# Local $H^1$–regularity and $H^{1/3-\delta}$–regularity up to the boundary in time dependent viscoplasticity.

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**Abstract.** Local and boundary regularity for quasistatic initial-boundary value problems from viscoplasticity is studied. The problems considered belong to a general class with monotone constitutive equations modelling materials showing kinematic hardening. A standard example is the Melan-Prager model. It is shown that the strain/stress/internal variable fields have the regularity $H^{\frac{4}{3}-\delta}/H^{\frac{1}{3}-\delta}/H^{\frac{1}{3}-\delta}$ up to the boundary. The proof uses perturbation estimates for monotone operator equations.

**Key words:** regularity, plasticity, viscoplasticity, maximal monotone operator, difference quotient technique, interpolation, model of Melan-Prager.

**AMS 2000 subject classification:** 35B65, 35D10, 74C10, 74D10, 35J25, 34G20, 34G25, 47H04, 47H05

## 1 Introduction

In this article we study interior and boundary regularity of solutions of quasistatic initial-boundary value problems from viscoplasticity. The models we study use constitutive equations with internal variables to describe the deformation behavior of metals at small strain.

We consider constitutive equations of monotone type, a rather broad class of constitutive equations introduced in [2], which generalizes the class of generalized standard materials introduced by Halphen and Nguyen Quoc Son [18]. The class includes the well known models of Prandtl-Reuss, Norton-Hoff and Melan-Prager [23, 25, 32, 34], to mention just a few. Precisely, we only study models of monotone type, for which the associated free energy is a positive definite quadratic form. Such models describe materials showing linear kinematic hardening. This excludes the models of Prandtl-Reuss and Norton-Hoff, but includes the model of Melan-Prager. For a larger number of examples of constitutive equations used in engineering and for details on the monotone type class we refer to [2] and also to [3, 4].

Our results can be briefly summarized as follows: We show for the strain field $u$ under suitable regularity assumptions on the volume force and the

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boundary data that the time derivative $\partial_t u$ belongs to $L^\infty(0,\infty;H^1(\Omega))$ and the space derivatives $\nabla_x u$ to $L^\infty(0,\infty;H^1_{\text{loc}}(\Omega))$. Concerning derivatives at the boundary we prove that the tangential derivatives $\partial_\tau u$ belong to $L^\infty(0,\infty;H^1(\Omega))$, whereas for the normal derivatives we can only show a weaker result. Namely, we show that $\nabla_x u$ belongs to $L^\infty(0,\infty;H^{1/3-\delta}(\Omega))$ for every $\delta > 0$. The stress field $T$ and the vector of internal variables $z$ have the same regularity as the $\nabla_x u$–field.

For the time dependent problem to the Norton-Hoff and Prandtl-Reuss laws it was shown in [8] that the stress field $T$ belongs to $L^\infty(0,\infty;H^1_{\text{loc}}(\Omega))$. In [14] this result is proved again using other methods and under different assumptions on the data. We are not aware of previous investigations of the regularity of the normal derivatives up to the boundary in the time dependent case, and we believe that our results proved in Section 3.3 are the first ones obtained. However, since the completion of this paper D. Knees [21] was able to improve the boundary regularity. Combining methods from our paper with new ones she proved that $\nabla_x u$, $T$ and $z$ belong to the space $L^\infty(0,\infty;H^{1/2-\delta}(\Omega))$. The model is more special than the one considered here and the domain is a cube, but it is intended to generalize the result.

For time independent problems results for the boundary regularity are known. In [33] it is shown for the stationary problem of elasto-plasticity with linear hardening in two space dimensions that the strain and stress fields belong to $H^2(\Omega)$ and $H^1(\Omega)$, respectively. For a stationary power-law model in the full three-dimensional case it is proved in [19, 20] that these fields belong to $H^{3/2-\delta}(\Omega)$ and $H^{1/2-\delta}(\Omega)$, whereas in [28] it is shown for a class of time discrete models, which includes a Cosserat model, that the displacement is in $H^2(\Omega)$ and the stress field in $H^1(\Omega)$. For local regularity results in the time independent case we refer to [7, 9, 12, 37, 38, 39, 40] and to [16] for a survey on other results.

We consider coefficients and constitutive functions, which depend on $x$. Our results thus generalize and extend the local regularity results for constant coefficients in the time dependent case obtained in [29].

In the remainder of this introduction we formulate the quasistatic initial-boundary value problem and state the main results.

Let $\Omega \subseteq \mathbb{R}^3$ be an open bounded set, the set of material points of the solid body. If not otherwise stated we assume that $\Omega$ has $C^1$–boundary. By $T_e$ we denote a positive number (time of existence), which can be chosen arbitrarily large. $\mathcal{S}^n$ denotes the set of symmetric $n \times n$–matrices. Unknown are the displacement $u(x,t) \in \mathbb{R}^3$ of the material point $x$ at time $t$, the Cauchy stress tensor $T(x,t) \in \mathcal{S}^3$ and the vector $z(x,t) \in \mathbb{R}^N$ of internal variables.

The model equations of the problem are

$$
\begin{aligned}
-\mathrm{div}_x T(x,t) &= b(x,t), && (1)\\
T(x,t) &= \mathcal{D}[x]\big(\varepsilon(\nabla_x u(x,t)) - Bz(x,t)\big), && (2)\\
\partial_t z(x,t) &\in g\big(x, -\nabla_z \psi\big(x, \varepsilon(\nabla_x u(x,t)), z(x,t)\big)\big)\\
&= g\big(x, B^T T(x,t) - L[x]\, z(x,t)\big), && (3)
\end{aligned}
$$

which must be satisfied in $\Omega \times [0, T_e)$. The initial condition and Dirichlet boundary condition are

$$
\begin{aligned}
z(x,0) &= z^{(0)}(x), \quad x \in \Omega, && (4)\\
u(x,t) &= \gamma(x,t), \quad (x,t) \in \partial\Omega \times [0, T_e)\,. && (5)
\end{aligned}
$$

Here we use the notation

$$
\mathrm{div}_x T = \big(\sum_{j=1}^{3} \partial_{x_j} T_{ij}\big)_{i=1,2,3}\,.
$$

$\nabla_x u(x,t)$ denotes the $3 \times 3$–matrix of first order partial derivatives. The strain tensor is

$$
\varepsilon(\nabla_x u(x,t)) = \frac{1}{2}\big(\nabla_x u(x,t) + (\nabla_x u(x,t))^T\big) \in \mathcal{S}^3,
$$

with the transposed matrix $(\nabla_x u)^T$. For every $x \in \Omega$, the elasticity tensor $\mathcal{D}[x] : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric mapping, which is positive definite, uniformly with respect to $x$. The linear mapping $B : \mathbb{R}^N \to \mathcal{S}^3$ assigns to the vector $z(x,t)$ the plastic strain tensor $\varepsilon_p(x,t) = Bz(x,t)$. The free energy is

$$
\psi(x, \varepsilon, z) = \frac{1}{2}\big(\mathcal{D}[x](\varepsilon - Bz)\big) \cdot (\varepsilon - Bz) + \frac{1}{2}(L[x]z) \cdot z, \tag{6}
$$

where $L[x]$ denotes a symmetric $N \times N$–matrix, which is positive definite, uniformly with respect to $x \in \Omega$, and where $A \cdot C = \sum_{i,j=1}^{n} a_{ij} c_{ij}$ denotes the scalar product of two $n \times n$–matrices $A$ and $C$. The assumptions for $\mathcal{D}$ and $L$ imply that $\psi$ is a positive definite quadratic form with respect to $(\varepsilon, z)$. Finally, we require that the nonlinear mapping $g : \Omega \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ satisfies

$$
\begin{aligned}
0 &\in g(x, 0), && (7)\\
0 &\le (z_1 - z_2) \cdot (y_1 - y_2), && (8)
\end{aligned}
$$

for all $x \in \Omega,\ z_i \in \mathbb{R}^N,\ y_i \in g(x, z_i),\ i = 1, 2$. This means that $g$ is monotone with respect to $z$. Given are the volume force $b(x,t) \in \mathbb{R}^3$, the boundary data $\gamma(x,t) \in \mathbb{R}^3$ and the initial data $z^{(0)}(x) \in \mathbb{R}^N$.

The equality sign in (3) results from (6) by a short computation, which yields with the transposed mapping $B^T : \mathcal{S}^3 \to \mathbb{R}^N$ that

$$-\nabla_z \psi(x, \varepsilon, z) = B^T T - L[x]\, z.$$

This completes the formulation of the initial-boundary value problem. (2) and (3) are the constitutive equations, which assign the stress $T(x,t)$ to the strain history $s \mapsto \varepsilon(\nabla_x u(x,s))$, $s \leq t$, and which model the viscoelastic material behavior of the solid body. Since we assume that $L[x]$ is positive definite, they belong to the class of monotone type with linear kinematic hardening, which is defined as follows.

**Definition 1.1.** *The class of models of monotone type consists of all constitutive equations, which can be written in the form* (2), (3) *with* $g$ *satisfying* (7), (8), *and with* $L[x]$ *being positive semi-definite for every* $x$. *The subclass of monotone type with linear kinematic hardening is formed by all such constitutive equations, for which* $L[x]$ *is positive definite, uniformly with respect to* $x$.

**Main results.** We use the following notations. For functions $w$ defined on $\Omega \times [0,\infty)$ we denote by $w(t)$ the mapping $x \mapsto w(x,t)$, which is defined on $\Omega$. The space $W^{m,p}(\Omega, \mathbb{R}^k)$ with $p \in [1,\infty]$ consists of all functions in $L^p(\Omega, \mathbb{R}^k)$ with weak derivatives in $L^p(\Omega, \mathbb{R}^k)$ up to order $m$. We set $H^m(\Omega, \mathbb{R}^k) = W^{m,2}(\Omega, \mathbb{R}^k)$. For the space of linear, symmetric mappings from a vector space $V$ to itself we write $\mathcal{LS}(V,V)$.

The basis for our regularity results is the existence theorem for the initial-boundary value problem (1) – (5), which is proved in [3] in the case where the coefficient functions $\mathcal{D}$, $L$ and the constitutive function $g$ are independent of $x$. It is shown in [30] that the proof generalizes immediately to $x$–dependent coefficient and constitutive functions satisfying some natural conditions. In the statement of this general existence theorem given below we use that for fixed $t$ the equations (1), (2) and (5) together form an elliptic boundary value problem, the Dirichlet problem of linear elasticity theory. The data of this problem are $b(t)$, $z(t)$ and $\gamma(t)$. For $(b(t), z(t), \gamma(t)) \in L^2(\Omega) \times L^2(\Omega) \times H^1(\Omega)$ this problem has a unique weak solution $(u(t), T(t)) \in H^1(\Omega) \times L^2(\Omega)$. Since this problem plays an important role throughout our investigations, we discuss it more precisely in Section 2.1. The existence theorem is

**Theorem 1.2** (Existence)**.** *Assume that the coefficient functions satisfy* $L \in L^\infty(\Omega, \mathcal{S}^N)$, $\mathcal{D} \in L^\infty(\Omega, \mathcal{LS}(\mathcal{S}^3, \mathcal{S}^3))$, *and that there is a constant* $c > 0$ *such that*

$$(\zeta, L[x]\, \zeta) \geq c|\zeta|^2, \quad (\sigma, \mathcal{D}[x]\, \sigma) \geq c|\sigma|^2, \quad \text{for all } x \in \Omega,\ \zeta \in \mathbb{R}^N,\ \sigma \in \mathcal{S}^3. \tag{9}$$

*Let the mapping* $g : \Omega \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ *satisfy the following three conditions:*

- $0 \in g(x, 0)$,
- $z \mapsto g(x, z) : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ *is maximal monotone,*
- *the mapping* $x \mapsto j_\lambda(x, z) : \mathbb{R}^3 \to \mathbb{R}^N$ *is measurable for all* $\lambda > 0$, *where* $z \mapsto j_\lambda(\cdot, z)$ *is the inverse of* $z \mapsto z + \lambda g(\cdot, z)$.

*Suppose that* $b \in W^{2,1}(0, T_e; L^2(\Omega, \mathbb{R}^3))$ *and* $\gamma \in W^{2,1}(0, T_e; H^1(\Omega, \mathbb{R}^3))$. *Finally, assume that* $z^{(0)} \in L^2(\Omega, \mathbb{R}^N)$ *and that there exists* $\zeta \in L^2(\Omega, \mathbb{R}^N)$ *such that*

$$\zeta(x) \in g(x, B^T T^{(0)}(x) - Lz^{(0)}(x)), \quad \text{a.e. in } \Omega, \tag{10}$$

*with the weak solution* $(u^{(0)}, T^{(0)}) \in H^1(\Omega) \times L^2(\Omega)$ *of the Dirichlet problem* (1), (2), (5) *of linear elasticity theory to the given data* $b(0)$, $z(0) = z^{(0)}$, $\gamma(0)$.

*Then to every* $T_e > 0$ *there is a unique solution*

$$(u, T) \in W^{1,\infty}(0, T_e; H^1(\Omega, \mathbb{R}^3)) \times W^{1,\infty}(0, T_e; L^2(\Omega, \mathcal{S}^3)), \tag{11}$$

$$z \in W^{1,\infty}(0, T_e; L^2(\Omega, \mathbb{R}^N)) \tag{12}$$

*of the initial-boundary value problem* (1) – (5).

The proof is briefly sketched in Section 2.1, since we need the methods from this proof throughout our investigations. Now we are in a position to state our main results.

**Theorem 1.3** (Interior regularity)**.** *Let all conditions of Theorem 1.2 be satisfied. Assume further that there are constants* $C$, $C_1$, $C_2$ *such that for every* $x \in \Omega$ *and every* $y \in \mathbb{R}^3$ *with* $x+y \in \Omega$, *for every* $z \in \mathbb{R}^N$ *and all* $\lambda > 0$ *the Yosida approximation* $z \mapsto g^\lambda(x, z)$ *of* $z \mapsto g(x, z)$ *and the mappings* $\mathcal{D}$, $L$ *satisfy*

$$\begin{aligned} |g^\lambda(x+y, z) - g^\lambda(x, z)| &\leq C|y||g^\lambda(x, z)|, & (13)\\ \|\mathcal{D}[x+y] - \mathcal{D}[x]\|_{\mathcal{LS}(\mathcal{S}^3, \mathcal{S}^3)} &\leq C_1|y|, & (14)\\ \|L[x+y] - L[x]\|_{\mathcal{S}^N} &\leq C_2|y|. & (15) \end{aligned}$$

*Suppose that* $b \in W^{2,1}(0, T_e; L^2(\Omega, \mathbb{R}^3))$, $\gamma \in W^{2,1}(0, T_e; H^1(\Omega, \mathbb{R}^3))$ *and* $z^{(0)} \in H^1(\Omega, \mathbb{R}^N)$.

*Then in addition to* (11), (12), *the solution of the problem* (1) – (5) *satisfies*

$$(u, T) \in L^\infty\big(0, T_e; H^2_{\text{loc}}(\Omega, \mathbb{R}^3)\big) \times L^\infty\big(0, T_e; H^1_{\text{loc}}(\Omega, \mathcal{S}^3)\big), \tag{16}$$

$$z \in L^\infty\big(0, T_e; H^1_{\text{loc}}(\Omega, \mathbb{R}^N)\big). \tag{17}$$

*Remark* 1.4. For the definition of the Yosida approximation we refer to [10]. If the function $g$ is univalued, then (13) is equivalent to

$$|g(x+y,z) - g(x,z)| \leq C|y||g(x,z)|.$$

This follows directly from the relation $g^\lambda(y,z) = g(y, j_\lambda(y,z))$, which holds in this case. In general we only have $g^\lambda(y,z) \subseteq g(y, j_\lambda(y,z))$.

Of course, (14), (15) mean that $\mathcal{D}$ and $L$ are Lipschitz continuous.

At the boundary the tangential derivatives are as regular as all derivatives in the interior. This is shown by the next theorem.

**Theorem 1.5** (Boundary regularity, tangential derivatives)**.** *Let all conditions of Theorem 1.3 be satisfied. Assume additionally that* $\partial\Omega \in C^2$ *and* $\gamma \in W^{2,1}(0,T_e; H^2(\Omega,\mathbb{R}^3))$.

*Then, for any vector field* $\tau \in C^1(\overline{\Omega},\mathbb{R}^3)$*, which is tangential at the boundary* $\partial\Omega$*, the solution of the problem* (1) – (5) *satisfies*

$$(\partial_\tau u, \partial_\tau T) \in L^\infty(0,T_e; H^1(\Omega,\mathbb{R}^3)) \times L^\infty(0,T_e; L^2(\Omega,\mathcal{S}^3)), \tag{18}$$

$$\partial_\tau z \in L^\infty(0,T_e; L^2(\Omega,\mathbb{R}^N)), \tag{19}$$

*where* $\partial_\tau$ *denotes derivation in the direction of the vector field.*

The next regularity result for normal derivatives at the boundary holds in Besov spaces. The definition of these spaces and a brief review of several basic properties needed in our investigations is given in Appendix A.

**Theorem 1.6** (Boundary regularity, all derivatives)**.** *Under the conditions of Theorem 1.5 the solution of the problem* (1) – (5) *satisfies*

$$(u,T) \in L^\infty(0,T_e; B^{5/4}_{2,\infty}(\Omega,\mathbb{R}^3)) \times L^\infty(0,T_e; B^{1/4}_{2,\infty}(\Omega,\mathcal{S}^3)), \tag{20}$$

$$z \in L^\infty(0,T_e; B^{1/4}_{2,\infty}(\Omega,\mathbb{R}^N)). \tag{21}$$

*For the Sobolev-Slobodeckij spaces* $H^s = B^s_{2,2}$ *we have the better result*

$$(u,T) \in L^\infty(0,T_e; H^{4/3-\delta}(\Omega,\mathbb{R}^3)) \times L^\infty(0,T_e; H^{1/3-\delta}(\Omega,\mathcal{S}^3)), \tag{22}$$

$$z \in L^\infty(0,T_e; H^{1/3-\delta}(\Omega,\mathbb{R}^N)), \tag{23}$$

*for every* $\delta > 0$.

This work is organized as follows. To prove regularity estimates we reduce the initial-boundary value problem (1) – (5) to an evolution equation with a maximal monotone evolution operator and use perturbation estimates for such equations. In Section 2.1 we review this reduction, which was previously used in [3] to show existence of solutions. We also state some results for evolution equations needed in the proof of Theorem 1.3. That proof is carried through in Section 2.2.

Section 3 contains the proofs of Theorems 1.5 and 1.6. In Section 3.1 we discuss the local transformation of the initial-boundary value problem to a domain with flat boundary and the reduction to an evolution equation. Using this evolution equation we verify Theorems 1.5 and 1.6 in Sections 3.2 and 3.3, respectively. For simplicity, we restrict ourselves in the verification to the constant coefficient case. Whereas the proof of Theorem 1.5 runs along the same lines as the proof of Theorem 1.3, we need new ideas to prove Theorem 1.6. We first obtain $H^{1/4}(\Omega)$–regularity for the stress field by using the already proved regularity of tangential derivatives and a special perturbation estimate for the evolution equation. Subsequently we improve the regularity to $H^{1/3-\delta}(\Omega)$ by interpolation and by employing a bootstrap argument.

The appendix contains a definition of Besov spaces used in our investigations and a proof of an estimate for distributions in $H^{-1}$ generated by $L^2$–functions.

For all our regularity results we need that the data $b$ and $\gamma$ have two time derivatives. In the special case when $g(x,z) = \partial_z \chi(x,z)$ with a convex function $z \mapsto \chi(x,z)$, that is for a generalized standard material, this condition can be weakened. In this case it suffices to assume that $b \in W^{1,2}(0,T_e; L^2(\Omega,\mathbb{R}^3))$ and $\gamma \in W^{1,2}(0,T_e; H^2(\Omega,\mathbb{R}^3))$ to conclude that the solution satisfies (22) and (23). The proof is sketched in [5].

# 2 Local regularity

## 2.1 Preliminaries

**Reduction to an evolution equation.** The proof of the existence and uniqueness Theorem 1.2 given in [3] is based on the reduction of the initial-value problem (1) – (5) to an evolution equation with monotone evolution operator. We need this evolution equation to prove the regularity results. In this section we recall this reduction, sketch the proof of Theorem 1.2 and state a few results obtained by specialization of results from the theory of evolution equations to our needs.

In the reduction we need several linear operators. To introduce these operators consider first the linear boundary value problem formed by the equations (1), (2) and (5), which we state here in a new notation: Assume that the functions $\hat{\varepsilon}_p \in L^2(\Omega, \mathcal{S}^3)$, $\hat{b} \in H^{-1}(\Omega,\mathbb{R}^3)$ and $\hat{\gamma} \in H^1(\Omega,\mathbb{R}^3)$ are given. Then the problem

$$-\mathrm{div}_x \tilde{T}(x) = \hat{b}(x), \tag{24}$$

$$\tilde{T}(x) = \mathcal{D}[x]\big(\varepsilon(\nabla_x \tilde{u}(x)) - \hat{\varepsilon}_p(x)\big), \tag{25}$$

$$\tilde{u}(x) = \hat{\gamma}(x), \qquad x \in \partial\Omega, \tag{26}$$

has a unique solution

$$(\tilde{u}, \tilde{T}) \in H^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3),$$

cf. [13, 22, 26, 27, 43]. If $\hat{\varepsilon}_p \in H^1(\Omega, \mathcal{S}^3)$, $\hat{b} \in L^2(\Omega, \mathbb{R}^3)$ and $\hat{\gamma} \in H^2(\Omega, \mathbb{R}^3)$, then the solution is of higher regularity. We have in this case that $(\tilde{u}, \tilde{T}) \in H^2(\Omega, \mathbb{R}^3) \times H^1(\Omega, \mathcal{S}^3)$, cf. [9, 13, 17, 22, 27, 43].

**Definition 2.1.** *Let the linear operator $P : L^2(\Omega, \mathcal{S}^3) \to L^2(\Omega, \mathcal{S}^3)$ be defined by*

$$P\hat{\varepsilon}_p = \varepsilon(\nabla_x \tilde{u}),$$

*where $(\tilde{u}, \tilde{T})$ is the solution of (24) – (26) to $\hat{b} = 0$, $\hat{\gamma} = 0$ and $\hat{\varepsilon}_p \in L^2(\Omega, \mathcal{S}^3)$. With the identity operator $I$ on $L^2(\Omega, \mathcal{S}^3)$ set $Q = I - P$.*

Since we assumed that $\mathcal{D} \in L^\infty(\Omega, \mathcal{LS}(\mathcal{S}^3, \mathcal{S}^3))$, we can associate to the elasticity tensor a bounded mapping from $\mathcal{LS}(L^2(\Omega, \mathcal{S}^3), L^2(\Omega, \mathcal{S}^3))$, again denoted by $\mathcal{D}$, which is given by

$$(\mathcal{D}\xi)(x) = \mathcal{D}[x]\xi(x), \quad \xi \in L^2(\Omega, \mathcal{S}^3), \quad x \in \Omega.$$

Similarly, by the assumption $L \in L^\infty(\Omega, \mathcal{S}^N)$ we can associate to this matrix function the bounded mapping $L \in \mathcal{LS}(L^2(\Omega, \mathbb{R}^N), L^2(\Omega, \mathbb{R}^N))$ given by

$$(L\xi)(x) = L[x]\xi(x), \quad \xi \in L^2(\Omega, \mathbb{R}^N), \quad x \in \Omega.$$

(9) implies that both mappings are positive definite. Therefore we can define a new scalar product on $L^2(\Omega, \mathcal{S}^3)$ by

$$[\xi, \zeta]_\Omega = (\mathcal{D}\xi, \zeta)_\Omega \,,$$

where $(\xi, \zeta)_\Omega$ denotes the standard scalar product of $L^2(\Omega)$. The norm associated to this scalar product is equivalent to the standard norm $\|\cdot\|_{L^2(\Omega)}$. By Lemma 2.2 in [3, p. 113] we have

**Lemma 2.2.** *(i) The operators $P$ and $Q$ are projectors on $L^2(\Omega, \mathcal{S}^3)$, which are orthogonal with respect to the scalar product $[\xi, \zeta]_\Omega$.*
*(ii) The operator $B^T \mathcal{D} Q B : L^2(\Omega, \mathbb{R}^N) \to L^2(\Omega, \mathbb{R}^N)$ is selfajoint and nonnegative with respect to the scalar product $(\xi, \zeta)_\Omega$.*

Now we reduce the initial-boundary value problem (1) – (5) to an evolution equation in a Hilbert space. If $z(t)$ is known, then the component $(u(t), T(t))$ of the solution of this initial-boundary value problem is obtained as unique solution of the boundary value problem (1), (2), (5). Due to the linearity we have

$$(u(t), T(t)) = (\tilde{u}(t), \tilde{T}(t)) + (v(t), \sigma(t)), \tag{27}$$

where $(v(t), \sigma(t))$ is the solution of (24) – (26) to the data $\hat{b} = b(t)$, $\hat{\gamma} = \gamma(t)$, $\hat{\varepsilon}_p = 0$, and $(\tilde{u}(t), \tilde{T}(t))$ is the solution of (24) – (26) to the data $\hat{b} = \hat{\gamma} = 0$, $\hat{\varepsilon}_p = Bz(t)$. By definition of $Q$ we have that $\tilde{T}(t) = -\mathcal{D}QBz(t)$. Insertion of this equation into (3) yields

$$\frac{\partial}{\partial t} z(t) \in G\big(-Mz(t) + B^T\sigma(t)\big), \tag{28}$$

with the mappings $M : L^2(\Omega, \mathbb{R}^N) \to L^2(\Omega, \mathbb{R}^N)$ and $G : L^2(\Omega, \mathbb{R}^N) \to 2^{L^2(\Omega,\mathbb{R}^N)}$ defined by

$$M = B^T\mathcal{D}QB + L, \tag{29}$$

$$G(\xi) = \{\zeta \in L^2(\Omega, \mathbb{R}^N) \mid \zeta(x) \in g(x, \xi(x)) \text{ a.e.}\}. \tag{30}$$

Since $\sigma$ is determined from the boundary value problem (24) – (26) to the data $b, \gamma$, it can be considered to be known. Therefore (28) is a non-autonomous evolution equation for $z$. We transform this equation to an autonomous equation with a maximal monotone evolution operator, since strong existence and perturbation theorems are mainly available for such equations. To this end define a function $d : [0, T_e] \to L^2(\Omega, \mathbb{R}^N)$ by

$$d = -Mz + B^T\sigma. \tag{31}$$

We insert this function into (28) and use the initial condition (4) to obtain the initial boundary value problem

$$\frac{d}{dt} d(t) + A\, d(t) \ni B^T\sigma_t(t), \tag{32}$$

$$d(0) = -Mz^{(0)} + B^T\sigma(0), \tag{33}$$

for $d$, where the operator $A$ is given by

$$A = MG.$$

The relation between $z$ and $d$ given in (31) is one-to-one, and the evolution equation (32) is equivalent to the equation (28). For the proof note that since $L \in \mathcal{LS}(L^2(\Omega, \mathbb{R}^N), L^2(\Omega, \mathbb{R}^N))$ is positive definite, we obtain as an immediate consequence of Lemma 2.2 (ii) that the mapping $M$ defined in (29) satisfies

$$M \in \mathcal{LS}(L^2(\Omega, \mathbb{R}^N), L^2(\Omega, \mathbb{R}^N))$$

and is positive definite. Therefore it has a selfadjoint, bounded inverse. Since we consider $\sigma$ to be known, (31) is a one-to-one relation between $z$ and $d$.

Moreover, these properties of $M$ allow to define a new scalar product on $L^2(\Omega, \mathbb{R}^N)$ and the associated norm by

$$[\![\xi, \zeta]\!]_\Omega = (M^{-1}\xi, \zeta)_\Omega\,, \quad |\xi|_{L^2(\Omega)} = [\![\xi, \xi]\!]_\Omega^{1/2}.$$

The associated norm is equivalent to the norm $\|\xi\|_{L^2(\Omega)}$.

**Theorem 2.3.** *For the operators* $G, A : L^2(\Omega, \mathbb{R}^N) \mapsto 2^{L^2(\Omega,\mathbb{R}^N)}$ *we have*
*(i)* $G$ *is maximal monotone with respect to the scalar product* $(\xi, \zeta)_\Omega$.
*(ii)* $A$ *is maximal monotone with respect to the scalar product* $[\![\xi, \zeta]\!]_\Omega$.

This theorem is proved in [3, pp. 116,117] in the case where the elasticity tensor $\mathcal{D}$, the matrix $L$ and the function $g$ are independent of $x$. The proof is extended in [30] to the $x$–dependent case.

**Lemma 2.4.** *If* $b$, $\gamma$ *and* $z^{(0)}$ *satisfy the assumptions of Theorem 1.2, then* $d^{(0)} = -Mz^{(0)} + B^T\sigma(0)$ *belongs to the domain of definition* $D(G) = D(A)$ *of* $G$ *and* $A$.

**Proof:** By (27) and the definition of $Q$ we have

$$\begin{aligned} d^{(0)} = -Mz^{(0)} + B^T\sigma(0) &= B^T\big(-\mathcal{D}QBz^{(0)} + \sigma(0)\big) - Lz^{(0)} \\ &= B^T T(0) - Lz^{(0)} = B^T T^{(0)} - Lz^{(0)}, \end{aligned}$$

with the function $T^{(0)}$ introduced in Theorem 1.2. The lemma follows from this equation, since (10) implies that the right hand side belongs to $D(G)$. The proof is complete □

The proof of Theorem 1.2 given in [3] and in [30] is based on Theorem 2.3 and Lemma 2.4. This theorem and the lemma show that (32), (33) is an initial value problem to an autonomous evolution equation with a maximal monotone evolution operator and initial data in the domain of definition of the evolution operator. By the standard theory of such initial value problems this implies that a unique strong solution $d$ exists, cf. [6, 10, 35, 36]. Since the relation (31) is invertible, we conclude that a unique solution $z$ of (28) exists. To construct the components $(u(t), T(t))$ of the solution of (1) – (5) we solve (24) – (26) with $\hat{b} = b(t)$, $\hat{\varepsilon}_p = Bz(t)$ and $\hat{\gamma} = \gamma(t)$ inserted. For details of the proof we refer to the cited references.

**Bounds for the Yosida approximation.** In our investigations we need the Yosida approximation of the operator $G$.

**Lemma 2.5.** *For* $\lambda > 0$ *the Yosida approximation of* $G$ *is given by*

$$G^\lambda(\xi) = \{\zeta \in L^2(\Omega, \mathbb{R}^N) \mid \zeta(x) \in g^\lambda(x, \xi(x)) \ a.e.\}.$$

**Proof:** By definition we have $G^\lambda = \frac{1}{\lambda}\big(I - (I + \lambda G)^{-1}\big)$. To prove the lemma we must therefore show that for all $\xi \in L^2(\Omega, \mathbb{R}^N)$ and for almost all $x \in \Omega$ we have

$$\frac{1}{\lambda}\Big(\xi(x) - \big((I + \lambda G)^{-1}\xi\big)(x)\Big) = g^\lambda(x, \xi(x)). \tag{34}$$

Let $j_\lambda(x, \cdot)$ denote the inverse of $z \mapsto z + \lambda g(x, z) : \mathbb{R}^N \to \mathbb{R}^N$. From the definition of $G$ it is immediately seen that for $\xi \in L^2(\Omega, \mathbb{R}^N)$ we have

$$\big((I + \lambda G)^{-1}\xi\big)(x) = j_\lambda(x, \xi(x)),$$

for almost all $x \in \Omega$. Thus

$$\frac{1}{\lambda}\Big(\xi(x) - \big((I+\lambda G)^{-1}\xi\big)(x)\Big) = \frac{1}{\lambda}\Big(\xi(x) - j_\lambda(x,\xi(x))\Big)$$
$$= \frac{1}{\lambda}\Big(I_{\mathbb{R}^N} - j_\lambda(x,\cdot)\Big)\xi(x) = g^\lambda(x,\xi(x)).$$

This is (34). □

Since by Theorem 2.3 the operator $G$ is maximal monotone, it follows that also the single valued operator $G^\lambda$ is maximal monotone, cf. [10, p. 28]. We define the operator $A^\lambda$ on $L^2(\Omega,\mathbb{R}^N)$ by

$$A^\lambda = MG^\lambda,$$

with $M$ defined in (29). $A^\lambda$ is maximal monotone with respect to the scalar product $[\![\xi,\zeta]\!]_\Omega$. This is seen from Theorem 2.3, since the proof applies with $G$ and $A$ replaced by $G^\lambda$ and $A^\lambda$. If we insert the evolution operator $A^\lambda$ in (32) for $A$, we obtain the evolution equation

$$\frac{d}{dt}d^\lambda(t) + A^\lambda d^\lambda(t) = B^T\sigma_t(t), \tag{35}$$

in $L^2(\Omega,\mathbb{R}^N)$, which we use in the proof of Proposition 2.14. To estimate the solution we need the following

**Lemma 2.6.** *For $\lambda > 0$, $\sigma \in W^{2,1}(0,T_e;L^2(\Omega))$ and $d^{(0)} \in L^2(\Omega,\mathbb{R}^N)$ let $d, d^\lambda : [0,T_e) \to L^2(\Omega,\mathbb{R}^N)$ be the solutions of the evolution equations* (32) *and* (35)*, repectively, to the initial condition*

$$d(0) = d^\lambda(0) = d^{(0)}. \tag{36}$$

*Then, for all $0 \le t \le T_e$,*

$$|MG^\lambda(d^\lambda(t))|_{L^2(\Omega)} + \sqrt{\frac{2}{\lambda}}\,|d^\lambda(t) - d(t)|_{L^2(\Omega)}$$
$$\le 2\Big(|MG^\lambda(d^{(0)}) + B^T\sigma_t(0)|_{L^2(\Omega)} + \int_0^t |B^T\sigma_{tt}(s)|_{L^2(\Omega)}ds\Big) + |B^T\sigma_t(t)|_{L^2(\Omega)}\,. \tag{37}$$

**Proof.** This is Theorem 4.1 in [30], where a proof can be found. In fact, inequality (37) is nothing but a specialization and combination of two well known results from the theory of evolution equations to monotone operators: The estimate for $MG^\lambda(d^\lambda(t))$ is obtained by specialization of [6, Theorem 2.2, p. 131] to (35), and the estimate for $d^\lambda(t) - d(t)$ follows by a slight modification of [36, Theorem IV.4.1]. □

Since the operator $M$ is symmetric and positive definite, it has a bounded inverse. From the equivalence of the norms $\|\xi\|_{L^2(\Omega)}$ and $|\xi|_{L^2(\Omega)}$ it thus follows that there is $c > 0$ such that $|MG(\xi)|_{L^2(\Omega)} \ge c\|G(\xi)\|_{L^2(\Omega)}$. Therefore

(37) can be used to bound $\|G^\lambda(d^\lambda(t))\|_{L^2(\Omega)}$. To bound this term independently of $\lambda$, we must estimate $\|G^\lambda(d^{(0)})\|_{L^2(\Omega)}$ by a constant independent of $\lambda$. To this end define the operator $G^0 : D(G) \to L^2(\Omega, \mathbb{R}^N)$ as follows: Since $G$ is maximal monotone, to every $z \in D(G)$ there is a unique element $\zeta \in G(z)$ such that $\|\zeta\|_{L^2(\Omega)} = \min\{\|\xi\|_{L^2(\Omega)} \mid \xi \in G(z)\}$. Now set $G^0(z) = \zeta$. The following is a standard result of the theory of monotone operators, cf. [10, p. 28]:

**Lemma 2.7.** *$z \in L^2(\Omega, \mathbb{R}^N)$ belongs to the domain of definition $D(G)$ if and only if $\limsup_{\lambda\to 0} \|G^\lambda(z)\|_{L^2(\Omega)} < \infty$. If $z \in D(G)$ then*

$$\lim_{\lambda \searrow 0} \|G^\lambda(z)\|_{L^2(\Omega)} \nearrow \|G^0(z)\|_{L^2(\Omega)} . \tag{38}$$

## 2.2 Proof of Theorem 1.3

Let $V \subset\subset \Omega$ be any fixed open subset of $\Omega$. Select a cut-off function $\varphi \in C_0^\infty(\Omega, \mathbb{R})$ with

$$\varphi \equiv 1 \text{ on } V, \quad 0 \le \varphi \le 1.$$

Let $h \in \mathbb{R}^3$ be a vector in the direction of the $i$–th coordinate axis. Thus, $h = \hat{h} e_i$ with $\hat{h} \ge 0$ and with the $i$–th unit vector $e_i \in \mathbb{R}^3$. If we choose $\hat{h}_0 > 0$ sufficiently small, then for all $0 \le \hat{h} \le \hat{h}_0$ the mapping $\phi_h : \Omega \to \Omega$ defined by

$$\phi_h(x) = x + h\varphi(x) \tag{39}$$

is invertible for all $0 \le \hat{h} \le \hat{h}_0$. With the solution $(u, T, z)$ of the initial-boundary value problem (1) – (5) given by Theorem 1.2 we set

$$(u_h, T_h, z_h)(x, t) = (u(\phi_h(x), t), T(\phi_h(x), t), z(\phi_h(x), t)). \tag{40}$$

Our goal is to show that there is a constant $C$ such that for all $0 < \hat{h} < \hat{h}_0$ and $0 \le t \le T_e$ the estimate

$$\|z_h(t) - z(t)\|_{L^2(\Omega)} \le C\hat{h}$$

holds. At the end of this section we show that the statement of Theorem 1.3 follows readily from this inequality. The idea of proof of this inequality is to derive an evolution equation for $z_h$ similar to the evolution equation (32) satisfied by $z$, and to use perturbation estimates for such evolution equations to estimate the difference $z - z_h$ of the solutions.

**The shifted initial-boundary value problem.** To carry out this program let

$$b_h(x, t) = b(\phi_h(x), t), \quad z_h^{(0)}(x) = z^{(0)}(\phi_h(x)).$$

The chain rule yields that $(u_h, T_h, z_h)$ solves the initial-boundary value problem

$$
\begin{aligned}
-\mathrm{div}_x T_h(x,t) &= b_h(x,t) - \hat{h}\big(\partial_{\xi_i} T(\xi,t)\nabla_x\varphi(x)\big)_{\xi=\phi_h(x)}\,, && (41)\\
T_h(x,t) &= \mathcal{D}[\phi_h(x)]\Big(\varepsilon\big(\nabla_x u_h(x,t)\big) - Bz_h(x,t) \\
&\qquad - \hat{h}\;\varepsilon\big(u_{\xi_i}(\xi,t)\otimes\nabla_x\varphi(x)\big)_{\xi=\phi_h(x)}\Big), && (42)\\
\partial_t z_h(x,t) &\in g\Big(\phi_h(x), B^T T_h(x,t) - L[\phi_h(x)]z_h(x,t)\Big), && (43)\\
z_h(x,0) &= z_h^{(0)}(x), && (44)\\
u_h(x,t) &= \gamma(x,t), \qquad x\in\partial\Omega, && (45)
\end{aligned}
$$

where $(a\otimes b)_{ij} = a_i b_j$ is the usual tensor product of two vectors. Note that since we do not yet know whether $\partial_{\xi_i} T$ exists in $L^2$, we must consider the term $\big(\partial_{\xi_i} T(\xi,t)\nabla_x\varphi(x)\big)_{\xi=\phi_h(x)}$ to be a distribution. This distribution, which we denote by $(\partial_{\xi_i} T)_h\nabla_x\varphi$ , and the time derivative $\partial_t(\partial_{\xi_i} T)_h\nabla_x\varphi$ of this distribution are defined by

$$
\begin{aligned}
&\langle(\partial_{\xi_i} T)_h\nabla_x\varphi,\chi\rangle \\
&\quad = -\int_{\mathbb{R}^4} T(y,t)\cdot\partial_{y_i}\Big((\chi\otimes\nabla_x\varphi)(\phi_h^{-1}(y),t)\frac{1}{1+\hat{h}\partial_{x_i}\varphi(\phi_h^{-1}(y))}\Big)d(y,t),\\
&\langle\partial_t(\partial_{\xi_i} T)_h\nabla_x\varphi,\chi\rangle \\
&\quad = -\int_{\mathbb{R}^4} \partial_t T(y,t)\cdot\partial_{y_i}\Big((\chi\otimes\nabla_x\varphi)(\phi_h^{-1}(y),t)\frac{1}{1+\hat{h}\partial_{x_i}\varphi(\phi_h^{-1}(y))}\Big)d(y,t).
\end{aligned}
$$

By (11) we have $T, \partial_t T \in L^\infty(0,T_e;L^2(\Omega,\mathcal{S}))$. Therefore in both formulas the test function $\chi$ can be chosen from the space $L^1(0,T_e;H_0^1(\Omega,\mathbb{R}^3))$, which implies that $(\partial_{\xi_i} T)_h\nabla_x\varphi$ and $\partial_t(\partial_{\xi_i} T)_h\nabla_x\varphi$ both belong to the space $L^\infty(0,T_e;H^{-1}(\Omega))$, and an obvious computation yields that there is a constant $C$ such that for all $0\le\hat{h}\le\hat{h}_0$

$$
\begin{aligned}
\|(\partial_{\xi_i} T)_h\nabla_x\varphi\|_{L^\infty(0,T_e;H^{-1}(\Omega))} &\le C, && (46)\\
\|\partial_t(\partial_{\xi_i} T)_h\nabla_x\varphi\|_{L^\infty(0,T_e;H^{-1}(\Omega))} &\le C. && (47)
\end{aligned}
$$

By (11) we also have that $u, \partial_t u\in L^\infty(0,T_e;H^1(\Omega,\mathbb{R}^3))$, which by a similar computation yields that there is a constant $C$ such that for all $0\le\hat{h}\le\hat{h}_0$ the function $\varepsilon\big(u_{\xi_i}(\xi,t)\otimes\nabla_x\varphi(x)\big)_{\xi=\phi_h(x)}$ , which we denote by $\varepsilon\big((u_{\xi_i})_h\otimes\nabla_x\varphi\big)$, satisfies

$$
\begin{aligned}
\|\varepsilon\big((u_{\xi_i})_h\otimes\nabla_x\varphi\big)\|_{L^\infty(0,T_e;L^2(\Omega))} &\le C, && (48)\\
\|\partial_t\varepsilon\big((u_{\xi_i})_h\otimes\nabla_x\varphi\big)\|_{L^\infty(0,T_e;L^2(\Omega))} &\le C. && (49)
\end{aligned}
$$

Finally, by the assumptions of Theorem 1.3 we have that the volume force and initial data satisfy $b\in W^{2,1}(0,T_e;L^2(\Omega,\mathbb{R}^3))$, $z^{(0)}\in H^1(\Omega,\mathbb{R}^N)$. From this and from the definition of $b_h$ and $z_h^{(0)}$ we obtain by a straightforward computation that $b_h \in W^{2,1}(0,T_e;L^2(\Omega,\mathbb{R}^3))$, $z_h^{(0)} \in H^1(\Omega,\mathbb{R}^N)$. Moreover, we obtain that there is a constant $C$ such that for all $0 \le \hat{h} \le \hat{h}_0$

$$\|b - b_h\|_{W^{2,1}(0,T_e;H^{-1}(\Omega))} \le \hat{h}C, \tag{50}$$

$$\|z^{(0)} - z_h^{(0)}\|_{L^2(\Omega)} \le \hat{h}C. \tag{51}$$

Whereas (51) is obvious, we prove the inequality (50) in Appendix B.

**Reduction of the shifted problem to an evolution equation.** If we consider the terms $(\partial_{\xi_i}T)_h$ and $\varepsilon\big((u_{\xi_i})_h\big)$ on the right hand sides of (41), (42) to be known, then these terms together with $b_h$, $z_h^{(0)}$ and $\gamma$ are the data of the linear elliptic boundary value problem for $(u_h,T_h)$ formed by the equations (41), (42) and (45). To study this problem, we state it with new notations: For given $\hat{b} \in H^{-1}(\Omega)$, $\hat{\varepsilon}_p \in L^2(\Omega,\mathcal{S}^3)$ and $\hat{\gamma} \in H^1(\Omega)$ consider the problem

$$-\mathrm{div}_x \tilde{T}_h(x) = \hat{b}(x), \tag{52}$$

$$\tilde{T}_h(x) = \mathcal{D}[\phi_h(x)](\varepsilon(\nabla_x \tilde{u}_h(x)) - \hat{\varepsilon}_p(x)), \tag{53}$$

$$\tilde{u}_h(x) = \hat{\gamma}(x), \quad x \in \partial\Omega. \tag{54}$$

The solution satisfies

$$(\tilde{u}_h, \tilde{T}_h) \in H^1(\Omega,\mathbb{R}^3) \times L^2(\Omega,\mathcal{S}^3).$$

We use this boundary value problem to define projection operators analogous to the operators $P$ and $Q$ from Definition 2.1.

**Definition 2.8.** *Let $(\tilde{u}_h,\tilde{T}_h)$ be the solution of (52) – (54) to $\hat{b}=0$, $\hat{\gamma}=0$ and $\hat{\varepsilon}_p \in L^2(\Omega,\mathcal{S}^3)$. We define a linear operator $P_h : L^2(\Omega,\mathcal{S}^3) \to L^2(\Omega,\mathcal{S}^3)$ by*

$$P_h\hat{\varepsilon}_p = \varepsilon(\nabla_x \tilde{u}_h).$$

*Furthermore, we define the linear operator $Q_h = I - P_h$ with the identity operator $I$ on $L^2(\Omega,\mathcal{S}^3)$.*

We follow the procedure from Section 2.1 and define selfadjoint operators $\mathcal{D}_h \in \mathcal{LS}(L^2(\Omega,\mathcal{S}^3),L^2(\Omega,\mathcal{S}^3))$, $L_h \in \mathcal{LS}(L^2(\Omega,\mathbb{R}^N),L^2(\Omega,\mathbb{R}^N))$ by

$$(\mathcal{D}_h\xi)(x) = \mathcal{D}[\phi_h(x)]\xi(x), \qquad \xi \in L^2(\Omega,\mathcal{S}^3), \quad x \in \Omega.$$
$$(L_h\xi)(x) = L[\phi_h(x)]\xi(x), \qquad \xi \in L^2(\Omega,\mathbb{R}^N), \quad x \in \Omega.$$

Both operators are uniformly bounded and uniformly positive definite with respect to $h$.

**Lemma 2.9.** *(i) The operators $P_h$ and $Q_h$ are orthogonal projectors with respect to the scalar product $[\xi,\zeta]_{\Omega,h} = (\mathcal{D}_h\xi,\zeta)_\Omega$ on $L^2(\Omega,\mathcal{S}^3)$.*
*(ii) The operator $B^T\mathcal{D}_hQ_hB : L^2(\Omega,\mathbb{R}^N) \to L^2(\Omega,\mathbb{R}^N)$ is selfajoint, uniformly bounded with respect to $h$ and non-negative with respect to the scalar product $(\xi,\zeta)_\Omega$.*

To reduce (41) – (43) to an evolution equation we consider the boundary value problem (52) – (54) with the data $\hat{h}\,(\partial_{\xi_i}T)_h\nabla_x\varphi$, $\hat{h}\,\varepsilon\big((u_{\xi_i})_h\otimes\nabla_x\varphi\big)$, $b_h$, $\gamma$ from (41), (42), (45) inserted. Specifically, we study the equations

$$
\begin{aligned}
-\mathrm{div}_x\,\sigma_h &= b_h - \hat{h}\,(\partial_{\xi_i}T)_h\nabla_x\varphi\,, && (55)\\
\sigma_h &= \mathcal{D}_h\big(\varepsilon(\nabla_x v_h) + \hat{h}\,\varepsilon\big((u_{\xi_i})_h\otimes\nabla_x\varphi\big)\big), && (56)\\
v_h &= \gamma, \quad (x,t)\in\partial\Omega\times(0,T_e). && (57)
\end{aligned}
$$

This is a boundary value problem with respect to $x$ depending on $t$ as a parameter. Since the coefficient functions $\mathcal{D}_h$ and $\nabla_x\varphi$ are independent of $t$, since the terms $(\partial_{\xi_i}T)_h$ and $\varepsilon\big((u_{\xi_i})_h\big)$ have one time derivative and $b_h$ and $\gamma$ have two, we can differentiate all equations of this problem once with respect to time. In this way we get a boundary value problem for $(\partial_t v_h,\partial_t\sigma_h)$, which has the form of (55) – (57) with $\hat{h}\,(\partial_{\xi_i}T)_h\nabla_x\varphi$, $\hat{h}\,\varepsilon\big((u_{\xi_i})_h\otimes\nabla_x\varphi\big)$, $b_h$, $\gamma$ replaced by their first time derivatives. Noting the estimates (46) – (50) and the assumption $\gamma\in W^{2,1}(0,T_e;H^1(\Omega))$, we conclude from these two boundary value problems for $(v_h,\sigma_h)$ and $(\partial_t v_h,\partial_t\sigma_h)$ by standard elliptic theory that $(v_h,\sigma_h)\in W^{1,\infty}(0,T_e;H^1(\Omega)\times L^2(\Omega))$.

With this solution $(v_h,\sigma_h)$ the solution $(u_h,T_h)$ of (41), (42), (45) can be decomposed in the form

$$
(u_h,T_h) = (\tilde{u}_h,\tilde{T}_h) + (v_h,\sigma_h), \tag{58}
$$

where $(\tilde{u}_h,\tilde{T}_h)$ is the solution of the boundary value problem (52) – (54) to the data $\hat{b}=0$, $\hat{\varepsilon}_p = Bz_h$, $\hat{\gamma}=0$. By definition of $Q_h$ we have $\tilde{T}_h = -\mathcal{D}_hQ_h\hat{\varepsilon}_p$, which implies $T_h = -\mathcal{D}_hQ_hBz_h + \sigma_h$ . Insertion of this equation into (43) yields the evolution equation

$$
\frac{\partial}{\partial t}z_h(t) \in G_h\big(-M_hz_h(t) + B^T\sigma_h(t)\big), \tag{59}
$$

for the function $t\mapsto z_h(t) : [0,T_e]\to L^2(\Omega,\mathbb{R}^N)$, with the mappings $M_h : L^2(\Omega,\mathbb{R}^N)\to L^2(\Omega,\mathbb{R}^N)$ and $G_h : L^2(\Omega,\mathbb{R}^N)\to 2^{L^2(\Omega,\mathbb{R}^N)}$ defined by

$$
\begin{aligned}
M_h &= B^T\mathcal{D}_hQ_hB + L_h\,, && (60)\\
G_h(\xi) &= \{\zeta\in L^2(\Omega,\mathbb{R}^N) \mid \zeta(x)\in g(\phi_h(x),\xi(x))\ \text{a.e.}\}. && (61)
\end{aligned}
$$

As in Section 2.1 we transform this non-autonomous evolution equation to an autonomous equation by inserting the function

$$
d_h = -M_hz_h + B^T\sigma_h \tag{62}
$$

into (59). Noting the initial condition (44), we obtain the initial value problem

$$\partial_t d_h(t) + A_h d_h(t) \ni B^T \partial_t \sigma_h(t)\,, \tag{63}$$

$$d_h(0) = -M_h z_h^{(0)} + B^T \sigma_h(0), \tag{64}$$

for $d_h : [0, T_e] \to L^2(\Omega, \mathbb{R}^N)$, where $A_h = M_h G_h$. Since $L_h$ is selfadjoint and positive definite, we conclude from Lemma 2.9 (ii) that $M_h$ defined in (60) belongs to the space $\mathcal{LS}(L^2(\Omega, \mathbb{R}^N), L^2(\Omega, \mathbb{R}^N))$ and is positive definite. Therefore this operator has a selfadjoint, positive definite inverse. Consequently, (62) is a one-to-one relation between $z_h$ and $d_h$. Moreover, we can define the scalar product and associated norm

$$[\![\xi, \zeta]\!]_{h,\Omega} = (M_h^{-1}\xi, \zeta)_\Omega, \quad |\xi|_{h,L^2(\Omega)} = [\![\xi, \xi]\!]_{h,\Omega}^{1/2}.$$

Since $M_h$ is uniformly bounded and uniformly positive definite with respect to $h$, it follows that there are constants $c_1, c_2 > 0$ such that

$$c_1 \|\xi\|_{L^2(\Omega)} \geq |\xi|_{h,L^2(\Omega)} \geq c_2 \|\xi\|_{L^2(\Omega)} \tag{65}$$

holds for all $\xi \in L^2(\Omega)$ and all $0 \leq \hat{h} \leq \hat{h}_0$. The operators $G_h$ and $A_h$ differ from $G$ and $A$ only by the shifting of the $x$-variable. The proof of Theorem 2.3 also holds for the shifted operators. We thus have

**Corollary 2.10.** *The operator $G_h : L^2(\Omega, \mathbb{R}^N) \to 2^{L^2(\Omega, \mathbb{R}^N)}$ is maximal monotone with respect to the scalar product $(\xi, \zeta)_\Omega$, the operator $A_h = M_h G_h : L^2(\Omega, \mathbb{R}^N) \to 2^{L^2(\Omega, \mathbb{R}^N)}$ is maximal monotone with respect to the scalar product $[\![\xi, \zeta]\!]_{h,\Omega}$.*

Let $G_h^\lambda$ be the Yosida approximations of $G_h$. The Yosida approximation is maximal monotone, and as in the proof of Lemma 2.5 we obtain that

$$G_h^\lambda(\xi) = \{\zeta \in L^2(\Omega, \mathbb{R}^N) \mid \zeta(x) \in g^\lambda(\phi_h(x), \xi(x)) \text{ a.e.}\}.$$

The assumption (13) yields for $z \in L^2(\Omega, \mathbb{R}^N)$ and $x \in \Omega$ that

$$\begin{aligned} &|g^\lambda(x, z(x)) - g^\lambda(\phi_h(x), z(x))| \\ &\qquad \leq C|\phi_h(x) - x|\,|g^\lambda(x, z(x))| \leq C\hat{h}\,|g^\lambda(x, z(x))|, \end{aligned}$$

since $|\phi_h(x) - x| = |h\varphi(x)| \leq |h| = \hat{h}$. Thus, for all $z \in L^2(\Omega, \mathbb{R}^N)$

$$\|G_h^\lambda(z) - G^\lambda(z)\|_{L^2(\Omega)} \leq C\hat{h}\|G^\lambda(z)\|_{L^2(\Omega)}\,. \tag{66}$$

Obviously, this estimate implies that if $\lim_{\lambda\to 0} \|G^\lambda(z)\|_{L^2(\Omega)} < \infty$, then also $\lim_{\lambda\to 0} \|G_h^\lambda(z)\|_{L^2(\Omega)} < \infty$. Invoking Lemma 2.7 two times, applied first to $G$ and then to $G_h^\lambda$, we obtain

$$D(A) = D(G) \subseteq D(G_h) = D(A_h). \tag{67}$$

**Lemma 2.11.** *If $b$, $\gamma$ and $z^{(0)}$ satisfy the assumptions of Theorem 1.2, then $d^{(0)} = -Mz^{(0)} + B^T\sigma(0)$ and $d_h^{(0)} = -M_h z_h^{(0)} + B^T\sigma_h(0)$ with $\sigma$, $\sigma_h$ defined in* (27) *and* (58)*, respectively, both belong to the domain of definition $D(G_h) = D(A_h)$ of $G_h$ and $A_h$.*

**Proof:** It follows immediately from Lemma 2.4 and (67) that $d^{(0)} \in D(A_h) = D(G_h)$. By (58) and the definition of $Q_h$ we have

$$\begin{aligned} d_h^{(0)} &= -M_h z_h^{(0)} + B^T\sigma_h(0) \\ &\qquad = B^T\big(-\mathcal{D}_h Q_h B z_h^{(0)} + \sigma_h(0)\big) - L_h z_h^{(0)} = B^T T_h(0) - L_h z_h^{(0)}. \end{aligned}$$

The function $T^{(0)}$ from assumption (10) satisfies $T_h(x,0) = T(\phi_h(x),0) = T^{(0)}(\phi_h(x))$. Because $(L_h z_h^{(0)})(x) = L[\phi_h(x)]z^{(0)}(\phi_h(x))$, by definition of $L_h$ and $z_h$, we therefore obtain for the function $\zeta$ from (10) that

$$\zeta(\phi_h(x)) \in g\big(\phi_h(x), T_h(x,0) - L_h z_h^{(0)}(x)\big) = g(\phi_h(x), d_h^{(0)}(x)), \quad \text{a.e. in } \Omega.$$

It is readily seen that $\zeta \in L^2(\Omega, \mathbb{R}^N)$ implies $\zeta \circ \phi_h \in L^2(\Omega, \mathbb{R}^N)$. Remembering (61), we together conclude that $d_h^{(0)} \in D(G_h) = D(A_h)$. $\square$

**Perturbation estimate for the elliptic problem.** The initial value problem (63), (64) for $d_h$ is the analogous problem to (32), (33) for $d$. We use these initial value problems to estimate the difference $d(t) - d_h(t)$. To this end we first prove perturbation estimates for the linear elliptic problems considered above.

**Lemma 2.12.** *(i) Let $(\tilde{u}, \tilde{T})$ denote the solution of the problem* (24) – (26) *to the data $\hat{b} \in H^{-1}(\Omega)$, $\hat{\varepsilon}_p \in L^2(\Omega)$, $\hat{\gamma} \in H^1(\Omega)$, and let $(\tilde{u}_h, \tilde{T}_h)$ be the solution of the problem* (52) – (54) *to the data $\hat{b}_1 \in H^{-1}(\Omega)$, $\hat{\varepsilon}_{p,1} \in L^2(\Omega)$, $\hat{\gamma}_1 \in H^1(\Omega)$. Then there are constants $C_1$ $C_2$, which are independent of $\hat{b}, \hat{\varepsilon}_p, \hat{\gamma}, \hat{b}_1, \hat{\varepsilon}_{p,1}, \hat{\gamma}_1$, such that*

$$\begin{aligned} &\|(\tilde{u}, \tilde{T}) - (\tilde{u}_h, \tilde{T}_h)\|_{H^1(\Omega)\times L^2(\Omega)} \\ &\quad \le \ \hat{h} C_1 \|(\hat{b}, \hat{\varepsilon}_p, \hat{\gamma})\|_{H^{-1}(\Omega)\times L^2(\Omega)\times H^1(\Omega)} \\ &\qquad + C_2 \|(\hat{b} - \hat{b}_1, \hat{\varepsilon} - \hat{\varepsilon}_{p,1}, \hat{\gamma} - \hat{\gamma}_1)\|_{H^{-1}(\Omega)\times L^2(\Omega)\times H^1(\Omega)} . \end{aligned} \tag{68}$$

*(ii) Let $P$, $P_h$, $Q$, $Q_h$ be the projection operators introduced in Definitions 2.1 and 2.8, respectively, and let $M$, $M_h$ be the operators defined in* (29) *and* (60)*, respectively. Then there exists a constant $C$ such that*

$$\|P - P_h\|_{\mathcal{L}} \le C\hat{h}, \quad \|Q - Q_h\|_{\mathcal{L}} \le C\hat{h}, \quad \|M - M_h\|_{\mathcal{L}} \le C\hat{h}, \tag{69}$$

*for all $0 \le \hat{h} \le \hat{h}_0$, where $\|\cdot\|_{\mathcal{L}}$ denotes the operator norm.*

**Proof.** We first assume that $\hat{b} = \hat{b}_1, \hat{\varepsilon}_p = \hat{\varepsilon}_{p,1}, \hat{\gamma} = \hat{\gamma}_1$. Then the function $(\overline{u}, \overline{T}) = (\tilde{u}, \tilde{T}) - (\tilde{u}_h, \tilde{T}_h)$ solves

$$\begin{aligned} -\mathrm{div}_x \overline{T}(x) &= 0, \\ \overline{T}(x) &= \mathcal{D}[x]\varepsilon(\nabla_x \overline{u}(x)) - (\mathcal{D}_h[x] - \mathcal{D}[x])(\varepsilon(\nabla_x \tilde{u}_h(x)) - \hat{\varepsilon}_p(x)), \\ \overline{u}(x) &= 0, \qquad x \in \partial\Omega. \end{aligned}$$

From the definition of $P$ and $Q$ we thus infer that

$$\begin{aligned} \varepsilon(\nabla_x \overline{u}) &= P\mathcal{D}^{-1}(\mathcal{D}_h - \mathcal{D})(\varepsilon(\nabla_x \tilde{u}_h) - \hat{\varepsilon}_p), & (70) \\ \overline{T} &= -\mathcal{D}Q\mathcal{D}^{-1}(\mathcal{D}_h - \mathcal{D})(\varepsilon(\nabla_x \tilde{u}_h) - \hat{\varepsilon}_p), & (71) \end{aligned}$$

with the inverse $\mathcal{D}^{-1}$ of $\mathcal{D}$, which exists since $\mathcal{D}[x]$ is uniformly positive definite for $x \in \Omega$. Hence, (70) and (71) imply

$$\begin{aligned} \|\varepsilon(\nabla_x \overline{u})\|_{L^2(\Omega)} &\leq \|P\|_{\mathcal{L}} \, \|\mathcal{D}^{-1}\|_{\mathcal{L}} \, \|(\mathcal{D}_h - \mathcal{D}\|_{\mathcal{L}} \, \|\varepsilon(\nabla_x \tilde{u}_h) - \hat{\varepsilon}_p\|_{L^2(\Omega)} \\ &\leq C\hat{h}\|(\hat{b}, \hat{\varepsilon}_p, \hat{\gamma})\|_{H^{-1}(\Omega)\times L^2(\Omega)\times H^1(\Omega)}\,, & (72) \\ \|\overline{T}\|_{L^2(\Omega)} &\leq \|\mathcal{D}\|_{\mathcal{L}} \, \|Q\|_{\mathcal{L}} \, \|\mathcal{D}^{-1}\|_{\mathcal{L}} \, \|(\mathcal{D}_h - \mathcal{D}\|_{\mathcal{L}} \, \|\varepsilon(\nabla_x \tilde{u}_h) - \hat{\varepsilon}_p\|_{L^2(\Omega)} \\ &\leq C\hat{h}\|(\hat{b}, \hat{\varepsilon}_p, \hat{\gamma})\|_{H^{-1}(\Omega)\times L^2(\Omega)\times H^1(\Omega)}\,, & (73) \end{aligned}$$

where we used the assumption (14) for the mapping $\mathcal{D}$ and the elliptic estimate $\|\varepsilon(\nabla_x \tilde{u}_h)\| \leq C\|(\hat{b}, \hat{\varepsilon}_p, \hat{\gamma})\|_{H^{-1}(\Omega)\times L^2(\Omega)\times H^1(\Omega)}$. Since $\overline{u} \in H_0^1(\Omega)$, we can apply Korn's inequality to conclude from (72) that

$$\|\overline{u}\|_{H^1(\Omega)} \leq C_1\|\varepsilon(\nabla_x \overline{u})\|_{L^2(\Omega)} \leq C_2\hat{h}\|(\hat{b}, \hat{\varepsilon}_p, \hat{\gamma})\|_{H^{-1}(\Omega)\times L^2(\Omega)\times H^1(\Omega)}\,. \quad (74)$$

After this preparation we can prove (69). Let $(\tilde{u}, \tilde{T})$ and $(\tilde{u}_h, \tilde{T}_h)$ be the solutions of the problems (24) – (26) and (52) – (54) to the initial data $\hat{b} = 0$, $\hat{\varepsilon}_p \in L^2(\Omega)$, $\hat{\gamma} = 0$. By definition of $P$ and $P_h$ we have $P\hat{\varepsilon}_p = \varepsilon(\nabla_x \tilde{u})$ and $P_h\hat{\varepsilon}_p = \varepsilon(\nabla_x \tilde{u}_h)$. The estimate (72) yields

$$\|(P - P_h)\hat{\varepsilon}_p\|_{L^2(\Omega)} = \|\varepsilon(\nabla_x \overline{u})\|_{L^2(\Omega)} \leq C\hat{h}\|\hat{\varepsilon}_p\|_{L^2(\Omega)}\,.$$

The first two estimates of (69) follow from this inequality and from $Q - Q_h = P_h - P$. To verify the last estimate of (69) compute

$$\begin{aligned} \|M - M_h\|_{\mathcal{L}} &= \|B^T\mathcal{D}QB - B^T\mathcal{D}_hQ_hB + L - L_h\|_{\mathcal{L}} \\ &\leq \|B^T(\mathcal{D} - \mathcal{D}_h)QB\|_{\mathcal{L}} + \|B^T\mathcal{D}_h(Q - Q_h)B\|_{\mathcal{L}} + \|L - L_h\|_{\mathcal{L}}\,. \end{aligned}$$

Noting the definitions of $\mathcal{D}_h$ and $L_h$, we see that the assumptions (14), (15) for $\mathcal{D}$ and $L$ together with the inequality for $Q - Q_h$ in (69) yield the bound $C\hat{h}$ for the right hand side. This proves (69).

To prove (68), write $(\hat{b}_1, \hat{\varepsilon}_{p,1}, \hat{\gamma}_1) = (\hat{b}, \hat{\varepsilon}_p, \hat{\gamma}) + (\hat{b}_1 - \hat{b}, \hat{\varepsilon}_{p,1} - \hat{\varepsilon}_p, \hat{\gamma}_1 - \hat{\gamma})$. For the solution of the problem (52) – (54) we then have the decomposition

$(\tilde{u}_h, \tilde{T}_h) = (\tilde{u}_{h,1}, \tilde{T}_{h,1}) + (\tilde{u}_{h,2}, \tilde{T}_{h,2})$, where $(\tilde{u}_{h,1}, \tilde{T}_{h,1})$ solves this problem to the data $\hat{b}, \hat{\varepsilon}_p, \hat{\gamma}$ and $(\tilde{u}_{h,2}, \tilde{T}_{h,2})$ solves the same problem to the data $\hat{b}_1 - \hat{b}, \hat{\varepsilon}_{p,1} - \hat{\varepsilon}_p, \hat{\gamma}_1 - \hat{\gamma}$ . By (73) and (74) we have

$$\begin{aligned}
&\|(\tilde{u}, \tilde{T}) - (\tilde{u}_h, \tilde{T}_h)\|_{H^1(\Omega)\times L^2(\Omega)} \\
&\quad \leq \|(\tilde{u}, \tilde{T}) - (\tilde{u}_{h,1}, \tilde{T}_{h,1})\|_{H^1(\Omega)\times L^2(\Omega)} + \|(\tilde{u}_{h,2}, \tilde{T}_{h,2})\|_{H^1(\Omega)\times L^2(\Omega)} \\
&\quad \leq \hat{h} C \|(\hat{b}, \hat{\varepsilon}_p, \hat{\gamma})\|_{H^{-1}(\Omega)\times L^2(\Omega)\times H^1(\Omega)} + \|(\tilde{u}_{h,2}, \tilde{T}_{h,2})\|_{H^1(\Omega)\times L^2(\Omega)} .
\end{aligned}$$

Elliptic theory yields the well known estimate

$$\|(\tilde{u}_{h,2}, \tilde{T}_{h,2})\|_{H^1(\Omega)\times L^2(\Omega)} \leq C \|(\hat{b} - \hat{b}_1, \hat{\varepsilon} - \hat{\varepsilon}_{p,1}, \hat{\gamma} - \hat{\gamma}_1)\|_{H^{-1}(\Omega)\times L^2(\Omega)\times H^1(\Omega)} ,$$

where $C$ can be chosen uniform with respect to $\hat{h}_0 \geq \hat{h} \geq 0$. The last to relations together yield (68). The proof of Lemma 2.12 is complete. □

**Estimation of the difference $d - d_h$ , part I.** The solution $d_h$ of the initial value problem (63), (64) differs from the solution $d$ of (32), (33), firstly because the initial data and the right hand sides of both problems are different, and secondly because the evolution operators $A_h$ and $A$ are different. The difference of the evolution operators is caused by the $x$–dependence of the coefficient functions $\mathcal{D}[x]$, $L[x]$ and $g(x, z)$ in the initial-boundary value problem (1) – (5). To separate both influences we estimate $d - d_h$ in the form

$$\|d(t) - d_h(t)\|_{L^2(\Omega)} \leq \|d(t) - \hat{d}_h(t)\|_{L^2(\Omega)} + \|\hat{d}_h(t) - d_h(t)\|_{L^2(\Omega)} ,$$

where $\hat{d}_h$ solves the initial value problem

$$\partial_t \hat{d}_h(t) + A_h \hat{d}_h(t) \ni B^T \partial_t \sigma(t) , \tag{75}$$

$$\hat{d}_h(0) = d(0) = -M z^{(0)} + B^T \sigma(0). \tag{76}$$

The evolution operator both in (63) and in (75) is $A_h$. If the coefficient functions would not depend on $x$, we would have $A = A_h$ and $d - \hat{d}_h = 0$, as will be seen below. Therefore $d - \hat{d}_h$ reflects the $x$–dependence of the coefficients. We first derive an estimate for $d_h - \hat{d}_h$; the estimate for $d - \hat{d}_h$, which is more difficult to get, is given subsequently.

**Lemma 2.13.** *There is a constant $C$ such that for all $0 \leq \hat{h} \leq \hat{h}_0$ the solutions $d_h$ of (63), (64) and $\hat{d}_h$ of (75), (76) satisfy*

$$\|d_h - \hat{d}_h\|_{L^\infty(0,T_e;L^2(\Omega))} \leq C \hat{h}. \tag{77}$$

**P**roof. (63), (64)] and (75), (76) are both initial value problems to the same maximal monotone evolution operator $A_h$. By Lemma 2.11 the initial data for both problems belong to the domain of definition of $A_h$. From the standard theory of evolution equations to monotone operators it thus follows that the solutions to these initial value problems satisfy the estimate

$$\begin{aligned} |d_h(t) - \hat{d}_h(t)|_{h,L^2(\Omega)} \;\le\;& |d_h(0) - d(0)|_{h,L^2(\Omega)} \\ &+ \int_0^t |B^T(\partial_t \sigma_h(s) - \partial_t \sigma(s))|_{h,L^2(\Omega)}\, ds\,, \end{aligned} \tag{78}$$

for all $0 \le t \le T_e$, cf. [10, Lemma 3.1 and Theorem 3.4, p. 64, 65] or [3, Theorem 2.8, p. 117]. To estimate the two terms on the right hand side of this inequality we note that the function $(v,\sigma)$ in (27) is by construction the solution of the boundary value problem (24) – (26) to the data $\hat{b} = b(t)$, $\hat{\varepsilon}_p = 0$, $\hat{\gamma} = \gamma(t)$, whereas $(v_h, \sigma_h)$ is the solution of (55) – (57). Therefore we can apply Lemma 2.12 to estimate the difference $\sigma_h - \sigma$. Combination of the inequality (68) from this lemma with the estimates (46) – (50) results in

$$\begin{aligned} &\|\sigma - \sigma_h\|_{W^{1,\infty}(0,T_e;L^2(\Omega))} \\ &\quad\le\; \hat{h}C_1\|(b,\gamma)\|_{W^{1,\infty}(0,T_e;L^2(\Omega)\times H^1(\Omega))} + C_2\|b - b_h\|_{W^{1,\infty}(0,T_e;H^{-1}(\Omega))} \\ &\qquad + \hat{h}C_2\Big(\|(\partial_{\xi_i}T)_h \nabla_x\varphi\|_{W^{1,\infty}(0,T_e;H^{-1}(\Omega))} \\ &\qquad\qquad + \|\varepsilon\big((u_{\xi_i})_h \otimes \nabla_x\varphi\big)\|_{W^{1,\infty}(0,T_e;L^2(\Omega))}\Big) \;\le\; \hat{h}C. \end{aligned} \tag{79}$$

Since $B^T$ is bounded, we infer from this estimate and from (65) that

$$\int_0^t |B^T(\partial_t\sigma_h(s) - \partial_t\sigma(s))|_{h,L^2(\Omega)}\, ds \le CT_e\hat{h}, \quad 0 \le t \le T_e\,. \tag{80}$$

From (64) and (76) we have

$$\begin{aligned} d_h(0) - d(0) &= -M_h z_h^{(0)} + B^T\sigma_h(0) + Mz^{(0)} - B^T\sigma(0) \\ &= (M - M_h)z_h^{(0)} + M(z^{(0)} - z_h^{(0)}) + B^T(\sigma_h(0) - \sigma(0)). \end{aligned}$$

Thus, by (51) and (69),

$$\|d_h(0) - d(0)\|_{L^2(\Omega)} \le C_1\hat{h} + \|B^T(\sigma_h(0) - \sigma(0))\|_{L^2(\Omega)}. \tag{81}$$

From the boundedness of $B^T$, from (79) and from the Sobolev imbedding theorem we conclude that

$$\|B^T(\sigma_h(0) - \sigma(0))\|_{L^2(\Omega)} \le C_2\|\sigma_h - \sigma\|_{W^{1,\infty}(0,T_e;L^2(\Omega))} \le C_3\hat{h}. \tag{82}$$

Noting (65), we see that (81) and (82) imply $|d_h(0) - d(0)|_{h,L^2(\Omega)} \le C_4\hat{h}$. We use this inequality and (80) to estimate the right hand side of (78) and observe again (65) to obtain (77). This proves the lemma. □

**Estimation of the difference** $d - d_h$ **, part II.** It remains to estimate the difference $d - \hat{d}_h$.

**Proposition 2.14.** *If* (66) *holds, then there is a constant* $C > 0$ *such that the solutions of the initial value problems* (32), (33) *and* (75), (76) *satisfy for all* $0 \leq \hat{h} \leq \hat{h}_0$ *the inequality*

$$\|d - \hat{d}_h\|_{L^\infty(0,T_e;L^2(\Omega))} \leq C\hat{h}. \tag{83}$$

**Proof.** Define $d_0 = d(0) = -Mz^{(0)} + B^T\sigma(0)$ and consider the four initial value problems

$$\frac{d}{dt}d(t) + A\,d(t) \ni B^T\sigma_t(t), \qquad d(0) = d_0, \tag{84}$$

$$\frac{d}{dt}d^\lambda(t) + A^\lambda d^\lambda(t) = B^T\sigma_t(t), \qquad d^\lambda(0) = d_0, \tag{85}$$

$$\frac{d}{dt}\hat{d}_h^\lambda(t) + A_h^\lambda \hat{d}_h^\lambda(t) = B^T\sigma_t(t), \qquad \hat{d}_h^\lambda(0) = d_0, \tag{86}$$

$$\frac{d}{dt}\hat{d}_h(t) + A_h\hat{d}_h(t)) \ni B^T\sigma_t(t), \qquad \hat{d}_h(0) = d_0, \tag{87}$$

where $A^\lambda = MG^\lambda$ and $A_h^\lambda = M_h G_h^\lambda$, with $M$ and $M_h$ defined in (29) and (60), respectively. The first initial value problem coincides with (32), (33), the last one coincides with (75), (76). We estimate the differences $d(t) - d^\lambda(t)$, $d^\lambda(t) - \hat{d}_h^\lambda(t)$ and $\hat{d}_h^\lambda(t) - \hat{d}_h(t)$ separately.

Consider first $d^\lambda(t) - \hat{d}_h^\lambda(t)$. Since $A_h^\lambda$ is monotone with respect to the scalar product $[\![\cdot,\cdot\cdot]\!]_{h,\Omega}$ and since the evolution equations in (85) and (86) have the same right hand sides, we have

$$\begin{aligned}
\big|d^\lambda(t) - \hat{d}_h^\lambda(t)\big|_{h,L^2(\Omega)} &\frac{d}{dt}\big|d^\lambda(t) - \hat{d}_h^\lambda(t)\big|_{h,L^2(\Omega)} \\
&= \frac{1}{2}\frac{d}{dt}|d^\lambda(t) - \hat{d}_h^\lambda(t)|^2_{h,L^2(\Omega)} \\
&= [\![\partial_t d^\lambda(t) - \partial_t\hat{d}_h^\lambda(t), d^\lambda(t) - \hat{d}_h^\lambda(t)]\!]_{h,\Omega} \\
&= -[\![A^\lambda d^\lambda(t) - A_h^\lambda\hat{d}_h^\lambda(t), d^\lambda(t) - \hat{d}_h^\lambda(t)]\!]_{h,\Omega} \\
&= -[\![A_h^\lambda d^\lambda(t) - A_h^\lambda\hat{d}_h^\lambda(t), d^\lambda(t) - \hat{d}_h^\lambda(t)]\!]_{h,\Omega} \\
&\qquad\qquad - [\![A^\lambda d^\lambda(t) - A_h^\lambda d^\lambda(t), d^\lambda(t) - \hat{d}_h^\lambda(t)]\!]_{h,\Omega} \\
&\leq \big|A^\lambda d^\lambda(t) - A_h^\lambda d^\lambda(t)\big|_{h,L^2(\Omega)}\ \big|d^\lambda(t) - \hat{d}_h^\lambda(t)\big|_{h,L^2(\Omega)}\,.
\end{aligned}$$

We divide this inequality by $2|d^\lambda(t) - \hat{d}_h^\lambda(t)|_{h,L^2(\Omega)}$, use the definitions of $A^\lambda$

and $A_h^\lambda$ and remember (65) to obtain

$$\begin{aligned}\frac{d}{dt}\big|d^\lambda(t)-\hat{d}_h^\lambda(t)\big|_{h,L^2(\Omega)} &\le \frac{1}{2}\big|(A^\lambda - A_h^\lambda)d^\lambda(t)\big|_{h,L^2(\Omega)} \\ &\le \frac{c_1}{2}\|(A^\lambda - A_h^\lambda)d^\lambda(t)\|_{L^2(\Omega)} \\ &= \frac{c_1}{2}\big\|(M-M_h)G^\lambda d^\lambda(t) + M_h(G^\lambda - G_h^\lambda)d^\lambda\big\|_{L^2(\Omega)}\,.\end{aligned}$$

We invoke (66) and (69) to conclude that

$$\frac{d}{dt}\big|d^\lambda(t)-\hat{d}_h^\lambda(t)\big|_{h,L^2(\Omega)} \le C_1\hat{h}\|G^\lambda d^\lambda(t)\|_{L^2(\Omega)} + C_2\hat{h}\|G^\lambda d^\lambda(t)\|_{L^2(\Omega)}\,. \quad (88)$$

Since the evolution equation in the initial value problem (85) coincides with (35), we can apply the inequality (37) from Lemma 2.6 to estimate the term $\|G^\lambda d^\lambda(t)\|_{L^2(\Omega)}$. Furthermore, since by Lemma 2.4 the initial data $d_0$ belong to the domain of definition of $G$, the relation (38) holds. Combination of this inequality and relation with (88) yields

$$\begin{aligned}&\frac{d}{dt}\big|d^\lambda(t)-\hat{d}_h^\lambda(t)\big|_{h,L^2(\Omega)} \\ &\quad\le \hat{h}K\Big(\|G^0 d_0\|_{L^2(\Omega)} + \|B^T\sigma_t(0)\|_{L^2(\Omega)} + \|B^T\sigma_t\|_{L^\infty(0,T_e;L^2(\Omega))} \\ &\qquad\qquad + \|B^T\sigma_{tt}\|_{L^1(0,T_e;L^2(\Omega))}\Big)\,. \quad (89)\end{aligned}$$

The bracket on the right hand side is independent of $h$ and $\lambda$. We denote it by $K_1$. Since $d(0)=\hat{d}_h(0)$, we infer by integration of (89) that

$$c_2\|d^\lambda(t)-\hat{d}_h^\lambda(t)\|_{L^2(\Omega)} \le \big|d^\lambda(t)-\hat{d}_h^\lambda(t)\big|_{h,L^2(\Omega)} \le \hat{h}KK_1T_e\,, \quad (90)$$

where we used (65).

Next we consider the differences $d(t)-d^\lambda(t)$ and $\hat{d}_h^\lambda(t)-\hat{d}_h(t)$. Since $d$ and $d^\lambda$ satisfy the initial boundary value problems (84) and (85) with coinciding initial data $d_0$, we can again apply Lemma 2.6. The inequality (37) combined with (38) yields

$$\|d(t)-d^\lambda(t)\|_{L^2(\Omega)} \le \sqrt{\frac{\lambda}{2}}CK_1\,, \quad (91)$$

with the constant $K_1$ as above. Since $\hat{d}_h$ and $\hat{d}_h^\lambda$ satisfy the initial boundary value problems (87) and (86), which have the same form as (84) and (85) with the operator $A$ replaced by $A_h$, and since $d_0 \in D(G_h)$, by Lemma 2.11, we obtain in the same way the estimate

$$\|\hat{d}_h(t)-\hat{d}_h^\lambda(t)\|_{L^2(\Omega)} \le \sqrt{\frac{\lambda}{2}}CK_1(h)\,, \quad (92)$$

however with a constant $K_1(h)$, which can depend on $h$. For given $h$ choose now $\lambda > 0$ such that $\sqrt{\frac{\lambda}{2}} \max\big(K_1, K_1(h)\big) \le \hat{h} K_1$. Then we obtain from (90) – (92) for all $0 \le t \le T_e$ that

$$\begin{aligned} \|d(t) - \hat{d}_h(t)\|_{L^2(\Omega)} &\le \|d(t) - d^\lambda(t)\|_{L^2(\Omega)} + \|d^\lambda(t) - \hat{d}_h^\lambda(t)\|_{L^2(\Omega)} \\ &\quad + \|\hat{d}_h(t) - \hat{d}_h^\lambda(t)\|_{L^2(\Omega)} \le \hat{h}(c_2^{-1} K K_1 T_e + 2CK_1). \end{aligned}$$

This proves the estimate (83) and completes the proof of Proposition 2.14. □

**End of the proof of Theorem 1.3.** Lemma 2.13 and Proposition 2.14 together imply

$$\|d_h - d\|_{L^\infty(0,T_e;L^2(\Omega))} \le C\hat{h}. \tag{93}$$

To derive an estimate for $z - z_h$ from this inequality note that the definitions of $d$ and $d_h$ in (31) and (62) yield

$$z - z_h = M_h^{-1}\big(d_h - d - B^T(\sigma_h - \sigma)\big) + \big(M_h^{-1} - M^{-1}\big)(d - B^T\sigma). \tag{94}$$

Applying the well known operator equation

$$\mathcal{A}_2^{-1} = \mathcal{A}_1^{-1} \sum_{m=0}^{\infty} \big((\mathcal{A}_1 - \mathcal{A}_2)\mathcal{A}_1^{-1}\big)^m$$

with $\mathcal{A}_1 = M$, $\mathcal{A}_2 = M_h$, we infer from (69) that

$$\|M_h^{-1} - M^{-1}\|_{\mathcal{L}} \le C_2\hat{h}. \tag{95}$$

Since (95) implies that $\|M_h^{-1}\|_{\mathcal{L}}$ is uniformly bounded with respect to $h$, we obtain by combination of (93) – (95) and (79) that

$$\|z(\cdot + h, \cdot\cdot) - z\|_{L^\infty(0,T_e;L^2(V))} \le \|z_h - z\|_{L^\infty(0,T_e;L^2(\Omega))} \le C_3\hat{h}, \tag{96}$$

where we also used the definition of $z_h$ in (39), (40). Therefore

$$(x,t) \to w_h(x,t) = \big(z(x+h,t) - z(x,t)\big)/\hat{h}$$

is uniformly bounded in $L^\infty\big(0,T_e;L^2(V)\big)$ with respect to $h$. Hence, there is a subsequence $w_{h_n}$, which converges weak–$*$ in the space $L^\infty\big(0,T_e;L^2(V)\big)$ to a limit function $w \in L^\infty\big(0,T_e;L^2(V)\big)$. For every test function $\varphi \in L^1\big(0,T_e;C_0^\infty(V,\mathbb{R}^N)\big)$ we compute with $V_{T_e} = V \times (0,T_e)$ that

$$\begin{aligned} (w,\varphi)_{V_{T_e}} &= \lim_{n\to\infty} (w_{h_n},\varphi)_{V_{T_e}} = \lim_{n\to\infty} \Big(\frac{z(\cdot + h_n, \cdot\cdot) - z}{\hat{h}_n}, \varphi\Big)_{V_{T_e}} \\ &= -\lim_{n\to\infty} \Big(z, \frac{\varphi - \varphi(\cdot - h_n, \cdot\cdot)}{\hat{h}_n}\Big)_{V_{T_e}} = -\Big(z, \frac{\partial}{\partial x_i}\varphi\Big)_{V_{T_e}}. \end{aligned}$$

We thus have $w = \partial_{x_i} z \in L^\infty\big(0, T_e; L^2(V)\big)$ for every open subset $V \subset\subset \Omega$. This proves (17).

To prove (16) we use that $(u(t), T(t))$ solves the elliptic boundary value problem (1), (2), (5) with $Bz \in L^\infty\big(0, T_e; H^1_{\text{loc}}(\Omega)\big)$, $b \in L^\infty\big(0, T_e; L^2(\Omega)\big)$ and $\gamma \in L^\infty\big(0, T_e; H^1(\Omega)\big)$. Elliptic regularity theory thus yields $(u, T) \in L^\infty\big(0, T_e; H^2_{\text{loc}}(\Omega) \times H^1_{\text{loc}}(\Omega)\big)$, cf. [17, Theorem 2.1, p. 30]. The proof of Theorem 1.3 is complete. □

# 3 Global regularity

## 3.1 Preliminaries

In this section we prove Theorems 1.5 and 1.6. For simplicity we give the proofs only in the special case where $\mathcal{D}$, $g$ and $L$ are independent of $x$. To verify the theorems in the general case, Proposition 2.14 can be carried over.

To prove both theorems we restrict the initial-boundary value problem (1) – (5) to a neighborhood of a boundary point and transform the restricted problem to a domain with straight boundary. Here we discuss this transformation and show how the transformed problem and the shifted transformed problem can be reduced to evolution equations, which we use in a similar manner as in Section 2 to estimate the difference between the solution and the shifted solution.

Let $x_0 \in \partial\Omega$. We choose a new coordinate system such that $x_0 = 0$ and such that the $x_1, x_2$–plane is tangential to the boundary at $x_0$. Since by assumption $\partial\Omega \in C^2$, there are a sufficiently small $\alpha > 0$ and a $C^2$–function $a$ such that a parametrization of the boundary $\partial\Omega$ in a neighborhood of $x_0$ is given by

$$(x_1, x_2) = x' \mapsto (x', a(x')) = (x', x_3) : \{|x'| = |(x_1, x_2)| < \alpha\} \subset \mathbb{R}^2 \to \partial\Omega.$$

Let $B \subset \mathbb{R}^3$ be a neighborhood $x_0$. For reasons which will become clear later, we choose it in the form

$$B = \big\{(x', x_3) \in \mathbb{R}^3 \;\big|\; |x'| < \alpha,\; a(x') - \beta < x_3 < a(x') + \beta\big\},$$

with $\beta > 0$ sufficiently small such that

$$\Omega \cap B = \big\{x \in \mathbb{R}^3 \;\big|\; |x'| < \alpha,\; a(x') < x_3 < a(x') + \beta\big\}.$$

Let $\Psi(x) = (x^{'}, x_3 - a(x^{'}))$. We define sets $D'$, $D$ and $\Gamma$ by

$$\begin{aligned} D' &= \Psi(B) = \{x \in \mathbb{R}^3 \mid |x'| < \alpha,\; -\beta < x_3 < \beta\}, \\ D &= \Psi(\Omega \cap B) = \{x \in \mathbb{R}^3 \mid |x'| < \alpha,\; 0 < x_3 < \beta\}, \\ \Gamma &= \Psi(\partial\Omega \cap B) = \{(x', 0) \in \mathbb{R}^3 \mid |x'| < \alpha\}. \end{aligned} \tag{97}$$

It is clear from the definition that $\Psi$ is a diffeomorphism from $\Omega \cap B$ to $D$. Now we consider the initial-boundary value problem (1) – (5) restricted to the domain $(\Omega \cap B) \times (0, T_e)$. Using $\Psi$ we transform this problem to the domain $D \times (0, T_e)$. Denoting the new coordinates by $(y,t) = (\Psi(x), t)$, we obtain the transformed problem

$$\begin{aligned} -\mathrm{div}^* T(y,t) &= b(y,t), && (98)\\ T(y,t) &= \mathcal{D}\Big(\varepsilon\big(\nabla^* u(y,t)\big) - Bz(y,t)\Big), && (99)\\ \partial_t z(y,t) &\in g\Big(B^T T(y,t) - Lz(y,t)\Big), && (100)\\ z(x,0) &= z^{(0)}(y), && (101)\\ u(y,t) &= \hat{u}(y,t) = \begin{cases} \gamma(y,t), & \text{for } y \in \Gamma,\\ u(y,t), & \text{for } y \in \partial D\backslash\Gamma, \end{cases} && (102) \end{aligned}$$

where the function $u|_{(\partial D\backslash\Gamma)\times[0,T_e]}$ on the right hand side of (102) is considered to be known from the existence theorem. The operators $\nabla^*$ and $\mathrm{div}^*$ are defined by

$$\begin{aligned} \nabla_y^* v(y) &= \nabla_x v(\Psi(x))\big|_{x=\Psi^{-1}(y)},\\ \mathrm{div}_y^* \sigma(y) &= \mathrm{div}_x \sigma(\Psi(x))\big|_{x=\Psi^{-1}(y)}. \end{aligned}$$

We introduce a new scalar product on $L^2(D)$ by

$$(\zeta_1, \zeta_2)_D^* = \int_D \zeta_1(y) \cdot \zeta_2(y)\, |\det(\nabla\Psi^{-1}(y))|\, dy,$$

with either $\zeta_i \in L^2(D, \mathbb{R}^3)$ or $\zeta_i \in L^2(D, \mathcal{S}^3)$, respectively. From the definition of $\Psi$ it follows that $\det(\nabla\Psi^{-1}(y)) = 1$, and therefore the $*$–scalar product is equal to the standard scalar product, but for more general transformations they would differ. For generality and clearness we thus stay with the $*$–notation. In this scalar product the operator $-\nabla^*$ is adjoint to $\mathrm{div}^*$. To see this, note that transformation of variables in the integral yields for $\sigma \in L^2(D, \mathcal{S}^3)$ with $\mathrm{div}^*\sigma \in L^2(\Omega, \mathbb{R}^3)$ and for all $v \in H_0^1(D, \mathbb{R}^3)$ that

$$(\mathrm{div}_y^* \sigma, v)_D^* = (\mathrm{div}_x(\sigma \circ \Psi), v \circ \Psi)_D = -(\sigma \circ \Psi, \nabla_x(v \circ \Psi))_D = -(\sigma, \nabla_y^* v)_D^*. \qquad (103)$$

Following the procedure in Section 2 we define the mapping $P : L^2(D, \mathcal{S}^3) \to L^2(D, \mathcal{S}^3)$ by

$$P\hat{\varepsilon}_p = \varepsilon\big(\nabla^* \tilde{v}\big), \qquad (104)$$

where $(\tilde{v}, \tilde{T}) \in H^1(D, \mathbb{R}^3) \times L^2(D, \mathcal{S}^3)$ solves the problem

$$\begin{aligned} -\mathrm{div}^* T(y) &= \hat{b}(y), && (105)\\ T(y) &= \mathcal{D}(\varepsilon(\nabla^* u(y)) - \hat{\varepsilon}_p(y)), && (106)\\ u(y) &= \hat{\gamma}(y), \quad y \in \partial D && (107) \end{aligned}$$

to the data $\hat{b} = \hat{\gamma} = 0$ and $\hat{\varepsilon}_p \in L^2(D, \mathcal{S}^3)$. We also define

$$Q = I - P, \quad M = B^T DQB + L.$$

Due to (103), the proof of Lemma 2.2 carries over to the present situation, and therefore $P$, $Q$ and $M$ have the same properties as the corresponding operators introduced in Definition 2.1 and in (29). In particular, $P$ and $Q$ are projectors orthogonal with respect to the scalar product

$$[\xi, \zeta]^*_D = (\mathcal{D}\xi, \zeta)^*_D,$$

and $M \in \mathcal{LS}(L^2(D), L^2(D))$ is a bounded, selfadjoint, positive operator with respect to the scalar product $(\xi, \zeta)^*_D$. Because of this, following the procedure in Section 2.1, we can define scalar products on $L^2(D, \mathbb{R}^N)$ with associated norms by

$$[\![\xi, \zeta]\!]^*_D = (M^{-1}\xi, \zeta)^*_D, \qquad |\xi|^*_{L^2(D)} = [\![\xi, \xi]\!]^{*1/2}_D,$$

$$[\![\![\xi, \zeta]\!]\!]_D = (M\xi, \zeta)^*_D, \qquad |\!|\!|\xi|\!|\!|_{L^2(D)} = [\![\![\xi, \xi]\!]\!]^{1/2}_D.$$

The associated norms are both equivalent to the norm $||\xi||_{L^2(D)}$.

With the operators $Q$ and $M$ we reduce the initial-boundary value problem (98) – (102) to an evolution problem in $L^2(D, \mathbb{R}^N)$ as follows. Let $(u, T, z)$ be the solution of this problem in $D \times (0, T_e)$. Then

$$(u(t), T(t)) = (\tilde{u}(t), \tilde{T}(t)) + (v(t), \sigma(t)) + (\hat{v}(t), \hat{\sigma}(t)),$$

where $(v(t), \sigma(t))$ is the solution of (105) – (107) to the data $\hat{b} = b(t)$, $\hat{\gamma} = 0$, $\hat{\varepsilon}_p = 0$, the function $(\hat{v}(t), \hat{\sigma}(t))$ is the solution of (105) – (107) to the data $\hat{b} = 0$, $\hat{\gamma} = \hat{u}(t)$, $\hat{\varepsilon}_p = 0$, and $(\tilde{u}(t), \tilde{T}(t))$ is the solution of (105) – (107) to the data $\hat{b} = \hat{\gamma} = 0$, $\hat{\varepsilon}_p = Bz(t)$. The definition of $Q$ implies $\tilde{T} = -\mathcal{D}QBz$, whence $T = -\mathcal{D}QBz + \sigma + \hat{\sigma}$. We insert this equation into (100). Together with the initial condition (101) we obtain the initial value problem

$$\frac{d}{dt} z(t) \in G\big( - Mz(t) + B^T(\sigma(t) + \hat{\sigma}(t))\big), \tag{108}$$

$$z(0) = z^{(0)}, \tag{109}$$

where the operator $G : L^2(D, \mathbb{R}^N) \to 2^{L^2(D, \mathbb{R}^N)}$ is given by

$$G(\xi) = \{\zeta \in L^2(D, \mathbb{R}^N) \mid \zeta(x) \in g(\xi(x)) \text{ a.e.}\}. \tag{110}$$

We set

$$f_1 = B^T(\sigma + \hat{\sigma}), \quad d = -Mz + f_1 \,. \tag{111}$$

If we insert $d$ into the initial value problem (108), (109), we obtain the equivalent problem

$$\frac{d}{dt} d(t) + A\, d(t) \ni f_{1,t}(t), \tag{112}$$

$$d(0) = d^{(0)} = -Mz^{(0)} + f_1(0)\,, \tag{113}$$

with the operator $A = MG$. The proof of Theorem 2.3 transfers immediately to the present situation. We therefore have

**Corollary 3.1.** *The operator $G$ is maximal monotone with respect to the scalar product $(\xi,\zeta)^*_D$ , the operator $A$ is maximal monotone with respect to the scalar product $[\![\xi,\zeta]\!]^*_D$ .*

**The shifted problem.** To prove that both tangential and normal derivatives are regular at the boundary we need to shift the solution of (98) – (102) by a vector $h$ and estimate the difference of the shifted and unshifted solution. Therefore we next consider the initial-boundary value problem solved by the shifted solution and reduce it to an evolution problem in $L^2(D,\mathbb{R}^N)$.

Let $V$ be an open neighborhood of 0 such that $V \subset\subset D'$ and let $\varphi \in C_0^\infty(D')$ satisfy

$$\varphi \equiv 1 \text{ on } V, \quad 0 \le \varphi \le 1.$$

For $\hat{h} > 0$ and $i = 1,2,3$ set $h = \hat{h}e_i \in \mathbb{R}^3$. Define

$$\phi_h(y) = y + h\varphi(y).$$

For all sufficiently small $\hat{h}$ the function $\phi_h$ maps $D$ into $D$ and is one-to-one. We only consider such small $\hat{h}$. Now set

$$(u_h, T_h, z_h)(y,t) = \big(u(\phi_h(y),t), T(\phi_h(y),t), z(\phi_h(y),t)\big).$$

The function $(u_h, T_h, z_h)$ is defined in $D \times [0,T_e]$ and solves the equations

$$\begin{aligned}
-\operatorname{div}^* T_h(y,t) &= b_h(y,t) - \hat{h}\big(\partial_{\xi_i} T(\xi,t)\nabla_y^*\varphi(y)\big)_{\xi=\phi_h(y)}, && (114)\\
T_h(y,t) &= \mathcal{D}\Big(\varepsilon\big(\nabla^* u_h(y,t)\big) - Bz_h(y,t) \\
&\qquad + \hat{h}\,\varepsilon\big(u_{\xi_i}(\xi,t)\otimes\nabla_y^*\varphi(y)\big)_{\xi=\phi_h(y)}\Big), && (115)\\
\partial_t z_h(y,t) &\in g\Big(B^T T_h(y,t) - Lz_h(y,t)\Big), && (116)\\
z_h(y,0) &= z_h^{(0)}(y). && (117)
\end{aligned}$$

For $x \in \partial D \setminus \Gamma$ we have $\phi_h(x) = x$; therefore the boundary conditions on $\partial D \times [0,T_e]$ are

$$u_h(y,t) = \hat{u}_h(y,t) := \begin{cases} u(\phi_h(y),t), & \text{for } y \in \Gamma, \\ u(y,t), & \text{for } y \in \partial D \backslash \Gamma. \end{cases} \tag{118}$$

Of course, $b_h(y,t) = b(\phi_h(y),t)$, and, as in Section 2.2, the second term $\big(\partial_{\xi_i} T(\xi,t)\nabla_y^*\varphi(y)\big)_{\xi=\phi_h(y)}$ on the right-hand side in (114) is a distribution.

The first two components of the solution of the problem (114) – (118) can be written as

$$(u_h(t), T_h(t)) = (\tilde{u}_h(t), \tilde{T}_h(t)) + (v_h(t), \sigma_h(t)) + (\hat{v}_h(t), \hat{\sigma}_h(t)) + (\tilde{v}_h(t), \tilde{\sigma}_h(t)),$$

where $(v_h(t), \sigma_h(t))$ is the solution of (105) – (107) to the data $\hat{b} = b_h(t)$, $\hat{\gamma} = 0$, $\hat{\varepsilon}_p = 0$, the function $(\tilde{u}_h(t), \tilde{T}_h(t))$ is the solution of (105) – (107) to the data $\hat{b} = \hat{\gamma} = 0$, $\hat{\varepsilon}_p = Bz_h(t)$, $(\hat{v}_h(t), \hat{\sigma}_h(t))$ solves (105) – (107) to the data $\hat{b} = \hat{\varepsilon}_p = 0$, $\hat{\gamma} = \hat{u}_h(t)$. The function $(\tilde{v}_h(t), \tilde{\sigma}_h(t))$ solves (105) – (107) to the data

$$\hat{b} = -\hat{h}\big(\partial_{\xi_i} T(\xi, t)\nabla_y^* \varphi(y)\big)_{\xi=\phi_h(y)}, \tag{119}$$

$$\hat{\varepsilon}_p = -\hat{h}\,\varepsilon\big(u_{\xi_i}(\xi, t) \otimes \nabla_y^* \varphi(y)\big)_{\xi=\phi_h(y)}, \tag{120}$$

$$\hat{\gamma} = 0. \tag{121}$$

Similarly as in the case of the unshifted problem we have $T_h = -\mathcal{D}QBz_h + \sigma_h + \hat{\sigma}_h + \tilde{\sigma}_h$. Insertion of this equation into (116) yields the initial value problem

$$\frac{d}{dt} z_h(t) \in G\big(-Mz(t) + B^T(\sigma_h(t) + \hat{\sigma}_h(t) + \tilde{\sigma}_h(t))\big) \tag{122}$$

$$z_h(0) = z_h^{(0)}, \tag{123}$$

with the operator $G$ from (110). Set

$$f_2 = B^T(\sigma_h + \hat{\sigma}_h + \tilde{\sigma}_h), \quad d_h = -Mz_h + f_2\,. \tag{124}$$

Insertion of the function $d$ into (122), (123) leads to the equivalent initial value problem

$$\frac{d}{dt} d_h(t) + Ad_h(t) \ni f_{2,t}(t), \tag{125}$$

$$d_h(0) = d_h^{(0)} = -Mz_h^{(0)} + f_2(0), \tag{126}$$

with the same operator $A$ as in (112).

### 3.2 Proof of Theorem 1.5 (Case of tangential derivatives)

Let $h$ be a tangential vector to $\Gamma$, whence $h = \hat{h}e_i$ with $i = 1, 2$. To estimate the difference $z(t) - z_h(t)$ in $L^2(D, \mathbb{R}^N)$ note that in this case we have $u(\phi_h(y), t) = \gamma_h(y, t)$ for $y \in \Gamma$. From the boundary conditions (102) and (118) we thus see that

$$u(y,t) - u_h(y,t) = \begin{cases} \gamma(y,t) - \gamma_h(y,t), & \text{for } y \in \Gamma, \\ 0, & \text{for } y \in \partial D \backslash \Gamma. \end{cases} \tag{127}$$

Having this boundary condition we can proceed as in the proof of Theorem 1.3. We sketch the details. As in Section 2.2 we see that the solutions of (125), (126) and (112), (113) satisfy the inequality

$$|d(t) - d_h(t)|^*_{L^2(D)} \le |d^{(0)} - d_h^{(0)}|^*_{L^2(D)} + \int_0^t |f_{1,t}(s) - f_{2,t}(s)|^*_{L^2(D)}\, ds \tag{128}$$

corresponding to (78). By definition of $f_1$ and $f_2$ in (111), (124) we have

$$|f_{2,t}(t) - f_{1,t}(t)|^*_{L^2(D)} \le C\big(\|\sigma_t(t) - \sigma_{h,t}(t)\|_{L^2(D)} + \|\hat\sigma_t(t) - \hat\sigma_{h,t}(t)\|_{L^2(D)} + \|\tilde\sigma_{h,t}(t)\|_{L^2(D)}\big)\,. \tag{129}$$

To estimate the terms on the right hand side we use the theory of elliptic boundary value problems, which yields

$$\|\hat\sigma_t(t) - \hat\sigma_{h,t}(t)\|_{L^2(D)} \le C\|u_t(t) - u_{h,t}(t)\|_{H^{\frac12}(\partial D)}\,.$$

Using (127) we infer that

$$\|\hat\sigma_t(t) - \hat\sigma_{h,t}(t)\|_{L^2(D)} \le C\|\gamma_{h,t}(t) - \gamma_t(t)\|_{H^{\frac12}(\Gamma)} \le C_1\|\gamma_{h,t}(t) - \gamma_t(t)\|_{H^1(\Gamma)}$$
$$\le C_2|h|\,\|\gamma_t\|_{L\infty(0,T_e;H^2(D))} \le C_3|h|\,\|\gamma\|_{W^{2,1}(0,T_e,H^2(D))}. \tag{130}$$

Just as in the proof of Lemma 2.13 we see that the inequalities

$$\|\sigma_t(t) - \sigma_{h,t}(t)\|_{L^2(D)} \le C\|b_t(t) - b_{h,t}(t)\|_{H^{-1}(D)}\,, \tag{131}$$
$$\le C\hat h\|b_t\|_{L^\infty(0,T_e;L^2(D))} \le C_1\hat h\|b\|_{W^{2,1}(0,T_e,L^2(D))}\,,$$
$$\|\tilde\sigma_{h,t}(t)\|_{L^2(D)} \le C\hat h, \tag{132}$$
$$|d^{(0)} - d_h^{(0)}|^*_{L^2(D)} \le C\hat h, \tag{133}$$

hold. Combination of (128) – (133) yields

$$\|d_t(t) - d_{h,t}(t)\|_{L^2(D)} \le C\hat h.$$

The rest of the proof of Theorem 1.5 follows the arguments given at the end of the proof of Theorem 1.3. We omit it. □

### 3.3 Proof of Theorem 1.6 (Case of normal derivatives)

Let $h$ be a normal vector to $\Gamma$, whence $h = \hat h e_3$. In this case we cannot use the inequality (128) to estimate the difference $z(t) - z_h(t)$ in $L^2(D, \mathbb{R}^N)$, as we did for tangential derivatives. Namely, the boundary conditions (102) and (118) imply

$$u(y,t) - u_h(y,t) = \begin{cases} \hat u(y,t) - \hat u_h(y,t), & \text{for } y \in \Gamma, \\ 0, & \text{for } y \in \partial D\backslash\Gamma, \end{cases} \tag{134}$$

and the estimate (130) would become

$$\|\hat\sigma_t(t) - \hat\sigma_{h,t}(t)\|_{L^2(D)} \le C\|u_t(t) - u_{h,t}(t)\|_{H^{\frac12}(\partial D)} = C\|\hat u_t(t) - \hat u_{h,t}(t)\|_{H^{\frac12}(\Gamma)},$$

where $\|\hat u_t(t) - \hat u_{h,t}(t)\|_{H^{\frac12}(\Gamma)}$ cannot be replaced by $\|\gamma_{h,t}(t) - \gamma_t(t)\|_{H^{\frac12}(\Gamma)}$. Instead, to show that the right hand side tends to zero for $h \to 0$ we would have to use the Sobolev imbedding theorem, which allows to estimate $\|\hat{u}_t(t) - \hat{u}_{h,t}(t)\|_{H^{\frac{1}{2}}(\Gamma)}$ by $\hat{h}^{\frac{1}{2}}\|\nabla_\Gamma u_t(t)\|_{H^1(\Omega)}$, where $\nabla_\Gamma$ denotes the tangential gradient. However, this requires an a-priori estimate for $\|\nabla_\Gamma u_t(t)\|_{H^1(\Omega)}$, which is not provided by the existence result stated in Theorem 1.2. Instead, we want to take advantage of the estimate for $\|\nabla_\Gamma u(t)\|_{H^1(\Omega)}$ in Theorem 1.5, which we have just derived. To this end we need a perturbation estimate, which does not involve a-priori estimates for time derivatives of the stress functions. Since in the non-autonomous initial value problems (108), (109) and (122), (123) the non-differentiated stress functions appear, a perturbation estimate for these initial value problems is required. The following Lemma from [30] provides such an estimate.

**Lemma 3.2.** *There is a constant $C$ such that for all $0 \leq \hat{h} \leq \hat{h}_0$ and all solutions $z(t)$ and $z_h(t)$ of the problems* (108), (109) *and* (122), (123), *respectively, we have*

$$\|z(t) - z_h(t)\|^2_{L^2(D)} \leq C\Big(\|z^{(0)} - z_h^{(0)}\|^2_{L^2(D)} + \int_0^t \|f_1(s) - f_2(s)\|_{L^2(D)} ds\Big). \tag{135}$$

**Proof.** For completeness we give this short proof. Since $t \mapsto z(t)$ and $t \mapsto z_h(t)$ are absolutely continuous functions, we get with the definition of the scalar product $[\![\xi, \zeta]\!]_D$ that

$$\begin{aligned}
\frac{d}{dt} |z(t) - z_h(t)|^2_{L^2(D)} &= 2[\![z_t(t) - z_{h,t}(t), z(t) - z_h(t)]\!]_D \\
&\leq -2\big(z_t(t) - z_{h,t}(t), (-Mz(t) + f_1(t)) - (-Mz_h(t) + f_2(t))\big)^*_D \\
&\qquad + 2(z_t(t) - z_{h,t}(t), f_1(t) - f_2(t))^*_D \\
&\leq 2\big(\|z_t(t)\|_{L^2(D)} + \|z_{h,t}(t)\|_{L^2(D)}\big)\|f_1(t) - f_2(t)\|_{L^2(D)}\,,
\end{aligned}$$

where we used that $z_t(t) \in G(-Mz(t)+f_1(t))$, $z_{h,t}(t) \in G(-Mz_h(t)+f_2(t))$, and noted that $G$ is monotone, by Corollary 3.1. Integration of the last inequality yields

$$\begin{aligned}
|z(t) - z_h(t)|^2_{L^2(D)} &\leq |z^{(0)} - z_h^{(0)}|^2_{L^2(D)} \\
&\quad +2\int_0^t \|f_1(s) - f_2(s)\|_{L^2(D)}\big(\|z_t(s)\|_{L^2(D)} + \|z_{h,t}(s)\|_{L^2(D)}\big)\, ds\,.
\end{aligned}$$

The statement of the lemma follows by combination of this inequality with equation (12) from Theorem 1.2, which yields with a constant $C$ independent of $h$ that

$$\|z_t(t)\|_{L^2(D)} + \|z_{h,t}(t)\|_{L^2(D)} \leq C\|z_t(t)\|_{L^2(D)} \leq C\|z\|_{W^{1,\infty}(0,T_e;L^2(\Omega))} < \infty.$$

□

To estimate the right hand side of (135) note that by definition of $f_1$ and $f_2$ in (111), (124) we have

$$\|f_2(t) - f_1(t)\|_{L^2(D)} \le C\big(\|\sigma(t) - \sigma_h(t)\|_{L^2(D)} + \|\hat\sigma(t) - \hat\sigma_h(t)\|_{L^2(D)} + \|\tilde\sigma_h(t)\|_{L^2(D)}\big)\,. \tag{136}$$

The theory of elliptic boundary value problems yields the standard estimate

$$\|\hat\sigma(t)-\hat\sigma_h(t)\|_{L^2(D)} \le C\|u(t)-u_h(t)\|_{H^{\frac12}(\partial D)} = C\|\hat u(t)-\hat u_h(t)\|_{H^{\frac12}(\Gamma)}\,, \tag{137}$$

where we applied (134) to get the last equality. The inequalities

$$\begin{aligned} \|\sigma(t) - \sigma_h(t)\|_{L^2(D)} &\le C\|b(t) - b_h(t)\|_{H^{-1}(D)}\,, && (138)\\ &\le C\hat h\|b\|_{L^\infty(0,T_e;L^2(D))} \le C_1\hat h\|b\|_{W^{2,1}(0,T_e,L^2(D))}\,, \\ \|\tilde\sigma_h(t)\|_{L^2(D)} &\le C\hat h, && (139)\\ \|z^{(0)} - z_h^{(0)}\|_{L^2(D)} &\le C\hat h, && (140) \end{aligned}$$

are obtained just as in Section 2.2. Combination of (136) – (140) and insertion into (135) yields

$$\|z(t) - z_h(t)\|^2_{L^2(D)} \le C\Big(\hat h^2 + \int_0^t \hat h + \|\hat u(s) - \hat u_h(s)\|_{H^{\frac12}(\Gamma)} ds\Big). \tag{141}$$

To estimate the right hand side we use the first estimate stated in the following lemma. The second estimate of the lemma is used later in the proof.

**Lemma 3.3.** *(i) There is a constant $C$ such that for all $v \in H^1(D)$ with $\nabla_\Gamma v \in H^1(D)$ and for all sufficiently small $\hat h > 0$*

$$\|v - v_h\|_{H^1(\Gamma)} \le \hat h^{1/2} C\|v\|_{H^2_\Gamma(D)}\,, \tag{142}$$

*where $\nabla_\Gamma v = (\partial_{x_1}v, \partial_{x_2}v)^T$, $v_h = v \circ \phi_h$ and $\|v\|^2_{H^2_\Gamma(D)} = \|v\|^2_{H^1(D)} + \|\nabla_\Gamma v\|^2_{H^1(D)}$.*
*(ii) There is a constant $C$ such that for all $w \in B^\beta_{2,2}(D)$ with $\beta \in (1, \frac32)$ and for all sufficiently small $\hat h > 0$ the inequality*

$$\|w - w_h\|_{L^2(\Gamma)} \le \hat h^{\beta-1/2} C\|w\|_{B^\beta_{2,2}(D)} \tag{143}$$

*holds, where $w_h = w \circ \phi_h$.*

**End of the proof of Theorem 1.6.** We apply (142) with $v = u(t)$ to obtain

$$\|\hat u(t) - \hat u_h(t)\|_{H^{\frac12}(\Gamma)} \le \|\hat u(t) - \hat u_h(t)\|_{H^1(\Gamma)} \le \hat h^{1/2} C\|u(t)\|_{H^2_\Gamma(D)}\,. \tag{144}$$

By (18) in Theorem 1.5 we have $u \in L^\infty(0,T_e;H^2_\Gamma(D))$. Hence, insertion of (144) into (141) results in

$$\begin{aligned}\|z(t)-z_h(t)\|^2_{L^2(D)} &\le C\Big(\hat{h}^2+\int_0^{T_e}\hat{h}+\hat{h}^{\frac12}\|u(s)\|_{H^2_\Gamma(D)}\,ds\Big)\\ &\le C_1\hat{h}^{\frac12}\big(1+\|u\|_{L^\infty(0,T_e;H^2_\Gamma(D))}\big)<C_2\hat{h}^{\frac12}. \qquad (145)\end{aligned}$$

We divide by $\hat{h}^{1/2}$ and note that $\phi_h(y)=y+h$ on $V\cap D$ to get for $0\le t\le T_e$

$$\frac{\|z(t)-z(\cdot+h,t)\|_{L^2(V\cap D)}}{\hat{h}^{1/4}}\le\sqrt{C_2}\,. \qquad (146)$$

Since $(u(t)-u_h(t),T(t)-T_h(t))$ solves the boundary value problem (105) – (107) to the data $\hat{b}=b(t)-b_h(t)+\hat{h}(\partial_{\xi_i}T)_h\,\nabla^*_y\varphi$, $\hat{\varepsilon}_p=B(z(t)-z_h(t))+\hat{h}\varepsilon((u_{\xi_i})_h\otimes\nabla^*_y\varphi)$, and to $\hat{\gamma}$ given by the right hand side of (134), we conclude in virtue of (144) and (145), applying the theory of elliptic boundary value problems in a by now standard way, that

$$\begin{aligned}&\|T(t)-T_h(t)\|_{L^2(D)}+\|u(t)-u_h(t)\|_{H^1(D)}\\ &\quad\le C\hat{h}\|b(t)\|_{L^2(D)}+\hat{h}^{\frac14}\sqrt{C_2}+C_3\hat{h}^{\frac12}\|u(t)\|_{H^2_\Gamma(D)} \qquad (147)\\ &\quad\le C_4\hat{h}^{\frac14}\big(1+\|b\|_{L^\infty(0,T_e;L^2(D))}+\|u\|_{L^\infty(0,T_e;H^2_\Gamma(D))}\big)\le C_5\hat{h}^{\frac14},\end{aligned}$$

for all $0\le t\le T_e$. It is a technical matter to prove analogous estimates for $h=\hat{h}e_3$ with $\hat{h}<0$. Hence, (146) and (147) imply by the definition of Besov spaces in Definition A.1 that

$$z\in L^\infty(0,T_e;B^{1/4}_{2,\infty}(V\cap D)),\quad (u,T)\in L^\infty(0,T_e;B^{5/4}_{2,\infty}(V\cap D)\times B^{1/4}_{2,\infty}(V\cap D)). \qquad (148)$$

To obtain (20) and (21) we transform $(u,T,z)$ from the domain $D$ back to $B\cap\Omega$ and extend this local regularity result to the global result on $\Omega$ by the usual technique.

To prove (22) and (23) we use a bootstrap argument. From (148) and from the embedding properties of Besov spaces in Lemma A.2 we infer that

$$\begin{aligned}&z\in L^\infty(0,T_e;B^{1/4-\delta}_{2,2}(V\cap D)),\\ &(u,T)\in L^\infty\big(0,T_e;B^{5/4-\delta}_{2,2}(V\cap D)\times B^{1/4-\delta}_{2,2}(V\cap D)\big), \qquad (149)\end{aligned}$$

for any $\delta>0$. Define $\alpha_1=5/4$ and apply (143) with $w=u$ and $\beta=\alpha_1-\delta$ to conclude from (149) that

$$\|\hat{u}(t)-\hat{u}_h(t)\|_{L^2(\Gamma)}\le\hat{h}^{\alpha_1-\delta-1/2}C\|u(t)\|_{B^{\alpha_1-\delta}_{2,2}(V\cap D)}.$$

We interpolate between the spaces $L^2(\Gamma)$ and $H^1(\Gamma)$ and apply (144) and the last inequality to obtain

$$\begin{aligned}\|\hat{u}(t)-\hat{u}_h(t)\|_{H^{\frac12}(\Gamma)} &\le \|\hat{u}(t)-\hat{u}_h(t)\|^{1/2}_{L^2(\Gamma)}\|\hat{u}(t)-\hat{u}_h(t)\|^{1/2}_{H^1(\Gamma)}\\ &\le \hat{h}^{(\alpha_1-\delta)/2}C\|u(t)\|^{1/2}_{B^{\alpha_1-\delta}_{2,2}(V\cap D)}\|u(t)\|^{1/2}_{H^2_\Gamma(D)}.\end{aligned}$$

By the previous argument we conclude that

$$z \in L^\infty(0, T_e; B_{2,2}^{\alpha_1/4-\delta/4-\delta_1}(V \cap D)),$$
$$(u, T) \in L^\infty\big(0, T_e; B_{2,2}^{1+\alpha_1/4-\delta/4-\delta_1}(V \cap D) \times B_{2,2}^{\alpha_1/4-\delta/4-\delta_1}(V \cap D)\big),$$

for any $\delta_1 > 0$. Further iteration yields a monotone increasing sequence $\{\alpha_n\}_{n=1}^\infty$ defined by $\alpha_{n+1} = 1 + \alpha_n/4$ such that

$$z \in L^\infty(0, T_e; B_{2,2}^{\alpha_n/4-\hat{\delta}}(V \cap D)),$$
$$(u, T) \in L^\infty\big(0, T_e; B_{2,2}^{1+\alpha_n/4-\hat{\delta}}(V \cap D) \times B_{2,2}^{\alpha_n/4-\hat{\delta}}(V \cap D)\big),$$

for any $\hat{\delta} > 0$. For $\alpha_n \leq \frac{4}{3}$ we have $\alpha_{n+1} - \alpha_n = 1 + \frac{\alpha_n}{4} - \alpha_n \geq 0$. Since the fixed point of the mapping $\alpha_n \mapsto 1 + \frac{\alpha_n}{4}$ is $\frac{4}{3}$, it follows that

$$\frac{5}{4} = \alpha_1 < \ldots < \alpha_n < \alpha_{n+1} \to \frac{4}{3}, \quad n \to \infty.$$

Therefore we conclude that

$$z \in L^\infty(0, T_e; B_{2,2}^{1/3-\delta}(V \cap D)),$$
$$(u, T) \in L^\infty\big(0, T_e; B_{2,2}^{1+1/3-\delta}(V \cap D) \times B_{2,2}^{1/3-\delta}(V \cap D)\big),$$

for any $\delta > 0$. Extension of this local regularity result to a global result yields (22), (23). This completes the proof of Theorem 1.6. □

**Proof of Lemma 3.3.** Inequality (142) follows immediately if we can show that

$$\|v - v_h\|_{L^2(\Gamma)} \leq \hat{h}^{\frac{1}{2}} \|u\|_{H^1(D)}, \quad \|\nabla_\Gamma (v - v_h)\|_{L^2(\Gamma)} \leq \hat{h}^{\frac{1}{2}} C \|v\|_{H^2_\Gamma(D)} . \tag{150}$$

To prove the first inequality in (150) let $\alpha, \beta > 0$ be the constants from the definition of $D$ in (97). Since $\phi_h(y) = y + \hat{h}\varphi(y)e_3$ and $0 \leq \varphi(y) \leq 1$ we have for $y' \in \mathbb{R}^2$ with $|y'| < \alpha$ and for $0 < \hat{h} \leq \beta$

$$\begin{aligned} |v(y', 0) - v_h(y', 0)| &= |v(y', 0) - v(y', \hat{h}\varphi(y', 0))| \\ &= \Big| \int_0^{\hat{h}\varphi(y',0)} \partial_{y_3} v(y', y_3) dy_3 \Big| \leq \hat{h}^{\frac{1}{2}} \Big( \int_0^{\hat{h}} \big|\partial_{y_3} v(y', y_3)\big|^2 dy_3 \Big)^{\frac{1}{2}}. \end{aligned}$$

We square both sides and integrate with respect to $y'$ over the ball $\{|y'| < \alpha\}$ to obtain the first inequality in (150).

To prove the second inequality in (150) we note that

$$\begin{aligned} \nabla_\Gamma v - \nabla_\Gamma v_h &= \nabla_\Gamma v - \big(\partial_{y_j}(v_i \circ \phi_h)\big)_{i=1,\ldots 3, j=1,2} \\ &= \nabla_\Gamma v - (\nabla_\Gamma v)_h - (\partial_{y_3} v)_h \otimes \hat{h} \nabla_\Gamma \varphi . \end{aligned} \tag{151}$$

If we replace $v$ by $\nabla_\Gamma v$ in the first inequality of (150) we obtain

$$\|\nabla_\Gamma v - (\nabla_\Gamma v)_h\|_{L^2(\Gamma)} \le \hat{h}^{\frac{1}{2}} \|\nabla_\Gamma v\|_{H^1(D)}. \tag{152}$$

To estimate the term $\|(\partial_{y_3} v)_h \otimes \hat{h}\nabla_\Gamma \varphi\|_{L^2(\Gamma)}$ we employ the change of variables formula given in [15, Theorem 2, p. 117]. For the convenience of the reader we state this theorem here.

**Theorem 3.4.** *Let $q : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \ge m$. Then for each $\mathcal{L}^n$-summable function $r : \mathbb{R}^n \to \mathbb{R}$,*

$$r|_{q^{-1}(y)} \text{ is } \mathcal{H}^{n-m} \text{ summable for } \mathcal{L}^m \text{ a.e. } y$$

*and*

$$\int_{\mathbb{R}^n} r(x)\, Jq(x)\, dx = \int_{\mathbb{R}^m} \Big[ \int_{q^{-1}(y)} r\ d\mathcal{H}^{n-m} \Big] dy.$$

*Here $Jq = \sqrt{\det\big((\nabla q)(\nabla q)^T\big)}$ is the Jacobian and $\mathcal{H}^s$ is the s-dimensional Hausdorff measure.*

We apply this theorem with $n = 2$, $m = 1$ and with

$$q(y') = \hat{h}\varphi(y', 0), \quad r(y') = \hat{h}\, \big|\partial_{y_3} v(y', \hat{h}\varphi(y', 0))\big|^2 \, \big|\nabla_\Gamma \varphi(y', 0)\big|.$$

Using that

$$Jq(y') = \sqrt{\det\big[\hat{h}^2 \big(\nabla_\Gamma \varphi(y', 0)\big) \cdot \big(\nabla_\Gamma \varphi(y', 0)\big)\big]} = \hat{h}\, |\nabla_\Gamma \varphi(y', 0)|\,,$$

and that $|(\partial_{y_3} v)_h \otimes \hat{h}\nabla_\Gamma \varphi| = \hat{h}\, |(\partial_{y_3} v)_h|\, |\nabla_\Gamma \varphi|$, we obtain in virtue of Theorem 3.4 that

$$\|(\partial_{y_3} v)_h \otimes \hat{h}\nabla_\Gamma \varphi\|^2_{L^2(\Gamma)} = \int_\Gamma \hat{h} |(\partial_{y_3} v)_h|^2 |\nabla_\Gamma \varphi|\, \hat{h} |\nabla_\Gamma \varphi|\, dy'$$
$$= \int_\Gamma r(y')\, Jq(y')\, dy' = \int_0^{\hat{h}} \Big[ \int_{q^{-1}(\xi)} r\ d\mathcal{H}^1 \Big] d\xi = \hat{h} \int_0^1 \Big[ \int_{q^{-1}(\hat{h}\zeta)} r\ d\mathcal{H}^1 \Big] d\zeta\,, \tag{153}$$

where

$$q^{-1}(\hat{h}\zeta) = \{ y' \in \Gamma \mid \varphi(y', 0) = \zeta \} = \ell_\zeta\,.$$

By choosing $\varphi$ suitably we can achieve that $\ell_\zeta$ is a smooth curve for every $\zeta \in (0, 1)$; we can even assume that $\ell_\zeta$ is a circle. Note that $\ell_\zeta$ is independent of $\hat{h}$ and encloses the set $V \cap \Gamma$. From the Sobolev imbedding theorem we thus conclude that

$$\int_{q^{-1}(\hat{h}\zeta)} r\ d\mathcal{H}^1 = \int_{\ell_\zeta} \hat{h}\, \big|\partial_{y_3} v(y', \hat{h}\zeta)\big|^2 \, \big|\nabla_\Gamma \varphi(y', 0)\big|\, ds_{y'}$$
$$\le \hat{h} C \|\partial_{y_3} v(\cdot, \hat{h}\zeta)\|^2_{L^2(\ell_\zeta)} \le \hat{h} C_1 \|\partial_{y_3} v(\cdot, \hat{h}\zeta)\|^2_{H^1(\Gamma)}\,,$$

with a constant $C_1$, which can be chosen independent of $\zeta$. Insertion into (153) yields

$$\|(\partial_{y_3} v)_h \otimes \hat{h}\nabla_\Gamma \varphi\|^2_{L^2(\Gamma)} \leq \hat{h} \int_0^1 \hat{h} C_1 \|\partial_{y_3} v(\cdot, \hat{h}\zeta)\|^2_{H^1(\Gamma)}\, d\zeta$$

$$= \hat{h} C_1 \int_0^{\hat{h}} \|\partial_{y_3} v(\cdot, \xi)\|^2_{H^1(\Gamma)}\, d\xi \leq \hat{h} C_1 \|v\|^2_{H^2_\Gamma(D)} .$$

Combining this estimate with (151) and (152) yields the second inequality in (150) and completes the proof of (142).

To verify (143) we observe that Taylor's formula yields for $w \in C^2(\overline{D}, \mathbb{R}^3)$ satisfying $\partial_{y_3} w|_\Gamma = 0$ that

$$\begin{aligned} |w(y',0) - w_h(y',0)| &= |w(y',0) - w(y', \hat{h}\varphi(y',0))| \\ &\leq \Big| \int_0^{\hat{h}} (\hat{h} - y_3) \partial^2_{y_3} w(y', y_3) dy_3 \Big| \\ &\leq \Big( \int_0^{\hat{h}} (\hat{h} - y_3)^2 dy_3 \Big)^{\frac{1}{2}} \Big( \int_0^{\hat{h}} |\partial^2_{y_3} w(y', y_3)|^2 dy_3 \Big)^{\frac{1}{2}} . \end{aligned}$$

We square both sides, integrate with respect to $y'$ over the ball $\{|y'| < \alpha\}$ and use that $\int_0^{\hat{h}} (\hat{h} - y_3)^2 dy_3 = \frac{1}{3}\hat{h}^3$ to obtain

$$\|w - w_h\|_{L^2(\Gamma)} \leq \frac{\hat{h}^{3/2}}{\sqrt{3}} \|w\|_{H^2(D)}.$$

This inequality holds for any $w$ from the closure $X^2 \subseteq H^2(D, \mathbb{R}^3)$ of the linear space $\{ w \in C^2(\overline{D}, \mathbb{R}^3) \mid \partial_{y_3} w|_\Gamma = 0 \}$ with respect to the $H^2$–norm. $X^2$ consists of all $w \in H^2(D, \mathbb{R}^3)$ satisfying $\partial_{y_3} w|_\Gamma = 0$ in the sense of traces. On the other hand, the first inequality in (150) yields for $w \in H^1(D, \mathbb{R}^3)$ that

$$\|w - w_h\|_{L^2(\Gamma)} \leq \hat{h}^{1/2} \|w\|_{H^1(D)}.$$

Let $\beta \in (1, 2)$. We interpolate between the last two inequalities with $\theta = \beta - 1$ and obtain

$$\|w - w_h\|_{L^2(\Gamma)} \leq \hat{h}^{(1-\theta)/2 + 3\theta/2} C \|w\|_{B^\beta_{2,2}(D)} = \hat{h}^{\beta - 1/2} C \|w\|_{B^\beta_{2,2}(D)} , \tag{154}$$

where we used that the interpolation space $X^\beta$ between $H^1(D) = B^1_{2,2}(D)$ and $X^2 \subseteq H^2(D) = B^2_{2,2}(D)$ is a subspace of $B^\beta_{2,2}(D)$. It is an easy corollary of [24, Theorem 11.5] that $X^\beta = B^\beta_{2,2}(D, \mathbb{R}^3)$ if $\beta \in (1, \frac{3}{2})$. This shows that (154) holds for all $w \in B^\beta_{2,2}(D, \mathbb{R}^3)$. The proof of Lemma 3.3 is complete. □

# A Besov spaces

Here we give the definition of Besov spaces $B^s_{p,\theta}(\Omega, \mathbb{R}^N)$ and state a few basic properties of these spaces, which we need in Section 3 to prove Theorem 1.6. Detailed expositions are given for example in [11, 31, 41, 42].

For $h \in \mathbb{R}^n$ and an open set $\Omega \subseteq \mathbb{R}^n$ we define

$$\Omega_h = \bigcap_{j=0}^{1} \{x \in \Omega \mid x + jh \in \Omega\}.$$

**Definition A.1.** *Let $1 \leq p, \theta \leq \infty$, $s \geq 0$ and let $\ell \in \mathbb{N}$ with $\ell > s$. The function $f$ belongs to the Besov (Nikol'skii–Besov) space $B^s_{p,\theta}(\Omega) = B^s_{p,\theta}(\Omega, \mathbb{R}^N)$ with order of smoothness $s$, if $f$ is measurable on $\Omega$ and satisfies*

$$\|f\|_{B^s_{p,\theta}(\Omega)} = \|f\|_{L^p(\Omega)} + \|f\|_{b^s_{p,\theta}(\Omega)} < \infty\,,$$

*where*

$$\|f\|_{b^s_{p,\theta}(\Omega)} = \begin{cases} \left(\displaystyle\int_{\mathbb{R}^n} \left(\frac{\|\triangle^\ell_h f\|_{L^p(\Omega_h)}}{|h|^s}\right)^\theta \frac{dh}{|h|^n}\right)^{1/\theta}, & \text{for } 1 \leq \theta < \infty\,, \\ \displaystyle\sup_{h \in \mathbb{R}^n, h \neq 0} \frac{\|\triangle^\ell_h f\|_{L^p(\Omega_h)}}{|h|^s}, & \text{for } \theta = \infty\,. \end{cases}$$

The $\ell$–th order difference operator $\triangle^\ell_h$ is defined by $\triangle_h f(x) = f(x + h) - f(x)$ and $\triangle^\ell_h f(x) = \triangle_h(\triangle^{\ell-1}_h f(x))$. Of course, the norm $\|f\|_{b^s_{p,\theta}(\Omega)}$ depends on the choice of $\ell$, but for $\ell > s$ all norms are equivalent, cf. [11]. There exist other equivalent norms on the space $B^s_{p,\theta}(\Omega)$, but this one is the most convenient for our purposes. The spaces $\mathcal{N}^s_p(\Omega) := B^s_{p,\infty}(\Omega)$ and $W^{s,p}(\Omega) := B^s_{p,p}(\Omega)$ are called in the literature Nikol'skii and Sobolev-Slobodeckij spaces, respectively.

**Lemma A.2.** *If $s \geq 0$, $0 < \epsilon < s$, $1 \leq p \leq \infty$, $1 \leq \theta_1 \leq \theta_2 \leq \infty$ one has*

$$B^{s+\epsilon}_{p,\infty}(\Omega) \subseteq B^s_{p,1}(\Omega) \subseteq B^s_{p,\theta_1}(\Omega) \subseteq B^s_{p,\theta_2}(\Omega) \subseteq B^s_{p,\infty}(\Omega) \subseteq B^{s-\epsilon}_{p,1}(\Omega).$$

Proofs of these embedding results can be found in [31, Section 6.2] or [42, Section 2.8].

Next we define the spaces $B^s_{p,p}(\partial\Omega) = W^{s,p}(\Omega)$. Assume that $\Omega$ is a bounded open subset of $\mathbb{R}^n$ such that $\partial\Omega \in C^m$. A family of pairs $\{(U_j, \phi_j)\}_{j=1}^k$ with $U_j \subseteq \partial\Omega$ and $\phi_j : U_j \to \mathbb{R}^{n-1}$ is called a $C^m$–atlas of $\partial\Omega$, if $\partial\Omega = \bigcup_{j=1}^k U_j$, if $\Gamma_j = \phi_j(U_j)$ is an open subset of $\mathbb{R}^{n-1}$ and if $\phi_j^{-1} : \Gamma_j \to U_j \subset \mathbb{R}^n$ is an $m$–times continuously differentiable parametrization of $U_j$ for every $j$. A set $\{\eta_j\}_{j=1}^k$ of functions $\eta_j : U_j \to \mathbb{R}$ is called a $C^m$–partition of unity on $\partial\Omega$ subordinate to the $C^m$–atlas $\{(U_j, \phi_j)\}_{j=1}^k$, if $\eta_j \circ \phi_j^{-1} \in C^m_0(\Gamma_j)$, if $\eta_j \geq 0$ and if $\sum_{j=1}^k \eta_j = 1$ on $\partial\Omega$.

**Definition A.3.** *Assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded open subset with $C^m$–boundary $\partial\Omega$. Let $\{\eta_j\}_{j=1}^k$ be a $C^m$–partition of unity on $\partial\Omega$ subordinate to a $C^m$–atlas $\{(U_j,\phi_j)\}_{j=1}^k$ with $\Gamma_j = \phi_j(U_j)$. For $m \geq s \geq 0$ and $1 \leq p \leq \infty$ the space $B^s_{p,p}(\partial\Omega,\mathbb{R}^N)$ consists of all functions $f : \partial\Omega \to \mathbb{R}^N$ such that*

$$\|f\|^p_{B^s_{p,p}(\partial\Omega)} = \sum_{j=1}^k \|(f\eta_j)\circ\phi_j^{-1}\|^p_{B^s_{p,p}(\Gamma_j)} < \infty\,.$$

For $m \in \mathbb{N}$, $1 \leq p \leq \infty$ and open sets $\Omega \subseteq \mathbb{R}^n$ with $C^m$–boundary we have

$$\mathrm{trace}_{\partial\Omega}\, B^m_{p,p}(\Omega) = \begin{cases} B^{m-1/p}_{p,p}(\partial\Omega), & \text{for } m > \frac{1}{p}, \\ L^1(\partial\Omega), & \text{for } m = p = 1, \end{cases}$$

with continuous trace operators $T : B^m_{p,p}(\Omega \to B^{m-1/p}_{p,p}(\partial\Omega)$ and $T : B^1_{1,1} \to L^1(\partial\Omega)$, cf. [11, Theorem 8, Section 5.5]. In particular, for $m = 1$ and $p = 2$ we obtain

$$\mathrm{trace}_{\partial\Omega}\, H^1(\Omega) = \mathrm{trace}_{\partial\Omega}\, B^1_{2,2}(\Omega) = B^{1/2}_{2,2}(\partial\Omega) = H^{1/2}(\partial\Omega).$$

# B Proof of inequality (50)

Here we give the proof of inequality (50).

**Lemma B.1.** *If $b \in W^{2,1}(0,T_e;L^2(\Omega,\mathbb{R}^3))$, then*

$$\|b - b_h\|_{W^{2,1}(0,T_e;H^{-1}(\Omega))} \leq \hat{h}C\|b\|_{W^{2,1}(0,T_e;L^2(\Omega))}\,,$$

*where $b_h = b \circ \phi_h$.*

**Proof.** In order to prove this lemma it suffices to show that

$$\|\partial_t^i(b - b_h)\|_{L^1(0,T_e;H^{-1}(\Omega))} \leq \hat{h}C\|\partial_t^i b\|_{L^1(0,T_e;L^2(\Omega))}\,,$$

for $i = 0,1,2$. We only consider the case $i = 0$, since the inequalities for $i = 1,2$ follow from this case if we replace $b$ by $\partial_t^i b$.

Since $b - b_h \in L^1(0,T_e;H^{-1}(\Omega,\mathbb{R}^3))$ is an element from the dual space of $L^\infty(0,T_e;H^1_0(\Omega,\mathbb{R}^3))$, the desired result follows immediately if we show that the inequality

$$|(b - b_h, v)_{\Omega\times(0,T_e)}| \leq \hat{h}C\|b\|_{L^1(0,T_e;L^2(\Omega))}\|v\|_{L^\infty(0,T_e;H^1_0(\Omega))} \tag{155}$$

holds for all $v \in L^\infty(0,T_e;H^1_0(\Omega,\mathbb{R}^3))$. To prove this inequality, we note that

$$\int_0^{T_e}\int_\Omega \big(b(x,t) - b(\phi_h(x),t)\big)v(x,t)dxdt$$
$$= \int_0^{T_e}\int_\Omega b(x,t)\big(v(x,t) - v(\phi_h^{-1}(x),t)\mathcal{I}_h(x)\big)dxdt, \tag{156}$$

where $\mathcal{I}_h(x) = |\det(\nabla\phi_h^{-1}(x))|$. Some considerations show that $\mathcal{I}_h(x) = 1 + O(\hat{h})$, uniformly with respect to $x$. Thus, from (156),

$$
\begin{aligned}
&|(b - b_h, v)_{\Omega\times(0,T_e)}| \\
&\quad \le \|b\|_{L^1(0,T_e;L^2(\Omega))} \sup_{0\le t\le T_e} \Big( \int_\Omega |v(x,t) - v(\phi_h^{-1}(x),t)\mathcal{I}_h(x)|^2 dx \Big)^{1/2}. \quad (157)
\end{aligned}
$$

The triangle inequality and the fundamental theorem of calculus yield

$$
\begin{aligned}
&\sqrt{\int_\Omega |v(x,t) - v(\phi_h^{-1}(x),t)\mathcal{I}_h(x)|^2 dx} \\
&\le \sqrt{\int_\Omega |v(x,t) - v(\phi_h^{-1}(x),t)|^2 dx\Big)} + \sqrt{\int_\Omega |v(\phi_h^{-1}(x),t)(1 - \mathcal{I}_h(x))|^2 dx} \\
&\le \sup_{x\in\Omega} |\phi_h^{-1}(x) - x| \sqrt{\int_\Omega \int_0^1 |\nabla v\big(x + s(\phi_h^{-1}(x) - x), t\big)|^2 ds\, dx} \\
&\qquad\qquad + \hat{h}C\|v(t)\|_{L^2(\Omega)} \\
&\le \hat{h}C\|v(t)\|_{H_0^1(\Omega)} \le \hat{h}C\|v\|_{L^\infty(0,T_e;H_0^1(\Omega))}\,.
\end{aligned}
$$

We use this inequality to estimate the right hand side of (157) and obtain (155). □

**Acknowledgement.** The authors thank Dorothee Knees for pointing out an error in the proof of Lemma 3.3 in a previous version of this manuscript.

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