An abstract setting for hamiltonian actions

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Abstract

In this paper we develop an abstract setup for hamiltonian group actions as follows: Starting with a continuous 2-cochain ω on a Lie algebra $\mathfrak{h}$ with values in an $\mathfrak{h}$-module V, we associate subalgebras $\mathfrak{sp}(\mathfrak{h},\omega) \supseteq \mathfrak{ham}(\mathfrak{h},\omega)$ of symplectic, resp., hamiltonian elements. Then $\mathfrak{ham}(\mathfrak{h},\omega)$ has a natural central extension which in turn is contained in a larger abelian extension of $\mathfrak{sp}(\mathfrak{h},\omega)$. In this setting, we study linear actions of a Lie group G on V which are compatible with a homomorphism $\mathfrak{g} \to \mathfrak{ham}(\mathfrak{h},\omega)$, i.e., abstract hamiltonian actions, corresponding central and abelian extensions of G and momentum maps $J\colon \mathfrak{g} \to V$.

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Introduction.

In [Br93] Brylinski describes how to associate to a connected, not necessarily finite-dimensional, smooth manifold M, endowed with a closed 2-form $\omega \in \Omega^2(M,\mathbb{R})$, a central extension of the Lie algebra $\mathfrak{ham}(M,\omega)$ of hamiltonian vector fields on M. If there exists an associated pre-quantum bundle P with connection 1-form θ and curvature ω (which is the case if ω is integral and M is smoothly paracompact), then Kostant's central extension ([Ko70]) is given by the short exact sequence

$$\mathbf{1} \to \mathbb{T} \hookrightarrow \mathrm{Aut}(P,\theta)_0 \twoheadrightarrow \mathrm{Ham}(M,\omega) \to \mathbf{1} \tag{1}$$

of groups, where $\mathrm{Aut}(P,\theta)_0$ is the group of those connection preserving automorphisms of P isotopic to the identity and $\mathrm{Ham}(M,\omega)$ is the group of hamiltonian diffeomorphisms of M. In general, neither $\mathrm{Ham}(M,\omega)$ nor $\mathrm{Aut}(P,\theta)_0$ carries a Lie group structure if M is not assumed to be compact.

However, we have shown in [NV03] that central Lie group extensions can be obtained as follows, even if M is infinite dimensional. Let Z be an abelian Lie group of the form $Z = \mathfrak{z}/\Gamma_Z$, where Γ_Z is a discrete subgroup of the Mackey complete space $\mathfrak{z}$, $\omega \in \Omega^2(M,\mathfrak{z})$ a closed 2-form, and (P,θ) a corresponding Z-pre-quantum bundle, i.e., P is a Z-principal bundle and $\theta \in \Omega^1(P,\mathfrak{z})$ a connection 1-form with $\mathtt{d}\theta = q_P^*\omega$, where $q_P\colon P \to M$ is the bundle projection. We call a smooth action of a (possibly infinite dimensional) connected Lie group G on a (possibly infinite dimensional) manifold M *hamiltonian* if the derived homomorphism maps into hamiltonian vector fields:

$$\zeta\colon \mathfrak{g} \to \mathfrak{ham}(M,\omega) := \{X \in \mathcal{V}(M)\colon (\exists f \in C^\infty(M,\mathbb{R}))\ i_X\omega = \mathtt{d}f\}.$$

Then the pullback of $\mathrm{Aut}(P,\theta)$ defines a central Lie group extension

$$\mathbf{1} \to Z \hookrightarrow \widehat{G}_{\mathrm{cen}} \twoheadrightarrow G \to \mathbf{1}, \tag{2}$$

i.e., $\widehat{G}_{\mathrm{cen}}$ carries a Lie group structure for which it is a principal Z-bundle over G. A Lie algebra 2-cocycle for the associated central Lie algebra extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathbb{R}$ is given by $(X,Y) \mapsto -\omega(\zeta(X),\zeta(Y))(m_0)$ for any fixed element $m_0 \in M$.

The main point of the present paper is to provide an abstract setting for this kind of hamiltonian group actions, momentum maps, and the associated central Lie group actions. As we shall see in the examples in Section 1, our setting is sufficiently general to cover various kinds of examples of different nature described below.

Starting with a continuous 2-cochain ω on a Lie algebra $\mathfrak{h}$ with values in a topological $\mathfrak{h}$-module V, we associate the subalgebras

$$\begin{aligned}\mathfrak{sp}(\mathfrak{h},\omega) &:= \{\xi \in \mathfrak{h}\colon \mathcal{L}_\xi\omega = 0, i_\xi \mathtt{d}_{\mathfrak{h}}\omega = 0\} = \{\xi \in \mathfrak{h}\colon \mathtt{d}_{\mathfrak{h}}(i_\xi\omega) = i_\xi \mathtt{d}_{\mathfrak{h}}\omega = 0\}\\ \supseteq \mathfrak{ham}(\mathfrak{h},\omega) &:= \{\xi \in \mathfrak{h}\colon i_\xi \mathtt{d}_{\mathfrak{h}}\omega = 0, (\exists v \in V)\ i_\xi\omega = \mathtt{d}_{\mathfrak{h}}v\}.\end{aligned}$$

of *symplectic*, resp., *hamiltonian elements* of $\mathfrak{h}$. Then $\mathfrak{ham}(\mathfrak{h},\omega)$ has a natural central extension

$$\widehat{\mathfrak{ham}}(\mathfrak{h},\omega) := \{(v,\xi) \in V \times \mathfrak{ham}(\mathfrak{h},\omega)\colon \mathtt{d}_{\mathfrak{h}}v = i_\xi\omega\}$$

by the trivial module $V^{\mathfrak{h}}$ which in turn is contained in the larger abelian extension of $\mathfrak{sp}(\mathfrak{h},\omega)$ by V defined by ω. To obtain the Lie bracket on $\widehat{\mathfrak{ham}}(\mathfrak{h},\omega)$, we observe that the space $V_\omega := \{v \in V\colon (\exists \xi \in \mathfrak{ham}(\mathfrak{h},\omega))\ \mathrm{d}_{\mathfrak{h}} v = i_\xi\omega\}$ of so-called *admissible elements* carries a Lie bracket analogous to the Poisson bracket:

$$\{v_1, v_2\} := \omega(\xi_2, \xi_1) \quad \text{for} \quad \mathrm{d}_{\mathfrak{h}} v_j = i_{\xi_j}\omega,$$

and $\widehat{\mathfrak{ham}}(\mathfrak{h},\omega)$ is a subalgebra of the Lie algebra direct sum $V_\omega \oplus \mathfrak{ham}(\mathfrak{h},\omega)$.

The classical example is given by $\mathfrak{h} = \mathcal{V}(M)$, $V = C^\infty(M,\mathbb{R})$ and a closed 2-form ω. In a similar spirit is the example arising from a Poisson manifold (M,Λ), where we put $\mathfrak{h} = \Omega^1(M,\mathbb{R})$, endowed with the natural Lie bracket (cf. Example 1.6), $V = C^\infty(M,\mathbb{R})$ and $\omega := \Lambda$, considered as a V-valued 2-cocycle on $\mathfrak{h}$. Of a different character are the examples obtained with $\mathfrak{h} = \mathcal{V}(M)$, $V := \overline{\Omega}^p(M,\mathbb{R}) := \Omega^p(M,\mathbb{R})/\mathrm{d}\Omega^{p-1}(M,\mathbb{R})$ and $\omega(X,Y) := [i_Y i_X \tilde{\omega}]$, where $\tilde{\omega}$ is a differential $(p+2)$-form on M. There are more examples arising from associative algebras in the spirit of non-commutative geometry.

In Section 2 we turn to *abstract hamiltonian actions* of a Lie group G on V, i.e., actions which are compatible with a homomorphism $\zeta\colon \mathfrak{g} \to \mathfrak{ham}(\mathfrak{h},\omega)$ and the $\mathfrak{h}$-action on V. Then the pullback extension $\widehat{\mathfrak{g}}_{\mathrm{cen}} := \zeta^*\widehat{\mathfrak{ham}}(M,\omega)$ defines a central extension of $\mathfrak{g}$ by $V^{\mathfrak{h}}$, and a *momentum map* is a continuous linear map $J\colon \mathfrak{g} \to V$ satisfying

$$\mathrm{d}_{\mathfrak{h}}(J(X)) = i_{\zeta(X)}\omega \quad \text{for} \quad X \in \mathfrak{g},$$

i.e., J defines a continuous linear section $\sigma := (J, \zeta, \mathrm{id}_{\mathfrak{g}})\colon \mathfrak{g} \to \widehat{\mathfrak{g}}_{\mathrm{cen}}$ of the associated central extension. The obstruction to the existence of an equivariant momentum map is measured by the 1-cocycle $\kappa\colon G \to C^1(\mathfrak{g}, V^{\mathfrak{h}}), \kappa(g) := g.J - J$ which in turn can be used to describe the adjoint action of the group G on the extended Lie algebra $\widehat{\mathfrak{g}}_{\mathrm{cen}}$.

In Section 3 we briefly discuss the existence of a central Lie group extension $\widehat{G}_{\mathrm{cen}}$ with Lie algebra $\widehat{\mathfrak{g}}_{\mathrm{cen}}$. Once a momentum map J is given for a hamiltonian G-action, the only condition for the existence of such an extension is the discreteness of the image Π_ω of a period homomorphism $\mathrm{per}_{\omega_{\mathfrak{g}}}\colon \pi_2(G) \to V^{\mathfrak{h}}$, associated to the Lie algebra cocycle $\omega_{\mathfrak{g}} := \zeta^*\omega$. If Π_ω is discrete, there also is an abelian Lie group extension $\widehat{G}_{\mathrm{ab}}$ of G integrating the Lie algebra extension $\widehat{\mathfrak{g}}_{\mathrm{ab}} := V \rtimes_{\omega_{\mathfrak{g}}} \mathfrak{g}$.

In many situations the period map $\mathrm{per}_{\omega_{\mathfrak{g}}}$ is quite hard to evaluate, but one may nevertheless show that its image is discrete. To illustrate this point,

we take in Section 4 a closer look at a smooth action of a connected Lie group G on a manifold M, leaving a closed $\mathfrak{z}$-valued 2-form ω invariant. If the group $S_\omega = \int_{\pi_2(M)} \omega$ of spherical periods of ω is discrete in V, then $Z := \mathfrak{z}/S_\omega$ is an abelian Lie group, and there exists a Lie group extension

$$\mathbf{1} \to Z \to \widehat{G}_{\mathrm{cen}} \to \tilde{G} \to \mathbf{1}$$

integrating the Lie algebra $\widehat{\mathfrak{g}}_{\mathrm{cen}}$. Here $q_G\colon \tilde{G} \to G$ denotes a simply connected covering group of G, so that $\widehat{G}_{\mathrm{cen}}$ can also be viewed as an extension of G by a 2-step nilpotent Lie group $\widehat{\pi}_1(G)$ which is a central extension of the discrete group $\pi_1(G) = \ker q_G$ by Z. Here we use that the universal covering manifold $q_M\colon \tilde{M} \to M$, endowed with $\tilde{\omega} := q_M^*\omega$, permits us to embed $\mathfrak{sp}(M,\omega)$ into $\mathfrak{sp}(\tilde{M},\tilde{\omega}) = \mathfrak{ham}(\tilde{M},\tilde{\omega})$, so that we actually obtain an abstract hamiltonian action of the universal covering group $\tilde{G}$ for the module $V := C^\infty(\tilde{M},\mathfrak{z})$ and the central extension $\widehat{G}_{\mathrm{cen}}$ acts by quantomorphisms on a Z-principal bundle P over $\tilde{M}$. This further leads to an abelian Lie group extension

$$\mathbf{1} \to C^\infty(M,Z)_0 \to \widehat{G}_{\mathrm{ab}} \to \tilde{G} \to \mathbf{1},$$

integrating the cocycle $\zeta^*\omega \in Z^2(\mathfrak{g}, C^\infty(M,\mathfrak{z}))$. In the short Section 5 we formulate the algebraic essence of the Noether Theorem in our context and the paper concludes with an appendix recalling some of the results from [Ne04] on the integrability of abelian Lie algebra extensions to Lie group extensions.

1 A Lie algebraic hamiltonian setup

Let V be a topological module of the topological Lie algebra $\mathfrak{h}$, i.e., the module operation $\mathfrak{h} \times V \to V, (\xi, v) \mapsto \xi.v$, is continuous. We write $(C^\bullet(\mathfrak{h},V), \mathtt{d}_\mathfrak{h})$ for the Chevalley–Eilenberg complex of continuous Lie algebra cochains with values in V, $Z^p(\mathfrak{h},V)$ denotes the space of p-cocycles, $H^p(\mathfrak{h},V)$ the cohomology space etc. We further write $\mathcal{L}_\xi$ for the operators defining the natural action of $\mathfrak{h}$ on the spaces $C^p(\mathfrak{h},V)$. We refer to [FF01] for the basic notation concerning continuous Lie algebra cohomology.

We first recall some results from [Ne06a] which we specialize here for 2-cochains. Fix a continuous 2-cochain $\omega \in C^2(\mathfrak{h},V)$. Then

$$\mathfrak{sp}(\mathfrak{h},\omega) := \{\xi \in \mathfrak{h} : \mathcal{L}_\xi\omega = 0, i_\xi \mathtt{d}_\mathfrak{h}\omega = 0\} = \{\xi \in \mathfrak{h}\colon \mathtt{d}_\mathfrak{h}(i_\xi\omega) = 0, i_\xi\mathtt{d}_\mathfrak{h}\omega = 0\}$$

is a closed subspace of $\mathfrak{h}$ and from

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]} \quad \text{and} \quad [\mathcal{L}_\xi, i_\eta] = i_{[\xi,\eta]}$$

it follows that $\mathfrak{sp}(\mathfrak{h}, \omega)$ is a Lie subalgebra of $\mathfrak{h}$, called the Lie algebra of *symplectic elements* of $\mathfrak{h}$. Since $\mathtt{d}_\mathfrak{h}\omega$ vanishes on $\mathfrak{sp}(\mathfrak{h}, \omega)$, the restriction of ω to this subalgebra is a Lie algebra 2-cocycle. Using the Cartan formula $\mathcal{L}_\xi = i_\xi \circ \mathtt{d}_\mathfrak{h} + \mathtt{d}_\mathfrak{h} \circ i_\xi$, we find for $\xi, \eta \in \mathfrak{sp}(\mathfrak{h}, \omega)$ the relation

$$i_{[\xi,\eta]}\omega = [\mathcal{L}_\xi, i_\eta]\omega = \mathcal{L}_\xi(i_\eta\omega) = \mathtt{d}_\mathfrak{h}(i_\xi i_\eta \omega) = \mathtt{d}_\mathfrak{h}(\omega(\eta, \xi)), \tag{3}$$

showing that the *flux homomorphism*,

$$f_\omega : \mathfrak{sp}(\mathfrak{h}, \omega) \to H^1(\mathfrak{h}, V), \quad \xi \mapsto [i_\xi\omega],$$

is a homomorphism of Lie algebras if $H^1(\mathfrak{h}, V)$ is endowed with the trivial Lie bracket. Its kernel is the ideal

$$\mathfrak{ham}(\mathfrak{h}, \omega) := \{\xi \in \mathfrak{sp}(\mathfrak{h}, \omega) : (\exists v \in V)\ i_\xi\omega = \mathtt{d}_\mathfrak{h} v\}$$

of *hamiltonian elements*. The set of *admissible vectors*

$$V_\omega := \{v \in V : (\exists \xi \in \mathfrak{ham}(\mathfrak{h}, \omega))\ \mathtt{d}_\mathfrak{h} v = i_\xi\omega\}$$

contains the subspace

$$V^\mathfrak{h} = \{v \in V : (\forall \xi \in \mathfrak{h})\ \xi.v = 0\} = H^0(\mathfrak{h}, V)$$

of $\mathfrak{h}$-invariant vectors. We then have a well-defined linear map

$$q : \mathfrak{ham}(\mathfrak{h}, \omega) \to \mathtt{d}_\mathfrak{h} V_\omega \cong V_\omega / V^\mathfrak{h}, \quad q(\xi) = i_\xi\omega$$

whose kernel is the radical $\mathrm{rad}(\omega, \mathtt{d}_\mathfrak{h}\omega) := \{\xi \in \mathfrak{h} \colon i_\xi\omega = 0, i_\xi \mathtt{d}_\mathfrak{h}\omega = 0\}$ of ω and $\mathtt{d}_\mathfrak{h}\omega$.

Proposition 1.1 (i) *The space V_ω of admissible vectors carries a Lie algebra structure defined by*

$$\{v_1, v_2\} := \xi_1.v_2 = -\xi_2.v_1 = -\omega(\xi_1, \xi_2) \quad \textit{for} \quad \mathtt{d}_\mathfrak{h} v_j = i_{\xi_j}\omega, j = 1, 2,$$

and for which $V^\mathfrak{h}$ is central. We have the following exact sequence of Lie algebras

$$\mathbf{0} \to \mathrm{rad}(\omega, \mathtt{d}_\mathfrak{h}\omega) \hookrightarrow \mathfrak{ham}(\mathfrak{h}, \omega) \xrightarrow{q} \mathtt{d}_\mathfrak{h}(V_\omega) \to \mathbf{0}. \tag{4}$$

(ii) *If* $\mathfrak{h}_\omega := \{\xi \in \mathfrak{h}\colon i_\xi \mathtt{d}\omega = 0, [\xi, \mathrm{rad}(\omega, \mathtt{d}_\mathfrak{h}\omega)] \subseteq \mathrm{rad}(\omega, \mathtt{d}_\mathfrak{h}\omega)\}$ *is the normalizer of* $\mathrm{rad}(\omega, \mathtt{d}_\mathfrak{h}\omega)$*, then* $\mathfrak{sp}(\mathfrak{h}, \omega) \subseteq \mathfrak{h}_\omega$ *and the set*

$$C^1(\mathfrak{h}, V)_\omega := \{i_\xi\omega\colon \xi \in \mathfrak{h}_\omega\} \subseteq C^1(\mathfrak{h}, V)$$

inherits a Lie algebra structure, defined by

$$[i_{\xi_1}\omega, i_{\xi_2}\omega] := i_{[\xi_1,\xi_2]}\omega.$$

For $\alpha_j = i_{\xi_j}\omega$*,* $j = 1, 2$*, we then have*

$$[\alpha_1, \alpha_2] = \mathcal{L}_{\xi_1}\alpha_2 - i_{\xi_2}\mathcal{L}_{\xi_1}\omega. \tag{5}$$

The maps

$$i^\omega : \mathfrak{h}_\omega \to C^1(\mathfrak{h}, V)_\omega, \quad \xi \mapsto i_\xi\omega \qquad \text{and} \qquad \mathtt{d}_\mathfrak{h}\colon V_\omega \to C^1(\mathfrak{h}, V)_\omega, \quad v \mapsto \mathtt{d}_\mathfrak{h}v$$

are homomorphisms of Lie algebras. In particular

$$\mathbf{0} \hookrightarrow \mathrm{rad}(\omega, \mathtt{d}_\mathfrak{h}\omega) \to \mathfrak{h}_\omega \xrightarrow{i^\omega} C^1(\mathfrak{h}, V)_\omega \to \mathbf{0} \tag{6}$$

is an exact sequence of Lie algebras extending (4)*.*

Proof. (i) First we note that $\{v_1, v_2\}$ is well-defined because the formula

$$\xi_1.v_2 = i_{\xi_1}\mathtt{d}_\mathfrak{h}v_2 = i_{\xi_1}i_{\xi_2}\omega = \omega(\xi_2, \xi_1)$$

shows that each choice of ξ_2 with $i_{\xi_2}\omega = \mathtt{d}_\mathfrak{h}v_2$ leads to the same value of $\omega(\xi_2, \xi_1)$, and a similar argument applies to ξ_1.

By (3), $\mathtt{d}_\mathfrak{h}\{v_1, v_2\} = i_{[\xi_1,\xi_2]}\omega$, so that V_ω is closed under the bracket $\{\cdot,\cdot\}$ and q is compatible with the brackets. It remains to verify the Jacobi identity in V_ω: For $\xi_j \in \mathfrak{ham}(\mathfrak{h}, \omega)$ with $\mathtt{d}_\mathfrak{h}v_j = i_{\xi_j}\omega$, $j = 1, 2, 3$, we have $\xi_j \in \mathfrak{sp}(\mathfrak{h}, \omega)$, so that

$$\begin{aligned} 0 = (\mathtt{d}_\mathfrak{h}\omega)(\xi_1, \xi_2, \xi_3) &= \sum_{\text{cycl.}} \xi_1.\omega(\xi_2, \xi_3) - \sum_{\text{cycl.}} \omega([\xi_1, \xi_2], \xi_3) \\ &\overset{(3)}{=} -2\sum_{\text{cycl.}} \omega([\xi_1, \xi_2], \xi_3) = 2\sum_{\text{cycl.}} \{\{v_1, v_2\}, v_3\}. \end{aligned}$$

(ii) The inclusion $\mathfrak{sp}(\mathfrak{h}, \omega) \subseteq \mathfrak{h}_\omega$ follows from (3), so that $\mathrm{rad}(\omega, \mathtt{d}_\mathfrak{h}\omega) \subseteq \mathfrak{h}_\omega$ is an ideal of $\mathfrak{h}_\omega$. Hence the set $C^1(\mathfrak{h}, V)_\omega \cong \mathfrak{h}_\omega/\mathrm{rad}(\omega, \mathtt{d}_\mathfrak{h}\omega)$ inherits a quotient Lie algebra structure for which i^ω is a morphism of Lie algebras. Further, $\mathtt{d}_\mathfrak{h}\colon V_\omega \to C^1(\mathfrak{h}, V)_\omega$ is a homomorphism of Lie algebras since for $\mathtt{d}_\mathfrak{h}v_j = i_{\xi_j}\omega$, $j = 1, 2$, equation (3) leads to $\mathtt{d}_\mathfrak{h}\{v_1, v_2\} = i_{[\xi_1,\xi_2]}\omega = [i_{\xi_1}\omega, i_{\xi_2}\omega] = [\mathtt{d}_\mathfrak{h}v_1, \mathtt{d}_\mathfrak{h}v_2]$. Now (5) follows from $[\alpha_1, \alpha_2] = i_{[\xi_1,\xi_2]}\omega = [\mathcal{L}_{\xi_1}, i_{\xi_2}]\omega = \mathcal{L}_{\xi_1}(i_{\xi_2}\omega) - i_{\xi_2}\mathcal{L}_{\xi_1}\omega$. ∎

The pullback by q of the central extension $V^{\mathfrak{h}} \hookrightarrow V_\omega \twoheadrightarrow \mathtt{d}_{\mathfrak{h}} V_\omega$ provides a central extension

$$\mathbf{0} \to V^{\mathfrak{h}} \hookrightarrow \widehat{\mathfrak{ham}}(\mathfrak{h},\omega) \twoheadrightarrow \mathfrak{ham}(\mathfrak{h},\omega) \to \mathbf{0}, \tag{7}$$

where

$$\widehat{\mathfrak{ham}}(\mathfrak{h},\omega) = \{(v,\xi) \in V_\omega \times \mathfrak{ham}(\mathfrak{h},\omega) : i_\xi \omega = \mathtt{d}_{\mathfrak{h}} v\}$$

is endowed with the Lie bracket

$$[(v_1,\xi_1),(v_2,\xi_2)] = (\{v_1,v_2\},[\xi_1,\xi_2]) = (-\omega(\xi_1,\xi_2),[\xi_1,\xi_2]).$$

We thus arrive at the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & V^{\mathfrak{h}} & \xrightarrow{\mathrm{id}} & V^{\mathfrak{h}} & \to & \mathbf{0} \\
 & & & & \downarrow{\scriptstyle inc.} & & \downarrow{\scriptstyle inc.} & & \\
\mathbf{0} & \to & \mathrm{rad}(\omega,\mathtt{d}_{\mathfrak{h}}\omega) & \to & \widehat{\mathfrak{ham}}(\mathfrak{h},\omega) & \longrightarrow & V_\omega & \to & \mathbf{0} \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathtt{d}_{\mathfrak{h}}} & & \\
\mathbf{0} & \to & \mathrm{rad}(\omega,\mathtt{d}_{\mathfrak{h}}\omega) & \to & \mathfrak{ham}(\mathfrak{h},\omega) & \xrightarrow{q} & \mathtt{d}_{\mathfrak{h}} V_\omega & \to & \mathbf{0}.
\end{array}$$

Lemma 1.2 *The subspace V_ω is an $\mathfrak{sp}(\mathfrak{h},\omega)$-submodule of V and the V-valued 2-cocycle ω on $\mathfrak{h}$ restricts to a V_ω-valued 2-cochain on $\mathfrak{sp}(\mathfrak{h},\omega)$.*

Proof. Let $\eta \in \mathfrak{sp}(\mathfrak{h},\omega)$ and $v \in V_\omega$ with $i_\xi\omega = \mathtt{d}_{\mathfrak{h}} v$ for some $\xi \in \mathfrak{ham}(\mathfrak{h},\omega)$. Then

$$\mathtt{d}_{\mathfrak{h}}(\mathcal{L}_\eta v) = \mathcal{L}_\eta \mathtt{d}_{\mathfrak{h}} v = \mathcal{L}_\eta i_\xi \omega = i_{[\eta,\xi]}\omega + i_\xi \mathcal{L}_\eta \omega = i_{[\eta,\xi]}\omega$$

implies $\mathcal{L}_\eta v \in V_\omega$. That $\omega(\xi,\eta) \in V_\omega$ for $\xi,\eta \in \mathfrak{sp}(\mathfrak{h},\omega)$ follows from (3). ∎

Proposition 1.3 *The central extension $\widehat{\mathfrak{ham}}(\mathfrak{h},\omega)$ is a Lie subalgebra of the abelian extension $V_\omega \rtimes_\omega \mathfrak{sp}(\mathfrak{h},\omega)$, defined by the Lie bracket*

$$[(v_1,\xi_1),(v_2,\xi_2)] = (\xi_1.v_2 - \xi_2.v_1 + \omega(\xi_1,\xi_2),[\xi_1,\xi_2]),$$

as well as of the central extension $V_\omega \times_{-\omega_0} \mathfrak{sp}(\mathfrak{h},\omega)$, ω_0 denoting the 2-cocycle ω on $\mathfrak{sp}(\mathfrak{h},\omega)$, considered as a cocycle with values in the trivial $\mathfrak{sp}(\mathfrak{h},\omega)$-module V_ω.

Proof. Both assertions follow from $\xi_1.v_2 = -\xi_2.v_1 = -\omega(\xi_1,\xi_2)$ for $\mathtt{d}_{\mathfrak{h}} v_j = i_{\xi_j}\omega$, $j = 1,2$. ∎

In the remainder of this section we describe several examples illustrating the abstract context described above. We start with an almost tautological example.

Example 1.4 Let $q\colon \widehat{\mathfrak{h}} \to \mathfrak{h}$ be a central extension of Lie algebras with kernel $\mathfrak{z}$. Then the adjoint action of $\widehat{\mathfrak{h}}$ on itself factors through a representation $\widehat{\mathrm{ad}}\colon \mathfrak{h} \to \mathrm{der}(\widehat{\mathfrak{h}})$, defined by $\widehat{\mathrm{ad}}(q(X))(Y) = [X, Y]$. Therefore $V := \widehat{\mathfrak{h}}$ carries a natural $\mathfrak{h}$-module structure. Moreover, the Lie bracket on $\widehat{\mathfrak{h}}$ defines an invariant 2-cocycle $\omega \in Z^2(\mathfrak{h}, V)^{\mathfrak{h}}$, determined by

$$\omega(q(X), q(Y)) := -[X, Y] = q(Y).X.$$

Clearly, $V^{\mathfrak{h}} = \mathfrak{z}(\widehat{\mathfrak{h}})$ is the center of the Lie algebra $\widehat{\mathfrak{h}}$ and $V_\omega = V$ follows from the fact that $i_{q(X)}\omega = \mathrm{d}_{\mathfrak{h}}X$ holds for each $X \in \widehat{\mathfrak{h}}$. This in turn implies that the "Poisson bracket" on $V = \widehat{\mathfrak{h}}$ is

$$\{X, Y\} = -\omega(q(X), q(Y)) = [X, Y],$$

so that we recover the Lie bracket on $\widehat{\mathfrak{h}}$. This shows that any central Lie algebra extension can be written as some V_ω, associated to an invariant 2-cocycle.

We also obtain $\mathfrak{h} = \mathfrak{sp}(\mathfrak{h}, \omega) = \mathfrak{ham}(\mathfrak{h}, \omega)$ and the corresponding central extension is

$$\begin{aligned}\widehat{\mathfrak{ham}}(\mathfrak{h}, \omega) &= \{(v, X) \in V \times \mathfrak{h}\colon i_X\omega = \mathrm{d}_{\mathfrak{h}}v\} = \{(Y, X) \in \widehat{\mathfrak{h}} \times \mathfrak{h}\colon i_X\omega = i_{q(Y)}\omega\} \\ &= \{(Y, X) \in \widehat{\mathfrak{h}} \times \mathfrak{h}\colon q(Y) - X \in \ker \widehat{\mathrm{ad}}\} \\ &= \{(Y, q(Y))\colon Y \in \widehat{\mathfrak{h}}\} + \{(0, X)\colon X \in \ker \widehat{\mathrm{ad}}\} \cong \widehat{\mathfrak{h}} \oplus q(\mathfrak{z}(\widehat{\mathfrak{h}})).\end{aligned}$$

If, in addition, $\mathfrak{z} = \mathfrak{z}(\widehat{\mathfrak{h}})$, then $q(\mathfrak{z}(\widehat{\mathfrak{h}})) = \{0\}$, and we simply obtain $\widehat{\mathfrak{h}} \cong \widehat{\mathfrak{ham}}(\mathfrak{h}, \omega)$, considered as a central extension of $\mathfrak{h} = \mathfrak{ham}(\mathfrak{h}, \omega)$.

Examples 1.5 (a) If (M, ω) is a finite-dimensional connected symplectic manifold, $\mathfrak{h} = \mathcal{V}(M)$ and $V = C^\infty(M, \mathbb{R})$, then $\mathfrak{sp}(\mathfrak{h}, \omega) = \mathfrak{sp}(M, \omega)$ is the Lie algebra of symplectic vector fields on M, $\mathfrak{ham}(\mathfrak{h}, \omega) = \mathfrak{ham}(M, \omega)$ is the Lie algebra of hamiltonian vector fields on M, and $\mathrm{rad}\,\omega = \mathrm{rad}(\omega, \mathrm{d}_{\mathfrak{h}}\omega) = \{0\}$ implies $\mathfrak{h}_\omega = \mathfrak{h}$. In this case

$$C^1(\mathfrak{h}, V)_\omega = \{i_\xi\omega\colon \xi \in \mathfrak{h}\} = \Omega^1(M, \mathbb{R})$$

is the space of all smooth 1-forms on M and the Lie bracket on this space is given by

$$[\alpha_1, \alpha_2] = \mathcal{L}_{\xi_1}\alpha_2 - i_{\xi_2}\mathcal{L}_{\xi_1}\omega$$

for $\alpha_j = i_{\xi_j}\omega$, $j = 1, 2$. We also have $V^{\mathfrak{h}} = \mathbb{R}$ (the constant functions), and $\widehat{\mathfrak{ham}}(\mathfrak{h}, \omega) \cong (C^\infty(M, \mathbb{R}), \{\cdot, \cdot\})$ is a central Lie algebra extension of $\mathfrak{ham}(M, \omega)$.

(b) If ω is a only a 2-form on M, $\mathfrak{h} = \mathcal{V}(M)$, and

$$\mathfrak{sp}(M, \omega) := \{X \in \mathcal{V}(M) \colon \mathcal{L}_X\omega = 0, i_X\mathtt{d}\omega = 0\}.$$

Considering $V = C^\infty(M, \mathbb{R})$ as above, we put

$$\mathfrak{ham}(M, \omega) := \{X \in \mathcal{V}(M) \colon \mathcal{L}_X\omega = 0, (\exists f \in V)\, i_X\omega = \mathtt{d}f\}.$$

Then $\mathfrak{h}_\omega$ may be a proper Lie subalgebra of $\mathcal{V}(M)$. In this case the space of admissible functions $V_\omega = \{f \in V \colon (\exists \xi \in \mathfrak{ham}(M, \omega))\ \mathtt{d}f = i_\xi\omega\}$ is an associative subalgebra of V because $\mathtt{d}f = i_\xi\omega$ and $\mathtt{d}g = i_\eta\omega$ imply

$$\mathtt{d}(fg) = f\mathtt{d}g + g\mathtt{d}f = i_{f\eta}\omega + i_{g\xi}\omega = i_{f\eta+g\xi}\omega,$$

and

$$\begin{aligned}\mathcal{L}_{f\eta}\omega = \mathtt{d}(i_{f\eta}\omega) = \mathtt{d}(fi_\eta\omega) &= \mathtt{d}f \wedge i_\eta\omega + f \cdot \mathtt{d}(i_\eta\omega)\\ &= \mathtt{d}f \wedge i_\eta\omega + f \cdot \mathtt{d}^2 g = i_\xi\omega \wedge i_\eta\omega = -\mathcal{L}_{g\xi}\omega\end{aligned}$$

show that $f\eta + g\xi \in \mathfrak{ham}(M, \omega)$. We thus obtain on V_ω the structure of a commutative Poisson algebra by $\{f_1, f_2\} := \omega(\xi_2, \xi_1)$ for $i_{\xi_j}\omega = \mathtt{d}f_j$ (cf. [Gra85]). Hamiltonian actions in this context are studied in [DTK07]. Our definition of $\mathfrak{sp}(M, \omega)$ does not coincide with the definition given there, where the condition $i_X\mathtt{d}\omega = 0$ is omitted. This leads to a larger Lie algebra to which the restriction of ω is not necessarily a cocycle.

For the case of closed 2-forms, the Lie bracket on V_ω already occurs in Fuchssteiner's paper [Fu82, p. 1085].

Example 1.6 It is interesting to compare Examples 1.5 with the situation arising from a Poisson manifold (M, Λ). Here Λ denotes the bivector field defining the Poisson bracket on $C^\infty(M, \mathbb{R})$ by $\{f, g\} := \Lambda(\mathtt{d}f, \mathtt{d}g)$. Then we associate to $\alpha \in \Omega^1(M, \mathbb{R})$ the vector field $X_\alpha = \Lambda^\sharp(\alpha)$, defined by $\beta(X_\alpha) =$

$\Lambda(\alpha,\beta)$ for $\beta \in \Omega^1(M,\mathbb{R})$. The vector fields of the form $X_{\mathtt{d}f}$ are called *hamiltonian*. Then

$$\begin{aligned}[\alpha,\beta] &:= \mathcal{L}_{X_\alpha}\beta - i_{X_\beta}\mathtt{d}\alpha = \mathcal{L}_{X_\alpha}\beta - \mathcal{L}_{X_\beta}\alpha - \mathtt{d}(\Lambda(\alpha,\beta))\\ &= i_{X_\alpha}\mathtt{d}\beta - i_{X_\beta}\mathtt{d}\alpha + \mathtt{d}(\Lambda(\alpha,\beta))\end{aligned}$$

defines a Lie bracket on the space $\Omega^1(M,\mathbb{R})$ of 1-forms on M for which the map

$$\Lambda^\sharp\colon \Omega^1(M,\mathbb{R}) \to \mathcal{V}(M), \quad \alpha \mapsto X_\alpha$$

is a homomorphism of Lie algebras ([Fu82], Thm. 1). We then have in particular

$$[\mathtt{d}f,\mathtt{d}g] = \mathcal{L}_{X_{\mathtt{d}f}}\mathtt{d}g = \mathtt{d}(i_{X_{\mathtt{d}f}}\mathtt{d}g) = \mathtt{d}\big(\Lambda(\mathtt{d}f,\mathtt{d}g)\big) = \mathtt{d}\{f,g\},$$

so that the exterior derivative

$$\mathtt{d}\colon (C^\infty(M,\mathbb{R}),\{\cdot,\cdot\}) \to (\Omega^1(M,\mathbb{R}),[\cdot,\cdot])$$

is a homomorphism of Lie algebras.

Using $\Lambda^\sharp$, we obtain the structure of an $\Omega^1(M,\mathbb{R})$-module on $C^\infty(M,\mathbb{R})$. Then the space $\mathfrak{X}^p(M)$ of sections of the bundle $\Lambda^p(T(M))$ can be viewed as a space of $C^\infty(M,\mathbb{R})$-valued Lie algebra p-cochains on $\Omega^1(M,\mathbb{R})$. We thus obtain a subcomplex of the corresponding Chevalley–Eilenberg complex, on which the differential is given by $f \mapsto -[\Lambda,f]$, where

$$[\cdot,\cdot]\colon \mathfrak{X}^p(M) \times \mathfrak{X}^q(M) \to \mathfrak{X}^{p+q-1}(M)$$

denotes the Schouten–Nijenhuis bracket ([Vai94], Prop. 4.3). In particular, the relation $[\Lambda,\Lambda] = 0$ implies that $\Lambda \in Z^2(\Omega^1(M,\mathbb{R}),C^\infty(M,\mathbb{R}))$ is a Lie algebra 2-cocycle.

With $\mathfrak{h} := \Omega^1(M,\mathbb{R})$, $V := C^\infty(M,\mathbb{R})$ and $\omega := -\Lambda$ we are now in the setting described above. Then $\mathfrak{sp}(\mathfrak{h},\omega)$ is a Lie subalgebra containing the Lie subalgebra of closed 1-forms. Indeed, the claim being local, it suffice to prove it for exact 1-forms, so that it follows from the relation

$$\begin{aligned}X_{\mathtt{d}f}(\Lambda(\mathtt{d}g,\mathtt{d}h)) &= \{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\}\\ &= \Lambda(\mathcal{L}_{X_{\mathtt{d}f}}\mathtt{d}g,\mathtt{d}h) + \Lambda(\mathtt{d}g,\mathcal{L}_{X_{\mathtt{d}f}}\mathtt{d}h).\end{aligned}$$

Since $\Lambda^\sharp$ is a homomorphism of Lie algebras, its kernel $\operatorname{rad}\omega$ is an ideal, so that $\mathfrak{h}_\omega = \mathfrak{h}$.

For each $\alpha \in \Omega^1(M,\mathbb{R})$ and $f \in V$ we have

$$(\mathtt{d}_{\mathfrak{h}} f)(\alpha) = X_\alpha.f = \mathtt{d}f(X_\alpha) = \Lambda(\alpha, \mathtt{d}f) = -(i_{\mathtt{d}f}\Lambda)(\alpha) = -\alpha(\Lambda^\sharp(\mathtt{d}f)),$$

showing that $V = V_\omega$, and the corresponding Lie algebra structure on V coincides with the given Poisson bracket. We also note that

$$\begin{aligned} V^{\mathfrak{h}} &= \{f \in V \colon (\forall \alpha \in \mathfrak{h})\ \mathtt{d}f(X_\alpha) = 0\} = \{f \in V \colon (\forall \alpha \in \mathfrak{h})\ \alpha(X_{\mathtt{d}f}) = 0\} \\ &= \{f \in V \colon X_{\mathtt{d}f} = 0\} = \mathfrak{z}(C^\infty(M,\mathbb{R})) \end{aligned}$$

is the center of the Poisson–Lie algebra $C^\infty(M,\mathbb{R})$, so that

$$\mathtt{d}_{\mathfrak{h}} V = \Lambda^\sharp(\mathtt{d}C^\infty(M,\mathbb{R})) \cong C^\infty(M,\mathbb{R})/\mathfrak{z}(C^\infty(M,\mathbb{R}))$$

is the set $\mathfrak{ham}(M,\Lambda)$ of hamiltonian vector fields of (M,Λ). This leads to the short exact sequence

$$\mathbf{0} \to V^{\mathfrak{h}} = \mathfrak{z}(C^\infty(M,\mathbb{R})) \to V = C^\infty(M,\mathbb{R}) \to \mathtt{d}_{\mathfrak{h}}(V) = \mathfrak{ham}(M,\Lambda) \to \mathbf{0},$$

and the map $q \colon \mathfrak{ham}(\Omega^1(M,\mathbb{R}),\Lambda) \to \mathtt{d}_{\mathfrak{h}}(V)$ corresponds to the homomorphism

$$\Lambda^\sharp \colon \mathfrak{ham}(\Omega^1(M,\mathbb{R}),\Lambda) \to \mathfrak{ham}(M,\Lambda), \quad \alpha \mapsto X_\alpha.$$

Example 1.7 (cf. [DKM90], [DV91]) In the context of non-commutative geometry, one considers the following situation: A is an associative, possibly non-commutative, algebra and $\mathfrak{h}$ is a Lie algebra acting by derivations on A. Then $(C^\bullet(\mathfrak{h},A), \mathtt{d}_{\mathfrak{h}})$ is a differential graded algebra which can be considered as a variant of the exterior algebra $(\Omega^\bullet(M,\mathbb{R}),\mathtt{d})$ of a smooth manifold M. In the abstract context, symplectic forms on A correspond to elements of $Z^2(\mathfrak{h},A)$.

(a) Matrix algebras are particularly simple examples. For $A = M_n(\mathbb{R})$, we consider $\mathfrak{h} := \mathrm{der}(A) \cong \mathfrak{sl}_n(\mathbb{R})$. Then $H^p(\mathfrak{h},A)$ vanishes for $p = 1,2$ by the Whitehead Lemmas. The Lie bracket

$$\omega(x,y) = [x,y]$$

defines an $\mathfrak{h}$-invariant A-valued 2-cocycle on $\mathfrak{h}$ and since $H^1(\mathfrak{h},A)$ vanishes, we have

$$\mathfrak{h} = \mathfrak{sp}(\mathfrak{h},\omega) = \mathfrak{ham}(\mathfrak{h},\omega).$$

The inclusion $\eta\colon \mathfrak{h} \to A$ satisfies $\mathtt{d}_{\mathfrak{h}}\eta = \omega$, and $i_x\omega = \operatorname{ad} x\colon \mathfrak{h} \to A$, so that, for $x \in \mathfrak{h}$, $a_x := -x \in A$ satisfies $\mathtt{d}_{\mathfrak{h}}a_x = \operatorname{ad} x = i_x\omega$. We thus get $A_\omega = A$ and $A^{\mathfrak{h}} = \mathbb{R}\mathbf{1}$. The Poisson bracket on $A_\omega = A$ coincides with the commutator bracket:

$$\{a, b\} = \omega(-a, -b) = [a, b].$$

(b) More generally, for any locally convex associative algebra A, the Lie algebra $\mathfrak{h} := A/\mathfrak{z}(A)$ acts by inner derivation on A, and the Lie bracket $\omega([a], [b]) := [a, b]$ is an element $\omega \in Z^2(\mathfrak{h}, A)$. It is a coboundary if and only if the central Lie algebra extension

$$\mathbf{0} \to \mathfrak{z}(A) \to A \to A/\mathfrak{z}(A) \to \mathbf{0}$$

splits.[1] Here we have $\mathfrak{h} = \mathfrak{ham}(\mathfrak{h}, \omega)$, $A = A_\omega$, $A^{\mathfrak{h}} = \mathfrak{z}(A)$, and

$$\{a, b\} = [a, b]$$

implies that $\widehat{\mathfrak{ham}}(\mathfrak{h}, \omega) \cong A$ (as a Lie algebra).

For any continuous linear splitting $\sigma\colon \mathfrak{h} \to A$ of the quotient map, we have

$$\mathtt{d}_{\mathfrak{h}}\sigma(x, y) = [\sigma(x), \sigma(y)] - [\sigma(y), \sigma(x)] - \sigma([x, y]) = 2\omega(x, y) - \sigma([x, y]),$$

so that $\tilde{\omega} := \omega - \mathtt{d}_{\mathfrak{h}}\sigma$ is a 2-cocycle equivalent to ω, as an A-valued cocycle, but whose values lie in the trivial $\mathfrak{h}$-module $\mathfrak{z}(A)$.

(c) Another, closely related example, arises for $\mathfrak{h} := \mathcal{V}(M)$, M a smooth manifold and the algebra $A := C^\infty(M, M_d(\mathbb{R}))$ for some $d > 0$. Then each closed 2-form $\omega \in \Omega^2(M, \mathbb{R})$ defines an A-valued 2-cocycle on $\mathfrak{h}$ because we may identify $C^\infty(M, \mathbb{R})$ with the center of A.

Examples arising from differential forms

Example 1.8 Let M be a finite-dimensional paracompact smooth manifold, $\mathfrak{z}$ be a Mackey complete locally convex space and

$$V := \overline{\Omega}^p(M, \mathfrak{z}) := \Omega^p(M, \mathfrak{z})/\mathtt{d}\Omega^{p-1}(M, \mathfrak{z}).$$

[1] A typical example where this is not the case is the algebra $A = B(H)$ of bounded operators on an infinite-dimensional complex Hilbert space because its center $\mathbb{C}\mathbf{1}$ is contained in the commutator algebra (cf. Cor. 2 to Probl. 186 in [Ha67]).

We write $[\alpha] = \alpha + \mathtt{d}\Omega^{p-1}(M,\mathfrak{z})$ for the elements of this space. In view of de Rham's Theorem, the subspace $\mathtt{d}\Omega^{p-1}(M,\mathfrak{z})$ of $\Omega^p(M,\mathfrak{z})$ consists of all closed p-forms for which the integrals over all smooth singular p-cycles vanish. Therefore it is closed and thus V inherits a natural Fréchet topology, turning it into a topological module of the Lie algebra $\mathfrak{h} := \mathcal{V}(M)$, acting by

$$X.[\alpha] = [\mathcal{L}_X\alpha] = [i_X\mathtt{d}\alpha + \mathtt{d}i_X\alpha] = [i_X\mathtt{d}\alpha].$$

For any $(p+2)$-form $\tilde{\omega} \in \Omega^{p+2}(M,\mathfrak{z})$, we now obtain a Lie algebra 2-cochain $\omega \in C^2(\mathcal{V}(M),V)$ by

$$\omega(X,Y) := [i_Y i_X \tilde{\omega}].$$

We may also define Lie subalgebras of *symplectic*, resp., *hamiltonian vector fields* on M by

$$\mathfrak{sp}(M,\tilde{\omega}) := \{X \in \mathcal{V}(M)\colon \mathcal{L}_X\tilde{\omega} = 0 = i_X\mathtt{d}\tilde{\omega}\}$$

and

$$\mathfrak{ham}(M,\tilde{\omega}) := \{X \in \mathfrak{sp}(M,\tilde{\omega})\colon i_X\tilde{\omega} \in \mathtt{d}\Omega^p(M,\mathfrak{z})\}.$$

If $\tilde{\omega}$ is non-degenerate in the sense that $i_v\tilde{\omega}(p) \neq 0$ for any non-zero $v \in T_p(M)$ and $\mathtt{d}\tilde{\omega} = 0$, then the pair $(M,\tilde{\omega})$ is called a *multisymplectic manifold*. For more details on this class of manifolds and some of its applications, we refer to [GIMM04, p.23], [CIL99] and [Ma88].

To understand the connection with our abstract algebraic setup, the following simple observation is quite useful:

Lemma 1.9 *If a differential form $\alpha \in \Omega^{p+1}(M,\mathfrak{z})$ on the finite-dimensional manifold M has the property that $i_X\alpha$ is exact for any vector field $X \in \mathcal{V}(M)$, then $\alpha = 0$.*

Proof. The case $p = \dim M$ is trivial, so that we may assume that $p < \dim M$. Let $X \in \mathcal{V}(M)$. Then there exists $\theta \in \Omega^{p-1}(M,\mathfrak{z})$ with $\mathtt{d}\theta = i_X\alpha$. For any $f \in C^\infty(M,\mathbb{R})$ the p-form $f\mathtt{d}\theta = i_{fX}\alpha$ is exact. In particular $\mathtt{d}f \wedge \mathtt{d}\theta = 0$ for any smooth function f, hence $\mathtt{d}\theta = 0$. Then α vanishes because the vector field X was arbitrary. ∎

As an immediate consequence of the preceding lemma, we see that $X.[\alpha] = [i_X\mathtt{d}\alpha] = 0$ holds for all $X \in \mathcal{V}(M)$ if and only if $\mathtt{d}\alpha = 0$, so that

$$V^{\mathfrak{h}} = H^p_{\mathrm{dR}}(M,\mathfrak{z})$$

(cf. [Ne06a, Lemma 23]).

Proposition 1.10 *For the Lie algebra 2-cochain $\omega \in C^2(\mathcal{V}(M), \overline{\Omega}^p(M,\mathfrak{z}))$, the following assertions hold:*

(a) *ω is a 2-cocycle if and only if $\tilde{\omega}$ is closed.*

(b) $\mathfrak{sp}(\mathfrak{h},\omega) = \mathfrak{sp}(M,\tilde{\omega})$.

(c) $\mathfrak{ham}(\mathfrak{h},\omega) = \mathfrak{ham}(M,\tilde{\omega})$.

Proof. First we derive some useful formulas. For $X, Y, Z \in \mathcal{V}(M)$, we have $\mathtt{d}\omega(X,Y,Z) = (i_X \mathtt{d}\omega)(Y,Z) = (\mathcal{L}_X\omega)(Y,Z) - \mathtt{d}(i_X\omega)(Y,Z)$. Further

$$\begin{aligned}(\mathcal{L}_X\omega)(Y,Z) &= \mathcal{L}_X\omega(Y,Z) - \omega([X,Y],Z) - \omega(Y,[X,Z]) \\ &= [\mathcal{L}_X i_Z i_Y \tilde{\omega} - i_Z i_{[X,Y]}\tilde{\omega} - i_{[X,Z]} i_Y \tilde{\omega}] = [i_Z i_Y \mathcal{L}_X \tilde{\omega}], \quad (8)\end{aligned}$$

and

$$\begin{aligned}\mathtt{d}(i_X\omega)(Y,Z) &= Y.\omega(X,Z) - Z.\omega(X,Y) - \omega(X,[Y,Z]) \\ &= [\mathcal{L}_Y i_Z i_X \tilde{\omega} - \mathcal{L}_Z i_Y i_X \tilde{\omega} - i_{[Y,Z]} i_X \tilde{\omega}] \\ &= [i_Z \mathcal{L}_Y i_X \tilde{\omega} - i_Z \mathtt{d}(i_Y i_X \tilde{\omega})] = [i_Z i_Y \mathtt{d}(i_X \tilde{\omega})].\end{aligned}$$

This leads to

$$\mathtt{d}\omega(X,Y,Z) = [i_Z i_Y (\mathcal{L}_X \tilde{\omega} - \mathtt{d}(i_X\tilde{\omega}))] = [i_Z i_Y i_X \mathtt{d}\tilde{\omega}]. \quad (9)$$

From (9) and Lemma 1.9 we now immediately derive (a). More precisely, we derive from Lemma 1.9 that for $X \in \mathcal{V}(M)$ the relation $i_X \mathtt{d}\omega = 0$ is equivalent to $i_X \mathtt{d}\tilde{\omega} = 0$ and from (8) that $\mathcal{L}_X\omega = 0$ is equivalent to $\mathcal{L}_X\tilde{\omega} = 0$. This proves (b).

To verify (c), we first note that for any $X \in \mathfrak{ham}(M,\tilde{\omega})$ and $\theta \in \Omega^p(M,\mathfrak{z})$ with $\mathtt{d}\theta = i_X\tilde{\omega}$ and $Y \in \mathcal{V}(M)$, we have

$$Y.[\theta] = [\mathcal{L}_Y\theta] = [i_Y \mathtt{d}\theta] = [i_Y i_X \tilde{\omega}] = \omega(X,Y) = (i_X\omega)(Y),$$

so that $X \in \mathfrak{ham}(\mathfrak{h},\omega)$. For the converse, let $X \in \mathfrak{ham}(\mathfrak{h},\omega)$ and $[\theta] \in V$ with $\mathtt{d}_{\mathfrak{h}}[\theta] = i_X\omega$. Then $i_Y(\mathtt{d}\theta - i_X\tilde{\omega})$ is exact for any vector field Y, hence $\mathtt{d}\theta = i_X\tilde{\omega}$ by Lemma 1.9 and thus $X \in \mathfrak{ham}(M,\tilde{\omega})$. ∎

Remark 1.11 The last part of the preceding proof shows in particular that the Lie algebra of admissible vectors in $V = \overline{\Omega}^p(M, \mathfrak{z})$ is

$$V_\omega = \{[\theta] \in V \colon (\exists X \in \mathfrak{ham}(M, \tilde{\omega}))\ \mathtt{d}\theta = i_X \tilde{\omega}\}$$

and its Lie bracket is

$$\{[\theta_1], [\theta_2]\} := \omega(X_2, X_1) = [i_{X_1} i_{X_2} \tilde{\omega}] \quad \text{for} \quad \mathtt{d}\theta_j = i_{X_j} \tilde{\omega}.$$

If θ_i is closed, then we may take $X_i = 0$, which shows that $H^p_{\mathrm{dR}}(M, \mathfrak{z})$ is central in the Lie algebra V_ω. In view of Proposition 1.10, we thus obtain the central Lie algebra extension

$$\widehat{\mathfrak{ham}}(M, \tilde{\omega}) := \{([\theta], X) \in V_\omega \times \mathfrak{ham}(M, \tilde{\omega}) \colon \mathtt{d}\theta = i_X \tilde{\omega}\}$$

of $\mathfrak{ham}(M, \tilde{\omega})$ by $H^p_{\mathrm{dR}}(M, \mathfrak{z})$, where the corresponding cocycle is the restriction of $-\omega$ to $\mathfrak{ham}(M, \tilde{\omega})$.

2 The momentum map

As in the previous section, let V a topological $\mathfrak{h}$-module and $\omega \in C^2(\mathfrak{h}, V)$. In addition, we now consider a continuous homomorphism of Lie algebras

$$\zeta : \mathfrak{g} \to \mathfrak{ham}(\mathfrak{h}, \omega).$$

Then we obtain via ζ a topological $\mathfrak{g}$-module structure on V defined by $X.v := \zeta(X).v$ for $X \in \mathfrak{g}, v \in V$ and the subspace V_ω is a $\mathfrak{g}$-submodule (Lemma 1.2).

Definition 2.1 The pullback $\widehat{\mathfrak{g}}_{\mathrm{cen}} := \{(v, X) \in V \times \mathfrak{g} : i_{\zeta(X)}\omega = \mathtt{d}_{\mathfrak{h}} v\}$ by ζ of the central extension $\widehat{\mathfrak{ham}}(\mathfrak{h}, \omega)$ in (7) is a central extension

$$\mathbf{0} \hookrightarrow V^{\mathfrak{h}} \to \widehat{\mathfrak{g}}_{\mathrm{cen}} \twoheadrightarrow \mathfrak{g} \to \mathbf{0}. \tag{10}$$

It can also be viewed more directly as the pullback of the central extension V_ω of $\mathtt{d}_{\mathfrak{h}}(V_\omega)$ by the homomorphism

$$\zeta_V \colon \mathfrak{g} \to \mathtt{d}_{\mathfrak{h}}(V), \quad X \mapsto i_{\zeta(X)}\omega.$$

A continuous linear map

$$J : \mathfrak{g} \to V_\omega \quad \text{with} \quad \mathtt{d}_\mathfrak{h}(J(X)) = i_{\zeta(X)}\omega \quad \text{for} \quad X \in \mathfrak{g} \tag{11}$$

is called a *momentum map* for ζ.

Momentum maps are in one-to-one correspondence with continuous linear sections of this extension because any continuous linear section $s\colon \mathfrak{g} \to \widehat{\mathfrak{g}}_{\mathrm{cen}}$ is of the form $s(X) = (J(X), X)$ for some momentum map J and vice versa. For any such section, we obtain a 2-cocycle by

$$\begin{aligned}\tau_J(X,Y) := \tau(X,Y) &:= [s(X), s(Y)] - s([X,Y]) \\ &= (\{J(X), J(Y)\}, [X,Y]) - (J([X,Y]), [X,Y]) \\ &= (X.J(Y) - J([X,Y]), 0) \in V^\mathfrak{h},\end{aligned}$$

satisfying

$$\widehat{\mathfrak{g}}_{\mathrm{cen}} \cong V^\mathfrak{h} \times_\tau \mathfrak{g}.$$

Lemma 2.2 *For a momentum map $J\colon \mathfrak{g} \to V_\omega$, the following are equivalent:*

(i) *J is a Lie algebra homomorphism.*

(ii) *J is $\mathfrak{g}$-equivariant.*

(iii) *$\tau_J = 0$.*

(iv) *$s = (J, \mathrm{id}_\mathfrak{g})\colon \mathfrak{g} \to \widehat{\mathfrak{g}}_{\mathrm{cen}}$ is a homomorphism of Lie algebras.*

Proof. (i) $\Leftrightarrow$ (ii): The $\mathfrak{g}$-equivariance of the momentum is equivalent to $X.J(Y) = J([X,Y])$ for $X, Y \in \mathfrak{g}$. Hence the assertion follows from

$$\{J(X), J(Y)\} = \omega(\zeta(Y), \zeta(X)) = (\mathtt{d}_\mathfrak{h} J(Y))(\zeta(X)) = X.J(Y). \tag{12}$$

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follow from the definition and the above formula for τ. ∎

Example 2.3 In the situation of Example 1.4, where $V = \widehat{\mathfrak{h}}$ was a central extension of $\mathfrak{h}$, we also have $\mathfrak{h} = \mathfrak{ham}(\mathfrak{h}, \omega)$. Then an equivariant momentum map $J\colon \mathfrak{h} \to \widehat{\mathfrak{h}}$ is the same as the splitting of the central Lie algebra extension $\widehat{\mathfrak{h}}$ of $\mathfrak{h}$ by $\mathfrak{z}$.

For two choices $J, J' \in C^1(\mathfrak{g}, V_\omega)$ of momentum maps for ζ the difference $J - J'$ has values in $V^\mathfrak{h}$, and this leads directly to the following:

Proposition 2.4 *The cohomology class $[\tau_J] \in H^2(\mathfrak{g}, V^{\mathfrak{h}})$ does not depend on the choice of the momentum map J. It is the obstruction to the existence of a $\mathfrak{g}$-equivariant momentum map for $\zeta : \mathfrak{g} \to \mathfrak{ham}(\mathfrak{h}, \omega)$. In particular, the central extension $\widehat{\mathfrak{g}}_{\rm cen}$ splits if and only if an equivariant momentum map exists.*

Remark 2.5 If we replace $\mathfrak{g}$ by the central extension $\widehat{\mathfrak{g}}_{\rm cen}$ and ζ by the homomorphism $\widehat{\zeta} : \widehat{\mathfrak{g}}_{\rm cen} \to \mathfrak{ham}(\mathfrak{h}, \omega), \widehat{\zeta}(v, X) = \zeta(X)$, we obtain the $\mathfrak{g}$-equivariant momentum map

$$\widehat{J} : \widehat{\mathfrak{g}}_{\rm cen} = V^{\mathfrak{h}} \times_\tau \mathfrak{g} \to V, \quad (v, X) \mapsto J(X) + v.$$

Indeed, since $v \in V^{\mathfrak{h}}$, we have $\mathtt{d}_{\mathfrak{h}}(\widehat{J}(v, X)) = \mathtt{d}_{\mathfrak{h}} J(X) = i_{\zeta(X)}\omega$, and the equivariance of $\widehat{J}$ follows from

$$\begin{aligned} X.\widehat{J}(v, Y) - \widehat{J}(\widehat{\mathrm{ad}}(X)(v, Y)) &= X.(J(Y) + v) - \widehat{J}(\tau(X, Y), [X, Y]) \\ &= X.J(Y) - J([X, Y]) - \tau(X, Y)) = 0, \end{aligned}$$

for $v \in V^{\mathfrak{h}}$ and $X, Y \in \mathfrak{g}$.

The pullback cochain $\omega_{\mathfrak{g}} := \zeta^*\omega \in C^2(\mathfrak{g}, V)$ is a $\mathfrak{g}$-invariant 2-cocycle with values in the $\mathfrak{g}$-module V_ω because $\zeta(\mathfrak{g}) \subseteq \mathfrak{sp}(\mathfrak{h}, \omega)$. With (12) it can be expressed by the momentum map as $\omega_{\mathfrak{g}}(X, Y) = Y.J(X)$.

Lemma 2.6 *In $C^2(\mathfrak{g}, V_\omega)$ we have $\tau_J = \mathtt{d}_{\mathfrak{g}} J + \omega_{\mathfrak{g}}$.*

Proof. This follows from the relation

$$(\mathtt{d}_{\mathfrak{g}} J)(X, Y) = X.J(Y) - Y.J(X) - J([X, Y]) = \tau_J(X, Y) - \omega_{\mathfrak{g}}(X, Y)$$

for $X, Y \in \mathfrak{g}$. ∎

Remark 2.7 If $\omega = \mathtt{d}_{\mathfrak{h}}\alpha$ is a coboundary and J a momentum map, the linear map $f_J = J + \zeta^*\alpha \in C^1(\mathfrak{g}, V)$ has the property that for all $X \in \mathfrak{g}$,

$$\begin{aligned} \mathcal{L}_{\zeta(X)}\alpha &= i_{\zeta(X)}\mathtt{d}_{\mathfrak{h}}\alpha + \mathtt{d}_{\mathfrak{h}}(i_{\zeta(X)}\alpha) = i_{\zeta(X)}\omega + \mathtt{d}_{\mathfrak{h}}(\zeta^*\alpha(X)) \\ &= \mathtt{d}_{\mathfrak{h}}(J(X) + \zeta^*\alpha(X)) = \mathtt{d}_{\mathfrak{h}}(f_J(X)), \end{aligned}$$

and Lemma 2.6 immediately yields $\tau_J = \mathtt{d}_{\mathfrak{g}} f_J$. In general, for any $f \in C^1(\mathfrak{g}, V)$ with $\mathcal{L}_{\zeta(X)}\alpha = \mathtt{d}_{\mathfrak{h}}\big(f(X)\big)$ for all $X \in \mathfrak{g}$, we have $(f - f_J)(\mathfrak{g}) \subseteq V^{\mathfrak{h}}$, so that $[\mathtt{d}_{\mathfrak{g}} f] = [\mathtt{d}_{\mathfrak{g}} f_J] = [\tau_J]$ in $H^2(\mathfrak{g}, V^{\mathfrak{h}})$.

Definition 2.8 Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and ρ_V a linear action of G on the $\mathfrak{h}$-module V. The G-action ρ_V is called an *abstract hamiltonian action* for the continuous 2-cochain $\omega \in C^2(\mathfrak{h}, V)$ if the derived $\mathfrak{g}$-action on V factors through $\mathfrak{ham}(\mathfrak{h}, \omega)$. This means there exists a Lie algebra homomorphism $\zeta : \mathfrak{g} \to \mathfrak{ham}(\mathfrak{h}, \omega)$ with $X.v = \zeta(X).v$. When the Lie algebra homomorphism ζ is given, we call ρ_V an *abstract hamiltonian G-action for ζ*.

One sees immediately that $V^G = V^{\mathfrak{g}} \supseteq V^{\mathfrak{h}}$. The $\mathfrak{g}$-invariant pullback cocycle $\omega_{\mathfrak{g}} \in Z^2(\mathfrak{g}, V_\omega)$ is also G-invariant:

$$g.\omega_{\mathfrak{g}}(X, Y) = \omega_{\mathfrak{g}}(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y). \tag{13}$$

Considering the natural G-action on $C^1(\mathfrak{g}, V)$ by $(g.c)(Y) = g.c(\mathrm{Ad}(g)^{-1}Y)$ and its infinitesimal version $(X.c)(Y) = X.c(Y) - c([X, Y])$, the Lie algebra 2-cocycle $\tau_J \in Z^2(\mathfrak{g}, V^{\mathfrak{h}})$ satisfies $\tau_J = \mathtt{d}_{\mathfrak{g}} J \in Z^1(\mathfrak{g}, C^1(\mathfrak{g}, V^{\mathfrak{h}}))$. As a $C^1(\mathfrak{g}, V)$-valued cocycle it is a coboundary, since $J \in C^1(\mathfrak{g}, V)$, but in general not as a $C^1(\mathfrak{g}, V^{\mathfrak{h}})$-valued 1-cocycle.

We define a $C^1(\mathfrak{g}, V)$-valued group cocycle by

$$\kappa = \mathtt{d}_G J : G \to C^1(\mathfrak{g}, V), \quad \kappa(g) = g.J - J.$$

Lemma 2.9 *$\kappa(g)(X) \in V^{\mathfrak{h}}$ for all $g \in G$ and $X \in \mathfrak{g}$.*

Proof. We have already seen above that G acts trivially on $V^{\mathfrak{g}} \supseteq V^{\mathfrak{h}}$. To see that all the maps $\kappa(g)$ have values in $V^{\mathfrak{h}}$, pick $\xi \in \mathfrak{h}$. It suffices to show that the function

$$F \colon G \to V, \quad g \mapsto \xi.\kappa(g)X = \xi.\big((g.J - J)(X)\big)$$

is constant because it vanishes in $\mathbf{1}$. Since G is connected, it suffices to see that for each $g \in G$ and $Y \in \mathfrak{g}$ we have

$$0 = T_g(F)(g.Y) = \xi.\big((g.(Y.J))(X)\big).$$

Since $(Y.J)(Z) = \tau(Y, Z) \in V^{\mathfrak{h}}$ for $Y, Z \in \mathfrak{g}$, and $(g.(Y.J))(X) = g.(Y.J)(\mathrm{Ad}(g)^{-1}X)$, this follows from the triviality of the action of G on $V^{\mathfrak{h}}$. ∎

The lemma implies that κ is a $C^1(\mathfrak{g}, V^{\mathfrak{h}})$-valued 1-cocycle on G. It measures the failure of the momentum map J to be G-equivariant because

$$g.\big(\kappa(g^{-1})(X)\big) = J(\mathrm{Ad}(g)X) - g.J(X).$$

Proposition 2.10 *The object measuring the failure of the momentum map $J \in C^1(\mathfrak{g}, V_\omega)$ to be G-equivariant is the group 1-cocycle*

$$\kappa : G \to C^1(\mathfrak{g}, V^{\mathfrak{h}}), \quad \kappa(g) = g.J - J.$$

Its cohomology class $[\kappa] \in H^1(G, C^1(\mathfrak{g}, V^{\mathfrak{h}}))$ does not depend on the choice of J. It is the obstruction for the existence of a G-equivariant momentum map for ζ.

Proof. The cohomology class $[\kappa] \in H^1(G, C^1(\mathfrak{g}, V^{\mathfrak{h}}))$ does not depend on the choice of the momentum map. Indeed, for two momentum maps J and J', we have $J - J' \in C^1(\mathfrak{g}, V^{\mathfrak{h}})$ and the corresponding group 1-cocycles κ and κ' satisfy $\kappa - \kappa' = \mathtt{d}_G(J - J')$.

If κ is a $C^1(\mathfrak{g}, V^{\mathfrak{h}})$-valued 1-coboundary on G, then there is an element $c \in C^1(\mathfrak{g}, V^{\mathfrak{h}})$ such that $\kappa(g) = g.c - c$. Then $J - c$ is a G-equivariant momentum map because, by definition, $\kappa(g) = g.J - J$. ∎

Proposition 2.11 *If ρ_V is an abstract Hamiltonian G-action for ζ, then the adjoint action of $\mathfrak{g}$ on $\widehat{\mathfrak{g}}_{\mathrm{cen}} = V^{\mathfrak{h}} \times_\tau \mathfrak{g}$ integrates to a smooth G-action, given by*

$$\widehat{\mathrm{Ad}}(g)(v, X) := \big(v + \kappa(g)(\mathrm{Ad}(g)X), \mathrm{Ad}(g)X\big) = \big(v - \kappa(g^{-1})(X), \mathrm{Ad}(g)X\big),$$

for $(v, X) \in \widehat{\mathfrak{g}}_{\mathrm{cen}}$ and $g \in G$. With respect to this action, the momentum map

$$\widehat{J} \colon \widehat{\mathfrak{g}}_{\mathrm{cen}} \to V, \quad (v, X) \mapsto J(X) + v$$

is G-equivariant.

Proof. First we recall from Lemma 2.9 that $\kappa(g)(\mathrm{Ad}(g)X) \in V^{\mathfrak{h}}$, so that $\widehat{\mathrm{Ad}}(g)$ defines a continuous linear automorphism of $\widehat{\mathfrak{g}}_{\mathrm{cen}}$. From the G-invariance of ω and the relation

$$\tau(X, Y) = X.J(Y) - J([X, Y]) = \omega_{\mathfrak{g}}(Y, X) - J([X, Y]),$$

we derive $(g.\tau - \tau)(X,Y) = (J - g.J)([X,Y]) = \mathtt{d}_{\mathfrak{g}}(\kappa(g))(X,Y)$, so that

$$\widehat{\mathrm{Ad}}(g)(v,X) = \big(v + \kappa(g)(\mathrm{Ad}(g)X), \mathrm{Ad}(g)X\big)$$

is a Lie algebra automorphism ([Ne06c, Lemma II.5]). Further, the cocycle property of κ implies that $\widehat{\mathrm{Ad}}$ defines a smooth action of G on $\widehat{\mathfrak{g}}_{\mathrm{cen}}$, and the corresponding derived action is

$$\widehat{\mathrm{ad}}(Y).(v,X) = ((Y.J)(X), [Y,X]) = (\tau_J(Y,X), [Y,X]) = [(0,Y),(v,X)],$$

which is the $\mathfrak{g}$-action on $\widehat{\mathfrak{g}}_{\mathrm{cen}}$ induced by the adjoint action of $\widehat{\mathfrak{g}}_{\mathrm{cen}}$. Since any representation of the connected group G is determined by its derived representation (cf. [Ne06b, Rem. II.3.7]), $\widehat{\mathrm{Ad}}$ is the unique smooth action of G on $\widehat{\mathfrak{g}}_{\mathrm{cen}}$, integrating the adjoint action $\widehat{\mathrm{ad}}$ of $\mathfrak{g}$ on $\widehat{\mathfrak{g}}_{\mathrm{cen}}$. Since κ has values in V^G, we obtain the relation

$$\kappa(g) \circ \mathrm{Ad}(g) = g^{-1}.\kappa(g) = J - g^{-1}.J = -\kappa(g^{-1}).$$

Finally, the G-equivariance of $\widehat{J}$ with respect to the G-action follows from the connectedness of G and the equivariance with respect to the $\mathfrak{g}$-action (Remark 2.5). ∎

We also note that the description of $\widehat{\mathfrak{g}}_{\mathrm{cen}}$ as $V^{\mathfrak{h}} \times_\tau \mathfrak{g}$, yields an identification of the affine space

$$\mathcal{A} := \{f \in \mathrm{Hom}(\widehat{\mathfrak{g}}_{\mathrm{cen}}, V^{\mathfrak{h}}) \colon f|_{V^{\mathfrak{h}}} = \mathrm{id}_{V^{\mathfrak{h}}}\}$$

with its translation space $C^1(\mathfrak{g}, V^{\mathfrak{h}})$, and the G-action on $\mathcal{A}$ induced by $\widehat{\mathrm{Ad}}$ thus corresponds to the affine action on $C^1(\mathfrak{g}, V^{\mathfrak{h}})$, defined by

$$(g * \alpha)(X) := \alpha(\mathrm{Ad}(g)^{-1}X) - \kappa(g)(X). \tag{14}$$

Example 2.12 (cf. [Ko70]) Let us take a closer look at the prototypical example for our setup. Let (M,ω) be a connected presymplectic manifold, i.e., ω is a closed 2-form, and $G \to \mathrm{Ham}(M,\omega)$ a hamiltonian action of the connected Lie group G on M, where the infinitesimal action is denoted $\zeta : \mathfrak{g} \to \mathfrak{ham}(M,\omega)$.

For $\mathfrak{h} = \mathcal{V}(M)$ and $V = C^\infty(M,\mathbb{R})$, we then have $V^{\mathfrak{h}} = \mathbb{R}$, and a momentum map $J \colon \mathfrak{g} \to V$, with $i_{\zeta(X)}\omega = \mathtt{d}(J(X))$ corresponds to a map $\mu \colon M \to \mathfrak{g}^*$

with $\mu(m)(X) = J(X)(m)$, i.e., μ is a momentum map for the hamiltonian G-action on (M, ω).

The function $\kappa(g)(X) = (g.J - J)(X)$ on M is constant and the so obtained map $\kappa : G \to \mathfrak{g}^*$ is a group 1-cocycle whose cohomology class is the obstruction for the existence of a G-equivariant momentum map $\mu : M \to \mathfrak{g}^*$. In any case there is an affine G-action on $\mathfrak{g}^*$, defined by $a_g(\alpha) = \mathrm{Ad}^*(g)\alpha - \kappa(g)$ for which μ is G-equivariant. The $\mathfrak{g}$-equivariance of the momentum map $\widehat{J}\colon \widehat{\mathfrak{g}}_{\mathrm{cen}} \to V$ implies its G-equivariance, so that the corresponding map $\widehat{\mu} : M \to \widehat{\mathfrak{g}}^*_{\mathrm{cen}}$ is G-equivariant. By identifying $\mathfrak{g}^*$ with the affine subspace $\{1\} \times \mathfrak{g}^* \subseteq (\widehat{\mathfrak{g}}_{\mathrm{cen}})^*$, we obtain the affine action (14): $g * \alpha := \mathrm{Ad}^*(g)\alpha - \kappa(g)$.

An equivariant cohomology picture of the momentum map

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. A *$G^\star$-module* is a topological super vector space Ω, endowed with a smooth G-action $\rho\colon G \to \mathrm{Aut}(\Omega)$ by automorphisms, a continuous odd derivation $\mathtt{d}_\Omega$ commuting with the G-action, and a G-equivariant map $\iota\colon \mathfrak{g} \to \mathrm{End}_1(\Omega)$ such that ι, together with the derived representation $\mathcal{L}_X := d\rho(X)$ turns $(\Omega, \mathtt{d}_\Omega)$ into a differential graded $\mathfrak{g}$-module.

If $(\Omega, \mathtt{d}_\Omega)$ is a $G^\star$-module, then the corresponding *Cartan complex* is the space

$$\mathrm{Pol}(\mathfrak{g}, \Omega)^G := \bigoplus_{n \in \mathbb{N}_0} \mathrm{Pol}^n(\mathfrak{g}, \Omega)^G$$

of continuous G-equivariant polynomial maps $\mathfrak{g} \to \Omega$, endowed with the differential

$$\mathtt{d}_G(f)(X) := \mathtt{d}_\Omega(f(X)) + \iota_X f(X) \quad \text{for } X \in \mathfrak{g}, f \in \mathrm{Pol}(\mathfrak{g}, \Omega)^G.$$

If Ω is $\mathbb{Z}$-graded, then the natural grading on the space of polynomials is defined by

$$\mathrm{Pol}(\mathfrak{g}, \Omega)^G_d = \bigoplus_{2n+k=d} \mathrm{Pol}^n(\mathfrak{g}, \Omega^k)^G.$$

Note that $\mathtt{d}_G^2(f)(X) = \mathcal{L}_X(f(X)) = 0$ vanishes because of the invariance, so that we obtain a cochain complex (cf. [GS99], Sect. 4.2). Its cohomology is called the G-equivariant cohomology of the $G^\star$-module Ω.

Elements of degree 2 in $\mathrm{Pol}(\mathfrak{g},\Omega)^G$ have the form

$$f=\omega+J,\quad \omega\in(\Omega^2)^G=\mathrm{Pol}^0(\mathfrak{g},\Omega^2)^G,\quad J\in\mathrm{Lin}(\mathfrak{g},\Omega^0)^G=\mathrm{Pol}^1(\mathfrak{g},\Omega^0)^G,$$

and the condition $\mathtt{d}_G f=0$ is equivalent to

$$\mathtt{d}_\Omega\omega=0\quad\text{ and }\quad \iota_X\omega=\mathtt{d}_\Omega J(X)\quad\text{ for }\quad X\in\mathfrak{g}. \tag{15}$$

The same remains true for G-equivariant momentum maps for Lie algebra 2-cocycles in the setup developed above. If $\mathfrak{h}$ and V are both smooth G-modules, so that the $\mathfrak{h}$-action on V is G-equivariant and the corresponding $\mathfrak{g}$-actions come from a continuous homomorphism $\zeta\colon\mathfrak{g}\to\mathfrak{h}$, then the Cartan-Eilenberg complex $(C^\bullet(\mathfrak{h},V),\mathtt{d}_\mathfrak{h})$ is a $G^\star$-module. Further, a 2-cochain for the corresponding Cartan complex is of the form $\omega+J$ for $\omega\in C^2(\mathfrak{h},V)^G$ and $J\in C^1(\mathfrak{g},V)^G$. By (15), the relation $\mathtt{d}_G(\omega+J)=0$ is equivalent to ω being a 2-cocycle and J an equivariant momentum map for ω.

3 Central Lie group extensions

Let V a topological $\mathfrak{h}$-module, $\omega\in C^2(\mathfrak{h},V)$ and $\zeta:\mathfrak{g}\to\mathfrak{ham}(\mathfrak{h},\omega)$ a continuous homomorphism of Lie algebras. Any momentum map J for ζ provides us with a continuous $V^\mathfrak{h}$-valued 2-cocycle $\tau(X,Y)=X.J(Y)-J([X,Y])$ on $\mathfrak{g}$, defining the central extension $\widehat{\mathfrak{g}}_{\mathrm{cen}}=V^\mathfrak{h}\times_\tau\mathfrak{g}$ from (10). The pullback $\omega_\mathfrak{g}=\zeta^*\omega$

$$\omega_\mathfrak{g}(X,Y):=\omega(\zeta(X),\zeta(Y))=\zeta(Y).J(X)=Y.J(X) \tag{16}$$

is a V_ω-valued 2-cocycle on $\mathfrak{g}$ and defines an abelian extension $\widehat{\mathfrak{g}}_{\mathrm{ab}}:=V_\omega\rtimes_{\omega_\mathfrak{g}}\mathfrak{g}$ of $\mathfrak{g}$ by V_ω. In view of Proposition 1.3, $\widehat{\mathfrak{g}}_{\mathrm{cen}}$ is a Lie subalgebra of $\widehat{\mathfrak{g}}_{\mathrm{ab}}$.

Now we assume, in addition, that V is Mackey complete and $\rho_V\colon G\to\mathrm{GL}(V)$ is an abstract hamiltonian action for ζ. Then all period integrals are defined (cf. Appendix 6), and we assume that the period group $\Pi_{\omega_\mathfrak{g}}$ is discrete, so that the Lie algebra $\widehat{\mathfrak{g}}_{\mathrm{ab}}$ integrates to an abelian Lie group extension, and we may compare the groups corresponding to the Lie algebras $\widehat{\mathfrak{g}}_{\mathrm{cen}}$ and $\widehat{\mathfrak{g}}_{\mathrm{ab}}$.

Proposition 3.1 (i) *The cocycles $\tau\in Z^2(\mathfrak{g},V^\mathfrak{h})$ and $\omega_\mathfrak{g}\in Z^2(\mathfrak{g},V_\omega)$ have the same period group $\Pi_\tau=\Pi_{\omega_\mathfrak{g}}\subset V^\mathfrak{h}$.*

(ii) *The flux homomorphisms $F_\tau\colon\pi_1(G)\to H^1(\mathfrak{g},V^\mathfrak{h})$ and $F_{\omega_\mathfrak{g}}:\pi_1(G)\to H^1(\mathfrak{g},V_\omega)$ both vanish.*

Proof. (i) In view of Lemma 2.6, $[\tau] = [\omega_{\mathfrak{g}}]$ in $H^2(\mathfrak{g}, V_\omega)$, so that their period homomorphisms coincide by [Ne04, Thm. 7.2].

(ii) The relation $\tau - \omega_{\mathfrak{g}} = \mathtt{d}_{\mathfrak{g}} J$ (Lemma 2.6) implies $\tau^{\mathrm{eq}} - \omega_{\mathfrak{g}}^{\mathrm{eq}} = \mathtt{d} J^{\mathrm{eq}}$ in $\Omega^2(G, V)$ (cf. Appendix 6), so that we also derive with $\mathcal{L}_{X_r} J^{\mathrm{eq}} = d\rho_V(X) \circ J^{\mathrm{eq}}$ for each loop γ in G

$$\int_\gamma i_{X_r} \tau^{\mathrm{eq}} - \int_\gamma i_{X_r} \omega_{\mathfrak{g}}^{\mathrm{eq}} = \int_\gamma i_{X_r}(\mathtt{d} J^{\mathrm{eq}}) = \int_\gamma \mathcal{L}_{X_r} J^{\mathrm{eq}} = d\rho_V(X). \int_\gamma J^{\mathrm{eq}},$$

and since this is a V-coboundary, for the inclusion $\iota\colon V^{\mathfrak{h}} \to V_\omega$, we have

$$\iota_* \circ F_\tau = F_{\omega_{\mathfrak{g}}}\colon \pi_1(G) \to H^1(\mathfrak{g}, V_\omega).$$

Hence it suffices to see that F_τ vanishes, but this follows from the existence of the smooth G-action on $\widehat{\mathfrak{g}}_{\mathrm{cen}}$, integrating the adjoint action of $\mathfrak{g}$ (Proposition 2.11 and [Ne02], Prop. 7.6). ∎

The next theorem provides a central Lie group extension similar to (1) in the introduction.

Theorem 3.2 *Suppose that ρ_V is an abstract hamiltonian G-action on V for the Lie algebra homomorphism $\zeta : \mathfrak{g} \to \mathfrak{ham}(\mathfrak{h}, \omega)$ such that the period group $\Pi := \Pi_{\omega_{\mathfrak{g}}} = \Pi_\tau$ is discrete. Then the following assertions hold:*

(1) *There exists a central Lie group extension $\widehat{G}_{\mathrm{cen}}$ of G by $Z := V^{\mathfrak{h}}/\Pi$, integrating the Lie algebra $\widehat{\mathfrak{g}}_{\mathrm{cen}} = V^{\mathfrak{h}} \times_\tau \mathfrak{g}$ from (10).*

(2) *The quotient Lie group $\widehat{G}_{\mathrm{ab}} := (V_\omega/\Pi \rtimes \widehat{G}_{\mathrm{cen}})/\overline{\Delta}_Z$ by the antidiagonal $\overline{\Delta}_Z := \{(z, z^{-1})\colon z \in Z\}$ is an abelian Lie group extension of G by the smooth G-module V_ω/Π, integrating the Lie algebra $\widehat{\mathfrak{g}}_{\mathrm{ab}} := V_\omega \rtimes_{\omega_{\mathfrak{g}}} \mathfrak{g}$.*

Proof. (1) The 2-cocycle $\tau \in Z^2(\mathfrak{g}, V^{\mathfrak{h}})$ has discrete period group Π and vanishing flux homomorphism F_τ (Proposition 3.1). Hence the central Lie algebra extension $\widehat{\mathfrak{g}}_{\mathrm{cen}} = V^{\mathfrak{h}} \times_\tau \mathfrak{g}$ integrates to a central Lie group extension $\widehat{G}_{\mathrm{cen}}$ of G by $Z = V^{\mathfrak{h}}/\Pi$ (Theorem 6.1).

(2) We know from Remark 6.3 in Appendix 6 below that $\widehat{G}_{\mathrm{ab}}$ is a Lie group extension of G by V_ω/Π. Its Lie algebra is isomorphic to $(V_\omega \rtimes \widehat{\mathfrak{g}}_{\mathrm{cen}})/\overline{\Delta}_{V^{\mathfrak{h}}}$, and this is the extension of $\mathfrak{g}$ by V_ω, defined by the cocycle τ, which is equivalent to $\omega_{\mathfrak{g}}$ (Lemma 2.6). ∎

Example 3.3 (Differential $(p+2)$-forms) Let $\rho\colon G \to \mathrm{Ham}(M,\tilde{\omega})$ be a smooth hamiltonian action in the context of Example 1.8. Then the induced G-action on $V = \overline{\Omega}^p(M,\mathfrak{z})$ is an abstract hamiltonian action for the V-valued 2-cocycle ω on $\mathcal{V}(M)$. Let $\Gamma_{\tilde{\omega}} = \int_{H_{p+2}(M)} \tilde{\omega} \subseteq \mathfrak{z}$ be the group of periods of $\tilde{\omega}$. Then the Period Formula in [Ne08, Thm. 3.18] shows that the image of the period map

$$\mathrm{per}_{\omega_\mathfrak{g}}\colon \pi_2(G) \to H^p_{\mathrm{dR}}(M,\mathfrak{z}) = V^\mathfrak{h}$$

is contained in

$$\Big\{[\theta]\colon \int_{H_p(M)} \theta \subseteq \Gamma_{\tilde{\omega}}\Big\} \cong \mathrm{Hom}(H_p(M), \Gamma_{\tilde{\omega}}),$$

and this group is discrete whenever $\Gamma_{\tilde{\omega}}$ is discrete and $H_p(M)$ is finitely generated (which is the case if M is compact).

In the following section we take a closer look at the case $p = 2$, where $\tilde{\omega}$ is a closed 2-form on M.

Example 3.4 (Continuous inverse algebras) If, in the setting of Example 1.7(b), A is a Mackey complete continuous inverse algebra, i.e., its unit group $A^\times$ is open and the inversion is continuous, then $A^\times$ is a locally exponential Lie group, $Z(A)^\times = Z(A^\times)$ is a Lie subgroup, and $G := A^\times/Z(A)^\times$ also carries a Lie group structure (cf. [GN08] for all that). The action of G on A by inner automorphisms is hamiltonian. Now the existence of the Lie group extension

$$\mathbf{1} \to Z(A)^\times \to A^\times \to G \to \mathbf{1}$$

implies that the corresponding period group Π_ω is contained in $\pi_1(Z(A)) = \{z \in Z(A)\colon \exp z = \mathbf{1}\}$ (cf. [Ne02], Prop. V.11), hence is in particular discrete.

Example 3.5 (Poisson manifolds) We consider the situation arising for a Poisson manifold (M,Λ) (Example 1.6). Suppose that $\rho\colon G \to \mathrm{GL}(V)$ is an abstract hamiltonian action, corresponding to the Lie algebra homomorphism $\zeta\colon \mathfrak{g} \to \mathfrak{ham}(\Omega^1(M,\mathbb{R}),\Lambda)$. Then $\mathfrak{g}$ acts on $V = C^\infty(M,\mathbb{R})$ by

$$\zeta_V\colon \mathfrak{g} \to \mathfrak{ham}(M,\Lambda), X \mapsto i_{\zeta(X)}\Lambda = \Lambda^\sharp(\zeta(X))$$

which implies that $\rho(G) \subseteq \mathrm{Ham}(M,\Lambda) \subseteq \mathrm{Aut}(M,\Lambda)$, so that we obtain a Hamiltonian action of G on (M,Λ). The central extension $\widehat{\mathfrak{g}}_{\mathrm{cen}}$ is the pullback of the central extension of $\mathfrak{ham}(M,\Lambda) \cong \mathfrak{d}_{\mathfrak{h}}(V)$ given by the Poisson–Lie algebra $V = C^\infty(M,\mathbb{R})$:

$$\widehat{\mathfrak{g}}_{\mathrm{cen}} \cong \{(f,X) \in C^\infty(M,\mathbb{R}) \times \mathfrak{g} \colon \zeta_V(X) = -\Lambda^\sharp(\mathfrak{d}f)\},$$

a central extension of $\mathfrak{g}$ by $\mathfrak{z}(C^\infty(M,\mathbb{R})) = V^{\mathfrak{h}}$.

We are interested in criteria for the corresponding period group $\Pi_\tau \subseteq \mathfrak{z}(C^\infty(M,\mathbb{R}))$ to be discrete. Let us assume that (M,Λ) admits a quantization line bundle $q\colon \mathbb{L} \to M$, i.e., $\mathbb{L}$ is a complex line bundle on which we have a covariant derivative ∇ for which the operators

$$\widehat{f}.s := \nabla_{X_{\mathfrak{d}f}} s + 2\pi i f \cdot s$$

define a homomorphism of Lie algebras

$$C^\infty(M,\mathbb{R}) \to \mathrm{End}(\Gamma\mathbb{L}), \quad f \mapsto \widehat{f}.$$

A characterization of Poisson manifolds for which such bundles exist is given in [Vai91] (see also [Vai94] and [Hu90]). It is equivalent to the existence of a closed 2-form λ representing an integral cohomology class and a vector field A for which the bivector field $\lambda^\sharp$ defined by $\lambda^\sharp(\alpha,\beta) := \lambda(\alpha^\sharp,\beta^\sharp)$ satisfies

$$\Lambda + L_A\Lambda = \lambda^\sharp.$$

Then λ is the curvature of the pair $(\mathbb{L},\nabla)$, hence a 2-cocycle describing the Lie algebra extension

$$\mathbf{0} \to \mathrm{End}_{C^\infty(M,\mathbb{R})}(\Gamma\mathbb{L}) \cong C^\infty(M,\mathbb{R}) \to \mathrm{dend}(\Gamma\mathbb{L}) \to \mathcal{V}(M) \to \mathbf{0},$$

where $\mathrm{dend}(\Gamma\mathbb{L})$ is the Lie algebra of derivative endomorphisms, i.e., those endomorphisms $D \in \mathrm{End}(\Gamma\mathbb{L})$ for which there exists a vector field $X \in \mathcal{V}(M)$, such that $[D,\nabla_X]$ is multiplication with a smooth function (cf. [Ko76]).

We conclude that the restriction of the $C^\infty(M,\mathbb{R})$-valued cocycle $\zeta_V^*\lambda$ is equivalent to $\zeta^*\tau$, hence leads to the same period homomorphism

$$\mathrm{per}\colon \pi_2(G) \to V^{\mathfrak{h}} \subseteq V.$$

Since the existence of the line bundle $\mathbb{L}$ with curvature λ implies that all periods of λ are integral, the discussion in Example 3.3 (specialized to $p = 0$),

implies that the period group of $\zeta^*\tau$ is discrete, so that Theorem 3.2 implies the existence of a central Lie group extension $\widehat{G}_{\mathrm{cen}}$ with Lie algebra $\widehat{\mathfrak{g}}_{\mathrm{cen}}$ acting by bundle automorphisms on $\mathbb{L}$. The crucial difference to the symplectic case is that in this situation the Lie algebra $\widehat{\mathfrak{g}}_{\mathrm{cen}}$ acts on $\mathbb{L}$ by vector fields not necessarily preserving the connection, resp., the covariant derivative.

Problem 3.6 Suppose that (M, Λ) is a Poisson manifold (Example 1.6). When is the Lie algebra $\Omega^1(M, \mathbb{R})$ integrable in the sense that it is the Lie algebra of some infinite-dimensional Lie group? Since it is an abelian extension of the Lie algebra $\operatorname{im} \Lambda^\sharp$ of vector fields, one would like to prove first that this Lie algebra is integrable and then try to use the results on abelian extensions in [Ne04], but the integrability of such Lie algebras of vector fields defined by integrable distributions is a difficult problem which is still open (cf. [Ne06b], Problem IV.13).

We can ask the same question for the Poisson–Lie algebra $C^\infty(M, \mathbb{R})$, which is a central extension of the Lie algebra of Hamiltonian vector fields. Under which conditions does any of these two Lie algebras integrate to a Lie group? If (M, Λ) is compact symplectic, then $\mathfrak{ham}(M, \Lambda)$ always does and its central extension $C^\infty(M, \mathbb{R})$ does at least if the cohomology class of the symplectic form has a discrete period group (cf. Example 3.3).

4 From symplectic to hamiltonian actions

Let $\mathfrak{z}$ be a Mackey complete locally convex space, M a connected smooth manifold (possibly infinite-dimensional) and $\omega \in \Omega^2(M, \mathfrak{z})$ a closed 2-form. Under the rather weak assumption that the group $S_\omega := \int_{\pi_2(M)} \omega$ of *spherical* periods of ω is discrete, we use a bypass through the simply connected covering $q_M \colon \tilde{M} \to M$ to associate to a smooth symplectic action

$$G \to \mathrm{Sp}(M, \omega) := \{\varphi \in \mathrm{Diff}(M) \colon \varphi^*\omega = \omega\}$$

of a connected Lie group G a hamiltonian action of the simply connected covering group $\tilde{G}$ on $\tilde{M}$ and further a corresponding central Lie group extension.

We now turn to the details. Let $Z = \mathfrak{z}/\Gamma_Z$, where $\Gamma_Z \subseteq \mathfrak{z}$ is a discrete subgroup. We write $q_Z \colon \mathfrak{z} \to Z$ for the quotient map and assume that the group S_ω of spherical periods of ω is contained in Γ_Z. Whenever S_ω is discrete, this is in particular satisfied if we put $\Gamma_Z := S_\omega$.

We write $\pi_1(M)$ for the group of deck transformations of $\tilde{M}$ over M, acting from the left. Then $\tilde{\omega} := q_M^*\omega$ is a closed $\mathfrak{z}$-valued 2-form on $\tilde{M}$, and since the natural homomorphism $\pi_2(\tilde{M}) \to \pi_2(M)$ is an isomorphism, it has the same group S_ω of spherical periods. Moreover, the Hurewicz homomorphism $\pi_2(\tilde{M}) \to H_2(\tilde{M})$ is an isomorphism, so that all periods of $\tilde{\omega}$ are contained in Γ_Z. If, in addition, M is smoothly paracompact, then this implies the existence of a pre-quantum principal Z-bundle $q_P\colon P \to \tilde{M}$ with a connection 1-form $\theta \in \Omega^1(P, \mathfrak{z})$ whose curvature is $\tilde{\omega}$, i.e., $q_P^*\tilde{\omega} = \mathtt{d}\theta$ (cf. [Br93]).

The following theorem is a slight generalization of Kostant's Theorem concerning finite-dimensional manifolds and the case $Z = \mathbb{T}$ ([Ko70], Prop. 2.2.1).

Theorem 4.1 *We have an abelian group extension*

$$\mathbf{1} \to C^\infty(\tilde{M}, Z) \cong \mathrm{Gau}(P) \to \mathrm{Aut}(P) \to \mathrm{Diff}(\tilde{M})_{[\tilde{\omega}]} \to \mathbf{1}$$

and a central extension

$$\mathbf{1} \to Z \to \mathrm{Aut}(P, \theta) \to \mathrm{Sp}(\tilde{M}, \tilde{\omega}) \to \mathbf{1}.$$

Proof. Smooth Z-bundles over $\tilde{M}$ are classified by the group

$$H^2(\tilde{M}, \Gamma_Z) \cong \mathrm{Hom}(\pi_2(M), \Gamma_Z) \hookrightarrow H^2_{\mathrm{dR}}(\tilde{M}, \mathfrak{z}) \cong \mathrm{Hom}(\pi_2(M), \mathfrak{z}).$$

Therefore $\varphi \in \mathrm{Diff}(\tilde{M})$ lifts to a bundle isomorphism of P if and only if $[\varphi^*\tilde{\omega}] = [\tilde{\omega}]$, which in turn is equivalent to $\varphi^*P \cong P$ as Z-bundles. This leads to the abelian extension, where $C^\infty(\tilde{M}, Z) \cong \mathrm{Gau}(P)$ acts on P by $\varphi_F(p) := p.F(q_P(p))$.

Any quantomorphism $\tilde{\varphi} \in \mathrm{Aut}(P, \theta)$ factors through an element of the group $\mathrm{Sp}(\tilde{M}, \tilde{\omega})$. To see that, conversely, each element φ of $\mathrm{Sp}(\tilde{M}, \tilde{\omega})$ lifts to a quantomorphism of P, we first note that the preceding paragraph yields the existence of a lift $\tilde{\varphi}$ to some bundle automorphism. Then $\tilde{\varphi}^*\theta - \theta$ can be written as $q_P^*\alpha$ for some $\alpha \in \Omega^1(\tilde{M}, \mathfrak{z})$, and we have

$$q_P^*\mathtt{d}\alpha = \mathtt{d}q_P^*\alpha = \tilde{\varphi}^*(\mathtt{d}\theta) - \mathtt{d}\theta = \tilde{\varphi}^*q_P^*\tilde{\omega} - q_P^*\tilde{\omega} = q_P^*(\varphi^*\tilde{\omega} - \tilde{\omega}),$$

so that α is closed if $\varphi \in \mathrm{Sp}(\tilde{M}, \omega)$. As $H^1_{\mathrm{dR}}(\tilde{M}, \mathfrak{z})$ vanishes, there exists a smooth function $f\colon \tilde{M} \to \mathfrak{z}$ with $\mathtt{d}f = \alpha$. For $F := q_Z \circ f \in C^\infty(\tilde{M}, Z)$ and the corresponding gauge transformation $\varphi_F \in \mathrm{Gau}(P)$, we then have

$$\varphi_F^*\theta - \theta = q_P^*\mathtt{d}f = q_P^*\alpha = \tilde{\varphi}^*\theta - \theta,$$

so that $\tilde{\varphi} \circ \varphi_F^{-1} \in \mathrm{Aut}(P, \theta)$ is a lift of φ. Now the observation that $\mathrm{Aut}(P, \theta)$ intersects $\mathrm{Gau}(P)$ in Z leads to the desired central extension. ∎

Now let G be a connected Lie group and $\rho\colon G \to \mathrm{Sp}(M,\omega)$ a homomorphism defining a smooth action of G on M preserving ω. Then there exists a unique smooth action $\tilde{\rho}\colon \tilde{G} \to \mathrm{Sp}(\tilde{M},\tilde{\omega})$ of the universal covering group $\tilde{G}$ of G on $\tilde{M}$. Further, $\tilde{\omega} = q_M^*\omega$ is invariant under $\pi_1(M)$. Let $\tilde{\mathrm{Sp}}(M,\omega) := N_{\mathrm{Sp}(\tilde{M},\tilde{\omega})}(\pi_1(M))$ denote the normalizer of $\pi_1(M)$ in $\mathrm{Sp}(\tilde{M},\tilde{\omega})$, which coincides with the set of all lifts of elements of $\mathrm{Sp}(M,\omega)$ to $\tilde{M}$. We thus obtain a short exact sequence

$$\mathbf{1} \to \pi_1(M) \to \tilde{\mathrm{Sp}}(M,\omega) \to \mathrm{Sp}(M,\omega) \to \mathbf{1}$$

with $\tilde{\rho}(\tilde{G}) \subseteq \tilde{\mathrm{Sp}}(M,\omega)$.

Since $\tilde{G}$ is connected and $\mathfrak{ham}(\tilde{M},\tilde{\omega}) = \mathfrak{sp}(\tilde{M},\tilde{\omega})$, the action of $\tilde{G}$ on $\tilde{M}$ is hamiltonian, and we derive from [NV03], Prop. 1.12 and Thm. 3.4, that the pullback $\widehat{G}_{\mathrm{cen}} := \tilde{\rho}^* \operatorname{Aut}(P,\theta)$ is a central Lie group extension of $\tilde{G}$ by Z, i.e., $\widehat{G}_{\mathrm{cen}}$ carries a Lie group structure which is a principal Z-bundle over $\tilde{G}$. A Lie algebra 2-cocycle representing the corresponding central Lie algebra extension $\widehat{\mathfrak{g}}_{\mathrm{cen}}$ of $\mathfrak{g}$ by $\mathfrak{z}$ is given in terms of the derived action

$$\tilde{\zeta}\colon \mathfrak{g} \to \mathfrak{ham}(\tilde{M},\tilde{\omega})$$

of $\tilde{\rho}$, resp., the derived action $\zeta\colon \mathfrak{g} \to \mathfrak{sp}(M,\omega)$ of ρ by

$$\tau(X,Y) = -\tilde{\omega}(\tilde{\zeta}(X),\tilde{\zeta}(Y))(\tilde{m}_0) = -\omega(\zeta(X),\zeta(Y))(m_0),$$

where $\tilde{m}_0 \in \tilde{M}$ and $m_0 \in M$ are points with $q_M(\tilde{m}_0) = m_0$. This further leads to an abelian Lie group extension

$$\mathbf{1} \to C^\infty(M,Z)_0 \to \widehat{G}_{\mathrm{ab}} \to \tilde{G} \to \mathbf{1},$$

integrating the cocycle $\zeta^*\omega \in Z^2(\mathfrak{g}, C^\infty(M,\mathfrak{z}))$.

Since the kernel of the universal covering homomorphism $q_G\colon \tilde{G} \to G$ is the abelian group $\pi_1(G)$, the group $\widehat{G}_{\mathrm{cen}}$ can also be viewed as an extension of G by the group $\widehat{\pi}_1(G)$, which is the inverse image of $\ker q_G \cong \pi_1(G)$ in $\widehat{G}_{\mathrm{cen}}$. As a central extension

$$\mathbf{1} \to Z \to \widehat{\pi}_1(G) \to \pi_1(G) \to \mathbf{1}$$

of an abelian group, this group is 2-step nilpotent. Since Z is divisible, all extensions of $\pi_1(G)$ which are abelian groups are trivial, so that $\widehat{\pi}_1(G)$ is characterized by its commutator map

$$C\colon \pi_1(G) \times \pi_1(G) \to Z,$$

([Bro82], Thm. 6.4). This map can be calculated by

$$C([\alpha],[\beta]) = q_Z\Big(\int_{\tilde\alpha\bullet\tilde\beta} \tau^{\mathrm{eq}}\Big),$$

where $\tau^{\mathrm{eq}} \in \Omega^2(\tilde G, \mathfrak{z})$ is the left invariant 2-form corresponding to τ and

$$\tilde\alpha \bullet \tilde\beta\colon [0,1]^2 \to \tilde G, \quad (t,s) \mapsto \tilde\alpha(t)\tilde\beta(s),$$

where $\tilde\alpha, \tilde\beta\colon [0,1] \to \tilde G$ are lifts of the smooth loops α, β in G, starting in $\mathbf{1}$ (cf. [Ne04], Cor. 6.5). Since the 2-form τ^{eq} on $\tilde G$ is the pullback of the corresponding form on G, we also have

$$C([\alpha],[\beta]) = q_Z\Big(\int_{\alpha\bullet\beta} \tau^{\mathrm{eq}}\Big) = q_Z\Big(\int_{\mathbb{T}^2} (\alpha\bullet\beta)^*\tau^{\mathrm{eq}}\Big), \quad \alpha\bullet\beta(t,s) := \alpha(t)\beta(s).$$

The orbit map $\rho^{m_0}\colon G \to M$ is equivariant with respect to the left action of G on itself, so that $(\rho^{m_0})^*\omega \in \Omega^2(G,\mathfrak{z})$ is the left invariant 2-form on G whose value in $\mathbf{1}$ is $-\tau$. Therefore $\tau^{\mathrm{eq}} = -(\rho^{m_0})^*\omega$, which leads to

$$C([\alpha],[\beta]) = q_Z\Big(-\int_{(\alpha\bullet\beta).m_0} \omega\Big),$$

where $(\alpha\bullet\beta).m_0$ can be considered as a smooth map $\mathbb{T}^2 \to M$.

Remark 4.2 We have introduced P as a Z-bundle over $\tilde M$, but we can also interprete it as a bundle over M, as follows. Let $\widehat\pi_1(M) \subseteq \mathrm{Aut}(P,\theta)$ denote the inverse image of the discrete subgroup $\pi_1(M) \subseteq \mathrm{Sp}(\tilde M, \tilde\omega)$. Then we have a short exact sequence

$$\mathbf{1} \to Z \to \widehat\pi_1(M) \to \pi_1(M) \to \mathbf{1},$$

and this group carries a natural Lie group structure for which Z is an open central subgroup, $\pi_1(M)$ is its group of connected components, and the action of this group on P is smooth. The orbit space of this action is $P/\widehat\pi_1(M) \cong \tilde M/\pi_1(M) \cong M$ and since the map to M has smooth local sections, P is a smooth $\widehat\pi_1(M)$-principal bundle over M. The extension $\widehat\pi_1(M)$ splits if and only if there exists a Z-prequantum bundle (P_M, θ_M) for (M,ω) (cf. [Ko70], Prop. 2.4.1).

5 The algebraic essence of the Noether Theorem

Given a hamiltonian G-action on a presymplectic manifold (M,ω) (ω is a closed 2-form on M), a momentum map $\mu : M \to \mathfrak{g}^*$ and a G-invariant function f on M, Noether's Theorem says that along the trajectories of the hamiltonian vector field $X_f := X_{\mathtt{d}f}$, the momentum map is constant. An important result of Marsden and Weinstein [MW74] concerns symplectic reduction in the context where two connected Lie groups G_1 and G_2 act in a hamiltonian way on the presymplectic manifold (M,ω) with momentum maps μ_1, μ_2, such that μ_2 is constant along the G_1-orbits. They show that μ_1 is constant along the G_2-orbits and the two actions commute. In the present section we show that the algebraic essence of this result can be formulated in our abstract setup of Lie algebra 2-cocycles.

Let V a topological $\mathfrak{h}$-module, $\omega \in Z^2(\mathfrak{h}, V)$ and $\zeta : \mathfrak{g} \to \mathfrak{ham}(\mathfrak{h},\omega)$ a continuous homomorphism of Lie algebras. The space $V^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant vectors for the topological $\mathfrak{g}$-module structure on V obtained via ζ contains $V^{\mathfrak{h}}$. The space of admissible $\mathfrak{g}$-invariant vectors $V_\omega^{\mathfrak{g}} = V^{\mathfrak{g}} \cap V_\omega$ is a Lie subalgebra of V_ω containing $V^{\mathfrak{h}}$ (cf. Proposition 1.1).

Proposition 5.1 *If $\xi \in \mathfrak{ham}(\mathfrak{h},\omega)$ satisfies $i_\xi\omega = \mathtt{d}_{\mathfrak{h}}v$ for some $v \in V^{\mathfrak{g}}$, then $\xi.J(X) = 0$ holds for all $X \in \mathfrak{g}$.*

Proof. Let $v \in V_\omega^{\mathfrak{g}}$ and $\mathtt{d}_{\mathfrak{h}}v = i_\xi\omega$. Then

$$\xi.J(X) = \big(\mathtt{d}_{\mathfrak{h}}J(X)\big)(\xi) = i_\xi i_{\zeta(X)}\omega = -i_{\zeta(X)}\mathtt{d}_{\mathfrak{h}}v = -\zeta(X).v = 0$$

for all $X \in \mathfrak{g}$. ∎

Proposition 5.2 *Let $\zeta_j : \mathfrak{g}_j \to \mathfrak{ham}(\mathfrak{h},\omega)$, $j = 1,2$, be two Lie algebra homomorphisms and $J_j : \mathfrak{g}_j \to V$, $j = 1,2$, corresponding momentum mappings such that J_2 takes values in $V^{\mathfrak{g}_1}$. Then J_1 takes values in $V^{\mathfrak{g}_2}$ and $[\zeta_1(\mathfrak{g}_1), \zeta_2(\mathfrak{g}_2)] \subseteq \operatorname{rad}\omega$.*

Proof. Let $X_1 \in \mathfrak{g}_1$ and $X_2 \in \mathfrak{g}_2$. Then $\zeta_1(X_1).J_2(X_2) = 0$. But $\zeta_2(X_2).J_1(X_1) = \omega(\zeta_1(X_1), \zeta_2(X_2)) = -\zeta_1(X_1).J_2(X_2) = 0$ and we further derive $0 = \mathtt{d}_{\mathfrak{h}}(\omega(\zeta_1(X_1), \zeta_2(X_2))) = -i_{[\zeta_1(X_1),\zeta_2(X_2)]}\omega$, so that $[\zeta_1(X_1), \zeta_2(X_2)] \in \operatorname{rad}\omega$. ∎

6 Appendix: Integration of abelian Lie algebra extensions

According to the general theory developed in [Ne02] and [Ne04], there are two obstructions for the integration of a Lie algebra cocycle $\omega \in Z^2(\mathfrak{g}, V)$ with values in a smooth Mackey complete G-module V to a Lie group extension of G by a quotient group $A = V/\Gamma_A$, where Γ_A is a discrete subgroup of V: the period map and the flux homomorphism. To define these homomorphisms, we associate to $\alpha \in C^p(\mathfrak{g}, V)$ the left equivariant V-valued p-form $\alpha^{\mathrm{eq}} \in \Omega^p(G, V)$, determined by

$$\alpha^{\mathrm{eq}}_{\mathbf{1}} = \alpha, \quad \lambda_g^* \alpha^{\mathrm{eq}} = \rho_V(g) \circ \alpha^{\mathrm{eq}},$$

where $\rho_V \colon G \to \mathrm{GL}(V)$ describes the G-module structure on V. We write $d\rho_V \colon \mathfrak{g} \to \mathfrak{gl}(V)$ for the corresponding derived representation.

The *period map* is the group homomorphism

$$\mathrm{per}_\omega : \pi_2(G) \to V^G, \quad \mathrm{per}_\omega([\sigma]) = \int_{\mathbb{S}^2} \sigma^* \omega^{\mathrm{eq}} \quad \text{for} \quad \sigma \in C^\infty(\mathbb{S}^2, G).$$

Its image Π_ω is called the *period group* of ω.

The *flux homomorphism* $F_\omega : \pi_1(G) \to H^1(\mathfrak{g}, V), [\gamma] \mapsto [I^\omega_\gamma]$, assigns to each piecewise smooth loop γ in G based at the identity, the cohomology class of the 1-cocycle

$$I^\omega_\gamma \colon \mathfrak{g} \to V, \quad I^\omega_\gamma(X) = -\int_\gamma i_{X_r} \omega^{\mathrm{eq}},$$

where $X_r \in \mathcal{V}(G)$ denotes the right invariant vector field with $X_r(\mathbf{1}) = X$.

Theorem 6.1 ([Ne04], Thm. 6.7) *For a Lie algebra 2-cocycle $\omega \in Z^2(\mathfrak{g}, V)$ with discrete period group Π_ω and vanishing flux homomorphism, the Lie algebra extension $\widehat{\mathfrak{g}} = V \rtimes_\omega \mathfrak{g}$ integrates to an abelian Lie group extension*

$$\mathbf{1} \to V/\Pi_\omega \hookrightarrow \widehat{G} \twoheadrightarrow G \to \mathbf{1}.$$

Definition 6.2 Let G be a group, A an abelian group which is a G-module and $Z \subseteq A^G$ a subgroup. Then we define the *Baer product* of a central extension $q_c \colon \widehat{G}_c \to G$ of G by Z and an abelian extension $q_a \colon \widehat{G}_a \to G$ of G by A by

$$\widehat{G}_c \otimes \widehat{G}_a := \widehat{G}/\{(z, z^{-1}) \colon z \in Z\},$$

where

$$\widehat{G} := \{(g_1, g_2) \in \widehat{G}_c \times \widehat{G}_a \colon q_c(g_1) = q_a(g_2)\}$$

is the fiber product of the two extensions, which is an abelian extension of G by the product module $Z \times A$, and the antidiagonal

$$\overline{\Delta}_Z := \{(z, z^{-1}) \colon z \in Z\} \subseteq Z \times A \subseteq \widehat{G}$$

is central in $\widehat{G}$.

Remark 6.3 (a) On the level of cocycles, the Baer product corresponds to the natural action map $H^2(G, Z) \times H^2(G, A) \to H^2(G, A)$, induced by the multiplication map $Z \times A \to A, (z, a) \mapsto za$.

(b) Suppose, in addition, G, Z and A above are Lie groups, where the action of G on A is smooth. Then the Baer product of two Lie group extensions $\widehat{G}_c$ and $\widehat{G}_a$ carries a natural structure of a Lie group extension of G by A. Here we use that the antidiagonal $\overline{\Delta}_Z$ is a Lie subgroup of $Z \times A$ with $Z \times A \cong \overline{\Delta}_Z \times A$, so that the factorization of $\overline{\Delta}_Z$ defines a Lie group extension.

(c) If the extension $\widehat{G}_a$ is trivial, i.e., of the form $\widehat{G}_a = A \rtimes G$, then we have in the notation of the preceding definition $\widehat{G} \cong A \rtimes \widehat{G}_c$, where $\widehat{G}_c$ acts on A through the quotient map q_c, and this leads to

$$\widehat{G}_c \otimes \widehat{G}_a \cong (A \rtimes \widehat{G}_c)/\overline{\Delta}_Z,$$

which is the abelian extension of G by A obtained from $\widehat{G}_c$ by the natural inclusion $Z \hookrightarrow A$.

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