Towards a Fully Space-Time Adaptive FEM for Magnetoquasistatics

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Abstract

This paper is concerned with fully space-time adaptive magnetic field computations. We describe a Whitney finite element method for solving the magnetoquasistatic formulation of Maxwell's equations on unstructured 3D tetrahedral grids. Spatial discretization is done by employing hierarchical tetrahedral $\boldsymbol{H}(\mathrm{curl})$-conforming elements proposed by Ainsworth and Coyle. For the time discretization, we use a newly constructed one-step Rosenbrock method ROS3PL with 3rd order accuracy in time. Adaptive mesh refinement and coarsening are based on hierarchical error estimators especially designed for Rosenbrock methods. An embedding technique is applied to get efficiency in time through variable time steps. Finally, we present numerical results for the benchmark problem TEAM 7.

Keywords: magnetoquasistatics, space-time adaptivity, Rosenbrock methods, hierarchical error estimator, TEAM 7 problem.

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1 Introduction

The magnetoquasistatic approximation (MQS) arises from Maxwell's equations by dropping the displacement current. This is reasonable for many electrical machines, generators and transformers which work in the low-frequency high-conductivity range. Wave propagation can then be neglected and vanishing tangential traces are used for artificial boundary conditions. The aim of our work is to develop a fully adaptive algorithm which provides numerical solutions to three-dimensional MQS problems. There is nowadays an increasing emphasis on all aspects of adaptively generating a space-time grid that evolves with the solution. Another challenge is to develop efficient higher-order one-step integration methods which can handle very stiff differential-algebraic electromagnetic problems and which allow us to accommodate a grid in each time step without any specific difficulties. A combined space-time adaptivity is widely used in computational fluid dynamics and thermodynamics, but it has received much less attention in MQS simulations. The authors are only aware of a recently published paper by Zheng, Chen, and Wang [19] where first-order approximations in time and space are used.

Here, we make use of hierarchical Whitney finite elements in space [3] and variable step-size one-step Rosenbrock methods in time [14, 15]. Implementations have been done in the KARDOS library [9, 2], which provides a suitable programming environment for adaptive algorithms to solve instationary PDEs. We wish to adaptively refine the space-time grid in order to capture local effects efficiently and to guarantee a prescribed accuracy for the approximate solution. This is described in the following Sections.

2 Numerical algorithm

There are different formulations of the MQS approximation. We use a magnetic vector potential $\boldsymbol{A}(\boldsymbol{x},t)$ as primary unknown. The equations can be written in the following form

$$\begin{aligned} \sigma\partial_t \boldsymbol{A} + \boldsymbol{\nabla}\times(\mu^{-1}\,\boldsymbol{\nabla}\times\boldsymbol{A}) &= \boldsymbol{J}_s, \text{ in } \Omega\times(0,T] \\ \boldsymbol{A}\times\boldsymbol{n} &= \boldsymbol{0}, \text{ on } \partial\Omega\times(0,T] \\ \boldsymbol{A}(\cdot,0) &= \boldsymbol{A}_0, \text{ on } \Omega\,. \end{aligned} \tag{1}$$

Since there may be insulating regions with $\sigma = 0$, system (1) is in general an elliptic-parabolic initial-boundary value problem. We consider inhomogeneous, linear and isotropic materials, i.e., $\sigma \geq 0$ and $\mu > 0$ are scalar functions of the spatial variable $\boldsymbol{x} \in \Omega$. The source term $\boldsymbol{J}_s(\boldsymbol{x},t)$ stands for the applied current density and has to satisfy the consistency condition $\nabla \cdot \boldsymbol{J}_s = 0$. The physically relevant quantities which can be derived from $\boldsymbol{A}$ are the magnetic induction $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ and the eddy current density $\boldsymbol{J}_E = -\sigma\partial_t\boldsymbol{A}$. The vector potential formulation (1) is widely used in electromagnetic computations due to its robustness. However, there are two essential difficulties: the uniqueness of $\boldsymbol{A}$ in parts of the domain where $\sigma = 0$, and the consistency of $\boldsymbol{J}_s$ which has to be ensured on each spatial mesh. For gauging, we use a tiny conductivity in the non-conducting regions, six orders of magnitude smaller than the minimum positive value for σ [17].

To discretize (1), we apply the adaptive Rothe method based on the discretization sequence first in time than in space, in contrast to the usual Method of Lines approach (see e.g. [14] and references therein). The spatial discretization is considered as a perturbation of the time integration process. It has to be controlled in an appropriate way within each time step.

2.1 Rosenbrock methods

To approximate the vector potential $\boldsymbol{A}(\cdot, t)$ defined in (1) by values $\boldsymbol{A}_n \approx \boldsymbol{A}(\cdot, t_n)$ at a certain time grid

$$0 = t_0 < t_1 < \cdots < t_n < \cdots < t_{M-1} < t_M = T\,, \tag{2}$$

we apply an s-stage one-step method of Rosenbrock type. This has the form

$$\boldsymbol{A}_{n+1} = \boldsymbol{A}_n + \sum_{i=1}^{s} m_i \boldsymbol{A}_{ni}, \tag{3}$$

with stage values $\boldsymbol{A}_{ni}$, $i = 1, \ldots, s$, determined from

$$\begin{aligned} \boldsymbol{\nabla}\times\left(\mu^{-1}\,\boldsymbol{\nabla}\times\boldsymbol{A}_{ni}\right) + \frac{\sigma}{\tau_n\gamma}\boldsymbol{A}_{ni} &= \boldsymbol{R}_{ni}, \quad \text{in } \Omega \\ \boldsymbol{A}_{ni}\times\boldsymbol{n} &= 0, \qquad \text{on } \partial\Omega \end{aligned} \tag{4}$$

with the time step $\tau_n = t_{n+1} - t_n$ and γ being the stability constant of the method. The right hand side $\boldsymbol{R}_{ni}$ is defined by

$$\boldsymbol{R}_{ni} = -\,\boldsymbol{\nabla}\times\left(\mu^{-1}\,\boldsymbol{\nabla}\times\boldsymbol{A}_i\right) + \boldsymbol{J}_s(\cdot, t_i) -\,\sigma\sum_{j=1}^{i-1}\frac{c_{ij}}{\tau_n}\boldsymbol{A}_{nj} + \tau_n\gamma_i\boldsymbol{\partial}_t J_s(\cdot, t_n),$$

where $\boldsymbol{A}_i = \boldsymbol{A}_n + \sum_{j=1,\ldots,i-1} a_{ij}\boldsymbol{A}_{nj}$ and $t_i = t_n + \alpha_i\tau_n$. Observe that the system (4) has to be solved successively for $i = 1, \ldots, s$. In addition, the stage values $\boldsymbol{A}_{ni}$ can be also used to derive approximations $\boldsymbol{Z}_n \approx \partial_t\boldsymbol{A}(\cdot, t_n)$ of the same order to the first derivative and therefore for the eddy current density $\boldsymbol{J}_E$. We compute

$$\boldsymbol{Z}_{n+1} = \boldsymbol{Z}_n + \sum_{i=1}^{s} m_i \left(\frac{1}{\tau} \sum_{j=1}^{i} (c_{ij} - s_{ij}) \boldsymbol{A}_{nj} + (d_i - 1) \boldsymbol{A}_n \right).$$

The stage number s and the defining formula coefficients m_i, c_{ij}, γ_i, a_{ij}, α_i, s_{ij}, d_i, and γ are chosen to obtain a desired order of consistency and good stability properties for differential-algebraic equations [14]. The A-stable Rosenbrock solver ROS3P from [15] was constructed for parabolic problems. For differential-algebraic equations (as (1) with $\sigma = 0$ somewhere), we would like to have also L-stability and the property of stiff accuracy. For this, we have designed a new Rosenbrock solver ROS3PL. The number of stages is $s = 4$, the order of the method is $p = 3$, and it fulfills additional conditions to avoid order reduction (see [15] for more details). The set of coefficients is given in Table 1.

Rosenbrock methods offer a simple way to estimate the local error. A second solution $\hat{\boldsymbol{A}}_{n+1}$ of inferior order, say $\hat{p}$, can be computed by replacing the original weights m_i by $\hat{m}_i$ in (3). In order to take into account the scale of the problem, the local error estimator is defined by the weighted root mean square norm

$$r_{n+1} = \left(\frac{\|\boldsymbol{A}_{n+1} - \hat{\boldsymbol{A}}_{n+1}\|^2_{L^2(\Omega)}}{ATOL + RTOL\, \|\boldsymbol{A}_{n+1}\|^2_{L^2(\Omega)}} \right)^{1/2}. \tag{5}$$

The tolerances $ATOL$ and $RTOL$ have to be selected carefully to furnish meaningful input for the error control. The estimator can be used to propose a new time step by

$$\tau_{n+1} = \frac{\tau_n}{\tau_{n-1}} \left(\frac{TOL_t r_n}{r_{n+1} r_{n+1}} \right)^{1/(\hat{p}+1)} \tau_n, \tag{6}$$

where TOL_t is a desired tolerance prescribed by the user [11]. If $r_{n+1} > TOL_t$ the step is rejected and redone. Otherwise the step is accepted and we advance in time. The order of the embedded solution of ROS3PL is $\hat{p} = 2$.

Rosenbrock methods have been successfully applied in [7] to nonlinear magnetic field problems.

2.2 Adaptive multilevel Whitney finite elements

For spatial adaptivity, a multilevel finite element method is used to solve the s linear systems (4) in each time step. The solution space is replaced

$\gamma = 4.358665215084590e-01$	
$a_{11} = 0.000000000000000e+00$	$\alpha_1 = 0.000000000000000e+00$
$a_{21} = 1.147140180139521e+00$	$\alpha_2 = 5.000000000000000e-01$
$a_{22} = 0.000000000000000e+00$	$\alpha_3 = 1.000000000000000e+00$
$a_{31} = 2.463070773030053e+00$	$\alpha_4 = 1.000000000000000e+00$
$a_{32} = 1.147140180139521e+00$	
$a_{33} = 0.000000000000000e+00$	
$a_{41} = 2.463070773030053e+00$	
$a_{42} = 1.147140180139521e+00$	
$a_{43} = 0.000000000000000e+00$	
$a_{44} = 0.000000000000000e+00$	
$c_{11} = 2.294280360279042e+00$	$s_{11} = 0.000000000000000e+00$
$c_{21} = 2.631861185781065e+00$	$s_{21} = 2.631861185781065e+00$
$c_{22} = 2.294280360279042e+00$	$s_{22} = 0.000000000000000e+00$
$c_{31} = 1.302364158113095e+00$	$s_{31} = 5.650974900540168e+00$
$c_{32} = -2.769432022251304e+00$	$s_{32} = 2.631861185781065e+00$
$c_{33} = 2.294280360279042e+00$	$s_{33} = 0.000000000000000e+00$
$c_{41} = 1.552568958732400e+00$	$s_{41} = 5.650974900540168e+00$
$c_{42} = -2.587743501215153e+00$	$s_{42} = 2.631861185781065e+00$
$c_{43} = 1.416993298352020e+00$	$s_{43} = 0.000000000000000e+00$
$c_{44} = 2.294280360279042e+00$	$s_{44} = 0.000000000000000e+00$
$\gamma_1 = 4.358665215084590e-01$	$d_1 = 0.000000000000000e+00$
$\gamma_2 = -6.413347849154100e-02$	$d_2 = 1.147140180139521e+00$
$\gamma_3 = 1.110281725125051e-01$	$d_3 = 2.294280360279042e+00$
$\gamma_4 = 0.000000000000000e-00$	$d_4 = 2.294280360279042e+00$
$m_1 = 2.463070773030053e+00$	$\hat{m}_1 = 2.346947683513665e+00$
$m_2 = 1.147140180139521e+00$	$\hat{m}_2 = 4.565305694518951e-01$
$m_3 = 0.000000000000000e+00$	$\hat{m}_3 = 5.694924394549457e-02$
$m_4 = 1.000000000000000e+00$	$\hat{m}_4 = 7.386849361662244e-01$

Table 1: Set of coefficients for ROS3PL

by a sequence of discrete spaces with successively increasing dimension to improve their approximation property. A posteriori error estimates provide the appropriate framework to determine where a mesh refinement is necessary and where degrees of freedom are no longer needed. Adaptive multilevel methods have proven to be a useful tool for drastically reducing the size of the arising linear algebraic systems and to achieve high and controlled accuracy of the spatial discretization [8, 14]. For stationary and time-harmonic Maxwell problems, they have been considered in [4, 5, 6, 18]. We extent this approach to MQS approximations (1) discretized in time by a variable step-size one-step Rosenbrock method.

Let $\mathcal{T}_h$ be an admissible tetrahedral mesh at $t = t_n$ and $\boldsymbol{W}_h^q$ be the associated $\boldsymbol{H}_0(\text{curl})$-conforming Whitney finite element space consisting of polynomials of order q on each finite element $T \in \mathcal{T}_h$. Then the Galerkin approximation $\boldsymbol{A}_{ni}^h \in \boldsymbol{W}_h^q$ of the stage values $\boldsymbol{A}_{ni}$, $i = 1, \ldots, s$, satisfies the weak formulation

$$b_n(\boldsymbol{A}_{ni}^h, \boldsymbol{V}^h) = (\boldsymbol{R}_{ni}, \boldsymbol{V}^h)\,, \qquad \forall\, \boldsymbol{V}^h \in \boldsymbol{W}_h^q, \tag{7}$$

where the bilinear $b_n(\cdot,\cdot)$ is defined as

$$b_n(\boldsymbol{A}_{ni}^h, \boldsymbol{V}^h) = (\mu^{-1}\, \boldsymbol{\nabla}\times \boldsymbol{A}_{ni}^h, \boldsymbol{\nabla}\times \boldsymbol{V}^h) + \left(\frac{\sigma}{\tau_n \gamma} \boldsymbol{A}_{ni}^h, \boldsymbol{V}^h\right)$$

and $(\cdot,\cdot)$ stands for the usual scalar product in $L^2(\Omega)$. As a basis in $\boldsymbol{W}_h^q$ we take the hierarchical tetrahedral basis functions proposed in [3]. The weak formulation (7) is equivalent to a linear system for each stage value $\boldsymbol{A}_{ni}^h$, $i = 1, ..., s$. Observe that the operator associated with the bilinear $b_n(\cdot,\cdot)$ is independent of the stage level i, and thus the calculation of the stiffness matrix is required only once within each time step. To solve the linear systems we use the AMG solver with Hiptmair smoother [13] implemented in the package ML of the Trilinos library [12].

After computing the approximate stage values $\boldsymbol{A}_{ni}^h$, a posteriori error estimates for the approximate Rosenbrock solution $A_{n+1}^h \in \boldsymbol{W}_h^q$ can be used to give specific assessment of the error distribution and to improve the spatial discretization. Hierarchical basis error estimators are well-known for standard conforming discretizations, e.g., [8, 14]. Considering a hierarchical decomposition

$$\boldsymbol{W}_h^{q+1} = \boldsymbol{W}_h^q \oplus \boldsymbol{Z}_h^{q+1}\,,$$

where $\boldsymbol{Z}_h^{q+1}$ is the subspace needed to extend the space $\boldsymbol{W}_h^q$ to higher order, the idea of a hierarchical error estimator is to bound the spatial error by evaluating its components in the space $\boldsymbol{Z}_h^{q+1}$ only. Hierarchical error estimators for problems as (7) are investigated in [5, 6]. Here, we want to estimate the error $\boldsymbol{A}_{n+1} - \boldsymbol{A}^h_{n+1}$ caused by the interpolation error of the initial value $\boldsymbol{A}_n$ and by the spatial approximation of all stage values $\boldsymbol{A}^h_{ni} \in \boldsymbol{W}^q_h$, $i = 1, \ldots, s$.

We define an a posteriori error estimator $\boldsymbol{E}^h_{n+1} \in Z^{q+1}_h$ as

$$\boldsymbol{E}^h_{n+1} = \boldsymbol{E}^h_{n0} + \sum_{i=1}^{s} m_i \boldsymbol{E}^h_{ni}\,, \tag{8}$$

with $\boldsymbol{E}^h_{n0}$ approximating the projection error of the initial value $\boldsymbol{A}_n$ in Z^{q+1}_h

$$b_n(\boldsymbol{E}^h_{n0}, \Phi) = b_n(\boldsymbol{A}_n - \boldsymbol{A}^h_n, \Phi), \quad \Phi \in Z^{q+1}_h \tag{9}$$

and $\boldsymbol{E}^h_{ni}$ estimating the spatial error of the stage value $\boldsymbol{A}^h_{ni}$

$$b_n(\boldsymbol{E}^h_{ni}, \Phi) = (\boldsymbol{R}^h_{ni}, \Phi) - b_n(\boldsymbol{A}^h_{ni}, \Phi), \quad \Phi \in Z^{q+1}_h \tag{10}$$

where

$$\boldsymbol{R}^h_{ni} = \boldsymbol{R}_{ni}(A^h_{n1} + E^h_{n1}, \ldots, A^h_{ni-1} + E^h_{ni-1}).$$

Considering the error estimators already computed takes into account the successive error transport within the sequence of stage problems. The local spatial error for a finite element $T \in \mathcal{T}_h$ can be estimated by computing the norm of $\boldsymbol{E}^h_{n+1}$ over T. For the overall spatial error, we define

$$|||\boldsymbol{E}^h_{n+1}||| = \left(\frac{\|\boldsymbol{E}^h_{n+1}\|^2_{L^2(\Omega)}}{ATOL + RTOL\, \|\boldsymbol{A}^h_{n+1}\|^2_{L^2(\Omega)}} \right)^{1/2}. \tag{11}$$

We have implemented our a posteriori error estimator for $q=1$. As proposed in [6], we further take advantage of a localization strategy. For this, we define a direct decomposition of the surplus space

$$Z^{q+1}_h = \sum_{edge\, e} Z^{q+1}_h(e) \oplus \sum_{face\, f} Z^{q+1}_h(f)$$

and end up with a sequence of scalar equations for each edge and of 2×2 linear systems for each face. From many practical computations, we have also experienced that using the simplified error estimator $\boldsymbol{E}^h_{n+1} \approx \boldsymbol{E}^h_{n0} + \boldsymbol{E}^h_{n1}/\gamma$, that is an error estimator for the embedded, locally second order Euler solution, is quite efficient [14]. For more details, we refer to the forthcoming paper [16].

A maximum selection strategy is used to mark elements for refinement. The iterative process estimate-refine-solve within a time step is continued until $|||E^h_{n+1}||| \le TOL_x$ with TOL_x being a prescribed tolerance for the spatial discretization error. To maintain the nesting property of the finite element subspaces, coarsening takes place only after an accepted time step before starting the multilevel process at a new time. The detailed algorithm of the spatial grid adaptation is fully described in [14]. For the convenience of the reader, in Fig. 1 the internal work steps are briefly illustrated.

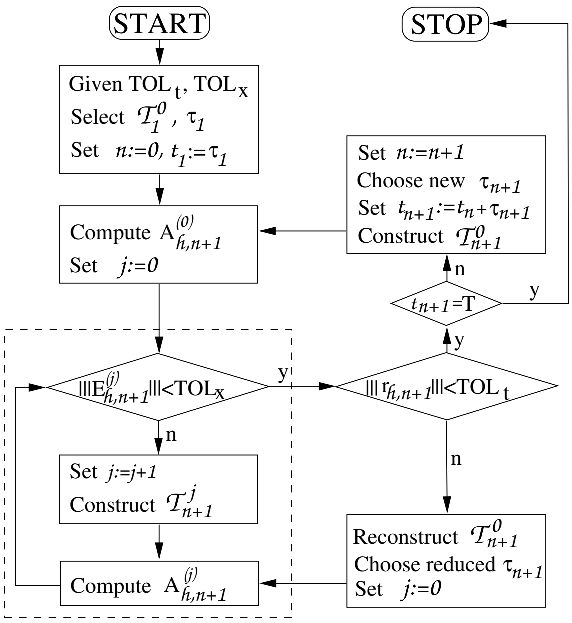

Figure 1: Flow chart for the time-space adaptive solver KARDOS.

3 Numerical results

We consider the TEAM 7 benchmark problem [10]. The problem consists of a rectangular aluminium plate, $\sigma = 3.526\,10^7\, S/m$, with eccentric rectangular cutout placed under an eccentrically positioned coil, Fig. 2. A sinusoidal current of $2742\,A$ and $50\,Hz$ flows through the coil and induces a current in the plate. As computational domain we use a cube with $1m$ edge length. Homogeneous boundary conditions, $\boldsymbol{n} \times \boldsymbol{A} = 0$, are taken. The results are simulated with KARDOS [2] and are visualized with AMIRA [1].

To check our implementation, we have first compared our results with experimental data. Fig. 3 reveals good agreement. In Fig. 4 various results illustrating the coupled time-space adaptivity are displayed. Comparisons with uniform approaches are given.

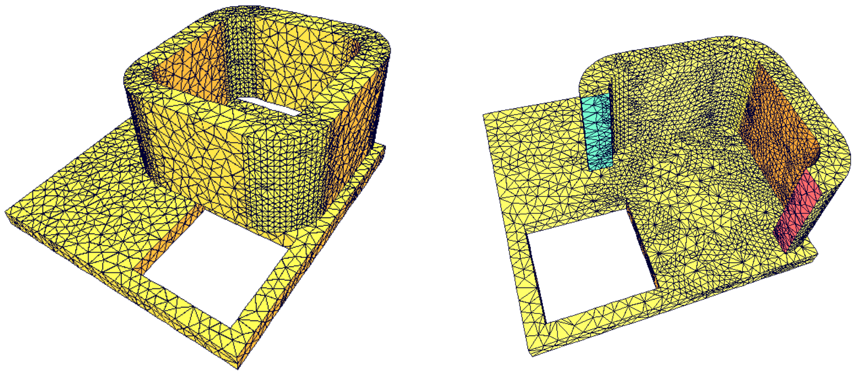

Figure 2: Coarse (left) and selected fine (right) tetrahedral meshes.

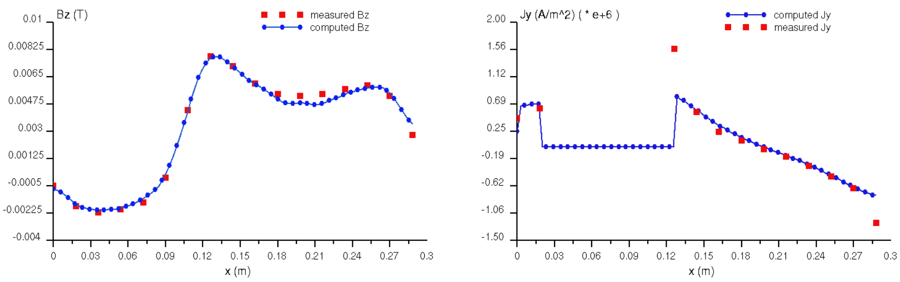

Figure 3: Comparison of computed and measured real part of $\boldsymbol{B}_z$ (left) and of $\boldsymbol{J}_{E,y}$ (right), see [10] for reference values.

4 Conclusion

We have combined variable step size one-step methods of Rosenbrock type and adaptive $\boldsymbol{H}$(curl)-conforming Whitney finite elements to solve linear three-dimensional magnetoquasistatics problems. Numerical investigations for the TEAM7 problem show the great potential of space-time adaptive methods with respect to reliability and efficiency.

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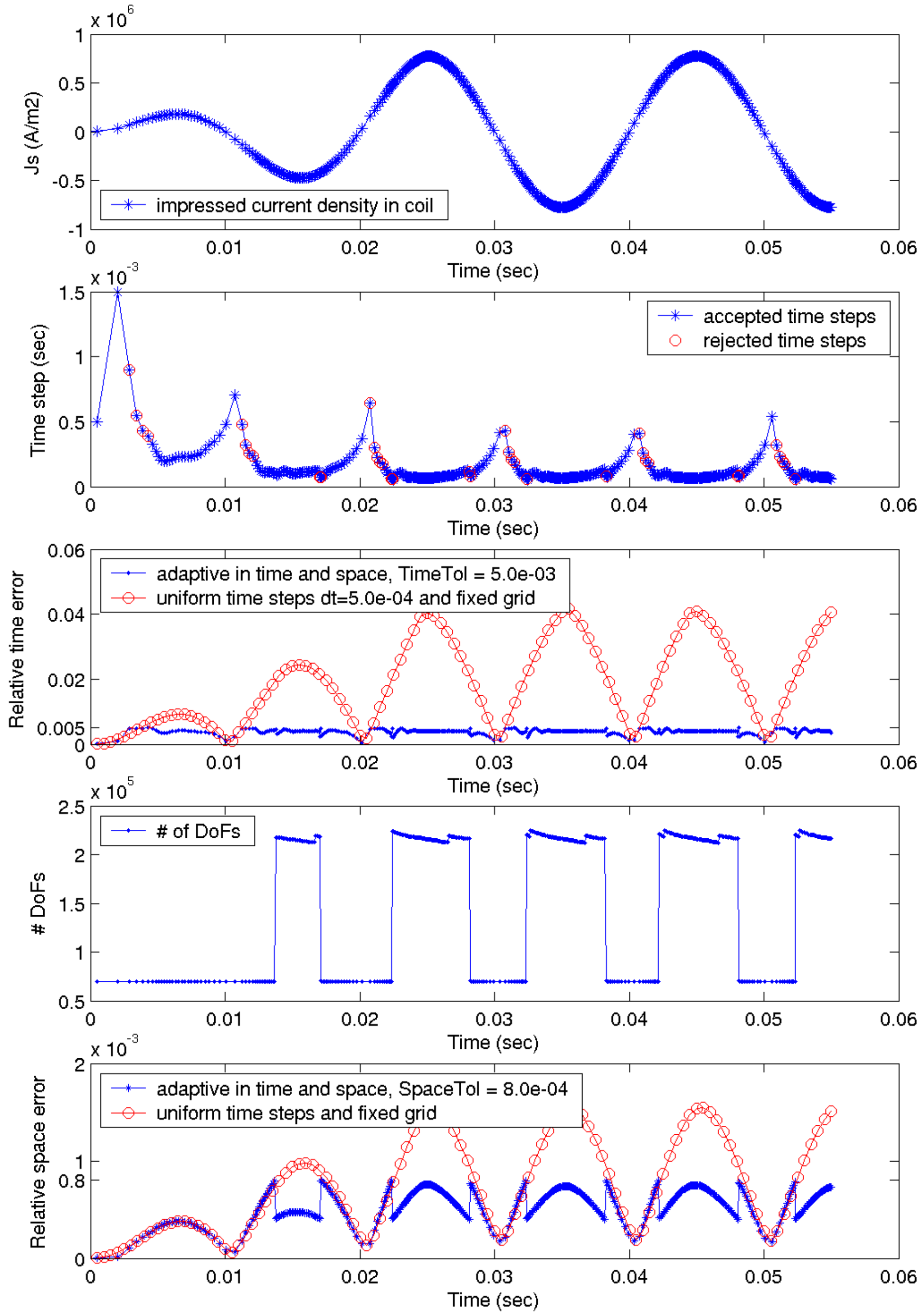

Figure 4: From above: (a) $J_s(t)$ in the coil, (b) typical evolution of time steps controlled by the estimator (5), (c) observed local time errors (5) for controlled and uniform step size, (d) evolution of spatial degrees of freedom necessary to reach $TOL_x = 0.0008$ in each time step, (e) comparison of estimated spatial errors (11) for fully adaptive and uniform approach.

References

[1] http://amira.zib.de/

[2] http://www.zib.de/Numerik/software/kardos

[3] M. Ainsworth, J. Coyle, *Hierarchic Finite Element Basis on Unstructured Tetrahedral Meshes*, Int. J. Num. Meth. Eng., 58(14), 2103-2130, 2003.

[4] F. Bachinger, U. Langer, J. Schöberle, *Efficient solvers for nonlinear time-periodic eddy current problems*, Comp. Vis. Sci., 9, 197-207, 2006.

[5] R. Beck, P. Deuflhard, R. Hiptmair, R.H.W. Hoppe, B. Wohlmuth, *Adaptive Multilevel Method for Edge Element Discretizations of Maxwell's Equations*, Surveys on Math. for Industry, 8, 271-312, 1998.

[6] R. Beck, R. Hiptmair, B. Wohlmuth, *Hierarchical error estimator for eddy current computation*, Proc. ENUMATH 99, Neittaanmäki,P. et al. (eds.), 110-120, World Scientific, Singapore, 1999.

[7] M. Clemens, M. Wilke, T. Weiland, *Linear-implicit time integration schemes for error-controlled transient nonlinear magnetic field simulations*, IEEE Trans. Magn., 39, 1175-1178, 2003.

[8] P. Deuflhard, P. Leinen, H. Yserentant, *Concepts of an adaptive hierarchical finite element code*, IMPACT Comp. Sci. Engrg., 1, 3-35, 1989.

[9] B. Erdmann, J. Lang, R. Roitzsch, *KARDOS User's Guide*, Technical Report ZR-02-42, ZIB, 2002.

[10] K. Fujiwara, T. Nakata, *Results for benchmark problem 7 (asymmetrical conductor with a hole)*, COMPEL, vol. 9, nr. 3, 137-154, 1990.

[11] K. Gustafsson, M. Lundh, G. Söderlind, *A PI stepsize control for the numerical solution of ordinary differential equations*, BIT, 28, 270-287, 1988.

[12] M. Heroux, R. Bartlett et al., *An Overview of Trilinos*, Sandia National Laboratories, Technical Report SAND2003-2927, 2003.

[13] R. Hiptmair, *Multigrid Method for Maxwell's Equations*, SIAM J. Num. Analysis, Vol. 36, No. 1, 204-225, 1999.

[14] J. Lang, *Adaptive Multilevel Solution of Nonlinear Parabolic PDE Systems. Theory, Algorithm, and Applications*, LNCSE vol. 16, Springer-Verlag, 2000.

[15] J. Lang, J. Verwer, *ROS3P - an Accurate Third-Order Rosenbrock Solver Designed for Parabolic Problems*, BIT 41, 730-737, 2001.

[16] J. Lang, D. Teleaga, *Adaptivity in space and time for magnetoquasistatics*, in preparation.

[17] J. Schöberl, S. Zaglmayr, *High order Nédélec elements with local complete sequence properties*, Int. J. Comp. Math. Electr. Electronic Eng., 24, 374-384, 2005.

[18] O. Sterz, A. Hauser, G. Wittum, *Adaptive Local Multigrid Methods for Solving Time-Harmonic Eddy-Current Problems*, IEEE Trans. Magn., 42, 309-318, 2006.

[19] W. Zheng, Z. Chen, L. Wang, *An adaptive finite element method for the $h-\psi$ formulation of time-dependent eddy current problems*, Numer. Math., 103, 667-689, 2006.