Solvable Subgroups of Locally Compact Groups

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Abstract. It is shown that a closed solvable subgroup of a connected Lie group is compactly generated. In particular, every discrete solvable subgroup of a connected Lie group is finitely generated. Generalizations to locally compact groups are discussed as far as they carry.
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A topological group G with identity component G_0 is said to be *almost connected* if G/G_0 is compact. We shall prove the following result.

Main Theorem. *A closed solvable subgroup of a locally compact almost connected group is compactly generated.*

This result belongs to a class of "descent" type results that are on record for compactly generated groups. The essay [8] provides a good background of their history. It follows, in particular, that *a discrete solvable subgroup of an almost connected locally compact group is finitely generated.*

Example S. The connected simple Lie group $\mathrm{PSL}(2,\mathbb{R})$ contains a discrete free group of infinite rank; such a closed subgroup is not compactly generated. □

We remark that a nonabelian free group is countably nilpotent (see e.g. [4], Definition 10.5); that is, the descending central series terminates at the singleton subgroup after ω steps. The Main Theorem therefore fails for transfinitely solvable subgroups in place of solvable ones.

The following example shows that subgroups of finitely generated solvable groups need not be finitely generated:

Example SOL. Let $\Gamma \subseteq \mathbb{Q} \rtimes \mathbb{Q}^\times$ be the subgroup generated by the two elements $a := (0,2)$ and $b := (1,0)$. Then

$$\Gamma \cong \left(\frac{1}{2^\infty}\cdot\mathbb{Z}\right) \rtimes \mathbb{Z},$$

is a 2-generator metabelian group, while the abelian subgroup $\frac{1}{2^\infty}\cdot\mathbb{Z} \times \{0\}$ is not finitely generated. □

Thus, in the Main Theorem, the hypothesis "G/G_0 compact" cannot be relaxed to "G/G_0 compactly generated".

For *abelian* subgroups the Main Theorem will allow us to derive a characteration theorem for compactly generated locally compact abelian groups as follows.

Theorem. *For a locally compact abelian group A the following conditions are equivalent:*

(1) *A is compactly generated.*
(2) *$A \cong \mathbb{R}^k \oplus C \oplus \mathbb{Z}^n$ for a unique largest compact subgroup C and natural numbers k, n.*
(3) *The character group $\widehat{A}$ is a Lie group.*
(4) *There is an almost connected locally compact group G and a closed subgroup H such that $A \cong H$.*

Proof. (1) $\Rightarrow$(2): See e.g. [3], Theorem 7.57(ii).

(2) $\Rightarrow$(3): $\widehat{A} \cong \widehat{\mathbb{R}^k} \oplus \widehat{C} \oplus \widehat{\mathbb{Z}^n} \cong \mathbb{R}^k \oplus D \oplus \mathbb{T}^n$ for a discrete abelian group D. This is a Lie group.

(3) $\Rightarrow$(2): If $\widehat{A}$ is a Lie group, then $(\widehat{A})_0$ is open and isomorphic to $\mathbb{R}^k \oplus \mathbb{T}^n$ for some k and n; it is divisible, whence $\widehat{A} \cong (\mathbb{R}^k \oplus \mathbb{T}^n) \oplus D$ for a discrete subgroup D. Hence $A = \widehat{\mathbb{R}^k} \oplus \widehat{D} \oplus \widehat{\mathbb{T}^n} \cong \mathbb{R}^k \oplus C \oplus \mathbb{Z}^n$ for the unique largest compact subgroup C of A.

(2) $\Rightarrow$(4): $A \subseteq \mathbb{R}^k \times C \times \mathbb{R}^n \cong \mathbb{R}^{k+n} \oplus C$, an almost connected locally compact group.

(4) $\Rightarrow$(1): Let G be an almost connected locally compact group and A a closed abelian subgroup. Then A is, in particular, solvable. Hence the Main Theorem provides the required implication. □

By comparison with Example SOL, the situation for abelian groups is distinctly simpler than it is for metabelian groups:

Corollary. (Morris' Theorem [5], [8]) *A closed subgroup of a compactly generated locally compact abelian group is compactly generated.*

Proof. We proved (2)$\Leftrightarrow$(3) in the Theorem independently of the Main Theorem. Thus if G is a locally compact compactly generated abelian group, then $\widehat{G}$ is an abelian Lie group. The character group $\widehat{A}$ of a closed subgroup A of G, by duality, is a quotient of the Lie group $\widehat{A}$ and thus is a Lie group. Hence A is compactly generated. □

As we now begin a proof of the main theorem we first reduce it to one on connected Lie groups and its closed subgroups:

Reduction. *The* Main Theorem *holds if every closed solvable subgroup H of a connected Lie group G is compactly generated.*

Proof. Indeed let G be an almost connected locally compact group and N a compact normal subgroup such that G/N is a Lie group. The existence of N is a consequence of Yamabe's Theorem saying that each almost connected locally compact group is a pro-Lie group ([9,10]). Then HN is a closed subgroup and HN/N is a closed solvable subgroup A of the Lie group $L = G/N$ with finitely many components. If our claim is true for connected Lie groups G, then $A \cap L_0$ is compactly generated. We may assume $L = L_0 A$. Then $A \cap L_0$ has finite index in A. Therefore $A = HN/N$ is compactly generated. Then HN is compactly generated. So H is compactly generated. (See [1], Chap. VII, §3, Lemma 3. Also see [8].) □

This reduction allows us to concentrate on connected Lie groups G and closed solvable subgroups H. Since any locally compact connected group, and so in particular every connected Lie group, is compactly generated we shall have to prove that $\pi_0(H) \stackrel{\text{def}}{=} H/H_0$ is finitely generated.

Lemma 1. *For a closed subgroup H of a connected* solvable *connected Lie group G any subgroup of $\pi_0(H)$ is finitely generated.*

Proof. This is proved in [7], Proposition 3.8. □

This shows that the two generator metabelian group Γ of Example SOL cannot be realized as $\pi_0(H)$ for a closed subgroup H of a connected solvable Lie group G—let alone be discretely embedded into G.

Lemma 2. *Let*

$$\mathbf{1} \to A \to B \xrightarrow{q} C \to \mathbf{1}$$

be a short exact sequence of groups. If A and C have the property that each subgroup is finitely generated, then B has this property as well.

Proof. Each subgroup $\Gamma \subseteq B$ is an extension of the finitely generated group $q(\Gamma)$ by the finitely generated group $A \cap \Gamma$, hence is finitely generated itself. □

Lemma 3. *Assume that the solvable Lie group G has the property that each subgroup of $\pi_0(G)$ is finitely generated. Let H be a closed subgroup of G. Then each subgroup of $\pi_0(H)$ is finitely generated.*

Proof. Let $q: G \to \pi_0(G)$ denote the quotient map. Then we have a short exact sequence

$$\mathbf{1} \to \pi_0(H \cap G_0) \to \pi_0(H) \to q(H) \to \mathbf{1}.$$

As a subgroup of $\pi_0(G)$, the group $q(H)$ has the property that all its subgroups are finitely generated, and the group $\pi_0(H \cap G_0)$ has this property by Lemma 1. Now Lemma 2 implies that each subgroup of $\pi_0(H)$ is finitely generated. □

Lemma 4. *If H is a closed solvable subgroup of $\mathrm{GL}_n(\mathbb{C})$, then each subgroup of $\pi_0(H)$ is finitely generated.*

Proof. Let S denote the Zariski closure of H. Then S is a solvable linear algebraic group, so that $\pi_0(S)$ is finite (see e.g. [6], Theorems 3.1.1 and 3.3.1). Since H is a closed subgroup of the Lie group S, the assertion follows from Lemma 3. □

In order to proceed we need a further line of lemmas. We shall call a Lie group *linear* if it has a faithful linear representation. The following statement is of independent interest.

Proposition 5. *A connected linear Lie group has a faithful linear representation with a closed image.*

Proof. By [2], Theorem IV.3 a connected Lie group G is linear if and only if it is isomorphic to a semidiret product $B\rtimes_\alpha H$ where B is a simply connected solvable Lie group and H is a linear reductive Lie group with compact center. We set $G = B\rtimes_\alpha H$ and deduce that the commutator subgroup G' equals $(G,B)\rtimes(H,H)$. From [2], Theorem IV.5 it follows that G' is closed in G. The quotient group G/G' is a direct product

$$\frac{B}{(G,B)} \times \frac{H}{(H,H)} \cong \frac{B}{(G,B)} \times \frac{Z(H)_0}{(Z(H)_0 \cap (H,H))},$$

where $B/(G,B)$ is a vector group and $Z(H)_0/(Z(H)_0 \cap (H,H))$ is a torus. This group has a representation mapping the vector group $B/(G,B)$ homeomorphically on a unipotent subgroup. That is, we have a representation $\rho\colon G \to \mathrm{GL}(W)$ such that

$$\ker\rho = (G,B)H \text{ and } \overline{\operatorname{im}\rho} = \operatorname{im}\rho, \tag{1}$$

the image being unipotent.

Now let $\pi\colon G \to \mathrm{GL}(V)$ be a faithful linear representation and define $\zeta = \pi \oplus \rho$. We shall show that ζ has a closed image. Suppose this is not the case. Then there is an $X \in \mathfrak{g}$ such that $T \overset{\text{def}}{=} \overline{\zeta(\exp\mathbb{R}{\cdot}X)}$ is a torus not contained in $\zeta(G)$ (see [2], Proposition XVI.2.3 and Theorem XVI.2.4). In the Appendix we shall show that, under any representation of a connected Lie group G, the commutator subgroup G' has a closed image. Thus $\zeta(G')$ is closed and $\zeta(Z(H))$ is compact since H has a compact center. Thus $\zeta(G'Z(H)) = \zeta(G')\zeta(Z(H))$ is closed and contained in $\zeta(G)$. Accordingly, X cannot be contained in $\mathfrak{g}' + \mathfrak{z}(\mathfrak{h}) = [\mathfrak{g},\mathfrak{b}] + \mathfrak{h}$. Thus by (1), $\exp\mathbb{R}{\cdot}X$ fails to be in $\ker\rho$. It follows that $\rho\circ\exp$ maps $\mathbb{R}{\cdot}X$ homeomorphically onto a unipotent one-parameter group. Then $\zeta\circ\exp$ maps $\mathbb{R}{\cdot}X$ homeomorphically as well, and that contradicts the fact that T is a torus. This contradiction proves the proposition. □

We now complete the proof of the Main Theorem by proving the last lemma:

Lemma 6. *Let G be a connected Lie group and H a closed solvable subgroup. Then H is compactly generated.*

Proof. Let $Z = Z(G)$ be the center of G. Then $A \overset{\text{def}}{=} \overline{ZH}$ is a closed solvable subgroup of G containing H. By Lemma 3 for H to be compactly generated it will suffice to show that all subgroups of $\pi_0(A) = A/A_0$ are finitely generated. Let A_1 be a subgroup of A containing A_0. Then A_1 is open in A, and so A_1Z is open and thus closed in A. Therefore

$$A_1/(A_1 \cap (A_0Z)) \cong A_1Z/A_0Z. \tag{1}$$

By the modular law,

$$A_1 \cap (A_0 Z) = A_0(A_1 \cap Z). \tag{2}$$

We have the following isomorphism of discrete groups

$$A_0(A_1 \cap Z)/A_0 \cong (A_1 \cap Z)/(A_0 \cap (A_1 \cap Z)) = (A_1 \cap Z)/(A_0 \cap Z). \tag{3}$$

Taking (1), (2) and (3) together we recognize the following exact sequence

$$\mathbf{1} \to \frac{A_1 \cap Z}{A_0 \cap Z} \to \frac{A_1}{A_0} \to \frac{A_1 Z}{A_0 Z} \to \mathbf{1}. \tag{4}$$

In order to show that A_1/A_0 is finitely generated it therefore suffices that
(a) $(A_1 \cap Z)/(A_0 \cap Z)$ is finitely generated,
(b) $(A_1 Z)/(A_0 Z)$ is finitely generated.

Ad (a): The center Z of the connected Lie group G is compactly generated. (Indeed the fundamental group $\pi_1(G/Z)$ is finitely generated abelian and $\pi_0(Z) = Z/Z_0$ is the kernel of the covering morphism $G/Z_0 \to G/Z$ and is therefore finitely generated as a quotient of $\pi_1(G/Z)$. Thus Z is compactly generated.) Since A_1 is open in A, the group $A_1 \cap Z$ is open in Z and thus compactly generated, and so (a) follows.

Ad (b): The adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}\,\mathfrak{g} \subseteq \mathrm{GL}(\mathfrak{g})$ induces a faithful linear representation of G/Z. Then by Lemma 4 and Proposition 5, A_1Z/Z, a closed solvable subgroup of G/Z, is compactly generated. Then the discrete factor group $A_1Z/A_0Z \cong (A_1Z/Z)/(A_0Z/Z)$ is finitely generated. Thus (b) is proved as well and this completes the proof of Lemma 6 and thereby the proof of the Main Theorem. □

Appendix

In the proof of Proposition 5 we used the following

Theorem A. *For any finite dimensional representation of a connected Lie group G, the image of the commutator subgroup is closed.*

Proof. It is no loss of generality to assume that G is simply connected. Then we have Levi decomposition $G = R \rtimes_\alpha S$ and $G' = (G, R) \rtimes S$. Let $\pi\colon G \to \mathrm{GL}(V)$ be a finite dimensional representation and let

$$V_0 = \{0\} \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

be a maximal flag of G-submodules of V such that all quotient modules V_{j+1}/V_j are simple. Since $\pi|S$ is a semisimple representation, we may choose S-invariant decompositions $V_j = V_{j-1} \oplus W_j$. Then

$$\pi(G) \subseteq G_F \overset{\mathrm{def}}{=} \{g \in \mathrm{GL}(V) : (\forall j) gV_j = V_j\},$$

and we have a semidirect decomposition $G_F = U_F \rtimes L_F$, where

$$U_F = \{g \in \mathrm{GL}(V) : (\forall j)(g-1)(V_j) = V_{j-1}$$

and $L_F = \prod_j \mathrm{GL}(W_j)$. Note also that $\pi(S) \subseteq L_F$. Furthermore, Theorem I.5.3.1 of [1] implies that the ideal $[\mathfrak{g}, \mathfrak{r}]$ acts trivially on each simple $\mathfrak{g}$-module and so $\pi((G,R)) \subseteq U_F$. Hence $\pi((G,R))$ is a unipotent analytic group and is therefore closed. Moreover, $\pi(S)$ is closed (see [2], Chapter XVI) and this shows that $\pi(G') \cong \pi((G,R)) \rtimes \pi(S)$ is closed. □

The proof of Theorem A can be derived from the theory of algebraic groups, since the commutator algebra of a linear Lie algebra is the Lie algebra of an algebraic group [6]. We gave a more direct proof inspired by the discussion of linear Lie groups in [2].

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