

Semi-bounded unitary representations of infinite-dimensional Lie groups

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In this note we introduce the concept of a semi-bounded unitary representations of an infinite-dimensional Lie group G . Semi-boundedness is defined in terms of the corresponding momentum set in the dual \mathfrak{g}' of the Lie algebra \mathfrak{g} of G . After dealing with some functional analytic issues concerning certain weak- $*$ -locally compact subsets of dual spaces, called semi-equicontinuous, we characterize unitary representations which are bounded in the sense that their momentum set is equicontinuous, we characterize semi-bounded representations of locally convex spaces in terms of spectral measures, and we also describe a method to compute momentum sets of unitary representations of reproducing kernel Hilbert spaces of holomorphic functions.

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Introduction

For infinite-dimensional Lie groups it seems quite hopeless to develop a general theory of unitary representations. This is mainly due to two obstacles which are non-existent for finite-dimensional Lie groups. The first one is that there is no group algebra such as $L^1(G)$ or $C^*(G)$, whose representations are in one-to-one correspondence with the continuous unitary representations of G . The second one is that there is no general structure theory for infinite-dimensional Lie groups, such as the Levi decomposition and the fine structure theory of semisimple Lie groups.

However, there are many interesting classes of infinite-dimensional Lie groups which possess a rich unitary representation theory. Many of these representations show up naturally in various contexts of mathematical physics (Mick87, Mick89, PS86, Go04, SeG81, Se58, Se78), where the Lie alge-

bra $\mathfrak{g} = \mathbf{L}(G)$ of the group under consideration often contains an element h , corresponding to the Hamiltonian of the underlying physical system, for which the spectrum of the operator $i \cdot \mathfrak{d}\pi(h)$ in the “physically relevant” representations (π, \mathcal{H}) is bounded from below. This suggests to study representations of infinite-dimensional Lie groups in terms of semi-boundedness properties of spectra.

Let G be a Lie group in the category of smooth manifolds modelled on locally convex spaces for which a smooth exponential function $\exp_G: \mathfrak{g} \rightarrow G$ exists^a (cf. Mil84, Ne06, GN08). For a unitary representation (π, \mathcal{H}) of G we write $\pi^v(g) := \pi(g)v$ for its orbits maps and call the representation (π, \mathcal{H}) *smooth* if the space

$$\mathcal{H}^\infty := \{v \in \mathcal{H} : \pi^v \in C^\infty(G, \mathcal{H})\}$$

of smooth vectors is dense in \mathcal{H} . Then all operators $i \cdot \mathfrak{d}\pi(x)$, $x \in \mathfrak{g}$, are essentially selfadjoint [Ne08, Lemma 3.6] and crucial information on their spectrum is contained in the momentum set I_π of the representation, a subset of the topological dual \mathfrak{g}' of the Lie algebra \mathfrak{g} of G . It is defined as the weak- $*$ -closed convex hull of the image of the momentum map on the projective space of \mathcal{H}^∞

$$\Phi_\pi: \mathbb{P}(\mathcal{H}^\infty) \rightarrow \mathfrak{g}' \quad \text{with} \quad \Phi_\pi([v])(x) = \frac{1}{i} \frac{\langle \mathfrak{d}\pi(x).v, v \rangle}{\langle v, v \rangle} \quad \text{for } [v] = \mathbb{C}v.$$

As a weak- $*$ -closed convex subset, the momentum set is completely determined by its support functional

$$s_\pi: \mathfrak{g} \rightarrow \mathbb{R} \cup \{\infty\}, \quad s_\pi(x) = -\inf\langle I_\pi, x \rangle = \sup(\text{Spec}(i \cdot \mathfrak{d}\pi(x))) \quad (1)$$

(cf. [Ne08, Lemma 3.7]).

It is now natural to study those representations for which s_π , resp., the set I_π , contains the most significant information. A natural regularity condition is that the function s_π is bounded on some non-empty open subset of \mathfrak{g} . We call such representations *semi-bounded*. Then the domain $s_\pi^{-1}(\mathbb{R})$ of s_π is a convex cone with non-empty interior and s_π is continuous on this open cone (cf. [Ne08, Prop. 4.8]). Since the momentum set I_π is invariant under the coadjoint action, the function s_π and its domain are invariant under the adjoint action. In Ne08 we described a framework which permits us to find C^* -algebras whose representations are in one-to-one correspondence with certain classes of semi-bounded representations of G .

^aWe shall keep this assumption throughout this paper.

In the present note we discuss several issues concerning semibounded representations. In Section 1 we briefly discuss the functional analytic background for semi-bounded representations. In particular we introduce the concept of a semi-equicontinuous set of linear functionals in such a way that semi-bounded representations are those whose momentum set is semi-equicontinuous. In many concrete situations, it is desirable to calculate momentum sets of representations by restricting the momentum map to quite small invariant submanifolds of projective space. The main result of Section 2 is a theorem which describes criteria under which the momentum set of a representation on a reproducing kernel Hilbert space of holomorphic functions can be determined directly from the reproducing kernel. Section 3 is devoted to a characterization of bounded representations as those for which the corresponding homomorphism $\pi: G \rightarrow U(\mathcal{H})$ is a smooth homomorphism of Lie groups, where the unitary group $U(\mathcal{H})$ carries its natural Banach–Lie group structure. This result is a quite straight forward consequence of the definitions if G is locally exponential, but for a general Lie group we have to use some more sophisticated arguments. Finally we explain in Section 4 that for the case where $G = (E, +)$ is the additive group of a locally convex space, semi-bounded representations are precisely those obtained by functional calculus from spectral measures supported by semi-equicontinuous subsets of the dual space.

For the future, the most important issue concerning semi-bounded representations is to understand the structural implications of the existence of semi-bounded representations, to determine which invariant semi-equicontinuous subsets of \mathfrak{g}' actually arise as momentum sets of unitary representations and to understand their convexity properties and the related symplectic geometry.

1. Semi-equicontinuous convex sets

Let E be a real locally convex space and E' its topological dual, i.e., the space of continuous linear functionals on E . We write $\langle \alpha, v \rangle = \alpha(v)$ for the natural pairing $E' \times E \rightarrow \mathbb{R}$ and endow E' with the weak- $*$ -topology, i.e., the coarsest topology for which all linear maps

$$\eta_v: E' \rightarrow \mathbb{R}, \quad \eta_v(\alpha) := \alpha(v)$$

are continuous. For a subset $X \subseteq E'$, the set

$$B(X) := \{v \in E: \inf \langle X, v \rangle > -\infty\}$$

is a convex cone which coincides with the domain of the *support function*

$$s_X: E \rightarrow \mathbb{R} \cup \{\infty\}, \quad s_X(v) := -\inf\langle X, v \rangle = \sup\langle X, -v \rangle$$

of X in the sense that $B(X) = s_X^{-1}(\mathbb{R})$. As a sup of a family of continuous linear functionals, the function s_X is convex, lower semicontinuous and positively homogeneous.

Remark 1.1. (a) The set X is weak- $*$ -bounded if and only if all functions η_v are bounded on X , i.e., $B(X) = E$. It is equicontinuous if and only if the function s_X is bounded on some 0-neighborhood in E . Each equicontinuous subset is in particular weak- $*$ -bounded, but the converse only holds if E is a barrelled space (Uniform Boundedness Principle) [Bou07, Ch. III, §4, no. 2, th.]. Recall that all Fréchet spaces are barrelled, but that also locally convex direct limits of barrelled spaces are barrelled.

(b) If $Y := \overline{\text{conv}}(X)$ denotes the weak- $*$ -closed convex hull of X , then $B(X) = B(Y)$, $s_X = s_Y$, and, using the Hahn–Banach Separation Theorem, Y can be reconstructed from s_Y by

$$Y = \{\alpha \in E' : (\forall v \in B(Y)) \alpha(v) \geq \inf\langle Y, v \rangle = -s_Y(v)\}.$$

Definition 1.2. We call X *semi-equicontinuous* if s_X is bounded on some non-empty open subset of E .

If X is semi-equicontinuous, then $B(X)$ clearly has interior points. We also have a partial converse:

Proposition 1.3. [Ne08, Thm. 4.10] *If E is barrelled, then a subset $X \subseteq E'$ is semi-equicontinuous if and only if $B(X)$ has interior points.*

Proposition 1.4. [Ne08, Props. 4.13, 4.4] *If X is a semi-equicontinuous weak- $*$ -closed convex subset, then the following assertions hold:*

- (a) *For each $v \in B(X)^0$, $\eta_v: X \rightarrow \mathbb{R}$ is a proper function. In particular, X is locally compact in the weak- $*$ -topology.*
- (b) $X = \{\alpha \in E' : (\forall v \in B(X)^0) \alpha(v) \geq \inf\langle X, v \rangle\}$.

Example 1.5. (a) If $W \subseteq E$ is an open convex cone and

$$W^* := \{\alpha \in E' : \alpha(W) \subseteq \mathbb{R}_+\}$$

its dual cone, then $B(W^*) = \overline{W}$ and $s_{W^*} = 0$ on W . Therefore W^* is semi-equicontinuous.

(b) Let Y be a topological space and $\omega: Y \rightarrow]0, \infty[$ a non-zero continuous function. Then

$$C_\omega(Y, \mathbb{R}) := \left\{ f \in C(Y, \mathbb{R}) : \sup \frac{|f|}{\omega} < \infty \right\}$$

is a Banach space with respect to the norm $\|f\| := \sup \frac{|f|}{\omega}$. Each element $y \in Y$ defines a continuous linear functional on this space by $\delta_y(f) := f(y)$, and the set

$$X := \{\delta_y : y \in Y\} \subseteq C_\omega(Y, \mathbb{R})'$$

is semi-equicontinuous. In fact, ω is positive on Y , and the open unit ball $B_1(\omega)$ around ω consists of non-negative functions. Therefore X is contained in the dual of an open cone, hence semi-equicontinuous.

Remark 1.6. (a) If $\varphi: E \rightarrow F$ is a continuous linear map between locally convex spaces and $X \subseteq F'$ is semi-equicontinuous, then the adjoint map $\varphi': F' \rightarrow E', \alpha \mapsto \alpha \circ \varphi$, maps X into a semi-equicontinuous subset because $s_{\varphi'(X)} = s_X \circ \varphi$ is bounded on some non-empty open subset of E .

(b) If $X \subseteq E'$ is semi-equicontinuous and $v \in B(X)^0$, then η_v is bounded from below, so that for some $c > 0$, the function $\omega := \eta_v + c$ is positive on X . For any other $w \in E$ there exists an $\varepsilon > 0$ with $v \pm \varepsilon w \subseteq B(X)$, so that there exists a $d > c$ with $\eta_{v \pm \varepsilon w}(X) \geq -d$. This implies that $|\eta_w| \leq \varepsilon^{-1}(\eta_v + d)$ on X , which in turn implies that $\eta_w \in C_\omega(X, \mathbb{R})$. We thus obtain a map

$$\eta: E \rightarrow C_\omega(X, \mathbb{R}), \quad w \mapsto \eta_w|_X$$

which is easily seen to be continuous with $\eta'(\delta_\alpha) = \alpha$ for each $\alpha \in X$.

This observation shows that any semi-equicontinuous set is contained in the image of $\{\delta_x : x \in X\}$ under the adjoint of some continuous linear map $\varphi: E \rightarrow C_\omega(X, \mathbb{R})$, and, in view of (a) and Example 1.5, we know that, conversely, all such sets are semi-equicontinuous.

2. Momentum sets of smooth unitary representations

Let G be a Lie group with a smooth exponential function $\exp_G: \mathfrak{g} \rightarrow G$, i.e., all the curves $\gamma_x(t) := \exp_G(tx)$, $x \in \mathfrak{g}$, are one-parameter groups with $\gamma'_x(0) = x$.

Now let (π, \mathcal{H}) be a smooth unitary representation of G . We then obtain for each $x \in \mathfrak{g}$ a unitary one-parameter group $\pi_x(t) := \pi(\exp_G(tx))$ and for each $v \in \mathcal{H}^\infty$ the derivative $d\pi(x)v := \left. \frac{d}{dt} \right|_{t=0} \pi(\exp_G(tx))v$ exists and defines an unbounded operator $d\pi(x): \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty \subseteq \mathcal{H}$. Since the map

$\mathcal{H}^\infty \rightarrow C^\infty(G, \mathcal{H}), v \mapsto \pi^v$ is equivariant with respect to the representation of G on $C^\infty(G, \mathcal{H})$ by $(g.f)(x) := f(xg)$, it easily follows that

$$\mathfrak{d}\pi: \mathfrak{g} \rightarrow \text{End}(\mathcal{H}^\infty)$$

defines a representation of \mathfrak{g} , called the *derived representation*.

We also have a natural action of G on the projective space $\mathbb{P}(\mathcal{H}^\infty)$ by $g.[v] := [\pi(g)v]$ and the coadjoint action of G on \mathfrak{g}' by $\text{Ad}^*(g).\alpha := \alpha \circ \text{Ad}(g)^{-1}$. An easy calculation now shows that the momentum map

$$\Phi_\pi: \mathbb{P}(\mathcal{H}^\infty) \rightarrow \mathfrak{g}', \quad \Phi_\pi([v])(x) = \frac{\langle \mathfrak{d}\pi(x)v, v \rangle}{i\langle v, v \rangle}$$

is G -equivariant. In particular, its image is G -invariant, and therefore I_π is invariant under the coadjoint action.

Proposition 2.1. [Ne08, Lemma 3.7] *For each $x \in \mathfrak{g}$, the closure of the operator $\mathfrak{d}\pi(x)$ on \mathcal{H}^∞ is the infinitesimal generator of the unitary one-parameter group $\pi_x(t) := \pi(\exp_G(tx))$ and satisfies*

$$\sup(\text{Spec}(i \cdot \mathfrak{d}\pi(x))) = s_{I_\pi}(x) = -\inf\langle I_\pi, x \rangle.$$

In terms of the momentum set, defined in the introduction, we now define:

Definition 2.2. A smooth representation (π, \mathcal{H}) is called *bounded*, resp., *semi-bounded* if its momentum set $I_\pi \subseteq \mathfrak{g}'$ is equicontinuous, resp., semi-equicontinuous.

Remark 2.3. (a) In view of Proposition 2.1, the convex cone $B(I_\pi)$ is the set of all elements of x for which the selfadjoint operator $i\mathfrak{d}\pi(x)$ is bounded above, which in turn is equivalent to the existence of an extension to a semigroup homomorphism

$$\widehat{\pi}_x: \mathbb{C}_+ := \mathbb{R} + i\mathbb{R}_+ \rightarrow B(\mathcal{H}), \quad \widehat{\pi}_x(z) := e^{z\mathfrak{d}\pi(x)}$$

(where the exponential is to be understood in terms of the functional calculus with respect to a spectral measure) which is strongly continuous and holomorphic on the open upper half plane (cf. Ne00).

(b) If (π, \mathcal{H}) is a bounded representation for which $\ker(\mathfrak{d}\pi) = I_\pi^\perp = \{0\}$, then $\|\mathfrak{d}\pi(x)\|$ defines a G -invariant norm on \mathfrak{g} . If G is finite-dimensional, then the existence of an invariant norm implies that the Lie algebra \mathfrak{g} is compact.

(c) If \mathfrak{g} is infinite-dimensional, then the existence of an invariant norm does not imply that the Lie bracket extends to the corresponding Banach completion. A simple example is the Lie algebra $(C^\infty(\mathbb{T}^2, \mathbb{R}), \{\cdot, \cdot\})$

of smooth functions on the 2-torus, endowed with the Poisson bracket with respect to the canonical symplectic form $\omega = dx \wedge dy$. Then the L^2 -inner product

$$(f, g) := \int_{\mathbb{T}^2} fg \cdot \omega$$

is invariant under the adjoint action of the corresponding Lie group $\text{Ham}(\mathbb{T}^2, \omega)$ (which is simply given by translation), but the Poisson–Lie bracket

$$\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}$$

is not continuous with respect to the L^2 -inner product.

Remark 2.4. Semi-bounded unitary representations of finite-dimensional Lie groups have been studied in some detail in Ne00, where it is shown that all these representations are direct integrals of irreducible semi-bounded representations [Ne00, Sect. XI.6]. and that, on the Lie algebra level, the irreducible representations are highest weight representations $(\pi_\lambda, \mathcal{H}_\lambda)$ [Ne00, Thms. X.3.9, XI.4.5]. If $[v_\lambda] \in \mathbb{P}(\mathcal{H}^\infty)$ is a highest weight vector, then the corresponding G -orbit $G.[v_\lambda]$ has the remarkable property that

$$\Phi(G.[v_\lambda]) = \text{Ext}(I_{\pi_\lambda}) = \text{Ad}^*(G)\Phi([v_\lambda]) \quad \text{and} \quad I_{\pi_\lambda} = \text{conv}(\Phi(G.[v_\lambda]))$$

[Ne00, Thm. X.4.1]. Moreover, two irreducible semi-bounded representations are equivalent if and only if the corresponding momentum sets, resp., the coadjoint orbits of extreme points coincide [Ne00, Thm. X.4.2].

One major feature of unitary highest weight representations is that the image of the highest weight orbit already determines the momentum set as the closed convex hull of its image. It is therefore desirable to understand in which situations smaller subsets of $\mathbb{P}(\mathcal{H}^\infty)$ already determine the momentum set. As we shall see below, this situation frequently occurs when \mathcal{H} consists of holomorphic functions on some complex manifold.

Definition 2.5. Let M be a complex manifold (modelled on a locally convex space) and $\mathcal{O}(M)$ the space of holomorphic complex-valued functions on M . We write \overline{M} for the conjugate complex manifold. A holomorphic function

$$K: M \times \overline{M} \rightarrow \mathbb{C}$$

is said to be a *reproducing kernel* of a Hilbert subspace $\mathcal{H} \subseteq \mathcal{O}(M)$ if for each $w \in M$ the function $K_w(z) := K(z, w)$ is contained in \mathcal{H} and satisfies

$$\langle f, K_z \rangle = f(z) \quad \text{for } z \in M, f \in \mathcal{H}.$$

Then \mathcal{H} is called the a *reproducing kernel Hilbert space* and since it is determined uniquely by the kernel K , it is denoted \mathcal{H}_K (cf. [Ne00, Sect. I.1]).

Now let G be a real Lie group and $\sigma: M \times G \rightarrow M, (m, g) \mapsto m.g$ be a smooth right action of G on M by holomorphic maps. Then $(g.f)(m) := f(m.g)$ defines a unitary representation of G on a reproducing kernel Hilbert space $\mathcal{H}_K \subseteq \mathcal{O}(M)$ if and only if the kernel K is invariant:

$$K(z.g, w.g) = K(z, w) \quad \text{for } z, w \in M, g \in G.$$

In this case we call \mathcal{H}_K a G -invariant reproducing kernel Hilbert space and write $(\pi_K(g)f)(z) := f(z.g)$ for the corresponding unitary representation of G on \mathcal{H}_K .

Lemma 2.6. *Let G be a Fréchet–Lie group, (π_K, \mathcal{H}_K) a unitary representation on a G -invariant reproducing kernel Hilbert space in $\mathcal{O}(M)$ and $\Omega := \{[K_m]: m \in M, K(m, m) > 0\}$. Then the following assertions hold:*

- (a) $K_z \in \mathcal{H}^\infty$ for each $z \in M$ and the representation of G on \mathcal{H}_K is smooth.
- (b) If $x \in \mathfrak{g}$ is such that the smooth action $M \times \mathbb{R} \rightarrow M, (m, t) \mapsto m.\exp_G(tx)$ extends to a holomorphic action of the upper half plane, then

$$\inf\langle I_\pi, x \rangle = \inf\langle \Phi_{\pi_K}(\Omega), x \rangle.$$

Proof. (a) For each $f \in \mathcal{H}$ and $z \in M$ we have

$$\langle f, g.K_z \rangle = \langle g^{-1}.f, K_z \rangle = (g^{-1}.f)(z) = f(z.g^{-1}),$$

which is a smooth function $G \rightarrow \mathbb{C}$. Hence the map $\alpha_z: G \rightarrow \mathcal{H}, g \mapsto g.K_z$, is weakly smooth. This implies that for each smooth map $h: \mathbb{R}^n \rightarrow G$ the composition $\alpha_z \circ h$ is weakly smooth, hence smooth by Grothendieck's Theorem (cf. [Sch54, p.146]). Now we apply [BGN04, Rem. 12.5] to see that α_z is smooth. This means that each K_z is a smooth vector, and since these elements span a dense subspace of \mathcal{H} , (a) follows.

(b) Let $(m, s) \mapsto m.s$ denote the holomorphic action of \mathbb{C}_+ on M extending the given action of \mathbb{R} . For $s \in \mathbb{C}_+$ we put $s^* := -\bar{s}$, which turns \mathbb{C}_+ into an involutive semigroup. For $z, w \in M$, the functions

$$f_1(s) := K(z.s, w) \quad \text{and} \quad f_2(s) := K(z, w.s^*).$$

Both are holomorphic on \mathbb{C}_+^0 , continuous on \mathbb{C}_+ and coincide on \mathbb{R} , so that they are equal [Ne00, Lemma A.3.6]. On the dense subspace

$$\mathcal{H}_K^0 := \text{span}\{K_z : z \in M\}$$

of \mathcal{H}_K we now obtain a representation of \mathbb{C}_+ by $(\widehat{\pi}_x(s).f)(m) := f(m.s)$ (cf. [Ne00, Prop. II.4.3]). Next we observe that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \Big|_{t=0} K(m.it, m.it) &= \frac{1}{2} \frac{d}{dt} \Big|_{t=0} K(m.2it, m) \\ &= i \frac{d}{dt} \Big|_{t=0} K(m.t, m) = i \frac{d}{dt} \Big|_{t=0} K_m(m.t) = i \frac{d}{dt} \Big|_{t=0} (\exp_G(tx).K_m)(m) \\ &= i \frac{d}{dt} \Big|_{t=0} \langle \pi_K(\exp_G(tx)).K_m, K_m \rangle = i \langle d\pi_K(x)K_m, K_m \rangle. \end{aligned}$$

Therefore [Ne00b, Prop. IV.1] implies that

$$\|\widehat{\pi}_x(a + ib)\| = e^{b \sup \langle \Phi_{\pi_K}(\Omega), -x \rangle}. \quad (2)$$

Clearly, $\sup \langle I_{\pi_K}, -x \rangle \geq \sup \langle \Phi_{\pi_K}(\Omega), -x \rangle$ and if the right hand side is infinite, then both are equal. Suppose that this is not the case, so that $\widehat{\pi}_x$ actually defines a representation of \mathbb{C}_+ by bounded operators on \mathcal{H}_K .

For each $f \in \mathcal{H}_K$ and $z, w \in M$ we then have $\widehat{\pi}_x(s).K_m = K_{m.s^*}$ [Ne00, Prop. II.4.3], so that $\langle \widehat{\pi}_x(s).K_m, f \rangle = \langle K_{m.s^*}, f \rangle = f(m.s^*)$ is holomorphic on \mathbb{C}_+^0 and continuous on \mathbb{C}_+ . Since the representation $\widehat{\pi}_x$ on \mathbb{C}_+ is locally bounded, [Ne00, Lemma IV.2.2] implies that $\widehat{\pi}_x : \mathbb{C}_+ \rightarrow B(\mathcal{H}_K)$ is strongly continuous and holomorphic on \mathbb{C}_+^0 . Now [Ne00, Lemma VI.5.2] shows that $\widehat{\pi}_x(s) = e^{s d\pi(x)}$ for each $s \in \mathbb{C}_+$, so that $id\pi(x)$ is bounded above with

$$\|\widehat{\pi}_x(i)\| = e^{\sup \text{Spec}(id\pi(x))} = e^{\sup \langle I_{\pi_K}, -x \rangle}.$$

Comparing with (2) now completes the proof. \square

Theorem 2.7. *Let G be a Fréchet–Lie group acting smoothly by holomorphic maps on the complex manifold M and $\mathcal{H}_K \subseteq \mathcal{O}(M)$ be a G -invariant reproducing kernel Hilbert space. If, for each $x \in B(I_{\pi_K})^0$, the action $(m, t) \mapsto m.\exp_G(tx)$ of \mathbb{R} on M extends to a holomorphic action of the upper half plane \mathbb{C}_+ , then*

$$I_{\pi_K} = \overline{\text{conv}}(\Phi(\{[K_m] : K(m, m) = \|K_m\|^2 > 0\})).$$

Proof. From the previous lemma, we obtain for each $x \in B(I_{\pi_K}^0)$ the relation $\inf \langle I_{\pi}, x \rangle = \inf \langle \Phi_{\pi_K}(\Omega), x \rangle$, so that the theorem follows from the reconstruction formula Proposition 1.4(b). \square

Remark 2.8. Since a holomorphic section of a vector bundle $\mathbb{V} \rightarrow M$ can always be identified with a holomorphic function on the total space of the dual bundle $\mathbb{V}' \rightarrow M$, any reproducing kernel Hilbert space of holomorphic sections can be realized as a reproducing kernel Hilbert space of holomorphic functions. Therefore the preceding theorem also applies to this more general situation.

3. Bounded representations

The main goal of this section is to prove the following theorem characterizing bounded representations. The main difficulty of the proof is to bridge the gap between the smoothness of a unitary representation as defined above and the smoothness of an action of G on the whole Hilbert space \mathcal{H} .

Theorem 3.1. *If (π, \mathcal{H}) is a smooth representation of the Lie group G with exponential function, then I_π is equicontinuous, i.e., π is bounded, if and only if $\pi: G \rightarrow U(\mathcal{H})$ is a morphism of Lie groups, where $U(\mathcal{H})$ carries its natural Banach–Lie group structure.*

Remark 3.2. For $x \in \mathfrak{g}$, the condition that $I_\pi(x)$ is a bounded subset of \mathbb{R} means that the unitary one-parameter group $\pi_x(t) := \pi(\exp_G tx) = e^{t\mathfrak{d}\pi(x)}$ is norm-continuous (cf. [Ne08, Lemma 3.7]). This is the special case $G = \mathbb{R}$ of Theorem 3.1.

In the following we write $x.g \in T_g(G)$ for the g -right translate of $x \in \mathfrak{g} \cong T_1(G)$.

Lemma 3.3. *Let $\sigma: G \times M \rightarrow M, (g, m) \mapsto g.m$, be a smooth action of the connected Lie group G on the smooth manifold M and*

$$\dot{\sigma}: \mathfrak{g} \rightarrow \mathcal{V}(M), \quad \dot{\sigma}(x)(m) := T_{(1,m)}(\sigma)(-x, 0)$$

the corresponding derived homomorphism of Lie algebras. If $f: G \rightarrow M$ is a smooth map with

$$f(\mathbf{1}) = m \quad \text{and} \quad T_g(f)(x.g) = -\dot{\sigma}(x)(f(g)) \quad \text{for } g \in G, x \in \mathfrak{g},$$

then $f(g) = g.m$ for each $g \in G$, i.e., f is the orbit map of m .

Proof. We consider the smooth map $h: G \rightarrow M, h(g) := g^{-1}.f(g) = \sigma_{g^{-1}}(f(g))$ and calculate for $x \in \mathfrak{g}$:

$$\begin{aligned} T_g(h)(xg) &= T_{f(g)}(\sigma_g^{-1})T_g(f)(x.g) + T_{f(g)}(\sigma_g^{-1})\dot{\sigma}(x)(f(g)) \\ &= -T_{f(g)}(\sigma_g^{-1})\dot{\sigma}(x)(f(g)) + T_{f(g)}(\sigma_g^{-1})\dot{\sigma}(x)(f(g)) = 0. \end{aligned}$$

Since G is connected, this implies that h is constant $h(\mathbf{1}) = m$, which implies the lemma. \square

Proposition 3.4. *Let $\pi_i: G \rightarrow \mathrm{GL}(E)$, $i = 1, 2$, be two representations of the connected Lie group G on the locally convex space E . We assume that π_1 is smooth and that for π_2 all orbit maps of element in the dense subspace $E^\infty \subseteq E$ are smooth, so that we obtain a homomorphism of Lie algebras*

$$\mathfrak{d}\pi_2: \mathfrak{g} \rightarrow \mathrm{End}(E^\infty), \quad \mathfrak{d}\pi_2(x)(v) := T_1(\pi_2^v)(x) \quad \text{for} \quad \pi_2^v(g) := \pi_2(g)v.$$

If these two representations are compatible in the sense that

$$\mathbf{L}(\pi_1)(x)v = \mathfrak{d}\pi_2(x)v \quad \text{for} \quad v \in E^\infty,$$

then $\pi_1 = \pi_2$.

Proof. We consider π_1 as a smooth action $\sigma(g)(v) := \pi_1(g)v$ of G on E . Then the corresponding homomorphism $\dot{\sigma}: \mathfrak{g} \rightarrow \mathcal{V}(V)$ is given by

$$\dot{\sigma}(x)(v) = -\mathbf{L}(\pi_1)(x)(v).$$

Next, let $v \in E^\infty$. For the smooth map $f: G \rightarrow E$, $f(g) := \pi_2(g)v$, we then have

$$T_g(f)(x.g) = \mathfrak{d}\pi_2(x)\pi_2(g)v = \mathbf{L}(\pi_1)(x)(\pi_2(g)v),$$

so that Lemma 3.3 implies that $\pi_2(g)v = f(g) = \pi_1(g)v$ for each $v \in E^\infty$. Since $\pi_1(g)$ and $\pi_2(g)$ are continuous operators on E and E^∞ is dense, it follows that $\pi_1(g) = \pi_2(g)$ for each $g \in G$. \square

Proof of Theorem 3.1. Suppose first that π is a morphism of Lie groups. Then $\mathcal{H}^\infty = \mathcal{H}$ and $\mathbf{L}(\pi) = \mathfrak{d}\pi: \mathfrak{g} \rightarrow \mathfrak{u}(\mathcal{H}) = \mathbf{L}(U(\mathcal{H}))$ is a continuous linear map. Hence $x \mapsto \|\mathfrak{d}\pi(x)\|$ defines a continuous seminorm on \mathfrak{g} . For each $v \in \mathcal{H}$ we now have $|\langle \mathfrak{d}\pi(x)v, v \rangle| \leq \|\mathfrak{d}\pi(x)\| \cdot \|v\|^2$, hence $|\langle I_\pi, x \rangle| \leq \|\mathfrak{d}\pi(x)\|$. Therefore I_π is equicontinuous.

Suppose, conversely, that I_π is equicontinuous. Then [Ne08, Lemma 3.7] implies that the essentially skew-adjoint operator $\mathfrak{d}\pi(x)$ on \mathcal{H}^∞ extends to a bounded operator, also denoted by $\mathfrak{d}\pi(x)$, on \mathcal{H} . Since the map $\mathfrak{d}\pi: \mathfrak{g} \rightarrow \mathrm{End}(\mathcal{H}^\infty)$ is a representation of \mathfrak{g} , the density of \mathcal{H}^∞ in \mathcal{H} and the estimate

$$\|\mathfrak{d}\pi(x)v\| \leq (\sup |\langle I_\pi, x \rangle|) \cdot \|v\|$$

imply that the map $\mathfrak{d}\pi: \mathfrak{g} \rightarrow B(\mathcal{H})$ also is linear, continuous and a homomorphism of Lie algebras.

Since homomorphisms of Lie groups are smooth if and only if they are smooth in an identity neighborhood, we may w.l.o.g. assume that G is connected. Let $q_G: \tilde{G} \rightarrow G$ be the simply connected covering group of G . Since the Banach–Lie group $U(\mathcal{H})$ is regular, the morphism $d\pi$ of Lie algebras integrates to a smooth group homomorphism

$$\tilde{\pi}: \tilde{G} \rightarrow U(\mathcal{H}) \quad \text{with} \quad \mathbf{L}(\tilde{\pi}) = d\pi$$

(Mil84). From Proposition 3.4 we now derive that $\tilde{\pi} = \pi \circ q_G$, and this implies that π is smooth. \square

If a unitary representation $\pi: G \rightarrow U(\mathcal{H})$ is a morphism of Lie groups, then it is in particular norm continuous, i.e., a morphism of topological groups. One may now ask under which circumstances, the norm continuity implies that π is smooth.

Proposition 3.5. *Let (π, \mathcal{H}) be a smooth unitary representation of the Lie group G which is norm continuous. Then π is a morphism of Lie groups if G is locally exponential or \mathfrak{g} is barrelled.*

Proof. (a) If G is locally exponential, then G and $U(\mathcal{H})$ are locally exponential Lie groups, and the smoothness of any continuous homomorphism follows from the Automatic Smoothness Theorem [Ne06, Thm. IV.1.18].

(b) Suppose that \mathfrak{g} is barrelled. For each $x \in \mathfrak{g}$, the unitary representation $\pi_x(t) := \pi(\exp_G(tx))$ is norm continuous, hence $I_\pi(x)$ is bounded. This implies that $I_\pi \subseteq \mathfrak{g}'$ is weak- $*$ -bounded, and since \mathfrak{g} is barrelled, it is equicontinuous (Remark 1.1). Now Theorem 3.1 shows that π is a morphism of Lie groups. \square

4. The abelian case

Let $G := (E, +)$ be a locally convex space, considered as a Lie group. We fix a weak- $*$ -closed convex semi-equicontinuous subset $X \subseteq E' = \mathfrak{g}'$ and recall from Proposition 1.4 that X is locally compact w.r.t. the weak- $*$ -topology. The following theorem characterizes the semibounded smooth unitary representations of G with $I_\pi \subseteq X$:

Theorem 4.1. *For a smooth representation (π, \mathcal{H}) of $(E, +)$, the following are equivalent:*

- (a) $I_\pi \subseteq X$.

(b) *There exists a holomorphic non-degenerate representation*

$$\widehat{\pi}: S := E + iB(X)^0 \rightarrow B(\mathcal{H})$$

of involutive semigroups (with respect to $(x + iy)^ := -x + iy$), satisfying $\pi(v)\widehat{\pi}(s) = \widehat{\pi}(v + s)$ for $v \in E$, $s \in S$ and*

$$\|\widehat{\pi}(x + iy)\| \leq e^{-\inf\langle X, y \rangle}.$$

(c) *There exists a Borel spectral measure P on the locally compact space X with $P(X) = \mathbf{1}$ and $P(e^{i\eta v}) = \pi(v)$ for each $v \in E$.*

Proof. (a) \Rightarrow (b): For each $z = x + iy \in S$ we define

$$\widehat{\pi}(x + iy) := \pi(x)e^{i\mathrm{d}\pi(y)} \in B(\mathcal{H}).$$

Since the spectral measures of the unitary one-parameter groups of the form $t \mapsto \pi(tv)$ commute pairwise, it follows that $\widehat{\pi}$ is a homomorphism of semigroups compatible with the involution. From

$$\|\widehat{\pi}(x + iy)\| = e^{\sup \mathrm{Spec}(i\mathrm{d}\pi(y))} = e^{-\inf\langle I_\pi, y \rangle}$$

it follows that $\widehat{\pi}$ is locally bounded. For each finite-dimensional subspace $F \subseteq E$ intersecting $B(X)^0$ non-trivially, we apply [Ne00, Section VI.5] to see that $\widehat{\pi}$ is holomorphic on $F + i(F \cap B(X)^0)$, hence holomorphic since it is locally bounded (He89).

(b) \Rightarrow (c) In view of [Ne08, Thm. 5.1], the representation of S comes from a representation of the C^* -algebra $C_0(X)$ with respect to the homomorphism $\gamma: S \rightarrow C_0(X), \gamma(s)(\alpha) := e^{i\alpha(s)}$. Since the representations of $C_0(X)$ are in one-to-one correspondence with spectral measures on X and π is uniquely determined by its compatibility with $\widehat{\pi}$, (c) follows.

(c) \Rightarrow (a): Since η_v is bounded from below on X by $-s_X(v)$, we derive from (c) that $\sup(\mathrm{Spec}(i\mathrm{d}\pi(v))) \leq s_X(v)$, which implies (a) (Proposition 1.4(b)). \square

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