Spatial discretization of Cuntz algebras

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Abstract

The (abstract) Cuntz algebra $\mathcal{O}_N$ is generated by non-unitary isometries and has therefore no intrinsic finiteness properties. To approximate the elements of the Cuntz algebra by finite-dimensional objects, we thus consider a spatial discretization of $\mathcal{O}_N$ by the finite section method. For we represent the Cuntz algebra as a (concrete) algebra of operators on $l^2(\mathbb{Z}^+)$ and associate with each operator A in this algebra the sequence $(P_n A P_n)$ of its finite sections. The goal of this paper is to examine the structure of the C^*-algebra $\mathcal{S}(\mathcal{O}_N)$ which is generated by all sequences of this form. Our main results are the fractality of a suitable restriction of the algebra $\mathcal{S}(\mathcal{O}_N)$ and a necessary and sufficient criterion for the stability of sequences in the restricted algebra. These results are employed to study spectral and pseudospectral approximations of elements of $\mathcal{O}_N$.

1 Introduction

Several classes of C^*-algebras are distinguished by intrinsic finiteness properties. These properties can be used in principle to approximate the elements of the algebra by finite-dimensional (or discrete) objects and, thus, to discretize the algebra. Good candidates for a discretization in that sense are AF-algebras and quasidiagonal algebras. N. Brown [5] has pointed out that the discretization procedure works particularly well for irrational rotation algebras in which case the discrete approximations can not only be constructed effectively but also own excellent convergence properties (for example, the sequence of the approximations is fractal in a sense which will be explained below).

At the other end of the scale there are C^*-algebras of infinite type which resist any intrinsic discretization. This fact justifies to consider another kind of approximation of the elements of a Cuntz algebra by finite rank operators, which we call *spatial approximation* and which is based on the finite sections method. Spatial approximation requires to represent the algebra *-isomorphically as a concrete C^*-algebra of operators on an infinite dimensional Hilbert space H. All concrete algebras considered in this paper will be separable; so we can assume that H is separable and, thus, isomorphic to the Hilbert space $l^2(\mathbb{Z}^+)$ of all sequences $(x_n)_{n\geq 0}$ of complex numbers such that

$$\|(x_n)\|^2 := \sum_{n\geq 0} |x_n|^2 < \infty.$$

The sequences

$$e_i := (0, \ldots, 0, 1, 0, 0, \ldots)$$

with the 1 standing at the ith position for a basis of $l^2(\mathbb{Z}^+)$ to which we refer as the standard basis. For n a positive integer, let P_n denote the operator

$$P_n : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+), \quad (x_k)_{k\geq 0} \mapsto (x_0, x_1, \ldots, x_{n-1}, 0, 0, \ldots),$$

and set $P_0 := 0$. Thus, P_n is the orthogonal projection of $l^2(\mathbb{Z}^+)$ onto the span of the first n elements of the standard basis.

For the finite section method for a bounded linear operator A on $l^2(\mathbb{Z}^+)$, one replaces the equation $Au = f$ on $l^2(\mathbb{Z}^+)$ by the sequence of the equations

$$P_n A P_n u_n = P_n f, \quad n = 1, 2, 3, \ldots \tag{1}$$

the solutions u_n of which are sought in $\operatorname{im} P_n$. The sequence $(P_n A P_n)$ is called the sequence of the finite sections of A. This sequence is said to be *stable* if there is an n_0 such that the operators $P_n A P_n : \operatorname{im} P_n \to \operatorname{im} P_n$ are invertible for $n \geq n_0$ and if their inverses are uniformly bounded. If the sequence $(P_n A P_n)$ is stable, then the operator A is invertible, the equations (1) possess unique solutions u_n for all $n \geq n_0$ and for all right hand sides $f \in l^2(\mathbb{Z}^+)$, and these solutions converge to the solution u of the equation $Au = f$.

Let $\mathcal{A}$ be a (for a moment not necessarily separable) C^*-subalgebra of the algebra $L(l^2(\mathbb{Z}^+))$ of all bounded linear operators on $l^2(\mathbb{Z}^+)$. We associate with each operator A in $\mathcal{A}$ the sequence $(P_n A P_n)$ of its finite sections and consider the C^*-algebra $\mathcal{S}(\mathcal{A})$ generated by this sequence. Spatial discretization of the concrete algebra $\mathcal{A}$ means to study the associated algebra $\mathcal{S}(\mathcal{A})$ of sequences. (To be precise: we are only interested in asymptotic properties of the sequences in $\mathcal{S}(\mathcal{A})$ which are encoded in the quotient algebra of $\mathcal{S}(\mathcal{A})$ modulo sequences tending to zero in the norm. It is this quotient algebra we are really interested in.)

Algebras of infinite type typically contain non-unitary isometries. The perhaps simplest example, the universal algebra generated by one isometry, is *-isomorphic to the smallest closed *-subalgebra $\mathcal{T}(C)$ of $L(l^2(\mathbb{Z}^+))$ which contains the operator

$$V : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+), \quad (x_k)_{k\geq 0} \mapsto (0, x_0, x_1, \ldots)$$

of forward shift. This is the contents of a theorem by Coburn [6]. The algebra $\mathcal{T}(C)$ is also known as the *Toeplitz algebra*, since each of its elements is of the form

$T(c) + K$ where $T(c)$ is a Toeplitz operator with continuous generating function c and K is a compact operator. The structure of the associated algebra $\mathcal{S}(\mathcal{T}(C))$ (factored by the zero sequences) and, thus, the stability of the finite sections method for operators in $\mathcal{T}(C)$ are fairly well understood. For a detailed account on the finite sections method for Toeplitz operators, presented in an algebraic language, see [3] and Section 1.4 in [9].

The aim of the present paper is to one step further and to examine the spatial discretization of algebras which are generated by a finite number of non-commuting non-unitary isometries, namely the Cuntz algebras. Recall that an *isometry* is an element s of a unital *-algebra for which s^*s is the identity element.

Let $N \geq 2$. The *Cuntz algebra* $\mathcal{O}_N$ is the universal C^*-algebra generated by N isometries $S_0, \ldots, S_{N-1}$ with the property that

$$S_0 S_0^* + \ldots + S_{N-1} S_{N-1}^* = I. \tag{2}$$

Cuntz algebras cannot be obtained as inductive limits of type I C^*-algebras. In particular, they cannot be approximated by finite dimensional algebras in the sense of AF-algebras. (For these and other facts, consult Cuntz' pioneering paper [7]. A nice introduction is also in [8].) The importance of Cuntz algebras in theory and applications cannot be overestimated. Let me only mention Kirchberg's deep result that a separable C^*-algebra is exact if and only if it embeds in the Cuntz algebra $\mathcal{O}_2$, and the role that representation of Cuntz algebras play in wavelet theory and signal processing (see [1, 4] and the references therein).

To discretize the Cuntz algebra $\mathcal{O}_n$ by the finite sections method, we have to represent this algebra as a C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$. Since Cuntz algebras are simple, every C^*-algebra which is generated by N isometries $S_0, \ldots, S_{N-1}$ which fulfill (2) is *-isomorphic to $\mathcal{O}_N$. Thus, $\mathcal{O}_N$ is *-isomorphically to the smallest C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$ which contains the operators

$$S_i : (x_k)_{k \geq 0} \mapsto (y_k)_{k \geq 0} \quad \text{with} \quad y_k := \begin{cases} x_r & \text{if } k = rN + i \\ 0 & \text{else} \end{cases} \tag{3}$$

for $i = 0, \ldots, N-1$. We denote the (concrete) Cuntz algebra generated by the operators S_i in (3) also by $\mathcal{O}_N$. It is this concrete Cuntz algebra for which we will examine the sequence algebra $\mathcal{S}(\mathcal{O}_N)$ (modulo zero sequences) in what follows. One should mention at this point that the abstract Cuntz algebra $\mathcal{O}_N$ has an uncountable set of equivalence classes of irreducible representations. Representations of $\mathcal{O}_N$ different from (3) will certainly lead to different algebras $\mathcal{S}(\mathcal{O}_N)$. The relations between these algebras are not yet clearly understood.

The paper is organized as follows. In Section 2 we collect some basic facts from numerical analysis, centered around the notions of stability and fractality. In Section 3 we examine the full algebra of the finite sections method for $\mathcal{O}_N$. We show that this algebra fails to be fractal. This observation suggests to pass to a restricted algebra of finite sections sequences, and it is in fact this restricted algebra which will be studied in what follows. In Section 4 we divide the stability problem into an invertibility problem in a certain quotient algebra of $\mathcal{S}(\mathcal{O}_N)$ and a lifting problem for an ideal of $\mathcal{S}(\mathcal{O}_N)$. The first of these problems is also solved in Section 4, whereas the solution of the lifting problem is subject of Section 6. The treatment of the lifting problem is prepared by Section 5 where we consider an algebra of *stratified Toeplitz operators* related to the Cuntz algebra and examine its structure. As already mentioned, in Section 6 we finish the proof of our stability result. These results are then applied to examine spectral approximation and Fredholm properties of sequences in (the restriction of) $\mathcal{S}(\mathcal{O}_N)$. We will see that a sequence in this algebra is Fredholm if and only if it is stable, which has remarkable consequences for the asymptotic behavior of the singular values of the nth approximant when n tends to infinity.

Coburn's already mentioned result suggests to consider the Toeplitz algebra as the Cuntz algebra $\mathcal{O}_1$. But one should have in mind that the main properties of $\mathcal{O}_1$ and of $\mathcal{O}_N$ for $N > 1$ are completely different from each other. For example, the compact operators $K(l^2(\mathbb{Z}^+))$ form a closed ideal of $\mathcal{O}_1$, and the quotient $\mathcal{O}_1/K(l^2(\mathbb{Z}^+))$ is isomorphic to $C(\mathbb{T})$, whereas $\mathcal{O}_N$ is simple if $N \geq 2$. These differences continue to the corresponding sequence algebras $\mathcal{S}(\mathcal{O}_1)$ and $\mathcal{S}(\mathcal{O}_N)$ for $N > 1$. A main point is that $\mathcal{S}(\mathcal{O}_1)$ contains an ideal which is constituted of two exemplars of the ideal $K(l^2(\mathbb{Z}^+))$, and the irreducible representations, W_1 and W_2 say, of $\mathcal{S}(\mathcal{O}_1)$ which come from this ideal are *sufficient* in the sense that a sequence $\mathbf{A} = (A_n)$ in $\mathcal{S}(\mathcal{O}_1)$ is stable if and only if $W_1(\mathbf{A})$ and $W_2(\mathbf{A})$ are invertible. It turns out that this fact implies an effective criterion to check the stability of a sequence in $\mathcal{S}(\mathcal{O}_1)$. In contrast to this, if $N > 1$, then $\mathcal{S}(\mathcal{O}_N)$ has only one non-trivial ideal. We will construct an injective representation of this ideal, and will then observe that this representation extends to a representation, W say, of $\mathcal{S}(\mathcal{O}_N)$ which is injective on all of $\mathcal{S}(\mathcal{O}_N)$. Thus, roughly speaking, our stability result will say that a sequence $\mathbf{A}$ in $\mathcal{S}(\mathcal{O}_N)$ is stable if and only if the operator $W(\mathbf{A})$ is invertible. At the first glance, this result might seem to be useless since the stability of $\mathbf{A}$ is not easier to check than the invertibility of $W(\mathbf{A})$. So why this effort, if many canonical homomorphisms on $\mathcal{S}(\mathcal{O}_N)$ own the same property as W: the identical mapping and the faithful representation via the GNS-construction, for example. What is important is the concrete form of the mapping W constructed below: it is defined by means of strong limits of operator sequences, and this special form implies an immediate proof of the fractality of the (restricted) algebra $\mathcal{S}(\mathcal{O}_N)$.

2 Preliminaries from numerical analysis

2.1 Algebras of matrix sequences

Let $\mathcal{F}$ denote the set of all bounded sequences $\mathbf{A} = (A_n)$ of matrices $A_n \in \mathbb{C}^{n\times n}$. Equipped with the operations

$$(A_n) + (B_n) := (A_n + B_n), \quad (A_n)(B_n) := (A_n B_n), \quad (A_n)^* := (A_n^*)$$

and the norm

$$\|\mathbf{A}\|_{\mathcal{F}} := \|A_n\|,$$

the set $\mathcal{F}$ becomes a C^*-algebra, and the set $\mathcal{G}$ of all sequences $(A_n) \in \mathcal{F}$ with $\lim \|A_n\| = 0$ forms a closed ideal of $\mathcal{F}$. The relevance of the algebra $\mathcal{F}^C$ and its ideal $\mathcal{G}$ in our context stems from the fact (following from a simple Neumann series argument) that a sequence $(A_n) \in \mathcal{F}$ is stable if, and only if, the coset $(A_n) + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{G}$. Thus, every stability problem is equivalent to an invertibility problem in a suitably chosen C^*-algebra.

Let further stand $\mathcal{F}^C$ for the set of all sequences $\mathbf{A} = (A_n)$ of operators $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$ with the property that the sequences $(A_n P_n)$ and $(A_n^* P_n)$ converge strongly. By the uniform boundedness principle, the quantity $\sup \|A_n P_n\|$ is finite for every sequence $\mathbf{A}$ in $\mathcal{F}^C$. This, if we identify each operator A_n on $\operatorname{im} P_n$ with its matrix representation with respect to the basis $e_0, \ldots, e_{n-1}$ of $\operatorname{im} P_n$, we can consider cF^C as a closed and symmetric subalgebra of $\mathcal{F}$ which contains $\mathcal{G}$ as its ideal. Note that the mapping

$$W : \mathcal{F}^C \to L(l^2(\mathbb{Z}^+)), \quad (A_n) \mapsto \text{s-lim}\, A_n P_n \tag{4}$$

is a *-homomorphism.

2.2 Discretization of concrete algebras

Let $\mathcal{A}$ be a (not necessarily separable) C^*-subalgebra of the algebra $L(l^2(\mathbb{Z}^+))$. We write D for the mapping of spatial (= finite sections) discretization, i.e.,

$$D : L(l^2(\mathbb{Z}^+)) \to \mathcal{F}, \quad A \mapsto (P_n A P_n), \tag{5}$$

and we let $\mathcal{S}(\mathcal{A})$ stand for the smallest closed C^*-subalgebra of the algebra $\mathcal{F}$ which contains all sequences $D(A)$ with $A \in \mathcal{A}$. Clearly, $\mathcal{S}(\mathcal{A})$ lies even in $\mathcal{F}^C$, and the mapping W in (4) induces a *-homomorphism from $\mathcal{S}(\mathcal{A})$ onto $\mathcal{A}$. On this level, one cannot say much about algebra $\mathcal{S}(\mathcal{A})$. The little one can say will follow easily from the following simple facts.

Proposition 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, $D : \mathcal{A} \to \mathcal{B}$ a linear contraction, and $W : \mathcal{B} \to \mathcal{A}$ a C^*-homomorphism such that $W(D(A)) = A$ for every $A \in \mathcal{A}$.*

Then

(a) D is an isometry, $D(\mathcal{A})$ is a closed linear subspace of $\mathcal{B}$, and $\operatorname{alg} D(\mathcal{A})$, the smallest closed C^*-subalgebra of $\mathcal{B}$ which contains $D(\mathcal{A})$, splits into the direct sum

$$\operatorname{alg} D(\mathcal{A}) = D(\mathcal{A}) \oplus (\ker W \cap \operatorname{alg} D(\mathcal{A})). \tag{6}$$

Moreover, for every $A \in \mathcal{A}$,

$$\|D(A)\| = \min_{K \in \ker W} \|D(A) + K\|. \tag{7}$$

(b) If $\mathcal{B} = \operatorname{alg} D(\mathcal{A})$, then $\ker W$ coincides with the quasicommutator ideal of $\mathcal{B}$, i.e., with the smallest closed ideal of $\mathcal{B}$ which contains all quasicommutators $D(A_1)D(A_2) - D(A_1A_2)$ with $A_1, A_2 \in \mathcal{A}$.

Proof. (a) Let $A \in \mathcal{A}$. The inequality

$$\|A\| = \|W(D(A))\| \leq \|D(A)\| \leq \|A\|$$

shows that D is an isometry; hence, $D(\mathcal{A})$ is a closed subspace of $\mathcal{B}$. Let $B \in D(\mathcal{A}) \cap \ker W$. Write $B = D(A)$ with $A \in \mathcal{A}$. From $W(B) = 0$ we get $A = W(D(A)) = W(B) = 0$, whence $B = 0$. Thus, $D(\mathcal{A}) \cap \ker W = \{0\}$.

Let $B \in \operatorname{alg} D(\mathcal{A})$. Then $W(B - D(W(B))) = W(B) - W(B) = 0$, hence

$$B = D(W(B)) + (B - D(W(B)) \in D(\mathcal{A}) + \ker W,$$

whence $\operatorname{alg} D(\mathcal{A}) = D(\mathcal{A}) + (\ker W \cap \operatorname{alg} D(\mathcal{A}))$. This proves (6). To check (7), let $A \in \mathcal{A}$ and $K \in \ker W$. Then

$$\|A\| = \|W(D(A) + K)\| \leq \|D(A) + K\|$$

which implies that $\|D(A)\| \leq \|D(A) + K\|$ since D is an isometry.

(b) Since W is a homomorphism and $W \circ D$ is the identity on $\mathcal{A}$, one has $D(A_1)D(A_2) - D(A_1A_2) \in \ker W$ for all $A_1, a_2 \in \mathcal{A}$. Thus, $\ker W$ contains the quasicommutator ideal. For the reverse inclusion, let $K \in \ker W$ and n a positive integer. Since $K \in \operatorname{alg} D(\mathcal{A})$, there are sums of products

$$K_n = \sum \prod D(A_{ij}^{(n)}) \quad \text{with} \quad A_{ij}^{(n)} \in \mathcal{A}$$

such that $\|K - K_n\| \leq 1/n$. Clearly, each K_n can be written as

$$K_n = D(\sum \prod A_{ij}^{(n)}) + Q_n$$

with an element Q_n in the quasicommutator ideal. From

$$\|K - Q_n\| \leq \|K - K_n\| + \|D(\sum \prod A_{ij}^{(n)})\|$$

and

$$\|D(\sum\prod A_{ij}^{(n)})\| \le \|\sum\prod A_{ij}^{(n)}\| = \|W(K_n)\| = \|W(K_n - K)\| \le \|K_n - K\|$$

we conclude that $\|K - Q_n\| \le 2\|K - K_n\| \le 1/n$. Thus, K can be approximated as closely as desired by elements in the quasicommutator ideal. Since the quasicommutator ideal is closed, the assertion follows. ■

We apply the preceding proposition in the following context:

- $\mathcal{A}$ is a C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$,
- $\mathcal{B} = \mathcal{S}(\mathcal{A})$,
- D is the restriction of the discretization (5) to $\mathcal{A}$, and
- W is the restriction of the homomorphism (4) to $\mathcal{S}(\mathcal{A})$.

Then Proposition 2.1 specializes to the following.

Proposition 2.2 *Let $\mathcal{A}$ be a C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$. Then the finite sections discretization $D : \mathcal{A} \to \mathcal{F}$ is an isometry, and $D(\mathcal{A})$ is a closed subspace of $\mathcal{S}(\mathcal{A})$. The algebra $\mathcal{S}(\mathcal{A})$ splits into the direct sum*

$$\mathcal{S}(\mathcal{A}) = D(\mathcal{A}) \oplus (\ker W \cap \mathcal{S}(\mathcal{A})),$$

and one has

$$\|D(A)\| = \min_{K \in \ker W} \|D(A) + K\|$$

for every operator $A \in \mathcal{A}$. Finally, $\ker W \cap \mathcal{S}(\mathcal{A})$ is equal to the quasicommutator ideal of $\mathcal{S}(\mathcal{A})$, i.e., to the smallest closed ideal of $\mathcal{S}(\mathcal{A})$ which contains all sequences $(P_nA_1P_nA_2P_n - P_nA_1A_2P_n)$ with operators A_1, $A_2 \in \mathcal{A}$.

We denote the ideal $\ker W \cap \mathcal{S}(\mathcal{A})$ of $\mathcal{S}(\mathcal{A})$ by $\mathcal{J}(\mathcal{A})$. Since the first item in the decomposition $D(\mathcal{A}) \oplus \mathcal{J}(\mathcal{A})$ of $\mathcal{S}(\mathcal{A})$ is isomorphic (as a linear space) to $\mathcal{A}$, a main part of the description of the algebra $\mathcal{S}(\mathcal{A})$ is to identify the ideal $\mathcal{J}(\mathcal{A})$.

Remark 2.3 Blackadar and Kirchberg gave an abstract characterization of C^*-subalgebras of $\mathcal{F}/\mathcal{G}$ as generalized inductive limits of finite-dimensional algebras, which they call MF-algebras (see the last sections of Blackadar's monograph [1]). In this sense, the ideal $\mathcal{J}(\mathcal{A})$ and its image $\mathcal{J}(\mathcal{A})/\mathcal{G}$ in $\mathcal{F}/\mathcal{G}$ can be considered as a measure for the deviation of an algebra from being an MF-algebra. For example, for the Toeplitz algebra $\mathcal{A} = \mathcal{T}(C)$ which is generated by one non-unitary isometry, one knows that $\mathcal{J}(\mathcal{A})/\mathcal{G}$ is *-isomorphic to the ideal $K(l^2(\mathbb{Z}^+))$ of the compact operators.

Note that since $\mathcal{S}(\mathcal{A})/\mathcal{J}(\mathcal{A})$ is canonically *-isomorphic to $\mathcal{A}$, the above construction implies a simple proof of the (well-known) fact that every C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$ is *-isomorphic to a quotient of an MF-algebra. ■

2.3 Fractal algebras of matrix sequences

The mapping which associates with every subalgebra $\mathcal{A}$ of $L(l^2(\mathbb{Z}^+))$ the sequence algebra $\mathcal{S}(\mathcal{A})$ does not mind the individual properties of $\mathcal{A}$. If $\mathcal{A}$ is separable, this fact can be compensated to some extent by passing to a *fractal restriction of* $\mathcal{S}(\mathcal{A})$, which is defined as follows.

Let $\eta : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence. By $\mathcal{F}_\eta$ we mean the set of all subsequences $(A_{\eta(n)})$ of sequences (A_n) in $\mathcal{F}$. As in Section 2.1, $\mathcal{F}_\eta$ can be made to a C^*-algebra in a natural way. The *-homomorphism

$$R_\eta : \mathcal{F} \to \mathcal{F}_\eta, \quad (A_n) \mapsto (A_{\eta(n)})$$

is called the restriction of $\mathcal{F}$ onto $\mathcal{F}_\eta$. It maps the ideal $\mathcal{G}$ of $\mathcal{F}$ onto a closed ideal $\mathcal{G}_\eta$ of $\mathcal{F}_\eta$. For every subset $\mathcal{S}$ of $\mathcal{F}$, we abbreviate $R_\eta \mathcal{S}$ by $\mathcal{S}_\eta$.

Let $\mathcal{S}$ be a C^*-subalgebra of $\mathcal{F}$. A *-homomorphism W from $\mathcal{S}$ into a C^*-algebra $\mathcal{B}$ is called *fractal* if, for every strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$, there is a mapping $W_\eta : \mathcal{S}_\eta \to \mathcal{B}$ such that $W = W_\eta R_\eta|_\mathcal{S}$. A C^*-subalgebra $\mathcal{S}$ of $\mathcal{F}$ is called *fractal*, if the canonical homomorphism

$$\mathcal{S} \to \mathcal{S}/(\mathcal{S} \cap \mathcal{G}), \quad \mathbf{A} \mapsto \mathbf{A} + (\mathcal{S} \cap \mathcal{G})$$

is fractal.

Thus, if $\mathcal{S}$ is a fractal algebra, then every sequence in $\mathcal{S}$ can be uniquely rediscovered from each of its (infinite) subsequences up to a sequence tending to zero. In that sense, the essential information on a sequence in $\mathcal{S}$ is already stored in each of its subsequences. These algebras were called *fractal* in [15] in order to emphasize exactly this self-similarity aspect.

If $\mathcal{A}$ is a C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$ and $\eta : \mathbb{N} \to \mathbb{N}$ a strongly monotonically increasing sequence, then we denote the restriction $R_\eta(\mathcal{S}(\mathcal{A}))$ of the associated sequence algebra by $\mathcal{S}_\eta(\mathcal{A})$. The following was proved in [12].

Theorem 2.4 *Let $\mathcal{A}$ be a separable C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$. Then there exists a strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$ such that the restricted algebra $\mathcal{S}_\eta(\mathcal{A})$ is fractal.*

Since the sequence η depends on $\mathcal{A}$, the fractal restrictions $\mathcal{S}_\eta(\mathcal{A})$ of $\mathcal{S}(\mathcal{A})$ will reflect the structure of $\mathcal{A}$ in a much higher extent than the full algebra $\mathcal{S}(\mathcal{A})$ of the finite sections discretization.

One should mention that there are further good reasons to be interested in fractal algebras. One of them is that sequences in fractal algebras exhibit an excellent asymptotic behavior. For example, if (A_n) is a sequence in a fractal algebra, then several important spectral quantities of the A_n (e.g., the set of the singular values, the pseudospectrum, the numerical range) converge with respect to the *Hausdorff metric* as n tends to infinity (see Chapter 3 in [9] and Section 6.4 below). Another reason came up in [13] where we observed that the property of fractality determines the ideal structure of the algebra to a large extent.

3 The full algebra $\mathcal{S}(\mathcal{O}_N)$

3.1 The full algebra of the finite sections method for operators in $\mathcal{O}_N$

In accordance with the above notations, let $\mathcal{S}(\mathcal{O}_N)$ denote the smallest closed subalgebra of $\mathcal{F}$ which contains all sequences (P_nAP_n) with A in the concrete Cuntz algebra $\mathcal{O}_N$. Since $(P_nAP_n)^* = (P_nA^*P_n)$, $\mathcal{S}(\mathcal{O}_N)$ is a C^*-algebra. The isometries S_i are defined as in (3). We further abbreviate $\Omega := \{0, 1, \ldots, N-1\}$.

Lemma 3.1 *$\mathcal{S}(\mathcal{O}_N)$ is the smallest C^*-subalgebra of $\mathcal{F}$ which contains all sequences $(P_nS_jP_n)$ with $j \in \Omega$.*

Proof. For a moment, let $\mathcal{S}'$ denote the smallest closed and symmetric subalgebra of $\mathcal{F}$ which contains all sequences $(P_nS_jP_n)$ with $j \in \Omega$. Evidently, $\mathcal{S}' \subseteq \mathcal{S}(\mathcal{O}_N)$. For the reverse inclusion, note first that

$$S_i^*S_j = 0 \quad \text{whenever } i \neq j. \tag{8}$$

Indeed, this follows for the operators (3) by straightforward calculation, but it also follows easily from the Cuntz axiom (2): Multiply (2) from the left by S_i^* and from the right by S_j and take into account that a sum of positive elements in a C^*-algebra is zero if and only if each of the elements is zero.

From (8) we conclude that every finite word with letters in the alphabet $\{S_1, \ldots, S_N, S_1^*, \ldots, S_N^*\}$ is of the form

$$S_{i_1}S_{i_2}\ldots S_{i_k}S_{j_1}^*S_{j_2}^*\ldots S_{j_l}^* \quad \text{with} \quad i_s,\, j_t \in \Omega \tag{9}$$

(Lemma 1.3 in [7]). Further one easily checks that

$$P_nS_j = P_nS_jP_n \quad \text{and} \quad S_j^*P_n = P_nS_j^*P_n \tag{10}$$

for every $j \in \Omega$ and every $n \in \mathbb{N}$. Thus, if A is any word of the form (9), then

$$P_nAP_n = P_nS_{i_1}P_n \cdot P_nS_{i_2}P_n \ldots P_nS_{i_k}P_n \cdot P_nS_{j_1}^*P_n \cdot P_nS_{j_2}^*P_n \ldots P_nS_{j_l}^*P_n \in \mathcal{S}'.$$

Since the set of all linear combinations of the words (9) is dense in $\mathcal{O}_N$, it follows that $\mathcal{S}(\mathcal{O}_N) \subseteq \mathcal{S}'$. ■

Recall that an element S of a C^*-algebra is called a partial isometry if $SS^*S = S$. If S is a partial isometry, then SS^* and S^*S are projections (i.e., self-adjoint idempotents), called the *range projection* and the *initial projection* of S, respectively. Conversely, if S^*S (or SS^*) is a projection for an element S, then S is a partial isometry. Recall also that projections P and Q are called orthogonal if $PQ = 0$.

Lemma 3.2 *Every sequence* $(P_n S_i P_n)$, $i \in \Omega$, *is a partial isometry in* $\mathcal{F}$, *and the corresponding range projections are orthogonal if* $i \neq j$. *Moreover,*

$$P_n S_i^* P_n S_j P_n = 0 \quad \text{if } i \neq j, \tag{11}$$

and

$$P_n S_0 P_n S_0^* P_n + \ldots + P_n S_{N-1} P_n S_{N-1}^* P_n = P_n. \tag{12}$$

Proof. The identities (10) imply that

$$P_n S_i S_i^* P_n = P_n S_i P_n S_i^* P_n \tag{13}$$

for every $i \in \Omega$ and every $n \in \mathbb{Z}^+$. The operators $S_i S_i^*$ are projections, and their matrices with respect to the standard basis of $l^2(\mathbb{Z}^+)$ are of diagonal form. Hence, the projections $S_i S_i^*$ and P_n commute, which implies that the left-hand side of (13) is a projection. Hence, $(P_n S_i P_n)$ is a partial isometry in $\mathcal{F}$, and $(P_n S_i S_i^* P_n)$ is the associated range projection.

Let $i \neq j$ be in Ω. The fact that the P_n and the $S_i S_i^*$ commute further implies together with (8) that

$$(P_n S_i S_i^* P_n)(P_n S_j S_j^* P_n) = P_n S_i S_i^* S_j S_j^* P_n = 0.$$

Multiplying $P_n S_i S_i^* P_n S_j S_j^* P_n = 0$ from the left by $P_n S_i^* P_n$ and from the right by $P_n S_j P_n$ yields (11). Finally, (12) follows by summing up the equalities (11) over $i \in \Omega$ and from axiom (2). ■

Thus, the generating sequences $(P_n S_i P_N)$, $i \in \Omega$, are still subject of the Cuntz axiom (2), but note they are partial isometries only and no longer isometries (which is not a surprise since the algebra $\mathcal{F}$, being a product of finite dimensional algebras, cannot contain non-unitary isometries).

The first assertion of the preceding lemma holds more generally.

Lemma 3.3 *Let* $i = (i_1, i_2, \ldots, i_k) \in \Omega^k$. *Every product*

$$(P_n S_{i_1} P_n S_{i_2} P_n \ldots P_n S_{i_k} P_n)$$

is a partial isometry in $\mathcal{F}$.

Indeed, from (10) we conclude that

$$P_n S_{i_1} \ldots S_{i_k} S_{i_k}^* \ldots S_{i_1}^* P_n = P_n S_{i_1} P_n \ldots P_n S_{i_k} P_n P_n S_{i_k}^* P_n \ldots P_n S_{i_1}^* P_n.$$

Since $S_{i_1} \ldots S_{i_k} S_{i_k}^* \ldots S_{i_1}^*$ is a projection of diagonal form, the assertion follows as in Lemma 3.2. ■

3.2 Initial projections

For $i = (i_1, i_2, \ldots, i_k) \in \Omega^k$, abbreviate $S_i := S_{i_1} S_{i_2} \ldots S_{i_k}$. By Lemma 3.3, the sequence

$$(P_n S_i P_n) = (P_n S_{i_1} P_n S_{i_2} P_n \ldots P_n S_{i_k} P_n)$$

is a partial isometry. We are going to determine its initial projection. The result will indicate that it is more natural to consider a certain restriction of the finite sections algebra $\mathcal{S}(\mathcal{O}_N)$ rather than the full algebra $\mathcal{S}(\mathcal{O}_N)$.

For every real number x, let $\{x\}$ denote the smallest integer which is greater than or equal to x.

Proposition 3.4 *For $i = (i_1, i_2, \ldots, i_k) \in \Omega^k$, set $v_{i,k} := i_1 + i_2 N + \ldots + i_k N^{k-1}$. Then*

$$P_n S_i^* S_i P_n = P_{\{(n - v_{i,k})/N^k\}}. \tag{14}$$

Proof. It follows from the definition of S_i that there are numbers $v_{i,k}$ and d_k such that

$$\begin{aligned} S_i = S_{i_1} \ldots S_{i_k} : \quad & (x_k)_{k \ge 0} \mapsto \\ & (\underbrace{0, \ldots, 0,}_{v_{i,k}} x_0, \underbrace{0, \ldots, 0,}_{d_k}, x_1, \underbrace{0, \ldots, 0,}_{d_k}, x_2, \ldots). \end{aligned}$$

The initial values

$$v_{(i_1),1} = i_1 \quad \text{and} \quad d_1 = N - 1$$

together with the recursions

$$i_1 + v_{(i_2, \ldots, i_{k+1}),k} \cdot N = v_{(i_1, \ldots, i_{k+1}),k+1} \quad \text{and} \quad N - 1 + d_k \cdot N = d_{k-1}$$

imply via induction that

$$v_{i,k} = i_1 + i_2 N + \ldots + i_k N^{k-1} \quad \text{and} \quad d_k = N^k - 1. \tag{15}$$

It follows that the jth component of S_i^* applied to $x = (x_k)_{k \ge 0}$ is

$$(S_i^* x)_j = x_{v_{i,k} + j(d_k + 1)} = x_{v_{i,k} + jN^k}.$$

Consequently,

$$P_n S_i^* P_n S_i P_n = P_j \quad \text{if} \quad v_{i,k} + (j-1)N^k < n \le v_{i,k} + jN^k,$$

whence $j = \{(n - v_{i,k})/N^k\}$. ■

3.3 The need of restrictions

We specialize the result of Proposition 3.4 to the case $k = 1$ and consider two examples. If $n = jN$ is a multiple of N, then the initial projections of $P_n S_i P_n$ are independent of i. Indeed, from

$$\{(n-i)/N\} = \{(jN-i)/N\} = \{j - i/N\} = j$$

we obtain

$$P_{jN} S_i^* P_{jN} S_i P_{jN} = P_j \quad \text{for all } i \in \Omega. \tag{16}$$

On the other hand, one has

$$P_n S_0^* P_n S_0 P_n - P_n S_1^* P_n S_1 P_n = \begin{cases} P_{j+1} - P_j & \text{if } n = jN+1, \\ 0 & \text{else.} \end{cases} \tag{17}$$

Thus, the sequence

$$(P_n S_0^* P_n S_0 P_n - P_n S_1^* P_n S_1 P_n)_{n \geq 1} \tag{18}$$

possesses both a subsequence consisting of zeros only (take $\eta(n) := nN$) and a subsequence consisting of non-zero projections (if $\eta(n) := nN + 1$). This shows that the algebra $\mathcal{S}(\mathcal{O}_N)$ cannot be fractal.

Moreover, a closer look reveals that also the restricted algebra $\mathcal{S}_\eta(\mathcal{O}_N)$, with $\eta(n) := nN$, cannot be fractal: For the sequence

$$(P_{nN}(S_0^*)^2 P_{nN} S_0^2 P_{nN} - P_{nN}(S_1^*)^2 P_{nN} S_1^2 P_{nN})_{n \geq 1} \tag{19}$$

one observes a similar unpleasant behavior as before. We will see later on that

$$\eta(n) := N^n \tag{20}$$

is the correct choice for the restriction η, since it will indeed guarantee the fractality of the restricted algebra $\mathcal{S}_\eta(\mathcal{O}_N)$.

It is interesting to observe that there are at least two further arguments which also suggest the choice (20) for the restriction η. The first one comes from asymptotic numerical analysis again. Consider the set of all sequences (A_n) in $\mathcal{F}$ with

$$\sup_{n \geq 0} \operatorname{rank} A_n < \infty.$$

The closure of this set in $\mathcal{F}$ is a two-sided ideal $\mathcal{K}$ of $\mathcal{F}$, the elements of which are called *compact sequences*. The ideal $\mathcal{K}$ plays a similar role for sequence algebras as the ideal of the compact operators does for operator algebras. In particular, there is a Fredholm theory for approximation sequences which parallels the common Fredholm theory for operators and which has remarkable consequences for the asymptotic behavior of singular values (see Chapter 6 in [9] and Chapters 4 and

5 in [14] for an introduction and Section 6.4 below for a closer look at Fredholm properties of finite sections sequences for operators in the Cuntz algebra).

The point now is that the sequences (18) and (19) are compact in this sense (and do not belong to the smaller ideal $\mathcal{G}$ of the zero sequences). Since the Cuntz algebras $\mathcal{O}_N$ do not possess non-zero compact operators at all, it seems to be not natural to consider discretizations which produce non-zero compact sequences.

The second argument comes from operator theory. The following lemma states that the choice (20) implies that, up to sequences in the ideal $\mathcal{G}_\eta$, the initial projections of the partial isometries $(P_{\eta(n)} S_i P_{\eta(n)})$ with $i \in \Omega^k$ only depend on the length k of the multi-index, not on the multi-index i itself.

Lemma 3.5 *Let $i \in \Omega^k$ and $n = N^j$ with $j \in \mathbb{Z}^+$. Then*

$$P_n S_i^* P_n S_i P_n = \begin{cases} 0 & \text{if } j < k \text{ and } N^j \leq v_{i,k}, \\ P_1 & \text{if } j < k \text{ and } N^j > v_{i,k}, \\ P_{N^{j-k}} & \text{if } j \geq k. \end{cases}$$

Proof. By (14), one has $P_n S_i^* P_n S_i P_n = P_r$ with

$$r = \{(N^j - v_{i,k})/N^k\} = \{N^{j-k} - v_{i,k}/N^k\}.$$

If $j \geq k$, then N^{j-k} is a positive integer, whereas $v_{i,k}/N^k \in [0, 1)$. Thus, in this case, $r = N^{j-k}$. Now let $j < k$. Then

$$(N^j - v_{i,k})/N^k \leq N^{j-k} < 1.$$

Thus, if $N^j - v_{i,k} > 0$, then $r = 1$. Finally, let $j < k$ and $N^j - v_{i,k} \leq 0$. Then we conclude from

$$(N^j - v_{i,k})/N^k \geq N^{j-k} - 1 > -1$$

that $r = 0$ (recall that we agreed upon $P_0 = 0$). ■

3.4 The restricted algebra $\mathcal{S}_\eta(\mathcal{O}_N)$

In what follows we will exclusively deal with the restricted algebra $\mathcal{S}_\eta(\mathcal{O}_N)$ where $\eta(n) = N^n$, as suggested by the arguments of the previous section.

Proposition 3.6 *The algebra $\mathcal{S}_\eta(\mathcal{O}_N)$ contains the ideal $\mathcal{G}_\eta$.*

Proof. In a first step we show that $\mathcal{S}_\eta(\mathcal{O}_N)$ contains all sequences of the form

$$(0, \ldots, 0, P_1, 0, 0, \ldots) \in \mathcal{F}_\eta \tag{21}$$

where P_1 stands at the kth position, k an arbitrary positive integer.

First let $k = 1$. Define 1-tuples

$$i_> := (0), \quad i_< := (1) \ \in \Omega^1.$$

Then $v_{i_>,1} = 0$ and $v_{i_<,1} = 1$, and for $j = 0$ one has

$$N^j > v_{i_>,1} = 0 \quad \text{but} \quad N^j \le v_{i_<,1} = 1.$$

By Lemma 3.5,

$$(P_{N^n} S^*_{i_>} P_{N^n} S_{i_>} P_{N^n})_{n \ge 0} = (P_1,\, P_{N^0},\, P_{N^1},\, P_{N^2},\, \ldots) \in \mathcal{S}_\eta(\mathcal{O}_N)$$

and

$$(P_{N^n} S^*_{i_<} P_{N^n} S_{i_<} P_{N^n})_{n \ge 0} = (0,\, P_{N^0},\, P_{N^1},\, P_{N^2},\, \ldots) \in \mathcal{S}_\eta(\mathcal{O}_N).$$

Hence, the sequence $(P_1, 0, 0, \ldots)$ belongs to $\mathcal{S}_\eta(\mathcal{O}_N)$. Let now $k \ge 2$ and set

$$i_> := (\underbrace{0,\, 0,\, \ldots,\, 0}_{k-2},\, 1,\, 0), \quad i_< := (\underbrace{0,\, 0,\, \ldots,\, 0}_{k-1},\, 1) \in \Omega^k.$$

Then $v_{i_>,k} = N^{k-2}$ and $v_{i_<,k} = N^{k-1}$, whence via Lemma 3.5,

$$(P_{N^n} S^*_{i_>} P_{N^n} S_{i_>} P_{N^n})_{n \ge 0} = (\underbrace{0,\, 0,\, \ldots,\, 0}_{k-1},\, P_1,\, P_{N^0},\, P_{N^1},\, P_{N^2},\, \ldots) \in \mathcal{S}_\eta(\mathcal{O}_N)$$

and

$$(P_{N^n} S^*_{i_<} P_{N^n} S_{i_<} P_{N^n})_{n \ge 0} = (\underbrace{0,\, 0,\, \ldots,\, 0}_{k},\, P_{N^0},\, P_{N^1},\, P_{N^2},\, \ldots) \in \mathcal{S}_\eta(\mathcal{O}_N).$$

Hence, the sequence $(0, \ldots, 0, P_1, 0, \ldots)$ with P_1 standing at the kth position belongs to $\mathcal{S}_\eta(\mathcal{O}_N)$, too.

In the next step we show that $\mathcal{S}_\eta(\mathcal{O}_N)$ contains all sequences of the form

$$(0,\, \ldots,\, 0,\, A,\, 0,\, 0,\, \ldots) \in \mathcal{F}_\eta \tag{22}$$

where A is an arbitrary $N^k \times N^k$ matrix standing at the kth position. Since all sequences of the form (21) belong to $\mathcal{S}_\eta(\mathcal{O}_N)$, it is sufficient to check that the set of all $n \times n$ matrices P_1, $P_n S_i P_n$ and $P_n S^*_i P_n$ with $i \in \Omega^k$ and $k \ge 1$ generates the algebra $\mathbb{C}^{n \times n}$ (of course, we will need this fact later on only for n being a power of N). Let $0 \le r,\, s < n$. Write r and s in the N-adic system as

$$r = i_1 + i_2 N + \ldots + i_k N^{k-1}, \qquad s = j_1 + j_2 N + \ldots + j_k N^{k-1}$$

with $i = (i_1, \ldots, i_k)$, $j = (j_1, \ldots, j_k) \in \Omega^k$. Then $r = v_{i,k}$ and $s = v_{j,k}$, and

$$\begin{aligned} P_n S_j P_n \cdot P_1 \cdot P_n S^*_i P_n (x_m)_{m=0}^{n-1} &= (\underbrace{0,\, \ldots,\, 0}_{v_{j,k}},\, x_{v_{i,k}},\, 0,\, \ldots,\, 0) \\ &= (\underbrace{0,\, \ldots,\, 0}_{s},\, x_r,\, 0,\, \ldots,\, 0). \end{aligned}$$

Thus, $P_n S_j P_n \cdot P_1 \cdot P_n S^*_i P_n$ is a matrix which takes the rth entry of a vector and writes it on the sth place. Since every matrix is a linear combination of matrices of this kind, the assertion of the second step follows.

Finally, every sequence in $\mathcal{G}_\eta$ can be approximated as closely as desired by finite sums of sequences of the form (22). Since $\mathcal{S}_\eta(\mathcal{O}_N)$ is closed, this implies $\mathcal{G}_\eta \subseteq \mathcal{S}_\eta(\mathcal{O}_N)$. ∎

4 The finite sections algebra $\mathcal{S}_N$

4.1 A distinguished ideal

By Proposition 3.6, one can form the quotient algebra $\mathcal{S}_\eta(\mathcal{O}_N)/\mathcal{G}_\eta$. We denote it by $\mathcal{S}_N$. Recall that $\mathcal{S}_N$ is generated by the partial isometries

$$s_i := (P_{N^n} S_i P_{N^n})_{n\geq 0} + \mathcal{G}_\eta, \quad i \in \Omega$$

and contains the identity element e of $\mathcal{F}_\eta/\mathcal{G}_\eta$. For each multi-index $i \in \Omega^k$ we further set

$$s_i := s_{i_1} s_{i_2} \dots s_{i_k}.$$

These elements are partial isometries by Lemma 3.3, and the initial projection of s_i does only depend on the length of i by Lemma 3.5. We denote the length of the multi-index i by $|i|$ and write p_k for the joint initial projection of all partial isometries s_i with length k. Further we write Ω_∞ for the set of all multi-indices (of arbitrary length). The following axioms collect the basic properties of these elements.

(A1) for every $i \in \Omega_\infty$, the coset s_i is a partial isometry, the initial projection of which depends on $|i|$ only: $s_i^* s_i = p_{|i|}$.

(A2) $s_0 s_0^* + s_1 s_1^* + \ldots + s_{N-1} s_{N-1}^* = e$.

All results in Section 4 will follow only from these two axioms. For later reference we list some further relations between partial isometries s_i and the projections p_k.

Lemma 4.1 *Let k, l positive integers and $i \in \Omega^k$. Then*
(a) $s_i = s_i p_k$ and $s_i^* = p_k s_i^*$.
(b) $s_i^* p_l s_i = p_{k+l}$.
(c) $p_k p_l = p_k$ if $k \geq l$.
(d) $p_l s_i = s_i p_{k+l}$.
(e) The generalized Cuntz condition $\sum_{i\in\Omega^k} s_i s_i^* = e$ holds for every $k \geq 1$.

Proof. Assertion (a) is the fact that p_k is the initial projection of s_i. For (b) write p_l as $s_j^* s_j$ with $j \in \Omega^l$. Then $s_i^* p_l s_i = s_i^* s_j^* s_j s_i = p_{k+l}$ by the definition of p_{k+l}. For (c), write p_k and p_l as $(s_0^*)^k s_0^k$ and $(s_0^*)^l s_0^l$. Since s_0^l is a partial isometry,

$$p_k p_l = (s_0^*)^k s_0^{k-l} s_0^l (s_0^*)^l s_0^l = (s_0^*)^k s_0^{k-l} s_0^l = p_k,$$

which gives (c). Assertions (b) and (c) imply that

$$\begin{aligned}(p_l s_i - s_i p_{k+l})^*(p_l s_i - s_i p_{k+l}) &= (s_i^* p_l - p_{k+l} s_i^*)(p_l s_i - s_i p_{k+l}) \\ &= s_i^* p_l s_i - s_i^* p_l s_i p_{k+l} - p_{k+l} s_i^* p_l s_i + p_{k+l} s_i^* s_i p_{k+l} \\ &= p_{k+l} - p_{k+l} - p_{k+l} + p_{k+l} p_k p_{k+l} = 0,\end{aligned}$$

whence (d) via the C^*-axiom. Finally, assertion (e) follows easily by induction. For $k = 1$, (e) reduces to $(A2)$. If assertion (e) holds for some $k \geq 1$, then it holds for $k + 1$ since

$$\sum_{i \in \Omega^{k+1}} s_i s_i^* = \sum_{(i_1, \tilde{i}) \in \Omega^{k+1}} s_{(i_1, \tilde{i})} s_{(i_1, \tilde{i})}^* = \sum_{i_1 \in \Omega^1} \sum_{\tilde{i} \in \Omega^k} s_{i_1} s_{\tilde{i}} s_{\tilde{i}}^* s_{i_1}^* = \sum_{i_1 \in \Omega^1} s_{i_1} s_{i_1}^* = e$$

by assumption and axiom (A2). ■

For every positive integer k, let $\mathcal{J}^{(k)}$ denote the smallest closed ideal of $\mathcal{S}_N$ which contains the projection $e - p_k$. By axiom (A1), every partial isometry s_i of length k is an isometry modulo $\mathcal{J}^{(k)}$. By Lemma 3.5,

$$e - p_1 = (0,\, P_N - P_1,\, P_{N^2} - P_N,\, P_{N^3} - P_{N^2},\, \ldots) + \mathcal{G}_\eta.$$

Proposition 4.2 *$\mathcal{J}^{(k)} = \mathcal{J}^{(1)}$ for every k.*

Proof. By Lemma 4.1 (c),

$$(e - p_k)(e - p_1) = e - p_k - p_1 + p_k p_1 = e - p_1.$$

Hence, $e - p_1 \in \mathcal{J}^{(k)}$, whence $\mathcal{J}^{(1)} \subseteq \mathcal{J}^{(k)}$. For the reverse inclusion recall from Lemma 4.1 (b) that

$$(s_0^*)^l (e - p_1) s_0^l = p_l - p_{l+1}$$

for every $l \in \mathbb{Z}^+$. Adding these identities for l between 0 and $k - 1$ gives $e - p_k$ on the right-hand side, whereas the element of the left-hand side belongs to $\mathcal{J}^{(1)}$. Thus, $e - p_k \in \mathcal{J}^{(1)}$, whence $\mathcal{J}^{(k)} \subseteq \mathcal{J}^{(1)}$. ■

In what follows we write $\mathcal{J}_N$ for the ideal $\mathcal{J}^{(1)}$ of $\mathcal{S}_N$. Note that *every* partial isometry s_i with $i \in \Omega_\infty$ is an isometry modulo $\mathcal{J}_N$.

Remark 4.3 The smallest closed ideal of $\mathcal{S}_N$ which contains the projection p_1 coincides with all of $\mathcal{S}_N$. Indeed, from $s_i = s_i s_i^* s_i = s_i p_1$ we conclude that every element s_i with $i \in \Omega$ belongs to this ideal. Since the s_i generate the algebra $\mathcal{S}_N$, the assertion follows.

We conclude this section with a further property of the p_n which will be needed in Section 4.

Lemma 4.4 *For each $j \in \mathcal{J}_N$, one has $\lim_{n\to\infty} \|p_n j\| = 0$.*

Proof. If j is of the form

$$s_i s_k^* (e - p_1) s_l s_m^* \quad \text{with} \quad |i| \leq |k| \tag{23}$$

then the assertion holds since

$$p_n s_i s_k^* (e - p_1) = s_i s_k^* p_{n+|i|-|k|} (e - p_1) = 0$$

for $n > |k| - |i|$. Hence, the assertion also holds if j is a linear combination of elements of the form (23). Since these linear combinations form a dense subset of $\mathcal{J}_N$ and $\|p_n\| = 1$ for each n, the assertion holds for every $j \in \mathcal{J}_N$. ■

4.2 Lifting $\mathcal{S}_N/\mathcal{J}_N$ to $\mathcal{S}_N$

Our further analysis of the algebra $\mathcal{S}_N$ is based on the following elementary fact, which can be considered as the simplest instance of a lifting theorem. It settles a condition which guarantees than every element which is invertible modulo an ideal can be lifted to an invertible element. The simple proof is in [9], Theorem 5.33.

Proposition 4.5 *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{I}$ a closed ideal of $\mathcal{A}$. Further suppose there is an unital *-homomorphism π from $\mathcal{A}$ into a unital C^*-algebra $\mathcal{B}$ such that the restriction of π onto $\mathcal{I}$ is injective. Then the following assertions are equivalent for every element $a \in \mathcal{A}$:*
(a) a is invertible in $\mathcal{A}$.
(b) The coset $a+\mathcal{I}$ is invertible in the quotient algebra $\mathcal{A}/\mathcal{I}$, and $\pi(a)$ is invertible in $\mathcal{B}$.

We shall apply this result with $\mathcal{A} := \mathcal{S}_N$ and $\mathcal{I} := \mathcal{J}_N$. By Proposition 4.5, the problem to derive a criterion for the invertibility of elements of $\mathcal{S}_N$ (and thus, for the stability of sequences in $\mathcal{S}_\eta(\mathcal{O}_N)$) splits into two separate tasks:

- to describe the quotient algebra $\mathcal{S}_N/\mathcal{J}_N$, and
- to construct an injective *-homomorphism on $\mathcal{J}_N$.

The solution of the first task is evident: The quotient algebra $\mathcal{S}_N/\mathcal{J}_N$ is generated by the cosets $s_i + \mathcal{J}_N$ with $i \in \Omega$. These cosets are isometries and they satisfy the Cuntz axiom

$$(s_0 + \mathcal{J}_N)(s_0 + \mathcal{J}_N)^* + \ldots + (s_{N-1} + \mathcal{J}_N)(s_{N-1} + \mathcal{J}_N)^* = e + \mathcal{J}_N.$$

By the universal property of Cuntz algebras, $\mathcal{S}_N/\mathcal{J}_N$ is *-isomorphic to the (abstract) Cuntz algebra $\mathcal{O}_N$. It is also not difficult to construct an isomorphism from $\mathcal{S}_N/\mathcal{J}_N$ onto the (concrete) Cuntz algebra $\mathcal{O}_N$ explicitly. For, let $W_\eta : \mathcal{F}_\eta \to L(l^2(\mathbb{Z}^+))$ denote the mapping which associates with each sequence in $\mathcal{F}_\eta$ its strong limit. Clearly, $_\eta$ is a *-homomorphism. Since the ideal $\mathcal{G}_\eta$ lies in the kernel of W_η, there is a correctly defined quotient homomorphism

$$W_\eta^{\mathcal{G}} : \mathcal{F}_\eta/\mathcal{G}_\eta \to L(l^2(\mathbb{Z}^+)), \quad \mathbf{A} + \mathcal{G}_\eta \mapsto W_\eta(\mathbf{A}). \tag{24}$$

Applying this homomorphism to both sides of the equality $s_0^* s_0 = p_1$ we get $S_0^* S_0 = W_\eta^{\mathcal{G}}(p_1)$, whence $W_\eta^{\mathcal{G}}(p_1) = I$. Hence, the ideal $\mathcal{J}_N$ lies in the kernel of $W_\eta^{\mathcal{G}}$, which implies that the quotient homomorphism

$$(\mathcal{S}_\eta(\mathcal{O}_N)/\mathcal{G}_\eta)/\mathcal{J}_N \to L(l^2(\mathbb{Z}^+)), \quad (\mathbf{A} + \mathcal{G}_\eta) + \mathcal{J}_N \mapsto W_\eta^{\mathcal{G}}(\mathbf{A} + \mathcal{G}) \tag{25}$$

is correctly defined, too; we denote it by $W^{\mathcal{J}}$.

Theorem 4.6 *$W^{\mathcal{J}}$ is a *-isomorphism from $\mathcal{S}_N/\mathcal{J}_N$ onto the (concrete) Cuntz algebra $\mathcal{O}_N$.*

Proof. The *-homomorphism $W^{\mathcal{J}}$ maps the generating cosets $s_i + \mathcal{J}_N$, $i \in \Omega$, to the generating operators S_i of the Cuntz algebra $\mathcal{O}_N$, respectively. Since both sets of generators consist of partial isometries which satisfy the (same) Cuntz axiom, the assertion follows form the universal property of Cuntz algebras again. ■

The following is an immediate consequence of this theorem and of the fact that $\mathcal{O}_N$ is a simple algebra.

Corollary 4.7 *The kernel of the restriction of the homomorphism $W_\eta^{\mathcal{G}}$ defined by* (24) *to the algebra $\mathcal{S}_N$ coincides with $\mathcal{J}_N$.*

The following fact sheds a first light on our second task. Some consequences of this fact are already discussed in the introduction.

Theorem 4.8 *Every proper closed ideal of $\mathcal{S}_N$ lies in $\mathcal{J}_N$.*

Proof. Let $\widetilde{\mathcal{J}}$ be a proper closed ideal of $\mathcal{S}_N$. Then $\mathcal{J}_N + \widetilde{\mathcal{J}}$ is a closed ideal of $\mathcal{S}_N$ with $\mathcal{J}_N \subseteq \mathcal{J}_N + \widetilde{\mathcal{J}} \subseteq \mathcal{S}_N$. Since the quotient $\mathcal{S}_N/\mathcal{J}_N$ is *-isomorphic to $\mathcal{O}_N$ and, hence, a simple algebra, one has either

- Case A: $\mathcal{J}_N + \widetilde{\mathcal{J}} = \mathcal{S}_N$, or
- Case B: $\mathcal{J}_N + \widetilde{\mathcal{J}} = \mathcal{J}_N$, i.e. $\widetilde{\mathcal{J}} \subseteq \mathcal{J}_N$.

We wish to exclude case A. Suppose we are in the situation of case A. Consider the ideals $\mathcal{I}_1 := \mathcal{J}_N/(\mathcal{J}_N \cap \widetilde{\mathcal{J}})$ and $\mathcal{I}_2 := \widetilde{\mathcal{J}}/(\mathcal{J}_N \cap \widetilde{\mathcal{J}})$ of $\mathcal{B} := \mathcal{S}_N/(\mathcal{J}_N \cap \widetilde{\mathcal{J}})$. These ideals have a trivial intersection, their sum is $\mathcal{B}$, and the algebra

$$\mathcal{B}/\mathcal{I}_1 = \left(\mathcal{S}_N/(\mathcal{J}_N \cap \widetilde{\mathcal{J}})\right) / \left(\mathcal{J}_N/(\mathcal{J}_N \cap \widetilde{\mathcal{J}})\right) \cong \mathcal{S}_N/\mathcal{J}_N$$

is still simple. Let W stand for the canonical homomorphism from $\mathcal{B}$ onto $\mathcal{B}/\mathcal{I}_2$ and write $\hat{a}$ for the coset of $a \in \mathcal{S}_N$ modulo $\mathcal{J}_N \cap \widetilde{\mathcal{J}}$. Since $W(\mathcal{B}) = W(\mathcal{I}_1)$, there is an element $\hat{\pi} \in \mathcal{I}_1$ such that $W(\hat{\pi}) = W(\hat{e})$. From

$$W(\hat{\pi}^2 - \hat{\pi}) = W(\hat{e}^2 - \hat{e}) = 0 \quad \text{and} \quad W(\hat{\pi}^* - \hat{\pi}) = W(\hat{e}^* - \hat{e}) = 0$$

we conclude that both $\hat{\pi}^2 - \hat{\pi}$ and $\hat{\pi}^* - \hat{\pi}$ belong to $\mathcal{I}_1 \cap \mathcal{I}_2$. Since this intersection is trivial, the element $\hat{\pi}$ is a (self-adjoint) projection. Moreover, since

$$W(\hat{a}\hat{\pi} - \hat{\pi}\hat{a}) = W(\hat{a})W(\hat{e}) - W(\hat{e})W(\hat{a}) = 0$$

for every element $\hat{a} \in \mathcal{B}$ we conclude as above that $\hat{\pi}$ lies in the commutant of $\mathcal{B}$. A similar reasoning shows finally that $\hat{\pi}$ is the identity element for $\mathcal{I}_1$. Similarly, $\hat{e} - \hat{\pi}$ belongs to $\mathcal{I}_2$ and is the identity element for $\mathcal{I}_2$.

Let $\pi \in \mathcal{J}_N$ be a representative of the coset $\hat{\pi}$. From Lemma 4.4 we infer

$$\|(e - p_n)\pi - \pi\| =\!\!\longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

whence

$$\|(\hat{e} - \widehat{p_n})\hat{\pi} - \hat{\pi}\| =\!\!\longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Since $\hat{\pi}$ is the identity element of $\mathcal{I}_1$ and $\hat{e} - \widehat{p_n} \in \mathcal{I}_1$, this implies

$$\|\hat{e} - \widehat{p_n} - \hat{\pi}\| =\!\!\longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Since $\hat{e} - \widehat{p_n}$ and $\hat{\pi}$ are commuting projections, this finally shows that $\hat{e} - \widehat{p_n} = \hat{\pi}$ for all sufficiently large n, say $n \geq k$. Consequently, for $n \geq k$, one has $\widehat{p_n} = \hat{e} - \hat{\pi} \in \mathcal{I}_2$, whence $p_n \in \widetilde{\mathcal{J}}$. Since $s_i = s_i p_k$ for all $i \in \Omega^k$ by Lemma 4.1, this implies $s_i \in \widetilde{\mathcal{J}}$ and, thus, the smallest closed ideal of $\mathcal{S}_N$ which contains all partial isometries s_i with $|i| = k$ lies in $\widetilde{\mathcal{J}}$. By Lemma 4.1 (e), this finally implies $e \in \widetilde{\mathcal{J}}$. Thus, $\widetilde{\mathcal{J}}$ is not a proper ideal of $\widetilde{\mathcal{J}}$. This contradiction excludes case A. ■

Corollary 4.9 *Every *-homomorphism on $\mathcal{S}_N$ which is injective on $\mathcal{J}_N$ is injective on all of $\mathcal{S}_N$.*

Indeed, if W is a *-homomorphism on $\mathcal{S}_N$ which is injective on $\mathcal{J}_N$, then its kernel is a proper ideal of $\mathcal{S}_N$. By Theorem 4.8, $\ker W \subset \mathcal{J}_N$. But $\mathcal{J}_N \cap \ker W = \{0\}$ by assumption. Hence, the kernel of W is trivial, and W is injective on $\mathcal{S}_N$. ■

In Section 6 we are going to construct an injective homomorphism on $\mathcal{J}_N$. We prepare this construction by a closer look at the Cuntz algebra and a related Toeplitz algebra in Section 5

5 Expectations of $\mathcal{O}_N$ and stratified Toeplitz operators

5.1 Block Toeplitz operators

By $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$ we denote the Hilbert space of all sequences $x = (x_n)_{n\geq 0}$ with values in $l^2(\mathbb{Z}^+)$ such that

$$\|x\|^2 := \sum_{n\geq 0} \|x_n\|^2 < \infty.$$

It will be convenient to identify the algebra of all bounded linear operators on $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$ with the minimal tensor product $L(l^2(\mathbb{Z}^+)) \otimes L(l^2(\mathbb{Z}^+))$. Let A be a bounded linear operator on $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$, and let $A_{ij} \in L(l^2(\mathbb{Z}^+))$ be the operator which maps the ith component of x to the jth component of Ax. Then A can be identified with the infinite matrix $(A_{ij})_{i,j\geq 0}$ in an evident way (but, of course, not every matrix with entries in $L(l^2(\mathbb{Z}^+))$ defines a bounded operator on $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+)))$.

The closure in $L(l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+)))$ of the set of all matrices with only finitely non-vanishing entries forms a closed ideal of this algebra which we identify with $K(l^2(\mathbb{Z}^+)) \otimes L(l^2(\mathbb{Z}^+))$ in a natural way.

To each function $a \in C(\mathbb{T}, L(l^2(\mathbb{Z}^+)))$ we associate its kth Fourier coefficient

$$a_k := \int_{\mathbb{T}} a(\lambda)\lambda^{-k}\, d\lambda \in L(l^2(\mathbb{Z}^+)), \quad k \in \mathbb{Z},$$

and consider the (infinite) Toeplitz matrix

$$T(a) := (a_{i-j})_{i,j\geq 0}.$$

Every Toeplitz matrix with a continuous generating function defines a bounded operator on $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$ which is a called a Toeplitz operator and denoted by $T(a)$ again, and

$$\|T(a)\| = \|a\|_\infty. \tag{26}$$

The Toeplitz operator $T(a)$ is invertible modulo $K(l^2(\mathbb{Z}^+)) \otimes L(l^2(\mathbb{Z}^+))$ if and only if the function a is invertible in $L(l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+)))$. For details see [11].

5.2 An algebra of stratified Toeplitz operators

For $i \in \Omega$, consider the infinite matrix

$$\Sigma_i := \begin{pmatrix} 0 & S_i & & & \\ & 0 & S_i & & \\ & & 0 & S_i & \\ & & & \ddots & \ddots \end{pmatrix} \tag{27}$$

with all entries left empty being zero, and for each multi-index $i \in \Omega^k$, let

$$\Sigma_i := \Sigma_{i_1} \dots \Sigma_{i_k}.$$

Clearly, every Σ_i is a Toeplitz matrix with continuous generating function. It defines a bounded operator on $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$, which we denote by Σ_i again. We let $\mathcal{T}_N$ refer to the smallest closed subalgebra of $L(l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+)))$ which contains all operators Σ_i and Σ_i^* with $i \in \Omega$.

One peculiarity of operators in $\mathcal{T}_N$ which will become more clear and important later on should be mentioned already here: Their matrix representation is stratified in the sense that all entries on the kth diagonal above the main diagonal are necessarily of the form $\Sigma_i \mathcal{O}_N^{par}$ with an multi-index i of length k. Thus, the only operator which can stand on each diagonal is the zero operator. (A curious consequence of this fact is that the only block Hankel operators which are contained in the Toeplitz algebra $\mathcal{T}_N$ are of the form of a diagonal matrix $\operatorname{diag}(A, 0, 0, \ldots)$ with $A \in L(l^2(\mathbb{Z}^+))$.)

One easily checks that the operators Σ_i are partial isometries which satisfy the Cuntz axiom

$$\sum_{i \in \Omega} \Sigma_i \Sigma_i^* = \sum_{i \in \Omega} \operatorname{diag}(S_i S_i^*, S_i S_i^*, \ldots) = \operatorname{diag}(I, I, \ldots).$$

Moreover, for each multi-index i, the initial projection $\Sigma_i^* \Sigma_i$ is equal to $I - \Pi_{|i|}$ where

$$\Pi_k := \operatorname{diag}(\underbrace{I, \ldots, I}_{k}, 0, 0, \ldots) \in \mathcal{T}_N \tag{28}$$

for $k \geq 1$. Thus, the algebra $\mathcal{T}_N$ contains the identity operator I and all projections Π_k, and the partial isometries Σ_i satisfy the axioms (A1) and (A2) in Section 4.1 in place of the s_i. Thus, all results of Section 4 will remain valid for the algebra $\mathcal{T}_N$ in place of $\mathcal{S}_N$ and for its ideal $\mathcal{C}_N$, which is the smallest closed ideal of $\mathcal{T}_N$ which contains the projection Π_1, in place of $\mathcal{J}_N$. In particular,

$$\Pi_n \in \mathcal{C}_N \quad \text{for each } n \geq 1 \tag{29}$$

and

$$\mathcal{T}_N / \mathcal{C}_N \cong \mathcal{O}_N. \tag{30}$$

5.3 Identification of $\mathcal{C}_N$

Our next goal is a description of the ideal $\mathcal{C}_N$ of $\mathcal{T}_N$. Let $\mathcal{O}_N^{par}$ refer to the smallest closed subalgebra of $\mathcal{O}_N$ which contains all products $S_i S_j^*$ with multi-indices i and j of the same length $|i| = |j|$. Here we allow multi-indices of length 0 and set $S_\emptyset := I$. Thus, $\mathcal{O}_N^{par}$ is a unital algebra. One easily checks that $\mathcal{O}_n$ is the closed span of the set of all products $S_i S_j^*$, whereas $\mathcal{O}_N^{par}$ is the closed span of all products $S_i S_j^*$ with $|i| = |j|$. The algebra $\mathcal{O}_N^{par}$ is known to be isomorphic to the UHF-algebra of type N^∞. In particular, $\mathcal{O}_2^{par}$ is the CAR-algebra. See [7, 8] for details.

We shall further make use of the following elementary observation. Let $\mathcal{A}$ be a C^*-subalgebra of a unital C^*-algebra $\mathcal{B}$ and let Σ be an isometry in $\mathcal{B}$. Then the mapping

$$\mathcal{A} \to \mathcal{B}, \quad A \mapsto \Sigma A \Sigma^* \tag{31}$$

is an injective *-homomorphism. Indeed, $\Sigma A_1 \Sigma^* \Sigma A_2 \Sigma^* = \Sigma A_1 A_2 \Sigma^*$ for $A_1, A_2 \in \mathcal{A}$, and if $\Sigma A \Sigma^* = 0$, then

$$0 = \Sigma^* \Sigma A \Sigma^* \Sigma = A.$$

We apply this observation to the algebras $\mathcal{A} := \mathcal{C}_N$ and $\mathcal{B} := L(l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+)))$ and to the isometry

$$\Sigma := \operatorname{diag}(I, S_0, S_0^2, S_0^3, \ldots).$$

Thus, $\mathcal{C}_N$ is *-isomorphic to $\Sigma\mathcal{C}_N\Sigma^*$. The latter algebra can be described as follows where we let $\Pi := \Sigma\Sigma^*$ be the image of the identity operator under the mapping (31).

Theorem 5.1 $\Sigma\mathcal{C}_N\Sigma^* = \Pi\,(K(l^2(\mathbb{Z}^+)) \otimes \mathcal{O}_N^{par})\,\Pi.$

Proof. Since the algebra $\mathcal{T}_N$ is generated by the partial isometries Σ_i, the algebra $\Sigma\mathcal{T}_N\Sigma^*$ is generated by the operators

$$\Sigma\Sigma_i\Sigma^* = \begin{pmatrix} 0 & S_iS_0^* & & & \\ & 0 & S_0S_i(S_0^*)^2 & & \\ & & 0 & S_0^2S_i(S_0^*)^3 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

All entries of this matrix belong to $\mathcal{O}_N^{par}$. Hence,

$$\Sigma\mathcal{T}_N\Sigma^* \subseteq \Pi\left(L(l^2(\mathbb{Z}^+)) \otimes \mathcal{O}_N^{par}\right)\Pi.$$

Further, the mapping (31) sends the generator Π_1 of the ideal $\mathcal{C}_N$ to itself. Since $\Pi_1\Pi = \Pi\Pi_1$, one has

$$\Sigma\Pi_1\Sigma^* = \Pi_1 \in \Pi\left(K(l^2(\mathbb{Z}^+)) \otimes \mathcal{O}_N^{par}\right)\Pi,$$

whence the inclusion

$$\Sigma\mathcal{C}_N\Sigma^* \subseteq \Pi\left(K(l^2(\mathbb{Z}^+)) \otimes \mathcal{O}_N^{par}\right)\Pi.$$

The reverse inclusion will follow once we have shown that for each $n \times n$-matrix $A := (A_{ij})$ with entries in $\mathcal{O}_N^{par}$, which we identify with an operator on the range of Π_n in the obvious way, the operator $\Pi A\Pi$ belongs to $\Sigma\mathcal{C}_N\Sigma^*$. Due to linearity we can further assume that only one of the entries of A, say A_{ij}, is different from 0. Finally, since $\mathcal{O}_N^{par}$ is spanned by products $S_lS_r^*$ with multi-indices of the same length, we can assume that only the ijth entry of A is different from zero and that this entry is $S_lS_r^*$ with $|r| = |l|$. Then $\Pi A\Pi$ is a matrix the only non-vanishing entry of which stands at the ijth position, and this entry is

$$S_0^i(S_0^*)^i\, S_lS_r^*\, S_0^j(S_0^*)^j. \tag{32}$$

Let

$$B := (\Pi_{i+1} - \Pi_i)(\Sigma_0^*)^i\Sigma_l\Sigma_r^*\Sigma_0^j(\Pi_{j+1} - \Pi_j).$$

This operator is in $\mathcal{C}_N$, all entries in the matrix representation of $\Sigma B\Sigma^*$ with exception of the ijth entry vanish, and the ijth entry coincides with (32). Thus, $\Sigma B\Sigma^* = \Pi A\Pi$, which finishes the proof. ∎

Next we will have a closer look at the ideal structure of $\mathcal{T}_N$.

Theorem 5.2 *The ideal $\mathcal{C}_N$ of $\mathcal{T}_N$ is simple.*

Proof. Let $\mathcal{R}$ be a closed ideal of $\mathcal{C}_N$. Then $\Pi_1\mathcal{R}\Pi_1$ is a closed ideal of $\Pi_1\mathcal{C}_N\Pi_1$. From Theorem 5.1 we infer that $\Pi_1\mathcal{C}_N\Pi_1$ is *-isomorphic to the algebra $\mathcal{O}_N^{par}$ which on its hand is known to be *-isomorphic to the UHF-algebra of type N^∞. Thus, $\Pi_1\mathcal{C}_N\Pi_1$ is isomorphic to the inductive limit

$$\mathbb{C} \to \mathbb{C}^{N\times N} \to \mathbb{C}^{N^2\times N^2} \to \mathbb{C}^{N^3\times N^3} \to \dots$$

with connecting maps

$$a \mapsto \operatorname{diag}(\underbrace{a,\, a,\, \dots, a}_{N}).$$

Being an inductive limit of simple algebras, the algebra $\Pi_1\mathcal{C}_N\Pi_1$ is simple. Hence,

$$\text{either} \quad \Pi_1\mathcal{R}\Pi_1 = \Pi_1\mathcal{C}_N\Pi_1 \quad \text{or} \quad \Pi_1\mathcal{R}\Pi_1 = \{0\}.$$

In the first case, $\Pi_1 \in \Pi_1\mathcal{R}\Pi_1 \subseteq \mathcal{R}$. Since Π_1 generates $\mathcal{C}_N$ as an ideal, we conclude $\mathcal{R} = \mathcal{C}_N$.

Assume now that $\Pi_1\mathcal{R}\Pi_1 = \{0\}$. Let $R = (R_{ij})_{ij\geq 0} \in \mathcal{R}$. Then, for arbitrary subscripts i_0, $j_0 \geq 0$, the matrix

$$(\Pi_{i_0+1} - \Pi_{i_0})R(\Pi_{j_0+1} - \Pi_{j_0})$$

has the entry $R_{i_0j_0}$ at the i_0j_0th position whereas all other entries are zero. Let k and l be multi-indices with $|k| = i_0$ and $|l| = j_0$. Then the matrix

$$\Sigma_k(\Pi_{i_0+1} - \Pi_{i_0})R(\Pi_{j_0+1} - \Pi_{j_0})\Sigma_l^* \tag{33}$$

has the entry $S_kR_{i_0j_0}S_l^*$ at the 00th position whereas all other entries are zero. Thus, the matrix (33) belongs to $\Pi_1\mathcal{R}\Pi_1$, whence $S_kR_{i_0j_0}S_l^* = 0$ by assumption. Since the S_i are isometries, this implies $R_{i_0j_0} = 0$, and since i_0 and j_0 were arbitrarily chosen, R is the zero matrix. Thus, $\mathcal{R}$ is the zero ideal in this case. ∎

As already mentioned, the Σ_i satisfy the axioms (A1) and (A2) and, thus, all results of Section 4 hold for the algebra $\mathcal{T}_N$ in place of $\mathcal{S}_N$ as well. In particular, every proper closed ideal of $\mathcal{T}_N$ lies in $\mathcal{C}_N$ by Theorem 4.8. Together with Theorem 5.2 this implies

Corollary 5.3 *$\mathcal{C}_N$ is the only non-trivial closed ideal of $\mathcal{T}_N$.*

5.4 Expectations on $\mathcal{O}_N$ and Toeplitz operators

There are at least two ways to associate with every element of the Cuntz algebra $\mathcal{O}_N$ a Toeplitz operator in $\mathcal{T}_N$. For facts cited without proof see [7, 8].

The first way is via a special expectation. Recall that the operators S_i and S_j^* with $i, j \in \Omega$ generate a dense subalgebra of $\mathcal{O}_N$. Each operator A in this algebra can be uniquely written as a finite sum

$$A = \sum_{k<0} (S_0^*)^{-k} A_k + A_0 + \sum_{k>0} A_k S_0^k \tag{34}$$

with "Fourier" coefficients $A_k \in \mathcal{O}_N^{par}$. For $k \in \mathbb{Z}$ and A as in (34, define $\Phi_k(A) := A_k$. Then $\|\Phi_k(A)\| \le \|A\|$, thus the Φ_k extend by continuity to bounded mappings from $\mathcal{O}_N$ onto $\mathcal{O}_N^{par}$. These mappings own the following properties:

- $\Phi_0 : \mathcal{O}_N \to \mathcal{O}_N^{par}$ is an expectation, i.e. $\Phi_0^2 = \Phi_0$,
- $\Phi_{k+1}(A) = \Phi_k(AS_0^*)$ if $k \ge 0$, and
- $\Phi_{k-1}(A) = \Phi_k(S_0 A)$ if $k < 0$.

We associate with each operator $A \in \mathcal{O}_N$ a formal matrix of operators on $l^2(\mathbb{Z}^+)$ by

$$\Psi(A) := \big((S_0^*)^i \Phi_0(S_0^i A (S_0^*)^j) S_0^j \big)_{i,j \ge 0} . \tag{35}$$

We will see in a moment that the formal matrix $\Psi(A)$ defines a bounded operator on $l^2(\mathbb{Z}^+,\, l^2(\mathbb{Z}^+))$ and that this operator is a Toeplitz operator in $\mathcal{T}_N$. The following example shows that this is at least true for operators in a dense subalgebra of $\mathcal{O}_N$.

Example 5.4 Let $A := S_l S_m^*$ with multi-indices l and m. Then $\Phi_0(S_0^i A (S_0^*)^j)$ is different from 0 only if $i + |l| = |m| + j$. In this case,

$$\Phi_0(S_0^i A (S_0^*)^j) = \Phi_0(S_0^i S_l S_m^* (S_0^*)^j) = S_0^i S_l S_m^* (S_0^*)^j,$$

whence

$$(S_0^*)^i \Phi_0(S_0^i A (S_0^*)^j) S_0^j = S_l S_m^* = A.$$

Thus, $\Psi(S_l S_m^*)$ is the matrix which has the entry $S_l S_m^*$ on its $i - j = |m| - |l|$th diagonal whereas all other entries are zero. In particular, $\Psi(S_l S_m^*)$ is a Toeplitz operator in $\mathcal{T}_N$.

Note that the mapping Ψ is not multiplicative. Indeed, $\Psi(S_k) = \Sigma_k$ for $k \in \Omega$ as we have just seen. Consequently, $\Psi(S_k^*)\Psi(S_k) = \Sigma_k^* \Sigma_k = I - \Pi_1$, which is different from $\Psi(S_k^* S_k) = \Psi(I) = I$. ■

A second way to associate with every operator in $\mathcal{O}_N$ a Toeplitz operator in $\mathcal{T}_N$ is via continuous functions. Let $\lambda \in \mathbb{T}$. Then the mapping $\rho_\lambda : S_i \mapsto \bar{\lambda} S_i$ extends to an automorphism of $\mathcal{O}_N$. Here, as usual, $\bar{\lambda}$ stands for the complex conjugate of λ; note that the mapping ρ_λ is defined in [8] without the bar. For each operator $A \in \mathcal{O}_N$, consider the function

$$f_A : \mathbb{T} \to \mathcal{O}_N, \quad \lambda \mapsto \rho_\lambda(A). \tag{36}$$

Lemma 5.5 *The function f_A is continuous for each $A \in \mathcal{O}_N$, and $\|f_A\|_\infty = \|A\|$.*

Proof. For each $\lambda \in \mathbb{T}$, one has $\|f_A(\lambda)\| = \|\rho_\lambda(A)\| \leq \|A\|$, whence $\|f_A\|_\infty \leq \|A\|$. Since $f_A(1) = A$, equality holds in this estimate. Since $A \mapsto f_A$ is a linear mapping, this implies

$$\|f_A - f_B\|_\infty = \|A - B\| \quad \text{for all } A,\, B \in \mathcal{O}_N. \tag{37}$$

Choose operators B_n in the dense subalgebra of $\mathcal{O}_N$ generated by the isometries S_i such that $\|A - B_n\| \to 0$ as $n \to \infty$. The functions f_{B_n} are evidently continuous. Being a uniform limit of continuous functions by (37), the function f_A is continuous. ∎

As in Section 5.1, we associate with the continuous function f_A the sequence of its Fourier coefficients and consider the associated Toeplitz operator $T(f_A)$ on $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$. From (26) and Lemma 5.5 we conclude that

$$\|T(f_A)\| = \|f_A\|_\infty = \|A\| \quad \text{for every } A \in \mathcal{O}_N. \tag{38}$$

Example 5.6 Let $A = S_l S_m^*$ with multi-indices l and m. Then

$$f_A(\lambda) = \lambda^{-|l|} S_l \lambda^{|m|} = \lambda^{|m|-|l|} A.$$

The $|m| - |l|$th Fourier coefficient of f_A is A whereas all other Fourier coefficients of this function vanish. Thus, $T(f_A)$ is the matrix which has the entry $S_l S_m^*$ on its $i - j = |m| - |l|$th diagonal whereas all other entries are zero. ∎

Since the products $S_l S_m^*$ span a dense subalgebra of $\mathcal{O}_N$ and since the mapping $A \mapsto T(f_A)$ is an isometry, we conclude from this example that $T(f_A)$ is a Toeplitz operator *in* $\mathcal{T}_N$ for every $A \in \mathcal{O}_N$.

We will see now that the two ways discussed above lead to the same goal.

Theorem 5.7 *The mapping Ψ is a linear contraction from $\mathcal{O}_N$ into $\mathcal{T}_N$. It coincides with the mapping $A \mapsto T(f_A)$.*

Proof. We learned from Examples 5.4 and 5.6 that $T(f_A) = \Psi(A)$ for $A = S_l S_m^*$. Since these products span a dense subalgebra of $\mathcal{O}_N$ and $A \mapsto T(f_A)$ and Ψ are linear mappings, this implies that

$$T(f_A) = \Psi(A) \in \mathcal{T}_N \quad \text{for all } A \text{ in a dense subalgebra of } \mathcal{O}_N. \tag{39}$$

Since the mapping $A \mapsto T(f_A)$ is an isometry, we get from (39) that

$$\|\Psi(A)\| = \|T(f_A)\| = \|A\|$$

for all A in a dense subalgebra of $\mathcal{O}_N$. Thus, the mapping mapping Ψ can be continued to a linear contraction from $\mathcal{O}_N$ into $\mathcal{T}_N$ (which, of course, coincides with the mapping $A \mapsto T(f_A)$). Since each entry of the matrix (35) depends continuously on A, this contractive continuation coincides with the formal matrix in (35). ∎

The classical Toeplitz algebra decomposes into the direct sum of the linear space $\{T(f) : f \in C(\mathbb{T})\}$ and the ideal of the compact operators. A similar decomposition holds for the algebra $\mathcal{T}_N$.

Theorem 5.8 $\mathcal{T}_N = \{T(f_A) : A \in \mathcal{O}_N\} \oplus \mathcal{C}_N = \{\Psi(A) : A \in \mathcal{O}_N\} \oplus \mathcal{C}_N$.

Proof. We reify Proposition 2.1 with the following algebras and mappings:

- $\mathcal{A}$ is the smallest closed subalgebra of $L(l^2(\mathbb{Z}, l^2(\mathbb{Z}^+)))$ which contains all operators of Laurent type represented by the two-sided infinite matrix

$$\begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & S_i & & & \\ & & 0 & S_i & & \\ & & & 0 & S_i & \\ & & & & \ddots & \ddots \end{pmatrix}$$

  with the zeros standing on the main diagonal.

- $\mathcal{B}$ is the algebra $\mathcal{T}_N$.

- D is the mapping $\mathcal{A} \to \mathcal{B},\ A \mapsto PAP$ where P is the orthogonal projection from $l^2(\mathbb{Z}, l^2(\mathbb{Z}^+))$ onto $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$ (the latter is identified with a closed subspace of the former in the obvious way).

- W is the mapping

$$\mathcal{B} \to \mathcal{A}, \quad B \mapsto \text{s-lim}_{n \to +\infty} V_n^* B V_n$$

  where V is the operator of forward shift on $l^2(\mathbb{Z}, l^2(\mathbb{Z}^+))$ and $V_n := V^n$ for $n \geq 0$, and where s-lim refers to the limit in the strong operator topology.

Then Proposition 2.1 implies that $\mathcal{T}_N = \{T(f_A) : A \in \mathcal{O}_N\} \oplus \ker W$, and it remains to verify that

$$\mathcal{C}_N \ (:= \text{closid}_{\mathcal{T}_N} \{\Pi_1\}) = \ker W. \tag{40}$$

Evidently, $\Pi_1 \in \ker W$, whence the inclusion $\mathcal{C}_N \subseteq \ker W$. To get the reverse implication, we show that the quasicommutator ideal of $\mathcal{T}_N$ lies in $\mathcal{C}_N$. Since the products $S_l S_m^*$ with multi-indices l, m span a dense subalgebra of $\mathcal{O}_N$, this fact will follow once we have shown that

$$\Psi(S_l S_m^*)\Psi(S_n S_r^*) - \Psi(S_l S_m^* S_n S_r^*) \in \mathcal{C}_N \tag{41}$$

for each choice of multi-indices l, m, n and r. Let Λ denote the operator of forward shift

$$\Lambda := \begin{pmatrix} 0 & & & \\ I & 0 & & \\ & I & 0 & \\ & & \ddots & \ddots \end{pmatrix} \in L(l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^*))) \tag{42}$$

and set $\Lambda_n := \Lambda^n$ for $n > 0$, $\Lambda_0 := I$ and $\Lambda_n := (\Lambda^*)^{-n}$ for $n < 0$. In Example 5.4 we have seen that

$$\Psi(S_l S_m^*) = S_l S_m^* \Lambda_{|m|-|l|} \tag{43}$$

(in this and the following equalities we consider $L(l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^*)))$ as an $L(l^2(\mathbb{Z}^+))$-module in an obvious way). Thus,

$$\begin{aligned} &\Psi(S_l S_m^*)\Psi(S_n S_r^*) - \Psi(S_l S_m^* S_n S_r^*) \\ &\quad = S_l S_m^* S_n S_r^* \left(\Lambda_{|m|-|l|}\Lambda_{|r|-|n|} - \Lambda_{|r|-|n|+|m|-|r|}\right). \end{aligned}$$

For arbitrary integers a, b one has

$$\Lambda_a \Lambda_b - \Lambda_{a+b} = \begin{cases} 0 & \text{if } \ a \le 0, \\ -\Pi_a \Lambda_{a+b} & \text{if } \ a > 0 \end{cases}$$

with Π_a defined as in (28). Hence,

$$\begin{aligned} &\Psi(S_l S_m^*)\, \Psi(S_n S_r^*) - \Psi(S_l S_m^* S_n S_r^*) \\ &\quad = \begin{cases} -S_l S_m^* S_n S_r^*\, \Pi_{|m|-|l|}\, \Lambda_{|r|-|n|+|m|-|r|} & \text{if } \ |m| - |l| > 0 \\ 0 & \text{if } \ |m| - |l| \le 0 \end{cases} \\ &\quad = \begin{cases} -\Pi_{|m|-|l|}\, \Psi(S_l S_m^* S_n S_r^*) & \text{if } \ |m| - |l| > 0 \\ 0 & \text{if } \ |m| - |l| \le 0. \end{cases} \end{aligned}$$

Since $\Pi_n \in \mathcal{C}_N$ for every $n \ge 1$ by (29), the inclusion (41) follows. ■

6 The lifting homomorphism

6.1 The algebra $(e - p_1)\mathcal{S}_N(e - p_1)$

In what follows is will be convenient to compare and to operate with multi-indices. Given multi-indices $i = (i_1, \ldots, i_k) \in \Omega^k$ and $j = (j_1, \ldots, j_l) \in \Omega^l$ we define their sum as the multi-index

$$i + j := (i_1, \ldots, i_k, j_1, \ldots, j_l) \in \Omega^{k+l}.$$

Differences of multi-indices i, k can be defined only if one of the multi-indices is a part of the other. Since addition of multi-indices is not commutative, we consider differences from the left and from the right. More precisely, we write $i \prec k$ if there is a multi-index j such that $i + j = k$, and we write $k \succ j$ if there is a multi-index i with $i + j = k$. The multi-indices j and i are uniquely determined, and we denote them by

$$j := (-i) + k \quad \text{and} \quad i := k - j,$$

respectively. Note that it follows from (11) that the product $s_i^* s_j$ is not zero only if $i \prec j$ or $j \prec i$. In the first case on gets

$$s_i^* s_j = s_i^* s_i s_{(-i)+j} = p_{|i|} s_{(-i)+j} = s_{(-i)+j} p_{|j|},$$

whereas in the second case

$$s_i^* s_j = (s_j s_{(-j)+i})^* s_j = s_{(-j)+i}^* s_j^* s_j = s_{(-j)+i}^* p_{|j|} = p_{|i|} s_{(-j)+i}^*.$$

Lemma 6.1 *Let i, j, k, l be multi-indices (not necessarily of the same length). Then the product*

$$(e - p_1) s_i s_j^* s_k s_l^*$$

can be written as

$$(e - p_1) s_r s_t^*$$

with multi-indices r and t such that $|r| \geq |t|$, or this product is zero.

Proof. The product $(e - p_1) s_i s_j^* s_k s_l^*$ is zero, or one has $j \prec k$ or $k \prec j$. By Lemma 4.1,

$$\begin{aligned}
(e - p_1) s_i s_j^* s_k s_l^* &= \begin{cases} (e - p_1) s_i s_j^* s_j s_{(-j)+k} s_l^* & \text{if } j \prec k \\ (e - p_1) s_i s_{(-k)+j}^* s_k^* s_k s_l^* & \text{if } k \prec j \end{cases} \\
&= \begin{cases} (e - p_1) s_i p_{|j|} s_{(-j)+k} s_l^* & \text{if } j \prec k \\ (e - p_1) s_i s_{(-k)+j}^* p_{|k|} s_l^* & \text{if } k \prec j \end{cases} \\
&= \begin{cases} s_i (p_{|i|} - p_{|i|+1}) p_{|j|} s_{(-j)+k} s_l^* & \text{if } j \prec k \\ s_i (p_{|i|} - p_{|i|+1}) p_{|j|} s_{(-k)+j}^* s_l^* & \text{if } k \prec j \end{cases} \\
&= \begin{cases} 0 & \text{if } |j| > |i| \\ s_i (p_{|i|} - p_{|i|+1}) s_{(-j)+k} s_l^* & \text{if } j \prec k,\ |j| \leq |i| \\ s_i (p_{|i|} - p_{|i|+1}) s_{(-k)+j}^* s_l^* & \text{if } k \prec j,\ |j| \leq |i| \end{cases} \\
&= \begin{cases} 0 & \text{if } |j| > |i| \\ (e - p_1) s_i s_{(-j)+k} s_l^* & \text{if } j \prec k,\ |j| \leq |i| \\ (e - p_1) s_i s_{(-k)+j}^* s_l^* & \text{if } k \prec j,\ |j| \leq |i| \end{cases} \\
&= \begin{cases} 0 & \text{if } |j| > |i| \\ (e - p_1) s_{i+((-j)+k)} s_l^* & \text{if } j \prec k,\ |j| \leq |i| \\ (e - p_1) s_i s_{l+((-k)+j)}^* & \text{if } k \prec j,\ |j| \leq |i|. \end{cases}
\end{aligned}$$

It remains to show that, whenever the product $(e - p_1) s_r s_t^*$ is not zero, then $|r| \geq |t|$. Assume that $|r| < |t|$. Then t can be written as $t_1 + t_2$ with $|t_1| > 0$ and $|t_2| = |r|$. From Lemma 4.1 (*d*) we conclude that

$$(e - p_1) s_r s_t^* = (e - p_1) s_r s_{t_2}^* s_{t_1}^* = s_r (p_{|r|} - p_{|r|+1}) s_{t_2}^* s_{t_1}^* = s_r s_{t_2}^* (e - p_1) s_{t_1}^*,$$

and $(e-p_1)s_{t_1}^* = 0$ by Lemma 4.1 (a). ∎

A repeated application of Lemma 6.1 yields the following.

Corollary 6.2 *Let i, j, k, l, $\ldots$, m, n be multi-indices (not necessarily of the same length). Then the product*

$$(e-p_1)s_i s_j^* s_k s_l^* \ldots s_m s_n^*(e-p_1)$$

can be written as

$$(e-p_1)s_r s_t^*(e-p_1)$$

with multi-indices r and t of the same length, or this product is zero.

Corollary 6.3 *Let $a \in \mathcal{S}_N$. Then $(e-p_1)a(e-p_1)$ can be approximated as closely as desired by linear combinations of elements of the form*

$$(e-p_1)s_r s_t^*(e-p_1)$$

with multi-indices r and t of the same length.

Let $\mathcal{S}_N^{par}$ stand for the smallest closed subalgebra of $\mathcal{S}_N$ which contains all products $s_i s_j^*$ with multi-indices i, j of the same length. Again we allow multi-indices of length zero, for which we set $s_\emptyset := e$.

Lemma 6.4 $\mathcal{S}_N^{par} = \operatorname{clos\,span}\{s_i s_j^* : |i| = |j|\}$.

Proof. Let i, j, k, l be multi-indices with $|i| = |j|$ and $|k| = |l|$. We have to show that the product $(s_i s_j^*)\,(s_k s_l^*)$ can be written as $s_r s_t^*$ with multi-indices r, t of the same length. This product is zero if not $jn \prec k$ or $k \prec j$. Let, for instance, $j \prec k$. Then

$$\begin{aligned} s_i s_j^* s_k s_l^* &= s_i s_j^* s_j s_{(-j)+k} s_l^* \\ &= s_i p_{|i|} s_{(-j)+k} s_l^* \\ &= s_i s_{(-j)+k} s_l^* \;=\; s_{i+((-j)+k)} s_l^* \end{aligned}$$

where $|i + ((-j)+k)| = |i| + |k| - |j| = |l|$. ∎

One can now state the assertion of Corollary 6.3 as follows:

$$\text{If } a \in \mathcal{S}_N \quad \text{then} \quad (e-p_1)a(e-p_1) \in (e-p_1)\,\mathcal{S}_N^{par}\,(e-p_1). \tag{44}$$

The mapping

$$\mathcal{S}_N \to \mathcal{S}_N^{par}, \quad a \mapsto (e-p_1)a(e-p_1) \tag{45}$$

is an expectation which is related with the expectation $\Phi_0 : \mathcal{O}_N \to \mathcal{O}_N^{par}$ as follows.

Proposition 6.5 *Let $A \in \mathcal{O}_N$, $a := (P_{N^n} A P_{N^n})_{n \ge 0} + \mathcal{G}_\eta$, and $i, j \in \mathbb{Z}^+$. Then*

$$\begin{aligned}
&(e-p_1)s_0^i a (s_0^*)^j (e-p_1) \\
&\quad = (e-p_1)\left((P_{N^n}\Phi_0(S_0^i A (S_0^*)^j) P_{N^n})_{n\ge 0} + \mathcal{G}_\eta\right)(e-p_1). \qquad (46)
\end{aligned}$$

Proof. Because of

$$\begin{aligned}
s_0^i a (s_0^*)^j &= (P_{N^n} S_0^i P_{N^n})(P_{N^n} A P_{N^n})(P_{N^n}(S_0^*)^j P_{N^n}) + \mathcal{G}_\eta \\
&= (P_{N^n} S_0^i P_{N^n} A P_{N^n}(S_0^*)^j P_{N^n}) + \mathcal{G}_\eta \\
&= (P_{N^n} S_0^i A (S_0^*)^j P_{N^n}) + \mathcal{G}_\eta,
\end{aligned}$$

It is sufficient to prove the assertion for $i = j = 0$. Thus, we have to show that

$$(e-p_1)a(e-p_1) = (e-p_1)\left((P_{N^n}\Phi_0(A)P_{N^n})_{n\ge 0} + \mathcal{G}_\eta\right)(e-p_1). \qquad (47)$$

Both sides of this equality depend linearly and continuously on A. It is thus sufficient to verify (47) for $A = S_k S_l^*$ with multi-indices k, l of arbitrary length. If k and l are of the same length, then $A \in \mathcal{O}_N^{par}$, whence $\Phi_0(A) = A$, whereas otherwise $\Phi_0(A) = 0$. Thus, the right-hand side of (47) is equal to $(e-p_1)a(e-p_1)$ if $|k| = |l|$ and zero otherwise. Since

$$P_{N^n} S_k S_l^* P_{N^n} = P_{N^n} S_k P_{N^n} \cdot P_{N^n} S_l^* P_{N^n},$$

the left-hand side of (47) is also zero whenever $|k| \ne |l|$. ∎

6.2 Definition of the lifting homomorphism

The desired lifting homomorphism will be defined explicitly by means of strong limits which involve the following reflection operators. For every positive integer n, let

$$R_n : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+), \quad (x_k)_{k\ge 0} \mapsto (x_{n-1}, x_{n-2}, \ldots, x_0, 0, 0, \ldots).$$

Proposition 6.6 *Let $a \in \mathcal{S}_N$ and write $(e-p_1)a(e-p_1)$ as $(A_{N^n}) + \mathcal{G}_\eta$. Then the strong limit*

$$\text{s-lim}_{n\to\infty} R_{N^n} A_{N^n} R_{N^n} \qquad (48)$$

exists, and the limit is independent of the choice of the representative of the coset $(e-p_1)a(e-p_1)$.

Proof. Evidently, the limit (48) is zero if (A_{N^n}) is a sequence in $\mathcal{G}_\eta$. This implies the independence of the choice of the representative.

By Corollary 6.3, it is sufficient to prove the existence of the strong limit (48) for sequences (A_{N^n}) which belong to the coset $e - p_1$ or to the coset $s_i s_j^*$ with multi-indices i, j of the same length.

Consider $e - p_1 = (P_{N^n}) - (P_{N^n} S_0^* P_{N^n})(P_{N^n} S_0 P_{N^n}) + \mathcal{G}_\eta$. For every $n \geq 1$, one has

$$P_{N^n} S_0^* P_{N^n} S_0 P_{N^n} = \operatorname{diag}(1,\, 1,\, \ldots 1,\, 0,\, 0,\, \ldots,\, 0)$$

with N^{n-1} ones followed by $N^n - N^{n-1}$ zeros. Hence,

$$R_{N^n}(P_{N^n} - P_{N^n} S_0^* P_{N^n} S_0 P_{N^n}) R_{N^n} = \operatorname{diag}(1,\, 1,\, \ldots 1,\, 0,\, 0,\, \ldots,\, 0)$$

with $N^n - N^{n-1}$ ones followed by N^{n-1} zeros, which implies that

$$R_{N^n}(P_{N^n} - P_{N^n} S_0^* P_{N^n} S_0 P_{N^n}) R_{N^n} \to I \quad \text{strongly.} \tag{49}$$

Next consider the sequence $(P_{N^n} S_i S_j^* P_{N^n})$ with multi-indices $i,\, j \in \Omega^k$. From Section 2.3 we infer that

$$\begin{aligned} S_i S_j^* : \quad & (x_r)_{r\geq 0} \mapsto \\ & (\underbrace{0,\, \ldots,\, 0}_{v_{i,k}},\, x_{v_{j,k}},\, \underbrace{0,\, \ldots,\, 0}_{d_k},\, x_{v_{j,k}+N^k},\, \underbrace{0,\, \ldots,\, 0}_{d_k},\, x_{v_{j,k}+2N^k},\, \ldots). \end{aligned} \tag{50}$$

In particular, $S_j S_j^*$ is the diagonal projection operator

$$S_j S_j^* = \operatorname{diag}(\underbrace{0,\, \ldots,\, 0}_{v_{j,k}},\, 1,\, \underbrace{0,\, \ldots,\, 0}_{d_k},\, 1,\, \underbrace{0,\, \ldots,\, 0}_{d_k},\, 1,\, \ldots). \tag{51}$$

For $n \geq k$, (50) implies that

$$\begin{aligned} P_{N^n} S_i S_j^* P_{N^n} : \quad & (x_r)_{r\geq 0} \mapsto \\ & (\underbrace{0,\, \ldots,\, 0}_{v_{i,k}},\, x_{v_{j,k}},\, \underbrace{0,\, \ldots,\, 0}_{d_k},\, x_{v_{j,k}+N^k},\, \ldots,\, x_{v_{j,k}+(N^{n-k}-1)N^k},\, \underbrace{0,\, \ldots,\, 0}_{N^k - v_{i,k} - 1}). \end{aligned} \tag{52}$$

Let V denote the shift operator

$$V : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+), \quad (x_k)_{k\geq 0} \mapsto (0,\, x_0,\, x_1,\, \ldots).$$

For every positive integer n, set $V_n := V^n$ and $V_{-n} := (V^*)^n$, and define $V_0 := I$. Then (50) shows that

$$S_i S_j^* = V_{v_{i,k}} V_{-v_{j,k}} S_j S_j^*,$$

whence

$$R_{N^n} P_{N^n} S_i S_j^* P_{N^n} R_{N^n} = R_{N^n} P_{N^n} V_{v_{i,k}} V_{-v_{j,k}} S_j S_j^* P_{N^n} R_{N^n}.$$

By (51), the operator $S_j S_j^*$ commutes with P_{N^n} which gives

$$\begin{aligned} & R_{N^n} P_{N^n} S_i S_j^* P_{N^n} R_{N^n} \\ & \quad = R_{N^n} P_{N^n} V_{v_{i,k}} V_{-v_{j,k}} P_{N^n} R_{N^n} \cdot R_{N^n} P_{N^n} S_j S_j^* P_{N^n} R_{N^n}. \end{aligned} \tag{53}$$

It is well known and easy to check that

$$R_{N^n}P_{N^n}V_{v_{i,k}}V_{-v_{j,k}}P_{N^n}R_{N^n} \to V_{v_{j,k}-v_{i,k}}$$

strongly as $n \to \infty$. According to (52), the second factor in (53) is equal to

$$R_{N^n}P_{N^n}S_jS_j^*P_{N^n}R_{N^n} = \operatorname{diag}(\underbrace{0,\,\ldots,\,0,}_{N^k-v_{j,k}-1}\,1,\,\underbrace{0,\,\ldots,\,0,}_{d_k}\,1,\ldots,\,1,\,\underbrace{0,\,\ldots,\,0}_{v_{j,k}}).$$

Since the number $N^k - v_{j,k} - 1$ of the leading zeros is independent of n, this factor converges strongly to

$$\operatorname{diag}(\underbrace{0,\,\ldots,\,0,}_{N^k-v_{j,k}-1}\,1,\,\underbrace{0,\,\ldots,\,0,}_{d_k}\,1,\,\underbrace{0,\,\ldots,\,0,}_{d_k}\ldots).$$

Using the notation

$$\Pi_k := \operatorname{diag}(1,\,\underbrace{0,\,\ldots,\,0,}_{d_k}\,1,\,\underbrace{0,\,\ldots,\,0,}_{d_k}\ldots),$$

we finally arrive at

$$\begin{aligned} R_{N^n}P_{N^n}S_jS_j^*P_{N^n}R_{N^n} &\to V_{v_{j,k}-v_{i,k}}V_{N^k-v_{j,k}-1}\Pi_k V^*_{N^k-v_{j,k}-1} \\ &= V_{N^k-v_{i,k}-1}\Pi_k V^*_{N^k-v_{j,k}-1} \end{aligned} \tag{54}$$

where the latter equality holds since $N^k - v_{j,k} - 1 \geq 0$. This settles the desired strong convergence. ■

For later use it will prove convenient to write the operator (54) in a different form. Note that

$$S_iS_j^* = V_{v_{i,k}}V_{-v_{j,k}}S_jS_j^* = V_{v_{i,k}}V_{-v_{j,k}}V_{v_{j,k}}\Pi_k V^*_{v_{j,k}} = V_{v_{i,k}}\Pi_k V^*_{v_{j,k}}.$$

A comparison with (54) suggests to introduce the *dual index* $\hat{i}$ of a multi-index $i = (i_1,\, i_2,\, \ldots,\, i_k)$ by

$$\hat{i} := (N-1-i_1,\, N-1-i_2,\, \ldots,\, N-1-i_k).$$

Evidently, $|\hat{i}| = |i| = k$, and one easily checks that

$$v_{\hat{i},k} = (N-1-i_1) + (N-1-i_2)N + \ldots + (N-1-i_k)N^{k-1} = N^k - 1 - v_{i,k}.$$

Hence,

$$V_{N^k-v_{i,k}-1}\Pi_k V^*_{N^k-v_{j,k}-1} = V_{v_{\hat{i},k}}\Pi_k V^*_{v_{\hat{j},k}} = S_{\hat{i}}S^*_{\hat{j}}.$$

For each $i \in \Omega$ we set $S_i^\sharp := S_{\hat{i}} = S_{N-1-i}$. Due to the universal property of Cuntz algebras, the mapping $^\sharp : S_i \mapsto S_{N-1-i}$ can be continued to an automorphism of $\mathcal{O}_N$.

Corollary 6.7 *If i and j are multi-indices of the same length then*

$$R_{N^n}P_{N^n}S_iS_j^*P_{N^n}R_{N^n} \to (S_iS_j^*)^\sharp \quad \text{strongly as } n \to \infty. \tag{55}$$

Note that the strong limit (55) need not to exist if the multi-indices are of different length; for example, the sequence

$$(R_{N^n}P_{N^n}S_0^*P_{N^n}R_{N^n})_{n\geq 0}$$

for the single isometry S_0 does not converge strongly.

We denote the strong limit (48) by $W_{00}(a)$ and consider W_{00} as a mapping from $\mathcal{S}_N$ into $L(l^2(\mathbb{Z}^+))$. More general, for $i,\, j \in \mathbb{Z}^+$, let $W_{ij} : \mathcal{S}_N \to L(l^2(\mathbb{Z}^+))$ refer to the operator

$$a \mapsto (S_{N-1}^*)^i\, W_{00}((e-p_1)s_0^i a (s_0^*)^j (e-p_1))S_{N-1}^j$$

which is consistent with the previous definition. With every element $a \in \mathcal{S}_N$, we associate the infinite matrix

$$\widetilde{W}(a) := (W_{ij}(a))_{i,\,j\geq 0} \tag{56}$$

the entries of which are operators on $L(l^2(\mathbb{Z}^+))$. We are going to show that the matrix (56) defines a linear bounded operator on $l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+))$ and that the mapping $\widetilde{W}$ is the desired injective lifting homomorphism. The following is the main result of this paper.

Theorem 6.8 *The mapping $\widetilde{W}$ defined by (56) is a *-isomorphism from $\mathcal{S}_N$ onto $\mathcal{T}_N$ which maps the ideal $\mathcal{J}_N$ onto $\mathcal{C}_N$.*

Corollary 6.9 *$\mathcal{J}_N$ is the only non-trivial closed ideal of $\mathcal{S}_N$.*

Indeed, this follows immediately from Theorem 6.8 and Corollary 5.3.

The proof of Theorem 6.8 will be given in the following section. Before, we consider a few examples which illustrate the action of $\widetilde{W}$. If $a = s_r$ with $r \in \Omega$, then

$$(e-p_1)s_0^i s_r (s_0^*)^j (e-p_1) = 0 \quad \text{for } i+1 \neq j.$$

If $i+1=j$ then we get with Corollary 6.7 that

$$\begin{aligned}(S_{N-1}^*)^i W_{00}((e-p_1)s_0^i s_r (s_0^*)^j (e-p_1))S_{N-1}^j \\ = (S_{N-1}^*)^i S_{N-1}^i S_{N-1-r} (S_{N-1}^*)^j S_{N-1}^j = S_{N-1-r} = S_r^\sharp.\end{aligned}$$

Thus,

$$\widetilde{W}(s_r) = \Sigma_{N-1-r} \quad \text{and, analogously,} \quad \widetilde{W}(s_r^*) = \Sigma_{N-1-r}^* \tag{57}$$

with Σ_i defined by (27). Let now $a = s_0^* s_0 = p_1$. If $i = 0$, one has for every j

$$(e-p_1)a(s_0^*)^j(e-p_1) = (e-p_1)p_1(s_0^*)^j(e-p_1) = 0,$$

and the same result follows if $i > 0$ and $i \neq j$:

$$(e - p_1)s_0^i s_0^* s_0 (s_0^*)^j (e - p_1) = (e - p_1)s_0^i (s_0^*)^j (e - p_1) = 0.$$

Let, finally, $i > 0$ and $i = j$. Then

$$(e - p_1)s_0^i s_0^* s_0 (s_0^*)^i (e - p_1) = (e - p_1)s_0^i (s_0^*)^i (e - p_1).$$

Applying the homomorphism W_{00} to the right-hand side of this equality and taking into account Corollary 6.7 we obtain the operator $S_{N-1}^i (S_{N-1}^*)^i$, whence

$$\widetilde{W}(p_1) = I - \Pi_1 \quad \text{and} \quad \widetilde{W}(e - p_1) = \Pi_1.$$

Note that $\widetilde{W}(s_r^*)\widetilde{W}(s_r) = I - \Pi_1 = \widetilde{W}(p_1) = \widetilde{W}(s_r^* s_r)$ in contrast to the mapping Ψ which does not act multiplicatively on these elements.

6.3 Proof of Theorem 6.8

It is not too hard to verify that the mapping $\widetilde{W}$ acts as a *-homomorphism on a *dense subalgebra* of $\mathcal{S}_N$. That it acts as a *-homomorphism on the whole algebra would easily follow from this fact if one would know that $\widetilde{W}$ is bounded. Conversely, the boundedness of $\widetilde{W}$ comes as a simple consequence of the fact that $\widetilde{W}$ acts as a *-homomorphism on $\mathcal{S}_N$. Unfortunately, neither the boundedness of $\widetilde{W}$ nor the homomorphy of $\widetilde{W}$ on all of $\mathcal{S}_N$ could be shown directly. Rather we have to prove both properties simultaneously by climbing step by step form small substructures of $\mathcal{S}_N$ to the whole algebra.

We will make use of the fact that the algebra $\mathcal{S}_N$ splits into the direct sum

$$\{(P_{N^n} A P_{N^n})_{n \geq 0} + \mathcal{G}_\eta : A \in \mathcal{O}_N\} \oplus \mathcal{J}_N. \tag{58}$$

Proposition 6.10 *The mapping* $\widetilde{W}$ *acts as a linear contraction on the first summand of* (58)*, and it maps this summand into* $\mathcal{T}_N$*.*

Proof. Let $A \in \mathcal{O}_N$ and $a := (P_{N^n} A P_{N^n})_{n \geq 0} + \mathcal{G}_\eta$. From equality (46) we conclude that

$$\begin{aligned} & W_{00}((e - p_1)s_0^i a (s_0^*)^j (e - p_1)) \\ & \quad = W_{00}\left((e - p_1)\ \left((P_{N^n} \Phi_0(S_0^i A (S_0^*)^j) P_{N^n})_{n \geq 0} + \mathcal{G}_\eta\right)\ (e - p_1)\right) \\ & \quad = \left(\Phi_0(S_0^i A (S_0^*)^j)\right)^\sharp \end{aligned}$$

Thus,

$$\begin{aligned} \widetilde{W}(a) &= \left((S_{N-1}^*)^i \left(\Phi_0(S_0^i A (S_0^*)^j)\right)^\sharp \ S_{N-1}^j\right)_{i,j \geq 0} \\ &= \left(\left((S_0^*)^i\, \Phi_0(S_0^i A (S_0^*)^j)\, S_0^j\right)^\sharp\right)_{i,j \geq 0} \\ &= \left((S_{N-1}^*)^i\, \Phi_0(S_0^i A (S_0^*)^j)^\sharp\, S_{N-1}^j\right)_{i,j \geq 0}. \end{aligned} \tag{59}$$

We claim that

$$\Phi_0(B)^\sharp = \Phi_0(B^\sharp) \quad \text{for every } B \in \mathcal{O}_N.$$

Since $\mathcal{O}_N$ is spanned by products $S_l S_m^*$ with multi-indices l, m and since the mappings Φ_0 and $\sharp$ are continuous, it is sufficient to check the claim for $B = S_l S_m^*$. If $|l| \neq |m|$, then both sides of the claimed identity are zero, whereas

$$\Phi_0(S_l S_m^*)^\sharp = (S_l S_m^*)^\sharp = S_{\hat{l}} S_{\hat{m}}^* = \Phi_0(S_{\hat{l}} S_{\hat{m}}^*) = \Phi_0((S_l S_m^*)^\sharp)$$

if $|l| = |m|$. This proves the claim and shows that (59) is equal to

$$\left((S_{N-1}^*)^i \, \Phi_0(S_{N-1}^i A^\sharp (S_{N-1}^*)^j) \, S_{N-1}^j\right)_{i,j \geq 0}.$$

Repeating the arguments from Example 5.4 one easily gets that the operators

$$(S_r^*)^i \, \Phi_0(S_r^i A^\sharp (S_r^*)^j) \, S_r^j$$

are independent of the choice of $r \in \Omega$. Thus, (59) further coincides with

$$\left((S_0^*)^i \, \Phi_0(S_0^i A^\sharp (S_0^*)^j) \, S_0^j\right)_{i,j \geq 0}$$

whence

$$\widetilde{W}(a) = \Psi(A^\sharp).$$

Now we conclude as follows. The mapping

$$a = (P_{N^n} A P_{N^n}) + \mathcal{G}_\eta \;\mapsto\; A = \text{s-lim} P_{N^n} A P_{N^n}$$

is a linear contraction by the Banach-Steinhaus theorem. As already mentioned, the mapping $A \mapsto A^\sharp$ is a linear contraction (and even an isometry) on $\mathcal{O}_N$, and from Theorem 5.7 we recall that the mapping $A^\sharp \mapsto \Psi(A^\sharp)$ is a linear contraction, too, with range in $\mathcal{T}_N$. ■

Now we consider the second summand in (58). Abbreviate $e - p_n$ to π_n and recall the definition (28) of Π_n.

Proposition 6.11 *The mapping $\widetilde{W}$ is a *-homomorphism from $\pi_n \mathcal{J}_N \pi_n$ into $\Pi_n \mathcal{C}_N \Pi_n$ for every $n \geq 1$.*

Proof. Let $a \in \mathcal{J}_N$. First we show that $\widetilde{W}(\pi_n a \pi_n) \in \Pi_n \mathcal{C}_N \Pi_n$. For, write $\pi_n = e - p_n = \sum_{i=0}^{n-1} (p_i - p_{i+1})$ with the convention $p_0 := e$. Then

$$\begin{aligned}
\pi_n a \pi_n &= \sum_{i,k=0}^{n-1} (p_i - p_{i+1}) a (p_k - p_{k+1}) \\
&= \sum_{i,k=0}^{n-1} (s_0^*)^i (e - p_1) s_0^i a (s_0^*)^k (e - p_1) s_0^k \\
&= \sum_{i,k=0}^{n-1} (s_0^*)^i (e - p_1) d_{ik} (e - p_1) s_0^k \qquad (60)
\end{aligned}$$

where the d_{ik} can be found in $\mathcal{S}_N^{par}$ by Corollary 6.3 and (44). Since $\mathcal{S}_N^{par}$ is spanned by the products $s_\mu s_\nu^*$ with multi-indices μ and ν of the same length, the assertion will follow once we have shown that

$$\widetilde{W}((s_0^*)^i(e-p_1)s_\mu s_\nu^*(e-p_1)s_0^k) \in \Pi_n \mathcal{C}_N \Pi_n \tag{61}$$

whenever $|\mu| = |\nu|$. The right-hand side of (61) is the matrix

$$\left((S_{N-1}^*)^r W_{00}((e-p_1)s_0^r(s_0^*)^i(e-p_1)s_\mu s_\nu^*(e-p_1)s_0^k(s_0^*)^t(e-p_1))S_{N-1}^t\right)_{r,t\geq 0}.$$

Only the ikth entry of this matrix is non-zero, and this entry equals

$$(S_{N-1}^*)^i W_{00}((e-p_1)s_0^i(s_0^*)^i(e-p_1)s_\mu s_\nu^*(e-p_1)s_0^k(s_0^*)^k(e-p_1))S_{N-1}^k.$$

By Proposition 6.6, this entry further coincides with

$$\begin{aligned}&(S_{N-1}^*)^i\, W_{00}((e-p_1)s_0^i(s_0^*)^i(e-p_1))\times \\ &\qquad \times W_{00}((e-p_1)s_\mu s_\nu^*(e-p_1))\, W_{00}((e-p_1)s_0^k(s_0^*)^k(e-p_1))\, S_{N-1}^k\end{aligned}$$

which on its hand is the same as

$$(S_{N-1}^*)^i S_{N-1}^i (S_{N-1}^*)^i S_{\hat{\mu}} S_{\hat{\nu}}^* S_{N-1}^k (S_{N-1}^*)^k S_{N-1}^k = (S_{N-1}^*)^i S_{\hat{\mu}} S_{\hat{\nu}}^* S_{N-1}^k$$

by Corollary 6.7. Thus, the right-hand side of (61) coincides with

$$(\Pi_i - \Pi i - 1)\Sigma_{N-1}\Psi(S_{\hat{\mu}} S_{\hat{\nu}}^*)\Sigma_{N-1}^*(\Pi_k - \Pi k - 1)$$

(with the convention $\Pi_0 := 0$). Clearly, this matrix is in $\Pi_n \mathcal{C}_N \Pi_n$, which proves (61).

It is now evident that the mapping $\widetilde{W} : \pi_n \mathcal{J}_N \pi_n \to \Pi_n \mathcal{C}_N \Pi_n$ is linear and symmetric. It remains to verify that this mapping is multiplicative,

$$\widetilde{W}(\pi_n a \pi_n)\, \widetilde{W}(\pi_n b \pi_n) = \widetilde{W}(\pi_n a \pi_n b \pi_n) \quad \text{for } a,\, b \in \mathcal{J}_N. \tag{62}$$

It is an elementary fact that a linear mapping W between algebras $\mathcal{A}$ and $\mathcal{B}$ which are spanned by subsets $\mathcal{L}$ and $\mathcal{M}$, respectively, is multiplicative if (and only if) $W(l)W(m) = W(lm)$ for each choice of elements $l \in \mathcal{L}$ and $m \in \mathcal{M}$. Thus, and by the first part of this proof, it is sufficient to verify the equality (62) in case

$$\pi_n a \pi_n = (s_0^*)^i(e-p_1)s_\mu s_\nu^*(e-p_1)s_0^k,$$

$$\pi_n b \pi_n = (s_0^*)^l(e-p_1)s_\lambda s_\tau^*(e-p_1)s_0^m$$

with multi-indices μ, ν, λ and τ such that $|\mu| = |\nu|$ and $|\lambda| = |\tau|$. From above we infer that only the ikth entry of the matrix $\widetilde{W}(\pi_n a \pi_n)$ is different from zero, and this entry is $(S_{N-1}^*)^i S_{\hat{\mu}} S_{\hat{\nu}}^* S_{N-1}^k$. Similarly, only the lmth entry of the matrix $\widetilde{W}(\pi_n b \pi_n)$ not zero, and this entry equals $(S_{N-1}^*)^l S_{\hat{\lambda}} S_{\hat{\tau}}^* S_{N-1}^m$. Consequently, the matrix $\widetilde{W}(\pi_n a \pi_n)\widetilde{W}(\pi_n b \pi_n)$ is not the zero matrix only if $k = l$. In this case, this matrix has at most one non-vanishing entry, namely the imth entry, which is

$$(S_{N-1}^*)^i S_{\hat{\mu}} S_{\hat{\nu}}^* S_{N-1}^k \cdot (S_{N-1}^*)^k S_{\hat{\lambda}} S_{\hat{\tau}}^* S_{N-1}^m. \tag{63}$$

Now we consider $\widetilde{W}(\pi_n a \pi_n b \pi_n)$. It is $\pi_n a \pi_n b \pi_n \neq 0$ only if $k = l$, in which case

$$\pi_n a \pi_n b \pi_n = (s_0^*)^i (e - p_1) s_\mu s_\nu^* (e - p_1) s_0^k \cdot (s_0^*)^k (e - p_1) s_\lambda s_\tau^* (e - p_1) s_0^m.$$

Further, the product

$$(e - p_1) s_0^r \pi_n a \pi_n b \pi_n (s_0^*)^t (e - p_1)$$

is not zero only if $r = i$ and $m = t$. Consequently, the matrix $\widetilde{W}(\pi_n a \pi_n b \pi_n)$ is not zero only if $k = l$. In this case, only the imth entry of this matrix does not vanish. This entry is

$$(S_{N-1}^*)^i S_{N-1}^i (S_{N-1}^*)^i S_{\hat{\mu}} S_{\hat{\nu}}^* S_{N-1}^k \cdot (S_{N-1}^*)^k S_{\hat{\lambda}} S_{\hat{\tau}}^* S_{N-1}^m (S_{N-1}^*)^m S_{N-1}^m,$$

which coincides with (63). ■

Corollary 6.12 *The mapping $\widetilde{W}$ is a linear contraction from $\mathcal{J}_N$ into $\mathcal{C}_N$.*

Proof. Let $j \in \mathcal{J}_N$. From Lemma 4.4 we infer that

$$\lim_{n\to\infty} \|j - \pi_n j \pi_n\| = 0 \quad \text{for each } j \in \mathcal{J}_N. \tag{64}$$

Hence, $(\pi_n j \pi_n)_{n\geq 1}$ is a Cauchy sequence. From Proposition 6.11 we further infer that $\widetilde{W} : \pi_n \mathcal{J}_N \pi_n \to \Pi_n \mathcal{C}_N \Pi_n$ is a *-homomorphism, hence contractive:

$$\|\widetilde{W}(\pi_n j \pi_n)\| \leq \|\pi_n j \pi_n\| \quad \text{for all}\, n \geq 1. \tag{65}$$

Since $\pi_n j \pi_n \in \pi_m \mathcal{J}_N \pi_m$ for $m \geq n$, we conclude that

$$\|\widetilde{W}(\pi_n j \pi_n) - \widetilde{W}(\pi_m j \pi_m)\| \leq \|\pi_n j \pi_n - \pi_m j \pi_m\|$$

whenever $m \geq n$. Hence, $(\widetilde{W}(\pi_n j \pi_n))_{n\geq 1}$ is a Cauchy sequence. Let J denote its limit. Since all entries of the matrix mapping $\widetilde{W}$ are continuous, we conclude from

$$\|\widetilde{W}(\pi_n j \pi_n) - J\| \to 0$$

that $J = \widetilde{W}(j)$.Now it is clear that $\widetilde{W}(j) \in \mathcal{C}_N$, and passing to the limit as $n \to \infty$ in (65) yields $\|\widetilde{W}(j)\| \leq \|j\|$ for every $j \in cJ_N$. ■

Corollary 6.13 *The mapping $\widetilde{W} : \mathcal{J}_N \to \mathcal{C}_N$ is a *-homomorphism.*

Proof. We have to show that $\widetilde{W}$ is a multiplicative mapping on $\mathcal{J}_N$. Let $j_1, j_2 \in \mathcal{J}_N$. By Lemma 4.4, $j_1 j_2 = \lim \pi_n j_1 \pi_n j_2 \pi_n$, and since $\widetilde{W}$ is continuous on $\mathcal{J}_N$,

$$\widetilde{W}(j_1 j_2) = \lim \widetilde{W}(\pi_n j_1 \pi_n j_2 \pi_n).$$

Since $\widetilde{W}$ is multiplicative on $\pi_n \mathcal{J}_N \pi_n$ by Proposition 6.11, this implies

$$\widetilde{W}(j_1 j_2) = \lim \widetilde{W}(\pi_n j_1 \pi_n)\, \widetilde{W}(\pi_n j_2 \pi_n) = \widetilde{W}(j_1)\widetilde{W}(j_2),$$

whence the assertion. ∎

Corollary 6.14 *The mapping $\widetilde{W}$ is bounded on all of $\mathcal{S}_N$.*

Proof. Let $(A_n) + \mathcal{G}_\eta \in \mathcal{S}_N$. In accordance with (58), we write this coset as

$$(A_n) + \mathcal{G}_\eta = ((P_{N^n} A P_{N^n}) + \mathcal{G}_\eta) + ((J_n) + \mathcal{G}_\eta) =: a + j \tag{66}$$

with $A :=$ s-lim$A_n P_{N^n}$ and $j = (J_n) + \mathcal{G}_\eta \in \mathcal{J}_N$. Then

$$\|a\| = \|(P_{N^n} A P_{N^n}) + \mathcal{G}_\eta\| \le \|A\| \le \|(A_n) + \mathcal{G}_\eta\|,$$

i.e., the first summand in (66) depends continuously on $(A_n) + \mathcal{G}_\eta$. From $\|a\| \le \|a + j\|$ we obtain $\|j\| \le \|a + j\| + \|a\| \le 2\,\|a + j\|$, whence

$$\|\widetilde{W}(a + j)\| \le \|\widetilde{W}(a)\| + \|\widetilde{W}(j)\| \le \|a\| + \|j\| \le 3\,\|a + j\|$$

due to Proposition 6.10 and Corollary 6.12. ∎

Proposition 6.15 *The mapping $\widetilde{W}$ is a *-homomorphism from $\mathcal{S}_N$ into $\mathcal{T}_N$.*

Proof. It is sufficient to verify that $\widetilde{W}$ is multiplicative on $\mathcal{S}_N$. We start with showing a partial multiplicativity result,

$$\widetilde{W}(a\pi_k) = \widetilde{W}(a)\Pi_k \quad \text{for each } a \in \mathcal{S}_N. \tag{67}$$

Indeed, the matrix representation of $\widetilde{W}(a\pi_k)$ is

$$\left((S_{N-1}^*)^i W_{00}((e - p_1) s_0^i a \underbrace{\pi_k (s_0^*)^j (e - p_1)}) S_{N-1}^j \right)_{i,j \ge 0},$$

and the underbraced expression equals

$$\begin{aligned}
\pi_k (s_0^*)^j (e - p_1) &= (e - p_k)(s_0^*)^j (e - p_1) \\
&= (e - p_k)(p_j - p_{j+1})(s_0^*)^j \\
&= \begin{cases} 0 & \text{if } \; k \le j \\ (p_j - p_{j+1} - p_k + p_k p_{j+1})(s_0^*)^j & \text{if } \; k > j \end{cases} \\
&= \begin{cases} 0 & \text{if } \; k \le j \\ (p_j - p_{j+1})(s_0^*)^j & \text{if } \; k > j \end{cases} \\
&= \begin{cases} 0 & \text{if } \; k \le j \\ (s_0^*)^j (e - p_1) & \text{if } \; k > j \end{cases}
\end{aligned}$$

whence the assertion (67). For the proof of the general assertion, let a, $b \in \mathcal{S}_N$. Since $\pi_k \in \mathcal{J}_N$ for every k by Proposition 4.2 and $\widetilde{W}$ is multiplicative on $\mathcal{J}_N$, we get

$$\widetilde{W}(a\pi_m b\pi_k) = \widetilde{W}(a\pi_m)\,\widetilde{W}(b\pi_k)$$

for all k, $m \geq 0$. By (67),

$$\widetilde{W}(a\pi_m b\pi_k) = \widetilde{W}(a)\Pi_m\,\widetilde{W}(b)\Pi_k.$$

For $m \to \infty$ we have $a\pi_m b\pi_k \to ab\pi_k$ by Lemma 4.4 (recall that $\pi_k \in \mathcal{J}_N$) and $\Pi_m \to I$ strongly. Thus, due to the continuity of $\widetilde{W}$,

$$\widetilde{W}(ab\pi_k) = \widetilde{W}(a)\,\widetilde{W}(b)\Pi_k.$$

Invoking (67) again and letting k tend to infinity, we arrive at the assertion. ∎

We prepare the proof of the next proposition by a simple lemma.

Lemma 6.16 *Every coset in $(e-p_1)\mathcal{S}_N(e-p_1)$ can be written in the form $(e-p_1)c(e-p_1)$ with $c = (P_{N^n}CP_{N^n})_{n\geq 0} + \mathcal{G}_\eta$ with $C \in \mathcal{O}_N^{par}$.*

Proof. The assertion holds for cosets of the form $(e-p_1)s_r s_t^*(e-p_1)$ with multi-indices r, t of the same length. Indeed, it follows immediately from the identities (10) that

$$(e-p_1)s_r s_t^*(e-p_1) = (e-p_1)\ \left((P_{N^n}S_r S_t^* P_{N^n})_{n\geq 0} + \mathcal{G}_\eta\right)\ (e-p_1).$$

Then, by Corollary 6.3, the assertion holds for all cosets in a dense subalgebra of $(e-p_1)\mathcal{S}_N(e-p_1)$. Let now a be an arbitrary element of $\mathcal{S}_N$. Let $((e-p_1)a_n(e-p_1))_{n\geq 1}$ be a sequence in this dense subalgebra which converges to $(e-p_1)a(e-p_1)$ in the norm. As we have just checked, there are operators $A_n \in \mathcal{O}_N^{par}$ such that

$$(e-p_1)a_n(e-p_1) = (P_{N^n}A_nP_{N^n})_{n\geq 0} + \mathcal{G}_\eta.$$

Applying the homomorphism W_{00} to both sides of this equality we obtain

$$W_{00}((e-p_1)a_n(e-p_1)) = A_n^\sharp$$

for every n. Since the sequence $((e-p_1)a_n(e-p_1))_{n\geq 1}$ converges, $(A_n^\sharp)$ is a Cauchy sequence. But then (A_n) is a Cauchy sequence; hence, convergent, too. Let A denote its limit. Then $A \in \mathcal{O}_N^{par}$ and, clearly, $(e-p_1)a(e-p_1) = (P_{N^n}AP_{N^n})_{n\geq 0} + \mathcal{G}_\eta$. ∎

Proposition 6.17 *The mapping $\widetilde{W}$ is injective on $\mathcal{J}_N$.*

Proof. The assertion will follow once we have shown that

$$\widetilde{W} \text{ is injective on } \pi_n \mathcal{J}_N \pi_n \text{ for every } n \geq 1. \tag{68}$$

Indeed, let (68) be satisfied, and let $j \in \mathcal{J}_N$ be an element with $\widetilde{W}(j) = 0$. Then $\Pi_n \widetilde{W}(j) \Pi_n = 0$ for every n. By (67), this implies $\widetilde{W}(\pi_n j \pi_n) = 0$ for every n. From (68) we infer that $\pi_n j \pi_n = 0$ for every n. Passage to the limit $n \to \infty$ yields $j = 0$, which implies the desired injectivity.

Further, (68) will follow once we have shown that

$$\widetilde{W} \text{ is injective on } \pi_1 \mathcal{J}_N \pi_1 = (e - p_1) \mathcal{J}_N (e - p_1). \tag{69}$$

Indeed, let (69) be satisfied, and let $j \in \mathcal{J}_N$ be such that

$$\widetilde{W}(\pi_n j \pi_n) = \Pi_n \widetilde{W}(j) \Pi_n = 0 \quad \text{for some } n \geq 0.$$

Then

$$(S_{N-1}^*)^i \, W_{00}((e - p_1) s_0^i j (s_0^*)^k (e - p_1)) \, S_{N-1}^k = 0$$

for all $i, \, k \leq n - 1$. Multiplication by S_{N-1}^i from the left and by $(S_{N-1}^*)^k$ from the right-hand side yields

$$S_{N-1}^i (S_{N-1}^*)^i \, W_{00}((e - p_1) s_0^i j (s_0^*)^k (e - p_1)) \, S_{N-1}^k (S_{N-1}^*)^k = 0 \tag{70}$$

for $i, \, k \leq n - 1$. Because of

$$S_{N-1}^i (S_{N-1}^*)^i = W_{00}((e - p_1) s_0^i (s_0^*)^i (e - p_1))$$

and by the multiplicativity of W_{00} on $(e - p_1) \mathcal{J}_N (e - p_1)$, we conclude from (70) that

$$W_{00}((e - p_1) s_0^i (s_0^*)^i (e - p_1) s_0^i j (s_0^*)^k (e - p_1) s_0^k (s_0^*)^k (e - p_1)) = 0.$$

Since

$$(e - p_1) s_0^i (s_0^*)^i (e - p_1) = (e - p_1) s_0^i (s_0^*)^i,$$

this finally implies that

$$W_{00}((e - p_1) s_0^i j (s_0^*)^k (e - p_1)) = 0 \quad \text{if } i, \, k \leq n - 1.$$

Via assumption (69), this gives

$$(e - p_1) s_0^i j (s_0^*)^k (e - p_1) = 0 \quad \text{if } i, \, k \leq n - 1 \tag{71}$$

(note that

$$\widetilde{W}((e - p_1) r (e - p_1)) = \begin{pmatrix} W_{00}(e - p_1) r (e - p_1) & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

for every element r of $\mathcal{J}_N$, which implies that W_{00} is injective whenever (69) holds). Multiplying (70) from the left by $(s_0^*)^i$ and by s_0^k from the right-hand side and taking into account that $(s_0^*)^i(e-p_1)s_0^i = p_i - p_{i+1}$ we obtain

$$(p_i - p_{i+1})j(p_k - p_{k+1}) = 0 \quad \text{if } i,\, k \leq n-1.$$

Summation over $0 \leq i,\, k \leq n-1$ gives $\pi_n j \pi_n = 0$, whence the injectivity of $\widetilde{W}$ on $\pi_n \mathcal{J}_N \pi_n$.

It remains to prove (69). Let $j \in \mathcal{J}_N$ and $\widetilde{W}((e-p_1)j(e-p_1)) = 0$. Employing Lemma 6.16, we can write $(e-p_1)j(e-p_1)$ as

$$(e-p_1)\left((P_{N^n} C P_{N^n})_{n\geq 0} + \mathcal{G}_\eta\right)(e-p_1) \quad \text{with } C \in \mathcal{O}_N^{par}.$$

Consequently,

$$0 = W_{00}((e-p_1)j(e-p_1)) = Phi_0(C^\sharp) = C^\sharp$$

since Φ is an expectation from $\mathcal{O}_N$ onto $\mathcal{O}_N^{par}$ and C belongs to the latter subalgebra. Thus, $C = 0$, which implies that $(e-p_1)j(e-p_1)$. The injectivity of $\widetilde{W}$ on $\pi_1 \mathcal{J}_N \pi_1$ follows. ■

Now we can finish the proof of Theorem 6.8 as follows. The mapping $\widetilde{W}$ is an injective *-homomorphism on $\mathcal{J}_N$ as we have just seen. Hence, by Corollary 4.9, $\widetilde{W}$ is an injective *-homomorphism on $\mathcal{S}_N$. The range of this homomorphism contains the generating operators Σ_k of $\mathcal{T}_N$; thus, $\widetilde{W}$ maps $\mathcal{S}_N$ onto $\mathcal{T}_N$. Since $\widetilde{W}$ maps the generating element $e - p_1$ of the ideal cJ_N to the generating element Π_1 of $\mathcal{C}_N$, it is further clear that $\widetilde{W}$ maps $\mathcal{J}_N$ onto $\mathcal{C}_N$. ■

6.4 Some consequences of Theorem 6.8

Stability. The assertion of Theorem 6.8 is equivalent to the following stability criterion.

Theorem 6.18 *A sequence* $\mathbf{A} = (A_n)$ *in* $\mathcal{S}_\eta(\mathcal{O}_N)$ *is stable if and only if the operator* $\widetilde{W}(\mathbf{A} + \mathcal{G}_\eta)$ *is invertible.*

Specifying this result to finite sections sequences for operators in the Cuntz algebra yields

Corollary 6.19 *Let* $A \in \mathcal{O}_N$. *Then the sequence* $(P_{N^n} A P_{N^n})_{n\geq 0}$ *is stable if and only if the (stratified) Toeplitz operator* $\Psi(A^\sharp) = T(f_{A^\sharp}) \in L(l^2(\mathbb{Z}^+, l^2(\mathbb{Z}^+)))$ *is invertible.*

Proof. Set $a := (P_{N^n} A P_{N^n})_{n\geq 0} + \mathcal{G}_\eta$. We have to show that $\widetilde{W}(a) = \Psi(A^\sharp)$. If A is a product $S_l S_m^*$ with multi-indices l and m, then this equality follows from (43) and (57) by noting that the latter implies

$$\widetilde{W}(s_l s_m^*) = S_{\hat{l}} S_{\hat{m}}^* \Lambda_{|\hat{m}|-|\hat{l}|}$$

due to the homomorphy of $\widetilde{W}$ and that

$$S_{\hat{l}} S_{\hat{m}}^* \Lambda_{|\hat{m}|-|\hat{l}|} = (S_l S_m^*)^\sharp \Lambda_{|m|-|l|}$$

with Λ defined by (42). The general case follows from this partial result since the products $S_l S_m^*$ span a dense subalgebra of $\mathcal{O}_N$. ■

Fractality. The following is certainly the most important consequence of Theorem 6.8. It can also serve as a perfect illustration to Theorem 2.4. The proof will follow directly from the special form of the homomorphism $\widetilde{W}$.

Corollary 6.20 *The algebra $\mathcal{S}_\eta(\mathcal{O}_N)$ is fractal.*

Proof. Recall that the entries of the matrix operator $\widetilde{W}((A_n)+\mathcal{G}_\eta)$ are defined by strong limits. Consequently, if only an (infinite) subsequence of (A_n) is known, one can nevertheless determine the operator $\widetilde{W}((A_n) + \mathcal{G}_\eta) \in \mathcal{T}_N$. Since $\widetilde{W} : \mathcal{S}_N \to \mathcal{T}_N$ is an isomorphism one can, thus, reconstruct the coset of (A_n) modulo $\mathcal{G}_\eta$ from each subsequence of (A_n). ■

Spectral approximation. As already mentioned, sequences in fractal algebras are distinguished by their excellent convergence properties. To mention only a few of them, let $\sigma(a)$ denote the spectrum of an element a of a C^*-algebra with identity element e, write $\sigma_2(a)$ for the set of the singular values of a, i.e., $\sigma_2(a)$ is the set of all non-negative square roots of elements in the spectrum of a^*a and finally, for $\varepsilon > 0$, let $\sigma^{(\varepsilon)}(a)$ refer to the ε-pseudospectrum of a, i.e. to the set of all $\lambda \in \mathbb{C}$ for which $a - \lambda e$ is not invertible or $\|(a - \lambda e)^{-1}\| \geq 1/\varepsilon$. Let further

$$d_H(M,\, N) := \max\{\max_{m\in M} \min_{n\in N} |m-n|,\, \max_{n\in N} \min_{m\in M} |m-n|\}$$

denote the Hausdorff distance between the non-empty compact subsets M and N of the complex plane.

Theorem 6.21 *Let (A_n) be a sequence in $\mathcal{S}_\eta(\mathcal{O}_N)$ and set $a := (A_n) + \mathcal{G}_\eta$. Then the following set-sequences converge with respect to the Hausdorff distance as $n \to \infty$:*
(a) $\sigma(A_n) \to \sigma(\widetilde{W}(a))$ if a is self-adjoint;
(b) $\sigma_2(A_n) \to \sigma_2(\widetilde{W}(a))$;
(c) $\sigma^{(\varepsilon)}(A_n) \to \sigma^{(\varepsilon)}(\widetilde{W}(a))$.

The proof follows immediately from the stability criterion in Theorem 6.8 above and from Theorems 3.20, 3.23 and 3.33 in [9]. Let us emphasize that in general one cannot remove the assumption $a = a^*$ in assertion (a), whereas (c) holds without any assumption. This observation is only one reason for the present increasing interest in pseudospectra. For detailed presentations of pseudospectra and their applications as well as of other spectral quantities see the monographs [2, 3, 9, 17] and the references therein.

Compactness and Fredholm properties. Recall the definition of the algebra $\mathcal{F}$ of all bounded sequences of matrices and of its ideal $\mathcal{K}$ ideal of the compact sequences from Section 3.3 and let $\mathcal{F}_\eta$ and $\mathcal{K}_\eta$ denote the corresponding restricted algebras.

Proposition 6.22 *The only compact sequences in $\mathcal{S}_\eta(\mathcal{O}_N)$ are the sequences in $\mathcal{G}_\eta$.*

Proof. By Corollary 6.9, $\mathcal{J}_N$ is the only non-trivial closed ideal of $\mathcal{S}_N$. Thus, the intersection $\mathcal{S}_N \cap (\mathcal{K}_\eta/\mathcal{G}_\eta)$ is either $\mathcal{S}_n$, $\mathcal{J}_N$, or $\{0\}$. Since already $\mathcal{J}_N$ contains cosets of non-compact sequences (e.g., the coset $e - p_1$), the assertion follows. ■

Corollary 6.23 *Every Fredholm sequence in $\mathcal{S}_\eta(\mathcal{O}_N)$ is stable.*

The Fredholm property of a sequence (A_n) of matrices can be expressed in terms of the singular values of the A_n. To specify this remark to sequences $(A_n) \in \mathcal{S}_\eta(\mathcal{O}_N)$, let

$$0 \le s_1(A_n) \le s_2(A_n) \le \ldots \le s_{N^n}(A_n) = \|A_n\| \tag{72}$$

be the decreasingly ordered sequence of the singular values of A_n. The following is then an immediate consequence of Corollary 6.14 in [14]. Note that the condition of essential fractality, which is assumed in Corollary 6.14, is evidently satisfied in the present context.

Proposition 6.24 *Let $(A_n) \in \mathcal{S}_\eta(\mathcal{O}_N)$. If the sequence (A_n) is stable, then the sequence $(s_1(A_n))$ is bounded below from zero by a positive constant. If the sequence (A_n) fails to be stable, then*

$$\lim_{n\to\infty} s_k(A_n) = 0 \quad \textit{for each} \quad k \ge 1.$$

An open question. In [9] there are considered discretizations by the finite sections method of several concrete algebras $\mathcal{A}$ of operators on $l^2(\mathbb{Z}^+)$. In all cases, we observed that the associated quasicommutator ideal $\mathcal{J}(\mathcal{A})$ is a direct (or c_0-) sum of elementary ideals (i.e. ideals which are isomorphic to the ideal of the compact operators on a Hilbert space). Above we have seen that the ideal $\mathcal{J}_N$ is isomorphic to $\mathcal{C}_N$ and, hence, to a corner of a tensor product of an AF-algebra by $K(l^2(\mathbb{Z}^+))$. Is this the archetypal picture of a fractal irreducible ideal in $\mathcal{F}$?

7 Spatial discretization of the extended Cuntz algebra

Here we consider the smallest C^*-subalgebra $\mathcal{O}_N^{ext}$ of $L(l^2(\mathbb{Z}^+))$ which contains the (concrete) Cuntz algebra $\mathcal{O}_N$ and the ideal $K(l^2(\mathbb{Z}^+))$ of the compact operators. We let $\mathcal{S}_\eta(\mathcal{O}_N^{ext})$ denote the smallest C^*-subalgebra of the sequence algebra $\mathcal{F}_\eta$ which contains all finite sections sequences $(P_{N^n}AP_{N^n})_{n\geq 0}$ of operators $A \in \mathcal{O}_N^{ext}$ and set $\mathcal{S}_N^{ext} := \mathcal{S}_\eta(\mathcal{O}_N^{ext})/\mathcal{G}_\eta$. Further, let

$$\mathcal{J}_N^{comp} := \{(P_{N^n}KP_{N^n})_{n\geq 0} + \mathcal{G}_\eta : K \in K(l^2(\mathbb{Z}^+))\}.$$

Proposition 7.1 *The set $\mathcal{J}_N^{comp}$ is a closed ideal of $\mathcal{S}_N^{ext}$, and every coset $a \in \mathcal{S}_N^{ext}$ can be uniquely written as $b + k$ with $b \in \mathcal{S}_N$ and $k \in \mathcal{J}_N^{comp}$.*

Proof. It is evident from the above definitions that $\mathcal{J}_N^{comp}$ is contained in $\mathcal{S}_N^{ext}$, and it is easy to see that $\mathcal{J}_N^{comp}$ is a closed ideal of $\mathcal{S}_N^{ext}$ (compare the proof of Theorem 1.19 in [9]). Since every sequence $(P_{N^n}KP_{N^n})_{n\geq 0}$ with a compact operator K is compact, one has $\mathcal{J}_N^{comp} \subset \mathcal{K}_\eta/\mathcal{G}_\eta$. From Proposition 6.22 we thus conclude that $\mathcal{S}_N^{ext} \cap \mathcal{J}_N^{comp}$ is the zero ideal, whence the assertion. ■

We let W be the mapping which associates with every sequence in $\mathcal{S}_\eta(\mathcal{O}_N^{ext})$ its strong limit. Clearly, W is a *-homomorphism which has the ideal $\mathcal{G}_\eta$ in its kernel. Thus, the quotient homomorphism

$$\mathcal{S}_N^{ext} \to L(l^2(\mathbb{Z}^+)), \quad (A_n)_{n\geq 0} + \mathcal{G}_\eta \mapsto W((A_n)_{n\geq 0})$$

is correctly defined; we denote it by W again. Further we extend the definition of the homomorphism $\widetilde{W}$ from $\mathcal{S}_N$ to the extended algebra $\mathcal{S}_N^{ext}$ by writing every element a of $\mathcal{S}_N^{ext}$ as $b+k$ with b and k as in Proposition 7.1 and setting $\widetilde{W}(a) := \widetilde{W}(b)$. It is easy to check that $\widetilde{W}$ is a *-homomorphism on the extended algebra $\mathcal{S}_N^{ext}$. Note that $\mathcal{J}_N$ is in the kernel of W and $\mathcal{J}_N^{comp}$ is in the kernel of (the extended) $\widetilde{W}$.

Remark 7.2 One can also give an explicit definition of the extended homomorphism $\widetilde{W}$. For, recall that $\widetilde{W}$ is defined on $\mathcal{S}_N$ via the homomorphism W_{00} which on its hand is defined on cosets $(P_{N^n}AP_{N^n}) + \mathcal{G}_N$ with $A \in \mathcal{O}_N^{par}$ by the strong limit

$$\text{s-lim}_{n\to\infty} R_{N^n}P_{N^n}AP_{N^n}R_{N^n}$$

(compare (55)). It is easy to check that this limit exists as well if A is a compact operator, in which case this limit is zero. ■

Theorem 7.3 *A sequence $\mathbf{A} = (A_n) \in \mathcal{S}_\eta(\mathcal{O}_N^{ext})$ is stable if and only if the operators $W(\mathbf{A} + \mathcal{G}_\eta)$ and $\widetilde{W}(\mathbf{A} + \mathcal{G}_\eta)$ are invertible.*

Proof. It is evident that the stability of $\mathbf{A}$ implies the invertibility of $W(\mathbf{A}+\mathcal{G}_\eta)$ and $\widetilde{W}(\mathbf{A}+\mathcal{G}_\eta)$. Conversely, assume that these operators are invertible for a sequence $\mathbf{A} \in \mathcal{S}_\eta(\mathcal{O}_N^{ext})$. Set $a := \mathbf{A}+\mathcal{G}_\eta$ and write a as $b+k$ with b and k as in Proposition 7.1. Then $\widetilde{W}(b) = \widetilde{W}(a)$ is invertible, whence the invertibility of b by Theorem 6.18. Thus, the sequence $\mathbf{A}$ is the sum of a stable sequence and of a sequence of the form $(P_{N^n}KP_{N^n})$ with a compact operator K. Now the assertion follows immediately from a general result on compact perturbations; see Corollary 1.22 in [9]. ■

The following is an immediate consequence of this stability result and of Theorem 1.69 in [9]. Note that the homomorphisms W and $\widetilde{W}$ are fractal in the sense of Definition 1.62 in [9]. This fact is evident for W, and it has been checked in the proof of Corollary 6.20 for $\widetilde{W}$.

Corollary 7.4 *The algebra $\mathcal{S}_\eta(\mathcal{O}_N^{ext})$ is fractal.*

As above, fractality has striking consequences for the asymptotic behavior of some spectral quantities.

Theorem 7.5 *Let (A_n) be a sequence in $\mathcal{S}_\eta(\mathcal{O}_N^{ext})$ and set $a := (A_n)+\mathcal{G}_\eta$. Then the following set-sequences converge with respect to the Hausdorff distance as $n \to \infty$:*
(a) $\sigma(A_n) \to \sigma(W(a)) \cup \sigma(\widetilde{W}(a))$ if a is self-adjoint;
(b) $\sigma_2(A_n) \to \sigma_2(W(a)) \cup \sigma_2(\widetilde{W}(a))$;
(c) $\sigma^{(\varepsilon)}(A_n) \to \sigma^{(\varepsilon)}(W(a)) \cup \sigma^{(\varepsilon)}(\widetilde{W}(a))$.

Finally we consider the Fredholm properties of sequences in $\mathcal{S}_\eta(\mathcal{O}_N^{ext})$. We use the notation (72) for the singular values.

Theorem 7.6 *Let (A_n) be a sequence in $\mathcal{S}_\eta(\mathcal{O}_N^{ext})$ and set $a := (A_n)+\mathcal{G}_\eta$.*
(a) The sequence (A_n) is Fredholm if and only if the operator $\widetilde{W}(a)$ is invertible. In this case, the operator $W(a)$ is Fredholm.
(b) Let (A_n) be a Fredholm sequence and $k := \dim\ker W(a)$. Then

$$\lim_{n\to\infty} \sigma_k(A_n) = 0 \quad \textit{and} \quad \liminf_{n\to\infty} \sigma_{k+1}(A_n) > 0. \tag{73}$$

Proof. For assertion (a), write a as $b+k$ with b and k as in Proposition 7.1. If a is invertible modulo $\mathcal{J}_N^{comp}$, then b is invertible modulo $\mathcal{J}_N^{comp}$; hence b is invertible due to Corollary 6.23. Thus, $\widetilde{W}(a) = \widetilde{W}(b)$ is invertible. Conversely, if $\widetilde{W}(a)$ is invertible, then $\widetilde{W}(b)$ is invertible, whence the invertibility of b via Theorem 6.18. But then $b+k = a$ is Fredholm.

Assertion (b) can be derived from Theorem 6.12 in [9]. But note that the assumption of a *standard* algebra which is made in [9] is not satisfied in the present context. A look at the proof of 6.12 in [9] shows that this assumption is not really needed. A short direct proof is in [16]. ■

8 Appendix

The algebra $\mathcal{F}$ of all bounded sequences of matrices is an algebra with polar decomposition in the following sense: A C^*-algebra $\mathcal{A}$ has the *polar decomposition property* if every element a of $\mathcal{A}$ can be written as $a = ru$ with a unitary element $u \in \mathcal{A}$ and a positive element $r \in \mathcal{A}$. It is well-known from Linear Algebra that every matrix $A \in \mathbb{C}^{n\times n}$ admits a polar decomposition. Thus, $\mathbb{C}^{n\times n}$ owns the polar decomposition property, and so does the algebra $\mathcal{F}$ and each quotient of $\mathcal{F}$.

If $\mathcal{A}$ is a unital C^*-algebra with polar decomposition property, then every element of $\mathcal{A}$ which is invertible from one side is invertible from both sides. Indeed, let $b \in \mathcal{A}$ be a left inverse of $a \in \mathcal{A}$, and write a as $a = ru$. From $ba = bru = e$ we get $br = u^*$ and $ubr = e$. Hence, r is invertible from the left-hand side. Taking adjoints in $ubr = e$ we get the invertibility of r from the right. Thus, r is invertible, and so is a. In particular, every isometry in an algebra with polar decomposition is unitary.

Corollary 8.1 *Algebras with polar decomposition cannot contain Cuntz algebras $\mathcal{O}_N$ with $N \geq 2$ as subalgebras.*

Indeed, let $\mathcal{A}$ be a C^*-algebra with polar decomposition, let $N \geq 2$, and assume there are isometries $s_0, \ldots, s_{N-1} \in \mathcal{A}$ with $s_0 s_0^* + \ldots + s_{N-1}s_{N-1}^* = e$. Then the s_i are unitaries, whence $s_i s_i^* = e$ for every i. Substituting $s_i s_i^* = e$ in the Cuntz axiom gives $Ne = e$, a contradiction.

Corollary 8.2 *For $N \geq 2$, there are no proper closed ideal $\mathcal{J}$ of $\mathcal{F}$ and no sequences $(S_n^{(0)})_{n\geq 1}, \ldots, (S_n^{(N-1)})_{n\geq 1}$ in $\mathcal{F}$ such that*

$$\big((S_n^{(i)})^* S_n^{(i)} - I_n\big) \in \mathcal{J} \quad \text{for all } i \tag{74}$$

and

$$\big(S_n^{(0)}(S_n^{(0)})^* + \ldots + S_n^{(N-1)}(S_n^{(N-1)})^* - I_n\big) \in \mathcal{J}. \tag{75}$$

This argument does not work if we replace $\mathcal{F}$ by a subalgebra, $\mathcal{B}$ say, since the factors of the polar decomposition need not belong to $\mathcal{B}$.

Note that, in contrast to Cuntz algebras, the Toeplitz algebra $\mathcal{T}(C)$, which is the universal algebra generated by one isometry, can be embedded in a quotient of $\mathcal{F}$. Indeed, let $\mathcal{S}(\mathcal{T}(C))$ stand for smallest closed subalgebra of $\mathcal{F}$ which contains all sequences (P_nAP_n) of finite sections of operators $A \in \mathcal{T}(C)$. One can show (see Theorem 1.53 in [9] for instance) that every sequence (A_n) in $\mathcal{S}(\mathcal{T}(C))$ can be uniquely written as

$$(A_n) = (P_nT(f)P_n) + (P_nKP_n) + (R_nLR_n) + (G_n)$$

where $T(f)$ is a Toeplitz operator with continuous generating function f, K and L are compact operators, and (G_n) is a sequence tending to zero in the norm (recall the definition of the operators R_n at the beginning of Section 6.2). It is not hard to check that the set

$$\mathcal{J} := \{(R_n L R_n) + (G_n) : K \text{ compact}, \ \|G_n\| \to 0\}$$

forms a closed ideal of $\mathcal{S}(\mathcal{T}(C))$ and that the quotient $\mathcal{S}(\mathcal{T}(C))/\mathcal{J}$ is isomorphic to $\mathcal{T}(C)$. In particular, $\mathcal{S}(\mathcal{T}(C))/\mathcal{J}$ contains (and is generated by) the non-unitary isometry $(P_n V P_n) + \mathcal{J}$ where V is the forward shift operator on $l^2(\mathbb{Z}^+)$. Note that $\mathcal{J}$ is contained in the ideal $\mathcal{K}$ of the compact sequences of $\mathcal{F}$ defined in Section 3.3.

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