. We present a reconstruction procedure for the location of an obstacle that is immersed in the fluid by using measurements at the boundary of the domain. These measurements are given by the Dirichlet data and the resulting Cauchy forces. The principal idea of our reconstruction method is based on the use of special solutions of the Stokes system.

As a second identification problem we show that special solutions could also be used to determine the (possibly varying) viscosity parameter in the Stokes equations.

The presented results are joint works with X. Li, G.Uhlmann and J.-N. Wang.

The Fujita-Kato approach to the equations of Navier-Stokes in the rotational setting

Matthias Hieber

TU Darmstadt
Schlossgartenstr. 7, 64289 Darmstadt, Germany
hieber@mathematik.tu-darmstadt.de

In this talk we consider the Navier-Stokes equations in the rotational setting. In the case where the underlying domain is all of $\mathbb{R}^3$, we show that there exists a unique, global mild solution to this problem for arbitrary speed of rotation, provided the initial data are sufficiently small in the $H^{1/2}$-norm. We study further the stability of particular stationary solutions in the case of the halfspace.

Asymptotic profile of the steady Stokes flow around a rotating obstacle

Reinhard Farwig

Fachbereich Mathematik, Technische Universität Darmstadt
64289 Darmstadt, Germany
farwig@mathematik.tu-darmstadt.de

Toshiaki Hishida

Graduate School of Mathematics, Nagoya University
Nagoya 464-8602, Japan
hishida@math.nagoya-u.ac.jp

Consider the motion of a viscous incompressible fluid around a rotating obstacle. One may expect that the rotation effect causes a certain anisotropic decay structure of the flow with respect to space variable. In this talk we find such a structure from asymptotic expansion of the steady Stokes flow at infinity.

On very weak solutions of the stationary Navier-Stokes equations

Hyunseok Kim

Sogang University
Department of Mathematics, Seoul, Republic of Korea
kimh@sogang.ac.kr

In this talk, we study the solvability of the stationary Navier-Stokes equations with irregular boundary data. If the boundary data is sufficiently regular, that is, belongs to the boundary Sobolev space of half order, the existence of a weak solution of the stationary Navier-Stokes equations has been well-known since the fundamental work by Leray [1]. The concept of very weak solutions has been introduced to solve the Navier-Stokes equations when the boundary data is irregular. The first existence result on very weak solutions was obtained by Marusic-Paloka [2] for boundary data in the Lebesgue space L^2. The purpose of the talk is not only to extend his existence result to the more general boundary data in Sobolev spaces of negative order but also to prove some regularity results on very weak solutions. Our results are based on recent results on very weak solutions of the Stokes equations by Galdi, Simader and Sohr [4] and Farwig, Galdi and Sohr [5].

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L_p-estimates for the Oseen system with slip boundary conditions

Paweł Konieczny

Institute of Applied Mathematics and Mechanics
Warsaw University
ul. Banacha 2, 02-097 Warszawa, Poland.
pk@mimuw.edu.pl

We investigate the Oseen system in a halfspace $\mathbb{R}^2_+$ considered with nonhomogeneous slip boundary conditions. This problem is the main contribution for considerations for the Oseen system in exterior domains. We show optimal L_p-estimates for the second gradient of the velocity – $\nabla^2 v$ and for the gradient of the corresponding pressure – ∇p. We use the Fourier Transform to obtain a system of ordinary differential equations and then we use the Multiplier Theorems of Marcinkiewicz and Lizorkin to show proper estimates.

Global DIV-CURL Lemma on bounded domains in $\mathbb{R}^3$

Hideo Kozono

Mathematical Institute, Tohoku University
Sendai 980-8578, Japan
kozono@math.tohoku.ac.jp

Taku Yanagisawa

Department of Mathematics, Nara Women's University
Nara 630-8506, Japan
taku@cc.nara-wu.ac.jp

We consider a global version of the Div-Curl lemma for vector fields in a bounded domain $\Omega \subset \mathbb{R}^3$ with the smooth boundary $\partial\Omega$. Suppose that $\{u_j\}_{j=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ converge to u and v weakly in $L^r(\Omega)$ and $L^{r'}(\Omega)$, respectively, where $1 < r < \infty$ with $1/r + 1/r' = 1$. Assume also that $\{\text{div } u_j\}_{j=1}^{\infty}$ and $\{\text{rot } v_j\}_{j=1}^{\infty}$ are bounded in $L^r(\Omega)$ and $L^{r'}(\Omega)$, respectively. If either $\{u_j \cdot \nu|_{\partial\Omega}\}_{j=1}^{\infty}$ is bounded in $W^{1-1/r,r}(\partial\Omega)$, or $\{v_j \times \nu|_{\partial\Omega}\}_{j=1}^{\infty}$ is bounded in $W^{1-1/r',r'}(\partial\Omega)$ (ν: unit outward normal to $\partial\Omega$), then it holds that

$$\int_\Omega u_j \cdot v_j dx \to \int_\Omega u \cdot v dx.$$

In particular, if either $u_j \cdot \nu = 0$ or $v_j \times \nu = 0$ on $\partial\Omega$ for all $j = 1, 2, \cdots$ is satisfied, then we have that

$$\int_\Omega u_j \cdot v_j dx \to \int_\Omega u \cdot v dx.$$

As an immediate consequence, we prove the well-known Div-Curl lemma for any open set in $\mathbb{R}^3$. The Helmholtz-Weyl decomposition for $L^r(\Omega)$ plays an essential role for the proof.

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Initial conditions of strong solutions of the Navier-Stokes equations

Petr Kučera

Czech Technical University
Prague
kucera@mat.fsv.cvut.cz

We show that there exist strong solutions of the Navier-Stokes initial boundary value problem solved in an arbitrary open domain such that their initial velocity is arbitrary large (in the norm $\|.\| + \|A^\alpha.\|$, where A^α is a fractional power of the Stokes operator A, $1/4 < \alpha \leq 1/2$), and they belong to an arbitrary chosen set U which is open in the domain of the operator A^γ, $1 > \gamma > 3/4$ at a time instant $\xi > 0$ which can be as small as we wish.

On the steady motion of an elastic body moving freely in a Navier-Stokes liquid under the influence of a constant body force

Mads Kyed

RWTH Aachen
Institut für Mathematik, Templergraben 55, 52062 Aachen, Germany
kyed@instmath.rwth-aachen.de

In my talk I will outline a proof of existence of a steady motion of an elastic body moving freely in a Navier-Stokes liquid under the influence of a constant body force. One may think of the free fall, due to gravity, of an elastic body in a liquid as an example hereof. When moving freely in a liquid, an elastic body will deform in response to the forces exerted on it by the fluid flowing past it. Furthermore, the body may, in addition to translation, perform a rotation. We shall say that the body is performing a steady free fall if, in a frame attached to the body, the time independent equation of motion possesses a solution. In my talk I will show that such a solution exists, provided the density of the elastic material is sufficiently small and the reference domain of the body satisfies a certain geometric property. This result is part of a joint work with G.P. Galdi.

The Hodge-Navier-Stokes equations in bounded Lipschitz domains

Sylvie Monniaux

Université Paul Cézanne - Aix-Marseille 3
Case cour A - FST Saint-Jérôme, Avenue Escadrille Normandie-Niémen
13397 Marseille Cédex 20, France
sylvie.monniaux@univ-cezanne.fr

In this talk, I will present some results obtained in collaboration with Marius Mitrea and Steve Hofmann from the University of Missouri, Columbia. The problem reads as follows. We consider Navier-Stokes equations in a bounded 3D domain Ω, and instead of Dirichlet boundary conditions, we are interested in the following so-called Hodge boundary conditions.

$$(HNS)\ \left\{\begin{array}{rcll} \dfrac{\partial u}{\partial t} - \Delta u + \nabla\tilde{\pi} + u \times \mathrm{curl\ u} & = & 0 & \text{in }]0,T] \times \Omega \\ \mathrm{div\ u} & = & 0 & \text{in }]0,T] \times \Omega \\ \nu \cdot u & = & 0 & \text{on}]0,T] \times \partial\Omega \\ \nu \times \mathrm{curl\ u} & = & 0 & \text{in }]0,T] \times \partial\Omega \\ u(0,x) & = & u_0(x), & x \in \Omega. \end{array}\right.$$

I will show that for all divergence-free vector field u_0 in the critical space $L^3(\Omega;\mathbb{R}^3)$ with sufficiently small norm in L^3, there exists a global smooth integral solution to (HNS). The strategy is first to study the linear part of the equation via the Hodge-Laplacian and prove that the operator involved generates an analytic semigroup in L^p for p in a suitable interval containing $[\frac{3}{2}, 3]$. Then, we apply the Fujita-Kato method, a fixed point method, to prove the existence of the solution.

On the existence of weak solutions to a stationary Navier-Stokes system coupled with an equation for the turbulent kinetic energy

Pierre-Étienne Druet

Weierstrass Institute for Applied Analysis and Stochastics, Berlin
druet@wias-berlin.de

Joachim Naumann

Department of Mathematics, Humboldt-University Berlin
jnaumann@math.hu-berlin.de

In a bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 2$ or $n = 3$) we consider the following model for the stationary turbulent motion of a viscous incompressible fluid:

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, && (7) \\ (\mathbf{u} \cdot \nabla)\mathbf{u} &= \operatorname{div}\Big((\nu + \nu_T(k)) D(\mathbf{u}) \Big) - \nabla p + \mathbf{f}, && (8) \\ \mathbf{u} \cdot \nabla k &= \operatorname{div}\Big((\nu + \nu_T(k)) \nabla k \Big) + \nu_T(k) D(\mathbf{u}) : D(\mathbf{u}) - g(k)\sqrt{k}, && (9) \end{aligned}$$

where: $\mathbf{u} = (u_1, \ldots, u_n)$ velocity, $p =$ pressure, k mean turbulent kinetic energy, $D(\mathbf{u}) = \frac{1}{2}\Big(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\Big)$ rate of strain, and $\nu = \text{const} > 0$ dynamical viscosity, $\nu_T(k) =$ coefficient of eddy viscosity, $\mathbf{f} =$ external force. The term $g(k)\sqrt{k}$ represents a generalization of the energy dissipation at small length scales.

The growth conditions on $\nu_T(k)$ that we consider include the classical model $\nu_T(k) = C_0\sqrt{k}$ ($C_0 = \text{const} > 0$).

We complete (1)-(3) by the following boundary conditions:

$$\mathbf{u} = 0, \quad k = k_0 \quad \text{on } \partial\Omega \quad (k_0 \geq 0 \text{ on } \partial\Omega). \tag{10}$$

We prove:

1. Existence of a weak solution to (1)-(4) where a defect measure occurs in the weak formulation of (3);
2. existence of a weak solution to (1)-(4) with $\operatorname{ess\,sup}_{\Omega} k < +\infty$ for small $\|\mathbf{f}\|$.

On the motion of several rigid bodies in an incompressible non-Newtonian fluid

Eduard Feireisl

Institute of Mathematics of the Academy of Sciences of the Czech Republic,
Zitná 25, 115 67 Praha 1, Czech Republic,
feireisl@math.cas.cz

Matthieu Hillairet

Université de Toulouse, IMT équipe MIP,
118, route de Narbonne, 31062 Toulouse cedex, France,

Šárka Nečasová

Institute of Mathematics of the Academy of Sciences of the Czech Republic,
Zitná 25, 115 67 Praha 1, Czech Republic,
matus@math.cas.cz

The motion of one or several rigid bodies in a viscous fluid occupying a bounded domain $\Omega \subset R^3$ represents an interesting theoretical problem featuring, among others, possible contacts of two or more solid objects. For viscous fluids, the standard hypothesis asserts the no-slip condition on the boundaries of the rigid bodies, meaning the velocity of the fluid coincides with that of the body. Accordingly, a collision of two rigid objects immersed in the fluid causes a singular situation when the velocity gradient necessarily exhibits abrupt changes resulting in extremely large stresses due to viscosity.

In work of Starovoitov see, [2], was shown that collisions, if any, must occur with the zero relative *translational* velocity as soon as the boundaries of the rigid objects are smooth and the gradient of the underlying velocity field is square integrable - a hypothesis satisfied by any Newtonian fluid flow of finite energy. The possibility or impossibility of collisions in viscous fluids is related to the properties of the velocity gradient.

Intuitively, the contacts should not occur in certain *non-Newtonian* fluids, where they can be eliminated by the phenomenon of shear-thickening. Motivated by this result, we consider the motion of several rigid bodies in a non-Newtonian fluid of a power-law type. Our main result establishes the existence of global-in-time solutions of the associated evolutionary system, where, in accordance with [2], collisions of two or more rigid objects do not appear in a finite time unless they were present initially.

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Low Mach number limits in the thermodynamics of compressible fluids

Antonin Novotny

Université du Sud Toulon-Var
BP 20132, 83957 La Garde, France
novotny@univ-tln.fr

The investigation of low Mach but also of low Froude and low Péclet number compressible flows is an actual research topic with numerous applications in meteorology, physics of atmosphere, astrophysics, acoustics, ...

In this contribution, we shall discuss some of singular limits arising in this context in the complete Navier-Stokes-Fourier system describing viscous compressible heat conducting flows or in the Navier-Stokes-Poisson system describing viscous compressible flows in isentropic regime. These investigations are carried out in the context of weak solutions on an arbitrary time interval and for ill-prepared initial data.

Multicomponent reactive flows: Global-in-time existence for large data

Hana Petzeltová

Institute of Mathematics, AS CR
Žitn a 25, 11567 Praha 1, Czech Republic
petzelt@math.cas.cz

Eduard Feireisl

Institute of Mathematics, AS CR
Žitná 25, 11567 Praha 1, Czech Republic
feireisl@math.cas.cz

Konstantina Trivisa

Department of Mathematics, University of Maryland
College Park, MD 20742-4015, U.S.A.
trivisa@math.umd.edu

We show the existence of weak solutions to a system describing multicomponent reactive flows. The main ingredients of our approach can be formulated as follows:

- A suitable *variational formulation* of the underlying physical principles based on the second law of thermodynamics, in particular, replacing the energy balance by the corresponding equation for the total entropy of the system.
- Physically grounded structural hypotheses imposed on the thermal equation of state for the pressure p. In particular, the effect of radiation, significant in the high temperature regime, is taken into account.
- *A priori* estimates based solely on boundedness of the initial energy and entropy of the system. As a matter of fact, this step requires the transport coefficients μ, κ, and D_k to be effective functions of the absolute temperature.
- The weak stability property of the effective viscous pressure established by Lions, its generalization to non-constant viscosity coefficients combined with the approach based on the oscillation defect measures.

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Properties of solutions to the Stokes problem in parabolically growing layer

Konstantin Pileckas

Faculty of Mathematics and Informatics, Vilnius University
Naugarduko Str., 24, Vilnius, 2006 Lithuania
pileckas@ktl.mii.lt

The Stokes problem is studied in the domain $\Omega \subset \mathbb{R}^3$ coinciding outside the ball $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$ with the parabolically growing layer $\mathbb{L} = \{x \in \mathbb{R}^3 : x' = (x_1, x_2) \in \mathbb{R}^2,\ |x_3| < h(x')\}$, where $h(x')$ is a smooth function,

$$h(x') \geq h_0 > 0 \quad \forall x' \in \mathbb{R}^2, \qquad h(x') = |x'|^\beta \equiv r^\beta,\ \beta \in (0,1), \quad \text{for} \quad r > R.$$

Asymptotics of a solution to the Stokes problem is constructed. In order to justify the asymptotics, coercive estimates are proved in the scale of weighted function spaces with the norm determined by a stepwise anisotropic distribution of weight factors.

These results were obtained jointly with L. Zaleskis.

Elliptic problems in the half space

Chérif Amrouche

Université de Pau et des Pays de l'Adour
Laboratoire de Mathématiques Appliquées
cherif.amrouche@univ-pau.fr

Yves Raudin

Université de Pau et des Pays de l'Adour
Laboratoire de Mathématiques Appliquées
yves.raudin@univ-pau.fr

The aim of this talk is the resolution of some elliptical problems in the half space $\mathbb{R}^N_+$, with $N \geq 2$. Using the Dirichlet and Neumann problems for the Laplace operator (see [1], [2]), we give existence, uniqueness and regularity results in L^p theory for the biharmonic and Stokes problems (see [3], [4] and [5]). For that, we consider data and give solutions which live in weighted Sobolev spaces. We assume that the boundary conditions are nonhomogeneous and we also take them in weighted Sobolev spaces. An important aspect of this study is the case of singular boundary conditions and the very weak solutions which correspond to it. We also treat the question of non standard boundary conditions.

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A direct approach to the vorticity transport & diffusion equation

Reimund Rautmann

University of Paderborn
Warburger Straße 100, 33098 Paderborn
rautmann@uni-paderborn.de

The evolution in time of the vorticity w of incompressible viscous flow in a bounded 3-dimensional domain Ω is governed by the initial boundary value problem of the vorticity transport & diffusion equation

$$(1) \qquad \begin{cases} \frac{\partial}{\partial t} w - \Delta w = w \cdot \bigtriangledown v - v \cdot \bigtriangledown w, \\ \text{div } w = 0, \\ w(0, \cdot) = w_0, \\ v(t, x) = (\text{rot}^{-1} w)(t, x), \end{cases}$$

the function $v(t, x)$ denoting the flow velocity.
The usual condition of adherence

$$(2) \qquad v(t, x) \mid_{\partial\Omega} = 0$$

for the flow velocity at the boundary $\partial\Omega$ constitutes a nonlocal boundary condition in terms of w. Equations (1) result from the equations of the Navier-Stokes initial boundary value problem for the function $v(t, x)$ by formal application of operator rot in case v being sufficiently smooth. Evidently system (1), (2) in itself does not imply the compatibility condition due to the pressure gradient occurring in the Navier-Stokes equations which there severely restricts the solutions initial regularity.

Having introduced suitable solution spaces in which the nonlocal boundary condition (2) for w holds true we get the uniqueness of generalized solutions to problem (1), (2) as well as the local in time existence of a unique strong solution which even exists globally in case of sufficiently small initial data.

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Large time regular solutions to the Navier-Stokes equations in cylindrical domains

Joanna Rencławowicz

Institute of Mathematics, Polish Academy of Sciences,
Śniadeckich 8, 00-956 Warsaw, Poland
jr@impan.gov.pl

Wojciech M. Zajączkowski

Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-956 Warsaw, Poland
wz@impan.gov.pl

We prove the large time existence of solutions to the Navier-Stokes equations with slip boundary conditions in a cylindrical domain. Assuming smallness of L_2-norms of derivatives of initial velocity with respect to variable along the axis of the cylinder, we are able to obtain estimate for velocity in $W_2^{2,1}$ without restriction on its magnitude. Then existence follows from the Leray-Schauder fixed point theorem.

Existence and regularity of solutions to the rotating or straining flows in the whole space

Okihiro Sawada

RIMS, Kyoto University,
Kitashirakawa, Sakyo-ku, Kyoto, 606-8502, Japan
sawada@kurims.kyoto-u.ac.jp

The local-in-time solutions to the Navier-Stokes equations in the whole space $\mathbb{R}^n$ are constructed when the initial velocity is given by $f + u_0$. Here, $f \in C^2_\sigma(\mathbb{R}^n)$ is a globally Lipschitz continuous function (may grow up at spatial infinity, linearly); u_0 is an initial disturbance in some function space, e.g. $u_0 \in L^q_\sigma(\mathbb{R}^n)$ with $q \in [n, \infty)$. For the case $f(x) = Mx$ with skew-symmetric matrix M the rotating fluid (or, rigid body) is demonstrated. When M is diagonal, this illustrates a straining flow. Our approach is based on the Ornstein-Uhlenbeck semigroup theory. It is also shown that when $f(x) = Mx$ the solution is unique, smooth, and satisfies the equations in the classical sense provided the pressure term is set suitably.

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On the Navier-Stokes equations around a rotating obstacle

Ana Leonor Silvestre

Instituto Superior Técnico, Technical University of Lisbon
Av. Rovisco Pais, 1049–001 Lisboa, Portugal
`Ana.Silvestre@math.ist.utl.pt`

Consider a Navier-Stokes liquid filling the whole 3–dimensional space exterior to a rotating body with constant angular velocity. This talk is devoted to the associated whole space problem, including the following topics

1. existence and uniqueness of steady-state solutions with finite energy, for certain conditions on the forcing function,
2. attainability of such steady solutions as limits of nonstationary ones,
3. nonlinear stability of these steady motions.

We present some recent results concerning the above questions. The study of the steady problem is based on the integral representation of the solution in terms of the corresponding fundamental solution, which allows to derive the asymptotic behavior of the velocity. The unsteady problems are studied within the framework of L^2 spaces.

A further property of the Cosserat operator and its application to regularity of weak solutions of Stokes' problem and related equations

Christian G. Simader

University of Bayreuth,
95440 Bayreuth, Germany
christian.simader@uni-bayreuth.de

For sake of simplicity we regard bounded domains $G \subset \mathbb{R}^n$ $(n \geq 2)$ with sufficiently smooth boundaries ∂G. Let for $1 < q < \infty$

$$L_0^q(G) := \left\{ p \in L^q(G) : \int_G p \, dy = 0 \right\}.$$

For $p \in L_0^q(G)$ we regard weak solutions $\underline{v} = (v_1, \ldots, v_n) \in \underline{H}_0^{1,q}(G) := \underline{H}_0^{1,q}(G)^n$ of the n weak Dirichlet problems

$$\sum_{i,k=1}^{n} \int_G \partial_i v_k \partial_i \phi_k =: \langle \nabla \underline{v}, \nabla \underline{\phi} \rangle = \langle p, \operatorname{div} \underline{\phi} \rangle := \int_G p \operatorname{div} \underline{\phi} \tag{11}$$

$\left(\underline{\phi} \in \underline{H}_0^{1,q'}(G), q' := \frac{q}{q-1} \right)$.

In the sense of a direct ($q = 2$ orthogonal) decomposition it holds

$$L_0^q(G) = A^q(G) \oplus B_0^q(G)$$

where $A^q(G) := \left\{ \Delta s : s \in H_0^{2,q}(G) \right\}$
and $B_0^q(G) := \left\{ p \in L_0^q(G) : \langle p, \Delta s \rangle = 0 \quad \forall s \in H_0^{2,q'}(G) \right\}$.

This decomposition is equivalent to the weak Dirichlet problem in L^q for $\Delta 2$. Clearly, for $p_0 = \Delta s \in A^q(G)$, the corresponding solution of (11) is given by $\underline{v}_0 := \nabla s \in \underline{H}_0^{1,q}(G)$. Now for $p_h \in B_0^q(G)$ let $\underline{v}_h$ denote the unique solution of (11) and let $Z_q p_h := \operatorname{div} \underline{v}_h$. Then it is readily seen that for the Cosserat operator holds true $Z_q : B_0^q(G) \to B_0^q(G)$. From an estimate by St. Weyers it follows easily that for $p_h \in B_0^q(G) \cap H^{k-1,q}(G)$ $(k \in \mathbb{N})$ that $Z_q(p_h) - \frac{1}{2} p_h \in B_0^q(G) \cap H^{k,q}(G)$ and with a constant $C_k = C(k, q, G) > 0$

$$\| Z_q(p_h) - \frac{1}{2} p_h \|_{H^{k,q}(G)} \leq C_k \| p_h \|_{H^{k-1,q}(G)} \tag{12}$$

for all $p_h \in B_0^q(G) \cap H^{k-1,q}(G)$. By means of this regularizing property it is easy to reduce the problem of regularity of weak solutions of Stokes', Stokes-like and Lamé's system to the well-known corresponding results for the scalar Laplacian and Bilaplacian equation.

Asymptotic dynamics of frequencies in global strong solutions of the Navier-Stokes equations

Zdeněk Skalák

Czech Technical University
Thákurova 7, 16612 Prague 6
skalak@mat.fsv.cvut.cz

If u is a global strong solution of the homogeneous Navier-Stokes equations then u decreases exponentially (that is $\lim_{t\to\infty} e^{\lambda t}||u(t)|| = 0$ for some $\lambda > 0$) if and only if lower frequencies in u die out quickly as t goes to infinity: $\lim_{t\to\infty} e^{\lambda t}||E_\lambda u(t)||/||u(t)|| = 0$, where $\{E_\lambda; \lambda \geq 0\}$ is the resolution of identity of the Stokes operator A_2. There is also a close relationship between the spectrum $\sigma(A_2)$ of A_2 and the class of exponentially decreasing global strong solutions: $\lambda > 0$ is from $\sigma(A_2)$ if and only if there exists a global strong solution u such that $\lim_{t\to\infty} e^{\mu t}||u(t)|| = 0$ for every $\mu \in (0, \lambda)$ and $\limsup_{t\to\infty} e^{\mu t}||u(t)|| > 0$ for every $\mu \in (\lambda, \infty)$.

On the other hand there exists a class of global strong solutions in which higher frequencies disappear asymptotically in time: $\lim_{t\to\infty} ||E_\lambda u(t)||/||u(t)|| = 1$ for every $\lambda > 0$.

Necessary and sufficient initial value conditions for local strong solutions of the Navier-Stokes equations

Hermann Sohr

University of Paderborn,
Department of Mathematics, 33098 Paderborn, Germany
hsohr@math.uni-paderborn.de

Reinhard Farwig

University of Darmstadt,
Department of Mathematics, 64289 Darmstadt, Germany
farwig@mathematik.tu-darmstadt.de

Since several years there have been developed various sufficient conditions on initial values u_0 for the existence of local strong solutions u of the Navier-Stokes equations. Some important results in this context are proved by

Kiselev/Ladyzhenskaya (1957) with $u_0 \in D(A)$,
Fujita/Kato (1964) with $u_0 \in D(A^{\frac{1}{4}})$, and by
Fabes/Jones/Rivière (1973), Kato (1984), Giga (1986) with $u_0 \in L^q_\sigma$, $q \geq 3$.

Our aim is to show that the condition

$$\int_0^\infty \|e^{-tA}u_0\|_q^s \, dt < \infty, \quad \frac{2}{s} + \frac{3}{q} = 1,$$

is sufficient *and* necessary for the existence of a local strong solution u. This condition is (strictly) weaker than the known conditions and yields the largest possible class of local strong solutions u with $u_{|t=0} = u_0$. As a corollary we obtain some further sufficient conditions on u_0 which are weaker than the known conditions. An equivalent formulation of this condition in Besov spaces of negative order is given by $u_0 \in \mathbb{B}_{q,s}^{-2/s}$. In these results the underlying domain $\Omega \subseteq \mathbb{R}^3$ is smooth and bounded. For completely general domains $\Omega \subseteq \mathbb{R}^3$ we can show that the condition $\int_0^\infty \|e^{-tA}u_0\|_q^s \, dt$ with $q = 4$, $s = 8$, is necessary and sufficient. The reason for this restriction is that in this case the theory of the Stokes operator A is available essentially only in L^2-spaces.

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The approximation of stationary flows in perturbed layers

Maria Specovius-Neugebauer

University of Kassel
Fachbereich 17, Mathematik, Universität Kassel, 34109 Kassel, Germany
specovi@mathematik.uni-kassel.de

Sergej Nazarov

Institute of Mechanical Engineering Problems, St. Petersburg, Russia
serna@snark.ipme.ru

To approximate solutions of boundary value problems in unbounded domains there exists mainly two options: either boundary integral methods or the construction of so called artificial boundary conditions. While the first method is often used for exterior boundary value problems, one is stuck with the second option if the boundary of the starting domain Ω is non compact. The basic idea there is to consider the boundary value problem on a bounded domain Ω_R, where R usually is a large parameter such that $\Omega = \bigcup_{R>R_0} \Omega_R$. On the truncation surface $\partial\Omega_R \setminus \partial\Omega$ an additional boundary condition has to be imposed. In the present lecture we present artificial boundary value conditions for the Stokes and Navier-Stokes problem with Dirichlet boundary conditions in a layer like domain, i.e $\Omega \subset \mathbb{R}^3$ is a domain with a smooth boundary $\partial\Omega$, and Ω coincides with the layer $\Lambda = \left\{x = (y, z) : y = (y_1, y_2) \in \mathbb{R}^2, |z| < \frac{1}{2}\right\}$ outside the ball $\mathbb{B}_{R_0} = \{x \in \mathbb{R}^3 : |x| < R_0\}$ of radius $R_0 > 1$. The domain Ω_R is defined as an intersection of Ω with a cylinder of radius R. The artificial boundary conditions on the lateral surface Γ_R involve the Steklov-Poincaré operator on a circle together with its inverse and thus turn out to be a combination of local and nonlocal boundary operators. Their construction is based on asymptotic representations which were obtained in [1],[2]. The proof for the existence and uniqueness of solutions to the approximation problem involves a non-trivial result on solutions to the continuity equation together with estimates which control the dependence of certain Sobolev norms on R.

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Recent progress in the mathematical theory of 1D barotropic flow and some historical remarks

Ivan Straškraba

Mathematical Institute of the Academy of Sciences of the Czech Republic
Žitná 25, 11567 Praha 1, Czech Republic
strask@math.cas.cz

In this talk, recent contributions to the 1D-theory of barotropic fluids will be presented. The governing equations used are of the following form:

$$\rho_t + (\rho u)_x = 0,$$
$$(\rho u)_t + (\rho u^2)_x - (\mu u_x - p(\rho))_x = \rho f.$$

Here $(x, t) \in (0, \ell) \times (0, T)$ with ℓ and T between 0 and ∞, according to the initial and boundary conditions chosen, ρ is the density of the fluid and u its velocity, both functions of x and t, while $p = p(\rho)$ represents the pressure as a given function of the density. Finally, μ is the viscosity coefficient, either a constant, or a function dependent of the density; f represents density of external force.

We are especially interested in the global in time behavior of solutions which are already proved to exist globally. Since an important role play boundary conditions, character of viscosity, eventually temperature participation, we will discuss all these cases separately. A special attention will be paid to whether the limit density may or may not vanish at infinity, and how eventually treat the singular case where the limit density really vanishes in some set. Since we know the limit, and are able, in the terms of the data, to recognize whether the limit density can vanish or not, we obtain quite complete theory of the asymptotic behavior of the evolution solutions in a sense.

Nevertheless, there remain still nontrivial questions for the future research. Notice, that despite of that the 1D model is far from the full reality, there are strong applications in the fluid flow description in tubes as for example in hydraulics, pumping systems etc., which are important in many areas of engineering industry.

This contribution can hardly be a complete survey of what has been done, since the literature about this field is enormous. It should be understood as a modest attempt to give an essence of the state and problems of the global behavior of "1D fluids".

Collisions in 3D Fluid Structure interactions problems

Takéo Takahashi

Institut Élie Cartan
Universit Henri Poincar Nancy 1. B.P. 239, F-54506 Vandoeuvre-ls-Nancy Cedex, France.
takeo8@gmail.com

Matthieu Hillairet

University of Toulouse
Laboratoire MIP, Universit Paul Sabatier Toulouse 3, 31062 TOULOUSE cedex 9, France
hillair@mip.ups-tlse.fr

We are interested by the contacts between rigid bodies moving into a viscous incompressible fluid. Recently, several studies proved lack of collision in fluid structure systems. In [4], E. Zuazua and J.L. Vazquez prove no collision can occur between particles for a 1D toy-model. Then, V.N. Starovoitov obtains a criterion for the velocity-field of solutions [3]. Namely, he proves no collision can occur if the gradient of the velocity-field is sufficiently integrable. Finally, two parallel studies [1, 2] proved a no collision result when there is only one body in a bounded (or partially bounded) two-dimensional cavity. In the first case, the author considers a rigid disk inside a bigger disk. In the second case, the author considers a rigid disk above a ramp. The aim of the present study is to extend these two-dimensional results to three-dimensional comparable configurations i.e., for a rigid sphere above a ramp in $\mathbb{R}^3$. More precisely we consider the system composed by a rigid ball moving into a viscous incompressible fluid, over a fixed horizontal plane. The equations of motion for the fluid are the Navier-Stokes equations and the equations for the motion of the rigid ball are obtained by applying Newtons laws. We show that for any weak solutions of the corresponding system satisfying the energy inequality, the rigid ball never touches the plane.

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On approximate approximations for the Stokes boundary value problem

Werner Varnhorn

University of Kassel
Fachbereich Mathematik, Universität Kassel,
34109 Kassel, Germany
varnhorn@mathematik.uni-kassel.de

The method of approximate approximations, introduced by Maz'ya, can also be used for the numerical solution of boundary integral equations. In this case, the matrix of the resulting algebraic system to compute an approximate source density depends only on the position of a finite number of boundary points and on the direction of the normal vector in these points (Boundary Point Method). We investigate this approach for the Stokes problem in the whole space and for the Stokes boundary value problem in a bounded convex domain $G \subset \mathbb{R}^2$, where the second part consists of three steps: In a first step the unknown potential density is replaced by a linear combination of exponentially decreasing basis functions concentrated near the boundary points. In a second step, integration over the boundary ∂G is replaced by integration over the tangents at the boundary points such that even analytical expressions for the potential approximations can be obtained. In a third step, finally, the linear algebraic system is solved to determine an approximate density function and the resulting solution of the Stokes boundary value problem. Even not convergent the method leads to an efficient approximation of the form $O(h^2) + \varepsilon$, where ε can be chosen arbitrarily small.

On the boundary regularity for suitable weak solutions to the non-stationary Navier-Stokes equations beyond Caffarelli, Kohn and Nirenberg

Jörg Wolf

Humboldt-University Berlin,
Unter den Linden 6, 10099 Berlin,
jwolf@mathematik.hu-berlin.de

Let $\Omega = \mathbb{R}^3_+$ and $0 < T < \infty$. We consider the non-stationary Navier-Stokes equations in the cylindrical domain $Q = \Omega \times (0,T)$

$$\begin{aligned} \nabla \cdot \boldsymbol{u} &= 0, &(13)\\ \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= \Delta u - \nabla p \quad \text{in} \quad Q, &(14)\\ \boldsymbol{u} &= 0 \quad \text{on} \quad \partial\Omega \times (0,T), &(15)\\ \boldsymbol{u}(0) &= \boldsymbol{u}_0 \quad \text{in} \quad \Omega, &(16) \end{aligned}$$

where: $\boldsymbol{u} = (u_1, u_2, u_3) =$ velocity field of the fluid, $p =$ pressure and $\boldsymbol{u}_0 = (u_{0,1}, u_{0,2}, u_{0,3})$ the initial distribution of the velocity.

We prove the existence of an absolute constant $\varepsilon_* > 0$, such that for a given suitable weak solution $(\boldsymbol{u}, p)$ to (1)-(4) satisfying the local energy inequality up to the boundary $\partial\Omega \times (0,T)$, *the condition*

$$R^{-1} \int\limits_{Q^+(X_0,R)} |\nabla \boldsymbol{u}|^2 dxdt \le \varepsilon_* \quad \textit{for some} \quad X_0 \in \partial\Omega \times (0,T), R > 0$$

implies $\boldsymbol{u} \in \boldsymbol{C}^{\gamma,\gamma/2}(\overline{Q^+(X_0,R/2)})$. In addition, we show that if the $\boldsymbol{L}^3$-norm of $\boldsymbol{u}$ is sufficiently small then $\boldsymbol{u}$ is Hölder continuous on $\overline{Q}$.

The Stationary Navier-Stokes Equation on the Whole Plane for External Force with Antisymmetry

Masao Yamazaki

Waseda University
Okubo, Shinjuku, Tokyo 169-8555 Japan
masao.yamazaki@waseda.jp

We are concerned with the unique existence of the solution of the stationary Navier-Stokes equation

$$-\Delta w + (w \cdot \nabla)w + \nabla p = \text{curl } F,$$
$$\text{div } w = 0.$$

In spite of the absence of the obstacle, typical decay rate of the linear Stokes equation is $|x|^{-1}$. The product of these function belongs only to weak-L 1 space, and hence it is hard to construct solution by iteration. In this talk we impose, together smallness and natural decay assumption, the following antisymmetry condition on the potential function F:

$$F(-x_1, x_2) = -F(x_1, x_2), \quad F(x_1, -x_2) = -F(x_1, x_2),$$
$$F(x_2, x_1) = -F(x_1, x_2), \quad F(-x_2, -x_1) = -F(x_1, x_2).$$

Then we show the existence of the stationary solution. The obtained solution belongs to C^α for all $\alpha \in [0, 1)$, and decays like $|x|^{-1}$.

We also show the uniqueness of stationary solutions $w(x)$ such that $\nabla w \in L^r$ with some $r \in (1, 2)$ and that w is small in the weak-L^2 space.

Finally, we show the stability of the stationary solution w under small initial perturbation in the weak-L^2 space.