Optimal Initial Value Conditions for the Existence of Local Strong Solutions of the Navier-Stokes Equations

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Consider the instationary Navier-Stokes system in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ with vanishing force and initial value $u_0 \in L^2_\sigma(\Omega)$. Since the work of Kiselev-Ladyzhenskaya [15] in 1963 there have been found several conditions on u_0 to prove the existence of a unique strong solution $u \in L^s\big(0,T;L^q(\Omega)\big)$ with $u(0) = u_0$ in some time interval $[0,T)$, $0 < T \le \infty$, where the exponents $2 < s < \infty$, $3 < q < \infty$ satisfy $\frac{2}{s} + \frac{3}{q} = 1$. Indeed, such conditions could be weakened step by step, thus enlarging the corresponding solution classes. Our aim is to prove the following optimal result with the weakest possible initial value condition and the largest possible solution class: Given u_0, q, s as above and the Stokes operator A_q, we prove that the condition $\int_0^\infty \|e^{-tA_q}u_0\|_q^s\,dt < \infty$ is necessary and sufficient for the existence of such a strong solution u. The proof rests on arguments from the recently developed theory of very weak solutions.

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1 Introduction

Throughout this paper we consider the instationary Navier-Stokes system

$$\begin{aligned} u_t - \Delta u + u\cdot\nabla u + \nabla p &= 0, \quad \operatorname{div} u = 0 \\ u\big|_{\partial\Omega} &= 0, \quad u(0) = u_0 \end{aligned} \tag{1.1}$$

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in a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ of class $C^{2,1}$ and a time interval $[0,T)$, $0 < T \leq \infty$, with initial value u_0, vanishing external force and viscosity $\nu = 1$. First we recall the definition of weak and strong solutions.

Definition 1.1 Given an initial value $u_0 \in L^2_\sigma(\Omega)$ a vector field

$$u \in L^\infty\big(0,T; L^2_\sigma(\Omega)\big) \cap L^2_{\text{loc}}\big([0,T); W^{1,2}_0(\Omega)\big) \tag{1.2a}$$

is called a *weak solution (in the sense of Leray and Hopf)* of the Navier-Stokes system (1.1) if the relation

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_\Omega \tag{1.2b}$$

holds for each test function $w \in C^\infty_0\big([0,T); C^\infty_{0,\sigma}(\Omega)\big)$, and if the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2\, d\tau \leq \frac{1}{2}\|u_0\|_2^2 \tag{1.2c}$$

is satisfied for all $t \in [0,T)$.

A weak solution u of (1.1) is called a *strong solution* if there exist exponents $2 < s < \infty$, $3 < q < \infty$ with $\frac{2}{s} + \frac{3}{q} = 1$ such that additionally *Serrin's condition*

$$u \in L^s\big(0,T; L^q(\Omega)\big) \tag{1.2d}$$

is satisfied.

Given a weak solution u of (1.1) we may assume without loss of generality that

$$u : [0,T) \to L^2_\sigma(\Omega) \quad \text{is weakly continuous,}$$

see [19, Theorem V, 1.3.1]. Moreover, there exists a distribution p on $(0,T)\times\Omega$, the so-called *associated pressure*, such that

$$u_t - \Delta u + u\cdot\nabla u + \nabla p = 0$$

in the sense of distributions [19, V.1.7].

Serrin's condition (1.2d) with $\frac{2}{s} + \frac{3}{q} = 1$ yields the regularity property

$$u \in C^\infty\big((0,T)\times\overline{\Omega}\big), \quad p \in C^\infty\big((0,T)\times\overline{\Omega}\big)$$

if $\partial\Omega$ is of class C^∞, see [19, Theorem V, 1.8.2]. Therefore, a strong solution is also called a *regular solution*.

The existence of at least one weak solution of (1.1) is well-known since the pioneering work of J. Leray [17] and E. Hopf [14]. To prove the existence of a strong solution $u \in L^s\big(0,T; L^q(\Omega)\big)$, $\frac{2}{s}+\frac{3}{q} = 1$, in some time interval $[0,T)$ we need besides the condition $u_0 \in L^2_\sigma(\Omega)$ a further regularity property of the initial value

u_0. The first sufficient condition on the initial value for a bounded domain seems to have been described in [15]. Since then many results on sufficient conditions on u_0 to guarantee the existence of local strong solutions were proved, see, e.g., [2], [8], [11], [13], [16], [18], [20], [21]. Indeed, during the last 40 years, the conditions on u_0 could be weakened step by step. The following result yields the weakest – necessary and sufficient – condition in this context. Here A_q denotes the Stokes operator on $L^q_\sigma(\Omega = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$, $1 < q < \infty$, where $C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega) : \operatorname{div} u = 0\}$, and e^{-tA_q}, $t \geq 0$, is the semigroup generated by A_q on $L^q_\sigma(\Omega)$.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, let $2 < s < \infty$, $3 < q < \infty$ satisfy $\frac{2}{s} + \frac{3}{q} = 1$ and let $u_0 \in L^2_\sigma(\Omega)$.*

(1) *The condition*

$$\int_0^\infty \|e^{-tA_q} u_0\|_q^s \, dt < \infty \tag{1.3}$$

is necessary and sufficient for the existence of a unique strong solution $u \in L^s\big(0,T; L^q(\Omega)\big)$ in some time interval $[0,T)$, $0 < T \leq \infty$, of the Navier-Stokes system (1.1) *with initial value $u(0) = u_0$.*

(2) *There exists a constant $\varepsilon_* = \varepsilon_*(\Omega, q) > 0$ with the following property: If*

$$\int_0^T \|e^{-tA_q} u_0\|_q^s \, dt \leq \varepsilon_* \quad \text{for some } 0 < T \leq \infty, \tag{1.4}$$

then (1.1) *has a unique strong solution u on the interval $[0,T)$ satisfying* (1.2a) – (1.2d) *and $u(0) = u_0$.*

To interpret the results of Theorem 1.2 let us recall some well-known facts on the Stokes operator. Let $A_q^\alpha : \mathcal{D}(A_q^\alpha) \subseteq L^q_\sigma(\Omega) \to L^q_\sigma(\Omega)$, $-1 \leq \alpha \leq 1$, denote the fractional powers of the Stokes operator. As is well-known,

$$\mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \subset \mathcal{D}(A_q^\alpha) \subseteq L^q_\sigma(\Omega)$$

for $0 \leq \alpha \leq 1$, and $(A_q^\alpha)^{-1} = A_q^{-\alpha}$. Moreover,

$$\|v\|_q \leq c\|A_\gamma^\alpha v\|_\gamma, \quad v \in \mathcal{D}(A_\gamma^\alpha), \quad 1 < \gamma \leq q, \quad 2\alpha + \frac{3}{q} = \frac{3}{\gamma}, \tag{1.5}$$

and

$$\|A_q^\alpha e^{-tA_q} v\|_q \leq c e^{-\delta t} t^{-\alpha} \|v\|_q, \quad v \in L^q_\sigma(\Omega),\ 0 \leq \alpha \leq 1,\ t > 0, \tag{1.6}$$

with constants $c = c(\Omega, q) > 0$, $\delta = \delta(\Omega, q) > 0$, see [1], [4], [9], [10], [12], [19], [21], [23].

The next remark yields some further aspects of this result.

Remark 1.3 (1) Since for $u_0 \in L^2_\sigma(\Omega)$

$$\|e^{-tA_q}u_0\|_q = \|A_q^\alpha e^{-tA_q} A_q^{-\alpha} u_0\|_q \le ct^{-\alpha} e^{-\delta t} \|A_q^{-\alpha} u_0\|_q$$

and $\|A_q^{-\alpha}u_0\|_q \le c\|u_0\|_2$, where $2\alpha + \frac{3}{q} = \frac{3}{2}$, we get that

$$\|e^{-tA_q}u_0\|_q^s \le c^s t^{-\alpha s} e^{-\delta s t} \|u_0\|_2^s$$

with constants $c = c(\Omega, q) > 0$. Hence $\|e^{-tA_q}u_0\|_q^s$ is well-defined and even continuous for every $t > 0$. Therefore, the integrability condition (1.3) is equivalent to the condition

$$\int_0^{T_0} \|e^{-tA_q}u_0\|_q^s \, dt < \infty \tag{1.7}$$

for each given $T_0 > 0$. If (1.3) is satisfied, the term $\int_0^{T_0} \|e^{-tA_q}u_0\|_q^s \, dt$ tends to 0 as $T_0 \to 0+$, hence (1.3) implies that condition (1.4) is always satisfied with some sufficiently small $T = T(\Omega, q, u_0) > 0$ depending on $u_0 \in L^2_\sigma(\Omega)$.

(2) Let $B^{2/s}_{q',s'}(\Omega)$ denote the usual Besov space ([22, Definition 4.2.1]) and let $\mathbb{B}^{-2/s}_{q,s}(\Omega)$ be the Besov space of solenoidal vector fields introduced in [3, (0.5), (0.6)], i.e., $\mathbb{B}^{-2/s}_{q,s}(\Omega) = \big(\mathbb{B}^{2/s}_{q',s'}(\Omega)\big)'$, $q' = \frac{q}{q-1}$, $s' = \frac{s}{s-1}$, means the dual space of

$$\begin{aligned} \mathbb{B}^{2/s}_{q',s'}(\Omega) &= B^{2/s}_{q',s'}(\Omega) \cap L^{q'}_\sigma(\Omega) = (L^{q'}_\sigma(\Omega), \mathcal{D}(A_{q'}))_{1/s,s'} \\ &= \{v \in B^{2/s}_{q',s'}(\Omega) : \operatorname{div} v = 0,\ N \cdot v|_{\partial\Omega} = 0\}, \end{aligned}$$

where $(\cdot,\cdot)_{1/s,s'}$ denotes the real interpolation space. Note that here $\frac{2}{s} < 1 - \frac{1}{q}$ and that $N \cdot v|_{\partial\Omega}$ is the normal component of v at $\partial\Omega$. Then the real interpolation method [22, Theorem 1.14.5] yields the equivalence of norms

$$\begin{aligned} \Big(\int_0^\infty \|e^{-tA_q}u_0\|_q^s \, dt\Big)^{1/s} &= \Big(\int_0^\infty \|A_q e^{-tA_q}(A_q^{-1}u_0)\|_q^s \, dt\Big)^{1/s} \\ \approx \|A_q^{-1}u_0\|_{(L^q_\sigma, \mathcal{D}(A_q))_{1-1/s,s}} &= \|u_0\|_{(A_q L^q_\sigma, L^q_\sigma)_{1-1/s,s}} \end{aligned}$$

where

$$A_q L^q_\sigma = \{v \in C^\infty_{0,\sigma}(\Omega)' : \exists c = c(v) \ge 0 : |\langle v, \varphi\rangle| \le c\,\|A_{q'}\varphi\|_{q'} \quad \text{for all } \varphi \in \mathcal{D}(A_{q'})\}$$

coincides with the dual space $\mathcal{D}(A_{q'})'$ and is equipped with the norm $\|v\|_{A_q L^q_\sigma} := \inf c(v)$ defined as the infimum of these constants $c(v)$. Moreover, duality theory [22, Theorem 1.11.2] and [3, Proposition 3.4] imply that

$$\begin{aligned} \Big(\int_0^\infty \|e^{-tA_q}u_0\|_q^s \, dt\Big)^{1/s} &\approx \|u_0\|_{(\mathcal{D}(A_{q'}), L^{q'}_\sigma)'_{1-1/s,s'}} = \|u_0\|_{(L^{q'}_\sigma, \mathcal{D}(A_{q'}))'_{1/s,s'}} \\ &\approx \|u_0\|_{(\mathbb{B}^{2/s}_{q',s'})'} \qquad = \|u_0\|_{\mathbb{B}^{-2/s}_{q,s}}\,. \end{aligned}$$

Hence we get an equivalent formulation of Theorem 1.2(1) in Besov spaces: The initial value u_0 admits a local strong solution $u \in L^s\big(0,T;L^q(\Omega)\big)$ of (1.1) on some interval $[0,T)$, $0 < T \le \infty$, if and only if $u_0 \in \mathbb{B}^{-2/s}_{q,s}(\Omega)$.

(3) In Theorem 1.2 let $3 < q \le s$ so that $s' \le q' < 2$. Then by [3] and [22, Theorem 4.6.1]

$$\begin{aligned}\mathbb{B}^{2/s}_{q',s'}(\Omega) = B^{2/s}_{q',s'}(\Omega) \cap L^{q'}_\sigma(\Omega) &\subset B^{2/s}_{q',q'}(\Omega) \cap L^{q'}_\sigma(\Omega)\\ &\subset H^{2/s}_{q'}(\Omega) \cap L^{q'}_\sigma(\Omega) =: \mathbb{H}^{2/s}_{q'}(\Omega) = \mathcal{D}(A^{1/s}_{q'})\end{aligned}$$

with continuous embeddings so that a duality argument yields the estimate

$$\|u_0\|_{\mathbb{B}^{-2/s}_{q,s}} \le c\|A_q^{-1/s}\, u_0\|_q.$$

By part (2) we conclude that

$$\Big(\int_0^\infty \|e^{-tA_q} u_0\|_q^s\, dt\Big)^{1/s} \le c\|A_2^{-1/s}\, u_0\|_q$$

with $c = c(\Omega, q) > 0$. Hence, if $3 < q \le s < \infty$ and $A_2^{-1/s}\, u_0 \in L^q(\Omega)$, Theorem 1.2(1) yields the existence of a unique strong solution $u \in L^s\big(0,T;L^q(\Omega)\big)$ in some interval $[0,T)$, $0 < T \le \infty$, of (1.1) with $u(0) = u_0$.

We note that conditions on initial values as weak as possible can be used in the regularity theory of weak solutions: For at least almost all $t \in (0,T)$ the term $u(t)$ plays the role of an initial value of a local strong solution which can be identified locally with u. By this idea several global regularity properties of u can be obtained, see [6], [7], [20].

Before coming to the proof of Theorem 1.2 let us explain some notations and the concept of very weak solutions. In this paper $\langle\cdot,\cdot\rangle_\Omega$ denotes the usual pairing of functions on Ω, and $\langle\cdot,\cdot\rangle_{\Omega,T}$ means the corresponding pairing on $[0,T)\times\Omega$. For $1 < q < \infty$ and $k \in \mathbb{N}$ we need the usual Lebesgue and Sobolev spaces $L^q(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and $W^{k,q}(\Omega)$ with norm $\|\cdot\|_{W^{k,q}(\Omega)} = \|\cdot\|_{k;q}$, respectively. Further we need the Bochner spaces $L^s\big(0,T;L^q(\Omega)\big)$, $1 < s < \infty$, with the norm $\|\cdot\|_{L^s(0,T;L^q)} = \|\cdot\|_{q,s} = \big(\int_0^T \|\cdot\|_q^s\, dt\big)^{1/s}$. Concerning smooth functions we use the spaces $C_0^\infty(\Omega)$ and $C_{0,\sigma}^\infty(\Omega) = \{u \in C_0^\infty(\Omega) : \operatorname{div} u = 0\}$. Then $L^q_\sigma(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_q}$ and $W_0^{1,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,q}(\Omega)}}$.

Let $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$ denote the Helmholtz projection, and let $A_q = -P_q\Delta : \mathcal{D}(A_q) \to L^q_\sigma(\Omega)$ be the Stokes operator with domain $\mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$ and range $\mathcal{R}(A_q) = L^q_\sigma(\Omega)$. Note that $P_q v = P_\gamma v$ for all $v \in L^q(\Omega) \cap L^\gamma(\Omega)$, $A_q v = A_\gamma v$ for $v \in \mathcal{D}(A_q) \cap \mathcal{D}(A_\gamma)$, $1 < q,\ \gamma < \infty$, and that $\|A_q^{1/2} v\|_q \approx \|v\|_{W^{1,q}(\Omega)}$ for all $v \in \mathcal{D}(A_q^{1/2}) = W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$.

Using the theory of maximal regularity, for every $f \in L^s\big(0,T;L^q_\sigma(\Omega)\big)$, $1 < q$, $s < \infty$, the instationary Stokes system

$$w_t + A_q w = f, \quad w(0) = 0, \tag{1.8}$$

has a unique solution $w \in C^0\big([0,T]; L^q_\sigma(\Omega)\big)$ with $w(t) \in \mathcal{D}(A_q)$ for a.a. $t \in [0,T]$ which additionally satisfies the *a priori* estimate

$$\|w_t\|_{L^s(0,T;L^q)} + \|A_q w\|_{L^s(0,T;L^q)} \le c\,\|f\|_{L^s(0,T;L^q)}, \tag{1.9}$$

with $c = c(\Omega, q, s) > 0$ independent of $T \in (0,\infty]$, see [12]. This solution has the representation

$$w(t) = \int_0^t e^{-(t-\tau)A_q} f(\tau)\,d\tau, \tag{1.10}$$

using the bounded analytic semigroup e^{-tA_q}, $t \ge 0$, on $L^q_\sigma(\Omega)$ generated by A_q.

Assume that $v \in C^\infty_{0,\sigma}(\Omega)'$ is a functional (distribution) well-defined for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$ such that the estimate

$$|\langle v, \varphi\rangle| \le c\,\|A^{1/2}_{q'}\varphi\|_{q'}$$

holds for all $\varphi \in \mathcal{D}(A^{1/2}_{q'})$ with a constant $c = c(v) > 0$ independent of φ. Then there exists a unique vector field $w \in L^q_\sigma(\Omega)$, also denoted by $A_q^{-1/2}P_q v$, such that

$$\langle v, A^{-1/2}_{q'}\psi\rangle = \langle w, \psi\rangle = \langle A_q^{-1/2}P_q v, \psi\rangle$$

for all $\psi \in L^{q'}_\sigma(\Omega)$. In particular, for $v = \operatorname{div} F = \Big(\sum_{i=1}^3 \partial F_{ij}/\partial x_i\Big)_{j=1}^3$, $F = (F_{ij})_{i,j=1}^3 \in L^q(\Omega)$, we obtain that $A_q^{-1/2}P_q \operatorname{div} F \in L^q_\sigma(\Omega)$ and

$$\|A_q^{-1/2}P_q \operatorname{div} F\|_q \le c\,\|F\|_q \tag{1.11}$$

with a constant $c = c(\Omega, q) > 0$. Note that $u = A_q^{-1/2}P_q \operatorname{div} F$ is the weak solution of the stationary Stokes system

$$-\Delta u + \nabla p = \operatorname{div} F, \quad \operatorname{div} u = 0 \text{ in } \Omega,\ u|_{\partial\Omega} = 0$$

in $L^q_\sigma(\Omega)$.

In the same way we obtain a unique vector field $A_q^{-1}P_q v \in L^q_\sigma(\Omega)$ defined by the relation

$$\langle v, A^{-1}_{q'}\psi\rangle = \langle A_q^{-1}P_q v, \psi\rangle$$

for all $\psi \in L^{q'}_\sigma(\Omega)$ if the estimate

$$|\langle v, \varphi\rangle| \le c\|A_{q'}\varphi\|_{q'}$$

holds for all $\varphi \in \mathcal{D}(A_{q'})$ with some constant $c = c(v) \ge 0$. For further details in this context we refer to [5] and [19, III 2.6].

Definition 1.4 Let $1 < s,\, q < \infty$ satisfy $\frac{2}{s} + \frac{3}{q} = 1$, let the space of initial values, $\mathcal{J}_\sigma^{q,s}(\Omega)$, be defined as the set of all $v \in C_{0,\sigma}^\infty(\Omega)'$ satisfying $A_q^{-1} P_q v \in L_\sigma^q(\Omega)$ and

$$\|v\|_{\mathcal{J}_\sigma^{q,s}(\Omega)} := \|A_q^{-1} P_q v\|_q + \int_0^\infty \|A_q e^{-tA_q} (A_q^{-1} P_q v)\|_q^s \, dt < \infty, \tag{1.12}$$

and let $u_0 \in \mathcal{J}_\sigma^{q,s}(\Omega)$. Then $u \in L^s\big(0,T; L^q(\Omega)\big)$ is called a *very weak solution* of the Navier-Stokes system (1.1) if

$$\langle u, w_t \rangle_{\Omega,T} - \langle u, \Delta w \rangle_{\Omega,T} - \langle uu, \nabla w \rangle_{\Omega,T} = \langle u_0, w(0) \rangle_\Omega \tag{1.13}$$

holds for all test functions $w \in C_0^1\big([0,T); C_{0,\sigma}^2(\overline{\Omega})\big)$; here $C_{0,\sigma}^2(\Omega) = \{v \in C^2(\overline{\Omega}) : \operatorname{div} v = 0, v|_{\partial\Omega} = 0\}$.

We note that Definition 1.4 is a special case of the concept of very weak solutions to inhomogeneous (Navier-)Stokes systems, see [5], [7]. In the corresponding definition of very weak solutions for the linear case (instationary Stokes system) where the nonlinear term $u \cdot \nabla u$ is absent, we have to omit in Definition 1.4 the restriction $\frac{2}{s} + \frac{3}{q} = 1$ and in (1.13) the term $\langle uu, \nabla w \rangle_{\Omega,T}$.

Theorem 1.5 ([5], [7]) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,1}$, let $1 < s$, $q < \infty$ satisfy $\frac{2}{s} + \frac{3}{q} = 1$ and let $u_0 \in \mathcal{J}_\sigma^{q,s}(\Omega)$. Then there exists a constant $\varepsilon_* = \varepsilon_*(\Omega, q) > 0$ with the following property: If for some $0 < T \leq \infty$*

$$\int_0^T \|A_q e^{-tA_q} \, (A_q^{-1} P_q u_0)\|_q^s \, dt \leq \varepsilon_*, \tag{1.14}$$

then the Navier-Stokes system (1.1) has a unique very weak solution $u \in L^s\big(0,T; L^q(\Omega)\big)$. Moreover, u has the representation

$$u(t) = E(t) - \int_0^t A_q^{1/2} \, e^{-(t-\tau)A_q} \, A_q^{-1/2} P_q \operatorname{div}(uu) d\tau, \quad 0 \leq t < T, \tag{1.15}$$

where $E(t) = A_q e^{-tA_q}(A_q^{-1} P_q u_0)$ is the unique very weak solution of the instationary Stokes system $E_t - \Delta E + \nabla p = 0$, $\operatorname{div} E = 0$ in $(0,T) \times \Omega$, $E|_{\partial\Omega} = 0$, $E(0) = u_0$. Finally, there exists a constant $C = C(\Omega, q) > 0$ such that

$$\|u\|_{L^s(0,T;L^q)} \leq C \|E\|_{L^s(0,T;L^q)}. \tag{1.16}$$

Theorem 1.5 is a special case of the general result on existence and uniqueness of very weak solutions to the inhomogeneous Navier-Stokes system. We will apply Theorem 1.5 for the proof of Theorem 1.2 only when $u_0 \in L_\sigma^2(\Omega)$. In that case (1.14) and (1.15) simplify and $E(t) = e^{-tA_q} u_0 = e^{-tA_2} u_0$.

2 Proof of Theorem 1.2

Proof of Theorem 1.2(2) First suppose that the condition

$$\int_0^T \|e^{-tA_q} u_0\|_q^s \, dt \le C \tag{2.1}$$

is satisfied for some constant $C > 0$ and $0 < T \le \infty$. Later on we will choose $C = \varepsilon_*$ sufficiently small.

Since by (1.5) with $2\alpha + \frac{3}{q} = \frac{3}{2}$, $0 < \alpha < \frac{3}{4}$,

$$\|A_q^{-1} P_q u_0\|_q \le c \|A_q^\alpha A_q^{-1} u_0\|_q \le c \|u_0\|_2,$$

we conclude with (1.6) from (2.1) that

$$\|A_q^{-1} P_q u_0\|_q + \int_0^\infty \|A_q \, e^{-tA_q} (A_q^{-1} P_q u_0)\|_q^s \, dt < \infty.$$

Therefore, u_0 lies in the set of admissible initial values, $\mathcal{J}_\sigma^{q,s}(\Omega)$. Hence Theorem 1.5 yields the existence of a unique very weak solution $u \in L^s\big(0, T; L^q(\Omega)\big)$ provided that $T > 0$ satisfies the condition (2.1) with $C = \varepsilon_*$ as in (1.14).

To prove that u is a strong solution on $(0, T)$ it remains to show that u satisfies (1.2a), (1.2c); then the variational equation (1.2b) will be an easy consequence of (1.13). Looking at the representation formula (1.15) note that $E(t) = e^{-tA_q} u_0 = e^{-tA_2} u_0$ is the weak solution of a homogeneous instationary Stokes system with initial value $E(0) = u_0 \in L_\sigma^q(\Omega) \subseteq L_\sigma^2(\Omega)$; thus

$$\begin{aligned} & E \in L^\infty\big(0, T; L^2(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0, T); W_0^{1,2}(\Omega)\big), \\ & \|E(t)\|_2 \le \|u_0\|_2, \quad \|\nabla E\|_{2,2} \le \|u_0\|_2 \, . \end{aligned} \tag{2.2}$$

Hence it suffices to analyze integrability properties of

$$\tilde{u}(t) := u(t) - E(t) = -\int_0^t A_q^{1/2} \, e^{-(t-\tau)A_q} \, A_q^{-1/2} P_q \, \mathrm{div}\,(uu) d\tau, \quad 0 \le t < T, \tag{2.3}$$

and also of $\nabla \tilde{u}(t)$, or equivalently of

$$A_q^{1/2} \tilde{u}(t) = -\int_0^t A_q^{1/2} \, e^{-(t-\tau)A_q} \, P_q \, \mathrm{div}\,(uu) d\tau. \tag{2.4}$$

To this reason we use the Yosida approximation of $\tilde{u}$, defined by

$$\tilde{u}_n = J_n \tilde{u}, \quad J_n = \big(I + \frac{1}{n} A_q^{1/2}\big)^{-1}, \quad n \in \mathbb{N},$$

so that $\tilde{u} = \tilde{u}_n + \frac{1}{n} A_q^{1/2} \tilde{u}_n$, see e.g. [19, II.3.4]. In order to smooth (2.3) we apply J_n and note that

$$\begin{aligned} J_n P_q \operatorname{div}(uu) &= J_n P_q \operatorname{div}(u\tilde{u}) + J_n P_q \operatorname{div}(uE) \\ &= J_n P_q (u \cdot \nabla \tilde{u}_n) + \frac{1}{n} A_q^{1/2} J_n A_q^{-1/2} P_q \operatorname{div}(u A_q^{1/2} \tilde{u}_n) + J_n P_q (u \cdot \nabla E). \end{aligned}$$

By the properties of the Yosida approximations – in particular J_n and $\frac{1}{n} A_q^{1/2} J_n$ are bounded on $L^q_\sigma(\Omega)$ uniformly with respect to $n \in \mathbb{N}$ – we get from (1.11) and Hölder's inequality with $\gamma = (\frac{1}{2} + \frac{1}{q})^{-1}$ that

$$\begin{aligned} \|J_n P_q \operatorname{div}(uu)\|_\gamma &\le c\big(\|u \cdot \nabla \tilde{u}_n\|_\gamma + \|A_q^{-1/2} P_q \operatorname{div}(u A_q^{1/2} \tilde{u}_n)\|_\gamma + \|u \cdot \nabla E\|_\gamma\big) \\ &\le c\|u\|_q\big(\|A_q^{1/2} \tilde{u}_n\|_2 + \|\nabla E\|_2\big). \end{aligned}$$

Using (1.5), (1.6) with $2\alpha + \frac{3}{2} = \frac{3}{\gamma}$ $\big(\alpha = \frac{3}{2q} < \frac{1}{2}\big)$, we obtain the estimate

$$\begin{aligned} \|A_q^{1/2} \tilde{u}_n(t)\|_2 &= \|A_q^{1/2} J_n \tilde{u}(t)\|_2 \\ &\le c \int_0^t \|A_q^\alpha\, A_q^{1/2}\, e^{-(t-\tau)A_q}\, J_n\, P_q \operatorname{div}(uu)\|_\gamma \, d\tau \\ &\le c \int_0^t (t-\tau)^{-\alpha-1/2} \|u\|_q \big(\|A_q^{1/2} \tilde{u}_n\|_2 + \|\nabla E\|_2\big)\, d\tau \end{aligned}$$

with a constant $c = c(\Omega, q) > 0$ independent of $n \in \mathbb{N}$. Next we apply the Hardy-Littlewood inequality with $\big(\frac{1}{2} - \alpha\big) + \frac{1}{2} = \frac{1}{2} + \frac{1}{s}$, see [19, Lemma II.3.3.2], and Hölder's inequality to see that

$$\|A_q^{1/2} \tilde{u}_n\|_{2,2} \le \tilde{c}\|u\|_{q,s}\big(\|A_q^{1/2} \tilde{u}_n\|_{2,2} + \|\nabla E\|_{2,2}\big)$$

with $\tilde{c} = \tilde{c}(\Omega, q) > 0$ independent of $n \in \mathbb{N}$. Using (1.16) and (1.4) we see that the constant $\varepsilon_* = \varepsilon_*(\Omega, q) > 0$ can be chosen in such a way that

$$\tilde{c}\|u\|_{q,s} = \tilde{c}\|u\|_{L^s(0,T;L^q)} \le \frac{1}{2}.$$

Then the absorption principle leads to the estimate

$$\|A_q^{1/2} \tilde{u}_n\|_{2,2} \le 2\tilde{c}\|u\|_{q,s}\, \|\nabla E\|_{2,2}.$$

As $n \to \infty$ we conclude that $A_q^{1/2} \tilde{u} \in L^2\big(0, T; L^2(\Omega)\big)$ and that

$$\|\nabla \tilde{u}\|_{2,2} = \|A_q^{1/2} \tilde{u}\|_{2,2} \le 2\tilde{c}\|u\|_{q,s}\, \|\nabla E\|_{2,2}.$$

In particular, due to (2.2) we get that

$$\nabla \tilde{u}, \nabla u \in L^2\big(0, T; L^2(\Omega)\big). \tag{2.5}$$

In the next step we show that

$$uu \in L^2\big(0,T;L^2(\Omega)\big). \tag{2.6}$$

For this purpose we write (2.3) in the form

$$\tilde{u}(t) = -\int_0^t e^{-(t-\tau)A_q} P_q(u\cdot\nabla u)\,d\tau\,,$$

choose $\alpha = \frac{3}{q}$, $\frac{1}{q_1} = \frac{1}{2} - \frac{1}{q}$, $\frac{1}{q_2} = \frac{1}{2} + \frac{1}{q}$ and use (1.5), (1.6) with $2\alpha + \frac{3}{q_1} = \frac{3}{q_2}$ to see that

$$\begin{aligned}\|\tilde{u}(t)\|_{q_1} &\le c\int_0^t \|A_{q_2}^{\alpha} e^{-(t-\tau)A_{q_2}} P_{q_2}(u\cdot\nabla u)\|_{q_2}\,d\tau\\ &\le c\int_0^t (t-\tau)^{-\alpha}\|u\cdot\nabla u\|_{q_2}\,d\tau\end{aligned}$$

with $c = c(\Omega,q) > 0$. Next we apply the Hardy-Littlewood inequality with $(1-\alpha)+\frac{1}{s_1} = \frac{1}{s_2}$ where $\frac{1}{s_1} = \frac{1}{2} - \frac{1}{s}$, $\frac{1}{s_2} = \frac{1}{2} + \frac{1}{s}$ so that $\alpha = 1 - \frac{2}{s} = \frac{3}{q}$ and obtain from (2.5) that

$$\|\tilde{u}\|_{q_1,s_1} \le c\,\|u\cdot\nabla u\|_{q_2,s_2} \le c\,\|u\|_{q,s}\,\|\nabla u\|_{2,2} < \infty.$$

The weak solution E, see (2.2), also satisfies $E \in L^{s_1}\big(0,T;L^{q_1}(\Omega)\big)$ since $\frac{2}{s_1} + \frac{3}{q_1} = \frac{3}{2}$. Hence we conclude that $u \in L^{s_1}\big(0,T;L^{q_1}(\Omega)\big)$, and Hölder's inequality proves (2.6) since

$$\|uu\|_{2,2} \le c\,\|u\|_{q,s}\|u\|_{q_1,s_1} < \infty.$$

In the final step we set $F = uu$ and conclude from (2.4) that u is a solution of the linear Stokes system

$$u_t + A_q u = -\operatorname{div} F, \quad u(0) = u_0 \tag{2.7}$$

in the weak sense. Since $\nabla u \in L^2\big(0,T;L^2(\Omega)\big)$ and $F \in L^2\big(0,T;L^2(\Omega)\big)$, classical linear theory yields $u \in L^\infty\big(0,T;L^2(\Omega)\big)$ and the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2\,d\tau \le \frac{1}{2}\|u_0\|_2^2 - \int_0^t \langle uu,\nabla u\rangle\,d\tau.$$

Moreover, $\langle uu,\nabla u\rangle = 0$ for a.a. $t \in (0,T)$ since $uu, \nabla u \in L^2(\Omega)$ for a.a. $t \in (0,T)$. This proves the energy inequality (1.2c) (indeed as an equality). The uniqueness of u, even within the class of weak solutions, follows from Serrin's uniqueness theorem, see [19, Theorem V.1.5.1].

Proof of Theorem 1.2(1) Suppose that (1.3) is satisfied. Then the function $t \mapsto \|e^{-tA_q}u_0\|_q^s$ is integrable on $(0,\infty)$, and there exists some $T \in (0,\infty]$ such that

(1.4) is satisfied. Now part (2) shows that condition (1.3) is sufficient for the existence of a strong solution.

Suppose that $u \in L^s\big(0,T; L^q(\Omega)\big)$ is a strong solution of (1.1) in some interval $[0,T)$, $0 < T \leq \infty$. Since by definition u is also a weak solution, Hölder's inequality easily implies that $F = uu \in L^2\big(0,T; L^2(\Omega)\big)$, see [19, p. 297]. Moreover, u can be considered as the weak solution of the linear system (2.7) so that $\tilde{u} = u - E$ has the representation (2.3) with $q = 2$, cf. [19, Theorem IV.2.4.1].

Using (1.5) with $2\alpha + \frac{3}{q} = \frac{3}{q/2}$, i.e. $\alpha = \frac{3}{2q}$, and (1.11) we conclude from (2.3) that

$$\|\tilde{u}(t)\|_q \leq c \int_0^t (t-\tau)^{-\alpha-1/2} \|uu\|_{q/2}\, d\tau, \quad 0 \leq t \leq T,$$

with $c = c(\Omega, q) > 0$. Then the Hardy-Littlewood inequality, using $(\frac{1}{2} - \alpha) + \frac{1}{s} = \frac{1}{s/2}$, $\alpha = \frac{1}{2}(1 - \frac{2}{s}) = \frac{3}{2q}$, implies that

$$\|\tilde{u}\|_{q,s} \leq c\, \|uu\|_{q/2,s/2} \leq c\, \|u\|_{q,s}^2 < \infty.$$

Hence $\tilde{u} \in L^s\big(0,T; L^q(\Omega)\big)$ and also $E = u - \tilde{u} \in L^s\big(0,T; L^q(\Omega)\big)$, which shows that the function $t \mapsto \|e^{-tA_2}u_0\|_q^s$ is integrable on $(0,T)$. By (1.5) with $2\alpha' + \frac{3}{q} = \frac{3}{2}$ ($\alpha' \in (0, \frac{3}{4})$) and (1.6) we see that

$$\|e^{-tA_2}u_0\|_q \leq c\, \|A_2^{\alpha'} e^{-tA_2} u_0\|_2 \leq c\, t^{-\alpha'} e^{-\delta t} \|u_0\|_2$$

with $c = c(\Omega, q) > 0$, $\delta = \delta(\Omega, q) > 0$. Therefore, the map $t \mapsto \|e^{-tA_2}u_0\|_q^s$ is integrable on $(0, \infty)$, i.e., condition (1.3) is satisfied.

Now the proof of Theorem 1.2 is complete. ■

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