Asymptotics of the solution to Robin problem.

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Abstract

Convergence of the solution to the exterior Robin problem to the solution of the Dirichlet problem, as the impedance tends to infinity, is proved. The rate of convergence is established. A method for deriving higher order terms of the asymptotics of the solution is given.

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1 Introduction

Consider the following problem:

$$Lu := \nabla^2 u = f \text{ in } D', \quad u(\infty) = 0 \tag{1}$$

$$u_N - \zeta u = 0 \text{ on } S, \tag{2}$$

where $D \subset \mathbb{R}^3$ is a bounded domain with an infinitely smooth boundary S, $D' := \mathbb{R}^3 | D$, N is an outer unit normal to S, $f \in L^2_0(D')$, ζ is a constant,

$\operatorname{Im}\zeta \neq 0$, or $\operatorname{Im}\zeta = 0$ and then $\zeta \geq 0$, see lemma 2 in Appendix. It is known that problem (1) - (2) has a solution, and the solution is unique under the above assumptions. For convenience of the reader a short proof is given in the Appendix. The problems we are studying in this paper are the following. The smoothness of the boundary can be assumed finite. Then, the number m in Theorem will not be arbitrary large: it depends on the smoothness of the boundary.

1. *Does u converge in some sense to the solution of the problem*

$$\nabla^2 v = f \text{ in } D', \quad v = 0 \text{ on } S; \quad v(\infty) = 0, \tag{3}$$

as $|\zeta| \to \infty$?

2. *At what rate does u converge to v as $|\zeta| \to \infty$?*

We assume throughout that $|\zeta| \to \infty$ means that $\zeta_1 := \operatorname{Re}\zeta \to +\infty$, and $\zeta_2 := \operatorname{Im}\zeta$ is bounded. It is a common belief that $u := u_\zeta \to v$ as $|\zeta| \to \infty$. We prove that this is correct and estimate the rate of convergence, namely, we prove that this rate is $O(\frac{1}{|\zeta|})$. We give a method for finding asymptotics of u_ζ as $|\zeta| \to \infty$.

We also prove that these conclusions hold for the problem in which ∇^2 is replaced by a more general elliptic operator of the second order.

Theorem 1. *Under the above assumptions one has*

$$u_\zeta = v + O\Big(\frac{1}{|\zeta|}\Big), \quad |\zeta| \to \infty, \tag{4}$$

where v solves the Dirichlet problem (3) *and* $O\big(\frac{1}{|\zeta|}\big) = \int_S \frac{\psi(s,\zeta)ds}{4\pi|x-s|}$, *where* $\|\psi(s,\zeta)\|_{H^m(S)} \leq \frac{c_m}{|\zeta|}$ *for sufficiently large* $|\zeta|$, $c_m = \text{const} > 0$, *and* $m \geq 0$ *is arbitrary large.*

Theorem 1 gives an exact description of the sense in which u_ζ converges to the solution v of the Dirichlet problem (3).

2 Proofs

Let

$$w := -\int_{D'} g(x,y)f(y)dy,$$

where $g(x,y)=\frac{1}{4\pi|x-y|}$. Then $u=w+\varphi$, where

$$\nabla^2\varphi=0 \text{ in } D', \quad \varphi(\infty)=0, \quad \big(\varphi_N-\zeta\varphi\big)\big|_S=-\big(w_N-\zeta w\big)\big|_S. \tag{5}$$

Let us look for φ of the form

$$\varphi(x)=\int_S g(x,s)\sigma(s)ds, \tag{6}$$

where σ is to be found from the boundary condition (5), φ solves equation (5), and φ vanishes at infinity. The boundary condition (5) yields (see e.g. [3])

$$\frac{A\sigma-\sigma}{2}-\zeta T\sigma=-h, \quad h:=(w_N-\zeta w)\big|_S, \tag{7}$$

where the operators A and T are defined as follows:

$$A\sigma:=2\int_S\frac{\partial g(s,t)}{\partial N_S}\sigma(t)dt, \quad T\sigma:=\int_S g(s,t)\sigma(t)dt. \tag{8}$$

Let us write equation (7) as:

$$T\sigma=-w+\tau w_N+\tau\,\frac{A\sigma-\sigma}{2}, \quad \tau:=\frac{1}{\zeta}=\frac{\zeta_1-i\zeta_2}{|\zeta|^2}:=\tau_1+i\tau_2, \tag{9}$$

where $\zeta_1=\operatorname{Re}\zeta$, $\zeta_2=\operatorname{Im}\zeta$. Let us assume, for example, that $\zeta_1>0$. If $\zeta_1<0$, the argument is similar. As $|\zeta|\to\infty$, $|\tau|\to 0$ and $\tau_1>0$. Let us prove that

$$\sigma=-T^{-1}w+O(\tau), \quad \tau\to 0. \tag{10}$$

The operator T is known to be an isomorphism between H^ℓ and $H^{\ell+1}$, where $H^\ell:=H^\ell(S)$ is the Sobolev space, and ℓ is an arbitrary real number if S is an infinitely smooth manifold (see e.g. [3]). Moreover, T is a positive selfadjoint compact operator in $H^0=L^2(S)$. Indeed, the Fourier transform of $\frac{1}{|x|}$ in $\mathbb{R}^3$ is

$$\int_{R^3}\frac{e^{i\xi\cdot x}}{|x|}\,dx=\frac{4\pi}{|\xi|^2},$$

so it is positive,

$$\frac{1}{4\pi|x|}=\frac{1}{(2\pi)^3}\int_{R^3}e^{-i\xi\cdot x}\,\frac{d\xi}{|\xi|^2}\,.$$

Therefore, if $\varphi \in L^2(S)$, then

$$\int_S \int_S ds\, dt\, \frac{\varphi(s)\,\overline{\varphi(t)}}{4\pi|s-t|} = \frac{1}{(8\pi)^3} \int_{R^3} \frac{d\xi}{|\xi|^2}\, |\tilde{\varphi}(\xi)|^2 > 0 \quad \text{if } \tilde{\varphi} \neq 0,$$

where the overbar stands for complex conjugate, and

$$\tilde{\varphi}(\xi) = \int_S \varphi(s) e^{i\xi\cdot s}\, ds = \int_{\mathbb{R}^3} e^{i\xi\cdot x}\, \delta_S(x)\varphi(s) dx.$$

Here $\delta_S(x)$ is the delta function supported on the surface S, and $\varphi(s)$ is the density function on S, so the integral on the right is defined as the integral on the left. In [2] one can find the formula:

$$\int_{R^n} |x|^\lambda\, e^{i\xi\cdot x}\, dx = 2^{\lambda+n}\, \pi^{n/2}\, \frac{\Gamma(\frac{\lambda+n}{2})}{\Gamma(-\frac{\lambda}{2})}\, \frac{1}{|\xi|^{\lambda+n}}\,,$$

where $\Gamma(z)$ is the Gamma-function, and $\lambda \neq 2m$, $m = 0, 1, 2, \dots$, $\lambda + n \neq -2p$, $p = 0, 1, 2, \dots$. If $\lambda = -1$ and $n = 3$, one obtains $\frac{1}{|x|} = \frac{4\pi}{|\xi|^2}$, i.e., the formula, given above.

Let us denote by $(p, q)_\ell$ the inner product in H^ℓ, and

$$(p, q) := (p, q)_0 = (p, q)_{L^2(S)}.$$

Since $T = T^* > 0$, there exists a unique square root $T^{1/2} > 0$, and

$$(T\sigma, \sigma) = \|T^{1/2}\sigma\|_0^2 = \|\sigma\|_{-1/2}^2\,,$$

since $T^{1/2} : H^\ell \to H^{\ell+1/2}$. Similarly

$$(p, q)_\ell = (T_p^{-\ell}, T_q^{-\ell})_0; \quad (T\sigma, \sigma)_\ell = \|T^{-\ell+1/2}\sigma\|_0^2 = \|\sigma\|_{\ell-1/2}^2\,. \tag{11}$$

Recall that T is surjective, as was stated below formula (10). Note that

$$|(p, q)_0| \leq \|p\|_\ell\, \|q\|_{-\ell}, \quad p \in H^\ell,\ q \in H^0, \tag{12}$$

so that the form $(p, q)_0$ extends to a pairing between H^ℓ and $H^{-\ell}$ for any $\ell \in \mathbb{R}$, provided that S is infinitely smooth, which we assume for simplicity, although S can be of finite smoothness, and then (12) holds for ℓ corresponding to the smoothness of S. The operator $T\sigma = \int_S \frac{\sigma(t)dt}{4\pi|s-t|}$ is an elliptic PDO (pseudodifferential operator of order -1 in $H^0 = L^2(S)$ (see [5]).

The Sobolev spaces H^s, $0 \leq s \leq 1$, can be defined as interpolation spaces, intermediate between $H^0 = L^2(S)$ and $H^1 = H^1(S)$, by the formula $H^s = D(T^s)$, where D is the domain of definition of the positive-definite unbounded selfadjoint operator T^{-s}, $D(T^{-1}) = H^1(S)$, $D(T^{-0}) = H^0(S) = L^2(S)$, (see, e.g., [1]).

Recall that $A : H^\ell \to H^{\ell+1}$ (see, e.g., [3]). The proof below uses an idea from [4].

Equation (9) can be considered as a singular perturbation problem, because the small parameter τ is in front of the "senior derivative". Indeed, the identity operator is a "senior derivative" compared with the operators T and A, which improve smoothness by one derivative.

Multiply (9) by σ in the H^ℓ inner product and get:

$$(T\sigma, \sigma)_\ell = -(w, \sigma)_\ell + \tau(w_N, \sigma)_\ell + \frac{\tau}{2}(A\sigma, \sigma)_\ell - \frac{\tau}{2}(\sigma, \sigma)_\ell \,. \tag{13}$$

The functions w and w_N belong to H^ℓ for any ℓ if S is infinitely smooth and $\operatorname{supp} f \subset D'$. Take the real part of (13) and use the fact that $\tau_1 > 0$. Then, (13) and (11) imply:

$$\|\sigma\|^2_{\ell-\frac{1}{2}} \leq \|w\|_{\ell+\frac{1}{2}} \|\sigma\|_{\ell-\frac{1}{2}} + |\tau| \; \|w_N\|_{\ell+\frac{1}{2}} \|\sigma\|_{\ell-\frac{1}{2}} + \frac{|\tau|}{2} \; \|A\sigma\|_{\ell+\frac{1}{2}} \; \|\sigma\|_{\ell-\frac{1}{2}} \,,$$

so $\sigma := \sigma_\tau$ satisfies the inequality:

$$\|\sigma\|_{\ell-\frac{1}{2}} \leq \|w\|_{\ell+\frac{1}{2}} + |\tau| \|w_N\|_{\ell+\frac{1}{2}} + \frac{|\tau|}{2} \|A\sigma\|_{\ell+\frac{1}{2}} \,. \tag{14}$$

We have

$$\|w\|_{\ell+\frac{1}{2}} + |\tau| \|w_N\|_{\ell+\frac{1}{2}} \leq c \,; \quad \|A\sigma\|_{\ell+\frac{1}{2}} \leq c \|\sigma\|_{\ell-\frac{1}{2}}, \tag{15}$$

where $c > 0$ stands for various constants independent of τ, $|\tau| \in (0, 1)$. It follows from (15) that

$$\|\sigma_\tau\|_{\ell-\frac{1}{2}} \leq c, \quad 0 < |\tau| < 1. \tag{16}$$

Let $T\nu = -w$. If $w \in H^{m+2}$, then $w_N \in H^{m+1}$, $\nu = -T^{-1}w \in H^{m+1}$, and $A\nu \in H^{m+2}$, so $w_N + \frac{A\nu - \nu}{2} \in H^{m+1}$.

We want to prove that the estimate

$$\sigma_\tau = \nu + O(|\tau|), \quad \tau \to 0, \tag{17}$$

holds in H^m, provided that $w \in H^{m+2}$.

Let $\sigma_\tau - \nu := \psi_\tau$ and write equation (9) as

$$(T + \frac{\tau}{2})\psi_\tau = \frac{\tau}{2}A\psi_\tau + \tau\big(w_N + \frac{A\nu - \nu}{2}\big).$$

Assume that $\tau > 0$. If $w \in H^{m+2}$, then $w_N + \frac{A\nu-\nu}{2} \in H^{m+1}$, as we have mentioned above. Applying the operator $(T + \frac{\tau}{2})^{-1}$ to the above equation, using the assumption $w_N + \frac{A\nu-\nu}{2} \in H^{m+1}$, the boundedness of the norm $||(T + \frac{\tau}{2})^{-1}A||_m < c$, where the constant $c > 0$ does not depend on τ, the boundedness of ψ_τ in H^m, and the fact that $T : H^m \to H^{m+1}$, one obtains estimate (17) for the remainder $||\psi_\tau||_m := ||\sigma_\tau - \nu||_m$.

The function ψ_τ is denoted $\psi(s, \zeta)$ in Theorem 1. Since we have assumed that $\operatorname{supp} f \subset D'$, the function w is infinitely smooth in a neighborhood of the boundary S, so our data is in H^{m+1} for any $m \geq 0$.

If S is assumed to be Lipschitz, rather than C^∞, then $m = 0$.

Since

$$u_\zeta = w + \int_S g(x,s)\,\sigma_\tau(s)ds, \quad \tau = \frac{1}{\zeta}\,, \tag{18}$$

the relation (17) implies

$$u_\zeta(x) = w(x) + \int_S g(x,s)\nu(s)ds + O(\frac{1}{|\zeta|}). \tag{19}$$

The function

$$v := w(x) + \int_S g(x,s)\nu(s)ds \tag{20}$$

solves problem (3).

Indeed,

$$\nabla^2 v = \nabla^2 w = f \text{ in } D', \quad v(\infty) = 0, \tag{21}$$

$$v\big|_S = w(s) + T\nu = 0. \tag{22}$$

The last relation holds because $T\nu = -w$. $\Box$

3 Generalizations

3.1 Suppose that the Laplace operator ∇^2 in (1) is replaced by a general selfadjoint second order elliptic differential expression L, and its fundamental solution $G(x,y)$, $LG_L = -\delta(x-y)$ in $\mathbb{R}^3$, defines a positive selfadjoint operator T_L in $H^\ell = H^\ell(S)$,

$$T_L\sigma := \int_S G_L(s,t)\sigma(t)dt,$$

where $T_L : H^\ell \to H^{\ell+1}$ is an isomorphism between H^ℓ and $H^{\ell+1}$. Then the arguments, given in Section 2, remain valid, and the theorem, similar to Theorem 1, holds.

If, for example

$$LG = \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial G}{\partial x_j}\big) - q(x)G,$$

where $a_{ij}(x)$ is a smooth, uniformly with respect to $x \in \mathbb{R}^3$ positive-definite matrix, $a_{ij}(x) = a_{ij}(\infty)$ for $|x| > R$, where $R > 0$ is an arbitrary large fixed number, $q(x)$ is a smooth function, $0 < b \le q(x)$, $q(x) = q(\infty)$ for $|x| > R$, then $|x - y|G(x, y) > 0$ as $|x - y| \to 0$, and

$$\int_S \int_S G(s,t)\varphi(t)\overline{\varphi(s)}dsdx > 0$$

if $\varphi \neq 0$. To prove this, one may take a ball $B_a \supset D$, $B_a := \{x : |x| \le a\}$, and define $G_a(x, y)$ as the unique solution to the problem

$$LG_a = -\delta(x-y) \text{ in } B_a, \quad G_a\big|_{\partial B_a} = 0.$$

Then, by the maximum principle,

$$0 \le G_a(x,y) < G(x,y), \quad x, y \in B_a,$$

and

$$G_a(x,y) = \sum_{j=1}^{\infty} \lambda_j^{-1}\psi_j(x)\,\overline{\psi_j(y)},$$

where λ_j are eigenvalues and ψ_j are orthonormal eigenfunctions of the Dirichlet operator L in $L^2(B_a)$, $\lambda_j > 0$. Thus

$$\int_S \int_S G_a(s,t)\varphi(t)\,\overline{\varphi(s)}dsdt = \sum_{j=1}^{\infty} \lambda_j^{-1}\Big|\int_S \varphi(t)\,\overline{\psi_j(t)}dt\Big|^2 \ge 0.$$

Indeed, if $\int_S \varphi(t)\,\overline{\psi_j(t)}dt = 0\ \forall j$, then

$$w(x) := \int_S G_a(x,t)\varphi(t)dt = 0 \text{ on } S \quad \text{and} \quad Lw = 0 \text{ in } D.$$

Since L is a positive-definite operator, it follows that $w = 0$ in D, and $w = 0$ in D'. Then $\varphi = 0$ by the jump formula for the conormal derivative of the potential of the single layer.

One can also prove an analog of Theorem 1 for operators L, which is not necessarily positive-definite. For example, if $L = \nabla^2 + k^2$, $k = \text{const} > 0$, and $\operatorname{Im}\zeta \le 0$, then, by Lemma 3 in Appendix, problem (1) – (2) with $L = \nabla^2 + k^2$ and the condition $u(\infty) = 0$ replaced by the radiation condition (A.6) has at most one solution. This solution converges, as $\zeta_1 = \operatorname{Re}\zeta \to +\infty$, $|\zeta_2| \le c$, $\zeta_2 := \operatorname{Im}\zeta$, to the solution of the Dirichlet problem in D' for the operator $L = \nabla^2 + k^2$, satisfying the radiation condition at infinity.

If k^2 is not a Dirichlet eigenvalue of the Laplacian in D, then the operator

$$T_L\sigma := \int_S \frac{e^{ik|s-t|}}{4\pi|s-t|}\,\sigma(t)dt$$

is an isomorphism of H^ℓ onto $H^{\ell+1}$, so that the equation

$$T_L\nu = -w,$$

similar to (19), is uniquely solvable in H^ℓ for any $w \in H^{\ell+1}$ and our proof of Theorem 1 remains valid.

If the diameter of D is sufficiently small, then k^2 is not a Dirichlet eigenvalue of the operator L in D. This case is discussed in [4], where the wave scattering by many small bodies was studied and the impedance boundary conditions were assumed at the boundaries of the small bodies.

However, the assumption that k^2 is not a Dirichlet eigenvalue of L in D is not necessary for the validity of a Theorem similar to Theorem 1. If k^2 is a Dirichlet eigenvalue of L in D, then the following change in the proof is needed: in place of $\frac{e^{ik|x-y|}}{4\pi|x-y|}$ one uses the kernel $g(x,y,k)$, which solves the equation $(\Delta + k^2)g = -\delta(x-y)$ in $\mathbb{R}^3\backslash B_\rho$, where $\rho > 0$ is a small number, $B_\rho = \{x : |x| \le \rho\} \subset D$, g satisfies the radiation condition, $g\big|_{\partial B_\rho} = 0$, and *$k^2$ is not a Dirichlet eigenvalue of the operator $L = \nabla^2 + k^2$ in $D\backslash B_\rho$*. Such a method was used in [3].

Let us explain what change in our proof should be made in the case

$$T_L\sigma = \int_S \frac{e^{ik|s-t|}}{4\pi|s-t|}\,\sigma(t)dt.$$

In this case the operator T_L is non-selfajoint and not positive. The key points are:

1) The operator $T_L = T + B$, where $T\sigma := \int_S \frac{\sigma(t)dt}{4\pi|s-t|}$, so $T = T^*$ is a positive operator,

and

2) The operator $B\sigma := \int_S \frac{e^{ik|s-t|}-1}{4\pi|s-t|}\,\sigma(t)dt$ is a smoothing operator, such that $T^{-1}B$ is compact in $H^0 = L^2(S)$ and $I + T^{-1}B$ is a boundedly invertible operator (i.e., its inverse is a bounded operator in H^0) because this operator is of Fredholm type and its null-space is trivial.

To prove the last statement, assume that $(I+T^{-1}B)\sigma = 0$. Then $T_L\sigma = 0$. This implies that $s(x) := \int_S \frac{e^{ik|x-t|}}{4\pi|x-t|}\,\sigma(t)dt = 0$ in D', and $s(x) = 0$ in D if k^2 is not a Dirichlet eigenvalue of the Laplacian in D. Thus, $\sigma = 0$ by the jump relation for the normal derivatives of the single-layer potential $s(x)$. If k^2 is a Dirichlet eigenvalue of the Laplacian in D, then, as has been already explained above, we replace $\frac{1}{4\pi|x-y|}$ by Green's function of the Dirichlet Laplacian in $R^3\backslash B_\rho$, where a ball B_ρ of radius ρ belongs to D and k^2 is not a Dirichlet eigenvalue of the Dirichlet Laplacian in $D\backslash B_\rho$. Such a ball alway exists (see [3]).

The equation, analogous to (9), is

$$T_L\sigma = -w + \tau w_N + \tau\,\frac{A\sigma}{2} - \tau\,\frac{\sigma}{2}\,,$$

where $T_L = T + B$, $T = T^* > 0$, B is a smoothing operator, $T^{-1}B$ is compact in H^ℓ, and the operator $I + T^{-1}B$ is boundedly invertible in $H^0 = L^2(S)$.

Using the argument, given in Section 2, one writes

$$T_L\sigma = T(I + T^{-1}B)\sigma,$$

denotes

$$(I + T^{-1}B)\sigma := \eta,$$

and get, as in (13) – (16) an estimate, analogous to (16) by taking into account that

$$\sigma = (I + K)\eta,$$

where $(I+T^{-1}B)^{-1} = I+K$, K is a compact smoothing operator in H^ℓ. We also use the following estimate:

$$-\mathrm{Re}\,\frac{\tau}{2}(\sigma,\eta)_\ell = -\mathrm{Re}\,\frac{\tau}{2}(\eta,\eta)_\ell - \frac{\tau}{2}(K\eta,\eta)_\ell \leq \frac{|\tau|}{2}\,\|K\eta\|_{\ell+\frac{1}{2}}\,\|\eta\|_{\ell-\frac{1}{2}} \leq c\tau\|\eta\|^2_{\ell-\frac{1}{2}}\,.$$

Here we have used the estimate

$$\|K\eta\|_{\ell+1/2} \leq c\|\eta\|_{\ell-1/2}\,,$$

which holds because K is a smoothing operator. The rest of the argument is similar to the one given in the proof of the estimate (16).

3.2 The method of the proof, given in Section 2, allows one to find asymptotics of σ_τ as $\tau \to 0$ provided that w and w_N are smooth.

4 An alternative approach to the derivation of the asymptotics

Let us write the boundary condition (2) as

$$u = \tau u_N, \quad \tau = \frac{1}{\zeta} = \tau_1 + i\tau_2, \quad \tau_1 > 0. \tag{23}$$

Denote by u_τ the unique solution to problems (1), (23) and let

$$\|u\|^2 := \int_{D'} \frac{(|\partial^2 u(x)|^2 + |\partial u(x)|^2 + |u(x)|^2)\,dx}{(1+|x|)^b}\,, \quad b > 1, \tag{24}$$

where ∂ stands for all first-order derivatives, so that $||u||$ is a weighted $H^2(D')$ Sobolev norm. Let us prove that

$$\limsup_{\tau\to 0} \|u_\tau\| \leq c\,, \quad c = \mathrm{const} > 0. \tag{25}$$

If (25) is false, then there is a sequence $\tau_n \to 0$ such that $\|u_{\tau_n}\| \to \infty$. Let $w_n := \frac{u_{\tau_n}}{\|u_{\tau_n}\|}$. Then $\|w_n\| = 1$. Thus

$$w_n \underset{H^2_{\mathrm{loc}}(D')}{\rightharpoonup} w, \quad \|\nabla^2 w_n\| = \left\|\frac{f}{\|u_{\tau_n}\|}\right\| \to 0,$$

where $w_n \underset{H^2_{\text{loc}}(D')}{\rightharpoonup} w$ denotes weak convergence. Therefore,

$$\nabla^2 w = 0 \text{ in } D', \quad w_n \underset{H^2_{\text{loc}}(D')}{\rightharpoonup} w, \quad \left\| \frac{\partial w_n}{\partial N} - \frac{\partial w}{\partial N} \right\|_{L^2(S)} + \|w_n - w\|_{L^2(S)} \underset{n\to\infty}{\longrightarrow} 0,$$

where $N = N_s$ is the unit normal to S at the point $s \in S$, pointing into D', and we have used the embedding theorem for the Sobolev space. This w solves the problem:

$$\Delta w = 0 \text{ in } D', \quad w|_S = 0, \quad w(\infty) = 0. \tag{26}$$

The last relation in (26) can be proved by passing to the limit in the formula

$$w_n(s) = \int_{|x|=R} \left[w_n(s) \frac{\partial g(x,s)}{\partial N_s} - g(x,s) \frac{\partial w_n(s)}{\partial N_s} \right] ds, \tag{27}$$

where $R > 0$ is sufficiently large, so that

$$\operatorname{supp} f \subset B_R = \{x : |x| \le R\}.$$

As $n \to \infty$ in (27), one gets

$$w(x) = \int_{|x|=R} \left[w(s) \frac{\partial g(x,s)}{\partial N_s} - w_{N_s}(s) g(x,s) \right] ds, \quad x \in B'_R = R^3 \backslash B_R. \tag{28}$$

From this formula the last relation in (26) follows immediately.

The only solution to (26) is

$$w = 0. \tag{29}$$

We now derive a contradiction by showing that

$$\|w\| = 1. \tag{30}$$

This contradiction will prove (25).

Since $\|w_n\| = 1$ and $w = 0$, equation (30) holds if

$$\lim_{n\to\infty} \|w_n - w\| = 0, \tag{31}$$

where the norm is defined in (24).

From the formulas (27) and (28) it follows that

$$\lim_{n\to\infty} \|w_n - w\|_{H^2(B'_R, \frac{1}{(1+|x|)^b})} = 0 \tag{32}$$

for R sufficiently large, such that $\operatorname{supp} f \subset B_R$.

In the region $B_R \backslash D$ one has

$$\begin{aligned} w_n(x) \;\; = \;\; & \int_{S_R} \Big[g(x,s)\, \frac{\partial w_n(s)}{\partial N} - w_n(s)\, \frac{\partial g(x,s)}{\partial N} \Big] ds \\ & - \int_S \Big[g(x,s)\, \frac{\partial w_n(s)}{\partial N} - w_n(s)\, \frac{\partial g(x,s)}{\partial N} \Big] ds \\ & + \int_{B_R \backslash D} g(x,y)\, f_n(y) dy, \quad x \in B_R \backslash D, \quad S_R = \{x : |x| = R\}. \end{aligned} \tag{33}$$

Since S and S_R are smooth, the surface integrals converge in $H^2_{\text{loc}}(D')$ to the function

$$\int_{S_R} \Big[g(x,s)\, \frac{\partial w(s)}{\partial N} - w(s)\, \frac{\partial g(x,s)}{\partial N} \Big] ds - \int_{S_R} \Big[g(x,s)\, \frac{\partial w(s)}{\partial N} - w(s)\, \frac{\partial g(x,s)}{\partial N} \Big] ds. \tag{34}$$

This is the function $w(x)$, as follows by Green's formula. The integral over $B_R \backslash D$ in (33) is a fixed function

$$\int_{B_R \backslash D} g(x,y) f(y) dy \in H^2_{\text{loc}}(D')$$

divided by a number $\|w_n\|$, and $\lim_{n\to\infty} \|w_n\| = \infty$. Thus, this integral converges to zero in $H^2\big(B_R \backslash D, \frac{1}{(1+|x|)^b}\big)$. Therefore the relation (32) is verified, and one gets a contradiction

$$1 = \lim_{n\to\infty} \|w_n\| = \|w\| = 0, \tag{35}$$

which proves inequality (25).

If (25) holds, then u_τ converges weakly in the norm (24) to an element v, while $\Delta u_\tau = f$ does not depend on τ, $\|\Delta u_\tau - f\|_{L^2(D')} = 0$. This implies strong convergence of u_τ to v in the norm (24). Passing to the limit in (23) as $\tau \to 0$, one gets for the limit v problem (3). Let us estimate the rate of convergence.

Let $u_\tau - v := z_\tau$. Then

$$\Delta z_\tau = 0 \text{ in } D', \quad z_\tau = \tau(u_{\tau N} - v_N), \quad z_\tau(\infty) = 0. \tag{36}$$

From (36) and either the integral representation for z_τ or from the a priori estimate of the solution to (36), one gets the following estimate:

$$\|z_\tau\|_{L^2(D', \frac{1}{(1+|x|)^b})} \le c\tau = O\big(\frac{1}{|\zeta|}\big). \tag{37}$$

This is an estimate of the type (4), but less precise than the one obtained in Theorem 1.

Appendix

Lemma 1. *Problem* (1) – (2) *has at most one solution if* $\operatorname{Im}\zeta \neq 0$*.*

Proof. Let w solve the homogeneous problems (1) – (2). We want to prove that $w = 0$. Multiply equation (1) (with $f = 0$) by $\overline{w}$, integrate over $D' \cap B_R := D'_R$, $B_R := \{x : |x| \le R\}$, assuming that the origin belongs to D, and then integrate by parts to get:

$$0 = -\int_{D'_R} |\nabla w|^2 \, dx - \zeta \int_S |w|^2 \, ds + \int_{S_R} \overline{w} w_r \, ds, \quad w_r := \frac{\partial w}{\partial r}. \tag{A.1}$$

Taking the imaginary part and using the relations

$$|w| = O\big(\frac{1}{|x|}\big), \quad |w_r| = O\big(\frac{1}{|x|^2}\big) \quad \text{as } |x| \to \infty,$$

one concludes that

$$\int_S |w|^2 \, ds = 0.$$

Thus $w = 0$ on S, and the boundary condition (2) implies $w_N = 0$ on S. Thus, $w = 0$ in D' by the uniqueness of the solution to the Cauchy problem for the Laplace equation. Lemma 1 is proved. □

Lemma 2. *If* $\operatorname{Im}\zeta = 0$ *and* $\zeta \ge 0$*, then problem* (1) – (2) *has at most one solution*

Proof. We argue as before, take the real part of (A.1), and use the relation

$$\lim_{R\to\infty}\int_{S_R} \overline{w}\, w_r\, ds = 0. \tag{A.2}$$

The result is

$$0 = \int_{D'} |\nabla w|^2\, dx + \zeta \int_S |w|^2\, ds. \tag{A.3}$$

If $\zeta = 0$, then (A.3) and the condition $w(\infty) = 0$ imply that $w = 0$ in D'. If $\zeta > 0$, then (A.3) implies $w = 0$ in D'. Lemma 2 is proved. □

Lemma 3. *Let*

$$(\nabla^2 + k^2)u = f \qquad \textit{in } D', \qquad f \in L^2_0(D'),\ k = \text{const} > 0, \tag{A.4}$$

$$u_N - \zeta u = 0 \qquad \textit{on } S, \tag{A.5}$$

$$\frac{\partial u}{\partial r} - iku = o\Big(\frac{1}{r}\Big), \qquad r \to \infty. \tag{A.6}$$

Assume that $\operatorname{Im}\zeta \le 0$. *Then problem* (A.4) – (A.6) *has at most one solution.*

A proof of Lemma 3 can be found in [3] and in [4].

References

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