Energy-Based Regularity Criteria for the Navier-Stokes Equations

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Abstract

We present several new regularity criteria for weak solutions u of the instationary Navier-Stokes system which additionally satisfy the strong energy inequality. (i) If the kinetic energy $\frac{1}{2}\|u(t)\|_2^2$ is Hölder continuous as a function of time t with Hölder exponent $\alpha \in (\frac{1}{2}, 1)$, then u is regular. (ii) If the dissipation energy satisfies the left-side condition $\liminf_{\delta\to 0} \delta^{-\alpha} \int_{t-\delta}^{t} \|\nabla u\|_2^2 \, d\tau < \infty$, $\alpha \in (\frac{1}{2}, 1)$, for all t of the given time interval, then u is regular. The proofs use local regularity results which are based on the theory of very weak solutions and on uniqueness arguments for weak solutions. Finally, in the last section, we mention a local space-time regularity condition.

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1 Introduction and main results

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, $[0, T)$ a time interval with $0 < T \le \infty$, and let $u_0 \in L^2_\sigma(\Omega)$ be some initial value. Then in $[0, T) \times \Omega$ we consider a weak solution u of the Navier-Stokes system

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p &= 0, \quad \operatorname{div} u = 0 \\ u\big|_{\partial\Omega} &= 0, \quad u\big|_{t=0} = u_0 \end{aligned} \tag{1.1}$$

with vanishing external force and with viscosity $\nu = 1$ as follows.

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Definition 1.1 A vector field

$$u \in L^\infty\big(0,T; L^2_\sigma(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0,T); W^{1,2}_0(\Omega)\big) \tag{1.2}$$

is called a *weak solution* of the system (1.1) if the relation

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_\Omega \tag{1.3}$$

is satisfied for all test functions $w \in C^\infty_0([0,T); C^\infty_{0,\sigma}(\Omega))$.

In this definition $\langle\cdot,\cdot\rangle_\Omega$ means the usual pairing of functions on Ω, $\langle\cdot,\cdot\rangle_{\Omega,T}$ means the corresponding pairing on $[0,T)\times\Omega$, $L^2_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2}$ with $C^\infty_{0,\sigma}(\Omega) = \{v \in C^\infty_0(\Omega); \operatorname{div} v = 0\}$ and $W^{1,2}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{W^{1,2}}}$. Finally $uu = (u_iu_j)^3_{i,j=1}$ for $u = (u_1,u_2,u_3)$ such that $u\cdot\nabla u = (u\cdot\nabla)u = \operatorname{div}(uu)$ when $\operatorname{div} u = 0$.

We may assume, without loss of generality, that

$$u : [0,T) \to L^2_\sigma(\Omega) \quad \text{is weakly continuous} \tag{1.4}$$

in Definition 1.1, with $u(0) = u_0$, see [14], p. 271. Moreover, there exists a distribution p, called an associated pressure, such that

$$u_t - \Delta u + u\cdot\nabla u + \nabla p = 0 \tag{1.5}$$

holds in the sense of distributions, see [14], p. 264.

Since the domain Ω is bounded, it is not difficult to prove the existence of a weak solution u as in Definition (1.1) which additionally satisfies the *strong energy inequality*

$$\frac{1}{2}\,\|u(t)\|_2^2 + \int_{t_0}^t \|\nabla u\|_2^2\,d\tau \le \frac{1}{2}\,\|u(t_0)\|_2^2 \tag{1.6}$$

for almost all $t_0 \in [0,T)$, including $t_0 = 0$, and all $t \in [t_0,T)$, see [14], p. 340. For further results in this context for unbounded domains we refer to [5].

A weak solution u of (1.1) is called *regular* in some interval $(a,b) \subseteq (0,T)$, $a<b$, if *Serrin's condition*

$$u \in L^s_{\mathrm{loc}}\big(a,b; L^q(\Omega)\big) \quad \text{with } 2<s<\infty,\ 3<q<\infty,\ \frac{2}{s}+\frac{2}{q} \le 1 \tag{1.7}$$

is satisfied, see [13], [14]. This means it holds

$$\|u\|_{L^s(a',b';L^q(\Omega))} = \Big(\int_{a'}^{b'} \|u\|_q^s\,dx\Big)^{1/s} < \infty$$

for each interval (a',b') with $a<a'<b'<b$; here $\|u\|_q = \big(\int_\Omega |u|^q\,dx\big)^{1/q}$.

The condition (1.7) implies that

$$u \in C^\infty\big((a,b)\times\overline{\Omega}\big), \quad p \in C^\infty\big((a,b)\times\overline{\Omega}\big), \tag{1.8}$$

if $\partial\Omega$ is of class C^∞; see e.g. [14], Theorem V, 1.8.2.

A time $t \in (0,T)$ is called a *regular point* of a weak solution u if u is regular in some interval $(a,b) \subseteq (0,T)$ with $a < t < b$.

Concerning a criterion based on the kinetic energy $\frac{1}{2}\,\|u(t)\|_2^2$, $t \in (0,T)$, we have the following result.

Theorem 1.2 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, and let $0 \le a < b \le T \le \infty$. Consider a weak solution u of the Navier-Stokes system* (1.1) *with $u_0 \in L^2_\sigma(\Omega)$, satisfying the strong energy inequality* (1.6).

Suppose $t \mapsto \frac{1}{2}\,\|u(t)\|_2^2$, $t \in (a,b)$, is Hölder continuous with exponent $\alpha \in (\frac{1}{2},1)$ in the sense that

$$\sup_{a<t<t'<b} \frac{|\frac{1}{2}\,\|u(t)\|_2^2 - \frac{1}{2}\,\|u(t')\|_2^2|}{|t-t'|^\alpha} < \infty. \tag{1.9}$$

Then u is regular in (a,b).

Remark 1.3 (1) An inspection of the proof of Theorem 1.2, which is based on Lemma 2.5 below, shows that at time t the left-side Hölder condition

$$\sup_{t-\delta<\tau<t} \frac{|\frac{1}{2}\|u(t)\|_2^2 - \frac{1}{2}\|u(\tau)\|_2^2|}{|t-\tau|^\alpha} < \infty, \quad 0<\delta<t<T,$$

suffices to get regularity of u in a neighborhood of t. Moreover, the supremum in this condition may be replaced by its infimum or even by the condition

$$\liminf_{\delta\to 0+} \frac{|\frac{1}{2}\,\|u(t-\delta)\|_2^2 - \frac{1}{2}\,\|u(t)\|_2^2|}{\delta^\alpha} < \infty. \tag{1.10}$$

However, if (1.10) is satisfied for some $\alpha \in (\frac{1}{2},1)$, then there exists $\alpha' \in (\frac{1}{2},\alpha)$ such that even

$$\liminf_{\delta\to 0+} \frac{|\frac{1}{2}\,\|u(t-\delta)\|_2^2 - \frac{1}{2}\,\|u(t)\|_2^2|}{\delta^{\alpha'}} = 0. \tag{1.11}$$

Hence (1.10) may be replaced by the formally stronger, but nevertheless equivalent condition (1.11). For a discussion of the limit case $\alpha = \frac{1}{2}$ we refer to Remark 1.5 (3) below.

(2) Conversely, if u is not regular at t, then the kinetic energy $\frac{1}{2}\|u(\tau)\|_2^2$ either has a jump downward at $t-$ or it is continuous, but not left-side Hölder continuous at t of order $\alpha \in (\frac{1}{2},1)$.

(3) Using the relation $c^2 - d^2 = (c-d)(c+d)$ and the energy inequality (1.6) with $t_0 = 0$, we see that the condition

$$\sup_{a<t<t'<b} \frac{|\,\|u(t)\|_2 - \|u(t'\|_2\,|}{|t-t'|^\alpha} < \infty \tag{1.12}$$

is sufficient for (1.9). Thus u in Theorem 1.2 is regular in (a,b) if (1.12) is satisfied for all $t \in (a,b)$ and $\alpha \in (\frac{1}{2}, 1)$. This condition means that $t \mapsto \frac{1}{2}\|u(t)\|_2$ is contained in the Hölder space $C^\alpha(a,b)$ for $\alpha \in (\frac{1}{2}, 1)$. The expression on the left-side side of (1.12) yields the seminorm $\|\cdot\|_{C^\alpha(a,b)}$ of this space.

Using a notation introduced in [8], Part 1, Section 9, Problem (3), and motivated by certain embedding estimates, we can write the Hölder space $C^\alpha(a,b)$ formally as the Lebesgue space $L^s(a,b) = C^\alpha(a,b)$ with negative exponent $s = -\frac{2}{\alpha}$. Then the condition $\frac{1}{2} < \alpha < 1$ can be written in the form $\frac{1}{2} < \frac{2}{s} + \frac{3}{2} < 1$, and the regularity condition (1.12) now reads as follows: If

$$u \in L^s_{\text{loc}}\big(a,b; L^2(\Omega)\big) \quad \text{with } -4 < s < -2,\ \frac{2}{s} + \frac{3}{2} < 1, \tag{1.13}$$

then u is regular in (a,b). Here $\|u\|_{L^s(a,b;L^2(\Omega))}$ is defined by the left-side side of (1.12).

Therefore (1.13) can be considered as an extension of Serrin's regularity condition (1.7) to negative exponents s. As expected in this context we see, if the space $L^q(\Omega)$ in (1.7) is replaced by the strictly larger space $L^2(\Omega)$ because of $2 < q$, we have to replace on the other hand the space $L^s(a,b)$ in (1.7) by the strictly smaller space $C^\alpha(a,b)$.

The next result concerns the dissipation energy $\int_{t-\delta}^t \|\nabla u\|_2^2\, d\tau$ in the interval $(t-\delta, t)$. In order to prove the regularity of u in (a,b) we need a certain smallness condition of this expression for all $t \in (a,b)$. Note that this condition is required only on the left-side side $(t-\delta, t)$ of t.

Theorem 1.4 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, and let $0 \le a < b < T$. Consider a weak solution u of the Navier-Stokes system* (1.1) *with $u_0 \in L^2_\sigma(\Omega)$, satisfying the strong energy inequality* (1.6)*.*

Let $\frac{1}{2} < \alpha < 1$ and assume that

$$\liminf_{\delta \to 0+} \frac{1}{\delta^\alpha} \int_{t-\delta}^t \|\nabla u\|_2^2\, d\tau < \infty \tag{1.14}$$

for each $t \in (a,b)$. Then u is regular in (a,b).

Remark 1.5 (1) By (1.14) u is regular in (a,b) if e.g. $\liminf_{\delta\to 0+}$ of the left-side mean value of $\|\nabla u\|_2^2$ is finite at each $t \in (a,b)$, i.e., if

$$\liminf_{\delta \to 0+} \frac{1}{\delta} \int_{t-\delta}^t \|\nabla u\|_2^2\, d\tau < \infty$$

holds for all $t \in (a,b)$. For example, this condition is satisfied if each $t \in (a,b)$ is a (left-side) Lebesgue point of $\|\nabla u\|_2^2$ in the sense that

$$\liminf_{\delta \to 0+} \frac{1}{\delta} \int_{t-\delta}^{t} \|\nabla u\|_2^2 \, d\tau = \|\nabla u(t)\|_2^2.$$

Note that this equation holds for almost all $t \in (0,T)$, see, e.g. [14], p. 341.

(2) Condition (1.14) for some $\alpha \in (\frac{1}{2}, 1)$ implies for any $\alpha' \in (\frac{1}{2}, \alpha)$ that

$$\liminf_{\delta \to 0+} \frac{1}{\delta^{\alpha'}} \int_{t-\delta}^{t} \|\nabla u\|_2^2 \, d\tau = 0. \tag{1.15}$$

Hence (1.14) may be replaced by the condition (1.15).

(3) It is known, see e.g. [Ga2], Theorem 6.4, that if $(0,t)$ is a maximal regularity interval of a weak solution u, then it necessarily holds $\|\nabla u(\tau)\|_2 \geq c(t-\tau)^{-1/4}$, $0 < \tau < t$, where $c = c(\Omega) > 0$. Hence (1.14) with $\alpha = \frac{1}{2}$ fails to imply regularity whereas (1.14) with a smallness condition could still lead to regularity when $\alpha = \frac{1}{2}$. Moreover, the estimate

$$2c^2 \leq \frac{1}{\delta^{1/2}} \int_{t-\delta}^{t} \|\nabla u(\tau)\|^2 \, d\tau \leq \frac{1}{2\delta^{1/2}} \big(\|u(t)\|_2^2 - \|u(t-\delta)\|_2^2 \big)$$

(for a.a. $\delta \in (0,t)$) shows that the condition (1.10) with $\alpha = \frac{1}{2}$ does not imply regularity. However, (1.10) with a smallness assumption could imply regularity when $\alpha = \frac{1}{2}$.

A third regularity criterion is not based on energy terms, but on local $L^s(L^q)$-norms of u in time and space; for details see Theorem 4.1 below. However, the idea of the proof is the same by identifying locally in space and time the given weak solution with a very weak one.

2 Preliminary local regularity results

Let $\Omega \subseteq \mathbb{R}^3$ and $0 < a < b \leq T$ be as in Section 1. We use the well-known spaces $L^q = L^q(\Omega)$, $1 < q < \infty$, with norm $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and pairing $\langle v, w \rangle_\Omega = \int_\Omega v \cdot w \, dx$ for $v \in L^q(\Omega)$, $w \in L^{q'}(\Omega)$, $q' = \frac{q}{q-1}$. Further we need the Bochner spaces $L^s\big(a,b; L^q(\Omega)\big)$, $1 < s < \infty$, with norm $\|\cdot\|_{L^s(a,b;L^q(\Omega))} = \big(\int_a^b \|\cdot\|_q^s \, dt \big)^{1/s}$ and corresponding pairing $\langle \cdot, \cdot \rangle_{\Omega,(a,b)}$. If $(a,b) = (0,T)$ we write $\langle \cdot, \cdot \rangle_{\Omega,(a,b)} = \langle \cdot, \cdot \rangle_{\Omega,T}$. We also use the usual smooth function spaces $C_0^\infty(\Omega)$, $C_{0,\sigma}^\infty(\Omega) = \{ v \in C_0^\infty(\Omega);\, \operatorname{div} v = 0\}$, and $L^q_\sigma = L^q_\sigma(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_q}$.

The proof of our theorems rests on a local existence result of regular solutions, which has been developed in the theory of very weak solutions, see [1] and [4]. In this context we use the Stokes operator

$$A_q = -P_q \Delta : \; D(A_q) \to L^q_\sigma(\Omega), \quad D(A_q) = L^q_\sigma \cap W_0^{1,q} \cap W^{2,q},$$

where $W_0^{1,q} = W_0^{1,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,q}}$, the Helmholtz projection $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$, and the semigroup $e^{-tA_q} : L^q_\sigma(\Omega) \to L^q_\sigma(\Omega)$, $0 \le t < \infty$, generated by A_q. See [3], [5], [9] – [12] and [15] concerning properties of these operators.

The following local result, see [6], Lemma 2.1, is essentially a consequence of [4], Theorem 1.

Lemma 2.1 ([6]) *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, let $4 < s < \infty$, $3 < q < 6$, $\frac{2}{s} + \frac{3}{q} = 1$, and $v_0 \in L^q_\sigma(\Omega)$. Then there is a constant $C = C(\Omega, q) > 0$ with the following property: If*

$$\int_0^{T_0} \|e^{-\tau A_q} v_0\|_q^s \, d\tau \le C \tag{2.1}$$

for some T_0, $0 < T_0 \le \infty$, then there exists a unique regular weak solution $v \in L^\infty\big(0, T_0; L^2_\sigma(\Omega)\big) \cap L^2_{\mathrm{loc}}([0, T_0); W_0^{1,2}(\Omega)\big)$ of the Navier-Stokes system

$$\begin{array}{rclrcl} v_t - \Delta v + v \cdot \nabla v + \nabla p &=& 0, & \operatorname{div} v &=& 0 \\ v\big|_{\partial\Omega} &=& 0, & v\big|_{t=0} &=& v_0, \end{array}$$

satisfying Serrin's condition $v \in L^s\big(0, T_0; L^q(\Omega)\big)$ and the energy inequality (1.6) *with $t_0 = 0$, u replaced by v.*

The following local regularity criterion is obtained when we apply (2.1) to a given weak solution u for appropriate initial values of the form $v_0 = u(t_0)$ with $0 < t_0 < T$. Then (2.1) is applied in the form

$$\int_0^{T_0 - t_0} \|e^{-\tau A_q} u(t_0)\|_q^s \, d\tau \le C, \quad t_0 < T_0 < T, \tag{2.2}$$

and the solution v in $[t_0, T_0] \times \Omega$ will be identified with u locally within $[t_0, T_0)$. See [6] concerning other results in this context.

Lemma 2.2 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, let $0 < T < \infty$, $1 \le s \le s_* < \infty$, $3 < q < 6$ satisfying*

$$\frac{2}{s} + \frac{3}{q} \ge 1, \quad \frac{2}{s_*} + \frac{3}{q} = 1, \tag{2.3}$$

and let u be a weak solution of the Navier-Stokes system (1.1) *with $u_0 \in L^2_\sigma(\Omega)$, satisfying the strong energy inequality* (1.6).

Then there is a constant $C = C(\Omega, q, s) > 0$ with the following property: If $0 < \delta < t < T$ and

$$\int_{t-\delta}^{t} \|u(\tau)\|_q^s \, d\tau \le C \delta^{1 - \frac{s}{s_*}}, \tag{2.4}$$

then u is regular in some interval $(t - \delta', t + \delta') \subseteq (0, T)$, $0 < \delta' < \delta$, in the sense that $u \in L^{s_*}\big(t - \delta', t + \delta'; \, L^q(\Omega)\big)$.

Remark 2.3 If (2.4) holds for each $t \in (0,T)$ with some $\delta = \delta(t) \in (0,t)$, then u is regular in $(0,T)$. In particular we obtain in the case $s = s_*$ that if the local left-side Serrin condition

$$\int_{t-\delta}^{t} \|u(\tau)\|_q^{s_*}\, d\tau \le C$$

holds for each $t \in (0,T)$ with some $\delta = \delta(t) \in (0,t)$, then u is regular in $(0,T)$. If $\frac{2}{s} + \frac{3}{q} > 1$, beyond Serrin's condition, then (2.4) requires a local left-side smallness condition depending on δ.

The next lemma shows that Serrin's regularity condition (1.7) can be extended to larger spaces L^q, L^s such that $\frac{2}{s} + \frac{3}{q} > 1$, if we additionally suppose a certain smallness condition on the norm $\|u\|_{L^s(0,T;L^q(\Omega))}$. The corresponding smallness criterion depends on the initial value u_0 which is supposed to belong to L^q_σ.

Lemma 2.4 *Let Ω, T, u be as in Lemma 2.2, assume additionally that $u_0 \in L^q_\sigma(\Omega)$, $3 < q < 6$, and let $1 \le s \le s_* < \infty$ with $\frac{2}{s} + \frac{3}{q} \ge 1$, $\frac{2}{s_*} + \frac{3}{q} = 1$. Then there is a constant $C = C(\Omega, q, s) > 0$ such that u is regular in $(0,T)$ in the sense $u \in L^{s_*}_{\text{loc}}\big((0,T); L^q(\Omega)\big)$ if*

$$\|u\|^s_{L^s(0,T;L^q(\Omega))} \le C \|u_0\|_q^{s-s_*} \tag{2.5}$$

is satisfied.

If $s = s_*$, then $\|u_0\|_q^{s-s_*} = 1$. Hence (2.5) can be applied with $(0,T)$ replaced by $(t_0, T_0) \subseteq (0,T)$, u_0 replaced by $u(t_0) \in L^q(\Omega)$ for almost all $t_0 \in (0,T)$, and we obtain again Serrin's regularity condition.

On the other hand, choose $s < s_*$ such that $\frac{2}{s} + \frac{3}{q} = \frac{3}{2}$. Then we know, see [14], p. 266, that $u \in L^s\big(0,T; L^q(\Omega)\big)$; this follows from (1.6) with $t_0 = 0$. Thus the left-side side of (2.5) is finite in this case; since $s - s_* < 0$, (2.5) yields the regularity of u if $\|u_0\|_q$ depending on $s < s_*$ is sufficiently small.

The proof of Lemma 2.4 shows that (2.5) can be replaced by the weaker condition (3.13), see Remark 3.1 below.

The next lemma yields the regularity of u in some point $t \in (0,T)$ if certain energy quantities of u are sufficiently small in a left-side neighborhood of t.

Lemma 2.5 *Let Ω, T, u, u_0 be as in Lemma 2.2. Then there is a constant $C = C(\Omega, s) > 0$ with the following property: If $0 < \delta < t < T$, $2 < s < 4$, and*

$$\text{(i)} \qquad \frac{1}{\delta^{s/4}} \int_{t-\delta}^{t} \|\nabla u\|_2^2\, \|u\|_2^{s-2}\, d\tau \le C \tag{2.6}$$

or

$$\text{(ii)} \qquad \frac{1}{\delta^{s/4}} \int_{t-\delta}^{t} \|\nabla u\|_2^2\, d\tau \le \frac{C}{\|u_0\|_2^{s-2}}\,, \tag{2.7}$$

then u is regular in some interval $(t-\delta',\, t+\delta') \subseteq (0,T)$, $0<\delta'<\delta$, in the sense that

$$u \in L^{s_*}\big(t-\delta',\, t+\delta';\, L^q(\Omega)\big), \tag{2.8}$$

where $4 < s_* < \infty$, $3 < q < 6$, $\frac{2}{s_*} + \frac{3}{q} = 1$.

Consider the case $T = \infty$. Then (1.6) with $t_0 = 0$ yields $\int_0^\infty \|\nabla u\|_2^2\, d\tau < \infty$, and there exists $\delta_0 > 0$ such that

$$\int_0^\infty \|\nabla u\|_2^2\, d\tau \le \frac{C}{\|u_0\|_2^{s-2}}\, \delta_0^{s/4}.$$

Then we conclude from (2.6) that u is regular in (δ_0, ∞). For each given $\delta_0 > 0$ we obtain the regularity of u in (δ_0, ∞) if $\|u_0\|_2$ is sufficiently small.

Using again (1.6) with $t_0 = 0$ we get from (2.6) that u is regular in (δ_0, ∞) with some $\delta_0 > 0$ if $\frac{1}{2}\, \|u_0\|_2^s \le C\delta_0^{s/4}$.

3 Proofs of the previous results

Proof of Lemma 2.2 (cf. [7], Corollary 1.4) Using (1.4) we see that $u(t_0) \in L^2_\sigma(\Omega)$ is well-defined for each $t_0 \in [0,T)$, and that by Definition 1.1 $\nabla u \in L^2\big(0,T; L^2(\Omega)\big)$. Therefore, applying Sobolev's embedding estimate with some constant $C = C(q) > 0$ in the form

$$\|u(t_0)\|_q \le C\|\nabla u(t_0)\|_2^\beta\, \|u(t_0)\|_2^{1-\beta}, \tag{3.1}$$

see [14], p. 52, with $3 < q < 6$, $0 \le \beta \le 1$, $\beta(\frac{1}{2} - \frac{1}{3}) + (1-\beta)\frac{1}{2} = \frac{1}{q}$, we get that $\nabla u(t_0) \in L^2(\Omega)$ and $u(t_0) \in L^q_\sigma(\Omega)$ are well-defined, and satisfy (3.1) for all $t_0 \in (0,T)\backslash N$ where $N \subseteq (0,T)$ is a null set.

First we assume that the condition (2.4) is satisfied with any given constant $C_1 > 0$, i.e., it holds

$$\int_{t-\delta}^t \|u(\tau)\|_q^s\, d\tau \le C_1 \delta^{1-\frac{s}{s_*}} \tag{3.2}$$

with $0 < \delta < t < T$, and with $1 \le s \le s_* < \infty$, $3 < q < 6$ as in (2.3).

Choose $\delta_0 \in (0,\delta]$ with $t + \delta_0 < T$, and set $T_0 = t + \delta_0$. Then for each $t_0 \in (t-\delta, t)\backslash N$ the expression

$$E(T_0, t_0) = \Big(\int_0^{T_0 - t_0} \|e^{-\tau A_q} u(t_0)\|_q^{s_*}\, d\tau\Big)^{\frac{s}{s_*}} \tag{3.3}$$

is well-defined, and obviously, there is at least one $t_0' \in (t-\delta, t)\backslash N$, such that

$$E(T_0, t_0') \le \frac{1}{\delta} \int_{t-\delta}^t E(T_0, t_0) dt_0; \tag{3.4}$$

the mean value in (3.4) cannot be strictly smaller than all these values $E(T_0, t_0')$. Since there is a constant $C_0 = C_0(\Omega, q) > 0$ such that

$$\|e^{-\tau A_q} v\|_q \le C_0 \|v\|_q, \quad v \in L^q_\sigma(\Omega),\ t \ge 0, \tag{3.5}$$

see [3], [11], [15], we obtain from (3.4) and (3.2) the estimate

$$\begin{aligned} E(T_0, t_0') &\le C_0^s \cdot \frac{1}{\delta}\big(T_0 - (t-\delta)\big)^{\frac{s}{s_*}} \int_{t-\delta}^{t} \|u(t_0)\|_q^s \, dt_0 \\ &\le C_0^s \, 2^{\frac{s}{s_*}} \, \delta^{\frac{s}{s_*}-1} \int_{t-\delta}^{t} \|u(t_0)\|_q^s \, dt_0 \\ &\le C_0^s \, 2^{\frac{s}{s_*}} \, \delta^{\frac{s}{s_*}-1} C_1 \delta^{1-\frac{s}{s_*}} \; = \; C_1 C_0^s 2^{\frac{s}{s_*}} . \end{aligned} \tag{3.6}$$

For a moment let $C_2 = C_2(\Omega, q) > 0$ denote the constant in Lemma 2.1. Then we set $C_1 = C_2^{\frac{s}{s_*}} C_0^{-s} 2^{-\frac{s}{s_*}}$ and take this constant as $C = C(\Omega, q, s)$ in Lemma 2.2, so that (3.6) yields

$$E(T_0, t_0')^{\frac{s_*}{s}} \le C_2.$$

Applying Lemma 2.1 in the formulation (2.2), now t, t_0, s replaced by τ, t_0', s_*, we obtain a weak solution v of the Navier-Stokes system

$$\begin{aligned} v_\tau - \Delta v + v \cdot \nabla v + \nabla p &= 0, \qquad \operatorname{div} v = 0 \\ v\big|_{\partial\Omega} &= 0, \qquad v\big|_{\tau = t_0'} = u(t_0') \end{aligned} \tag{3.7}$$

in $[t_0', T_0) \times \Omega$, satisfying Serrin's condition

$$v \in L^{s_*}\big(t_0', T_0; L^q(\Omega)\big), \tag{3.8}$$

and the energy inequality in the form

$$\frac{1}{2}\,\|v(\tau)\|_2^2 + \int_{t_0'}^{\tau} \|\nabla v\|_2^2 \, d\sigma \le \frac{1}{2}\,\|u(t_0')\|_2^2, \quad t_0' \le \tau < T_0. \tag{3.9}$$

Since u satisfies the strong energy inequality (1.6) for almost all $t_0 \in [0, T)$, we may assume without loss of generality that the null set $N \subseteq (0, T)$ is chosen in such a way that for each $t_0 \in (0, T) \backslash N$ in (3.3) the inequality (1.6) is satisfied. Thus also u satisfies the inequality

$$\frac{1}{2}\,\|u(\tau)\|_2^2 + \int_{t_0'}^{\tau} \|\nabla u\|_2^2 \, d\sigma \le \frac{1}{2}\,\|u(t_0')\|_2^2, \quad t_0' \le \tau < T_0. \tag{3.10}$$

Using Serrin's uniqueness criterion, see [13], [14], V, Theorem 1.5.1, we obtain that $u = v$ on $[t_0', T_0)$. Setting $\delta' = t - t_0' < \delta$ we conclude that u belongs to Serrin's class $L^{s_*}\big(t - \delta', t + \delta'; L^q(\Omega)\big)$. ∎

Proof of Lemma 2.4 Let $C_1 = C_1(\Omega, q) > 0$ be the constant in (2.1), and let $C_2 = C_2(\Omega, q, s) > 0$ be the constant in (2.4). Since by (3.5) for $T_0 > 0$

$$\int_0^{T_0} \|e^{-\tau A_q} u_0\|_q^{s_*} \, d\tau \leq T_0 C_0^{s_*} \|u_0\|_q^{s_*},$$

Lemma 2.1 implies the following: For

$$T_0 = C_1 C_0^{-s_*} \|u_0\|_q^{-s_*} \tag{3.11}$$

there exists a weak solution $v \in L^{s_*}\big(0, T_0; L^q(\Omega)\big)$ of the Navier-Stokes system

$$\begin{aligned} v_t - \Delta v + v \cdot \nabla v + \nabla p &= 0, \qquad \operatorname{div} v = 0 \\ v\big|_{\partial\Omega} &= 0, \qquad v\big|_{t=0} = u_0, \end{aligned}$$

in $[0, T_0) \times \Omega$ satisfying the energy inequality $\frac{1}{2} \|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2 \, d\tau \leq \frac{1}{2} \|u_0\|_2^2$, $0 \leq t < T_0$. If $T_0 \geq T$, then Serin's uniqueness condition as in the previous proof leads to $u = v$ in $(0, T)$, and u is regular in $(0, T)$. In the case $0 < T_0 < T$ we obtain $u = v$ with the corresponding regularity first of all only in $(0, T)$. Next we consider the interval $[T_0, T)$, set $\delta = \frac{1}{2} T_0$, $C_3 = C_2(\frac{1}{2} C_1 C_0^{-s_*})^{1-s/s_*}$, and conclude from Lemma 2.2, (2.4): If

$$\int_{t-\delta}^{t} \|u(\tau)\|_q^s \, d\tau \leq C_2 \delta^{1-\frac{s}{s_*}} = C_3 \|u_0\|_q^{s-s_*}, \tag{3.12}$$

then u is regular in some interval $(t - \delta', t + \delta') \subseteq (0, T)$, $0 < \delta' < \delta$, in the sense that $u \in L^{s_*}\big(t - \delta', t + \delta'; L^q(\Omega)\big)$. Finally, if $\int_0^T \|u(\tau)\|_q^s \, d\tau \leq C_3 \|u_0\|_q^{s-s_*}$, then (3.12) holds for all $t \in [T_0, T)$, and this proves Lemma 2.4. ■

Remark 3.1 The proof of Lemma 2.4 shows that the result can be improved as follows: Let $\Omega, T, u, u_0, q, s, s_*$ be as in this lemma, and let $T_0 \in (0, T)$ be defined by (3.11). Then there is a constant $C = C(\Omega, q, s) > 0$ such that u is regular in $(0, T)$ in the sense $u \in L^{s_*}_{\text{loc}}\big((0, T); L^q(\Omega)\big)$, if

$$\sup_{T_0 \leq t < T} \int_{t - \frac{1}{2} T_0}^{t} \|u\|_q^s \, d\tau \leq C \|u_0\|_q^{s - s_*} \tag{3.13}$$

is satisfied.

Proof of Lemma 2.5 Applying Sobolev's embedding estimate (3.1) we obtain with $0 < \delta < t < T$, $2 < s < 4$, $3 < q < 6$, $0 \leq \beta \leq 1$, $\beta(\frac{1}{2} - \frac{1}{3}) + (1 - \beta)\frac{1}{2} = \frac{1}{q}$, $\beta s = 2$, the inequality

$$\begin{aligned} \int_{t-\delta}^{t} \|u\|_q^s \, d\tau &\leq C_1 \int_{t-\delta}^{t} \|\nabla u\|_2^{\beta s} \, \|u\|_2^{(1-\beta)s} \, d\tau \\ &= C_1 \int_{t-\delta}^{t} \|\nabla u\|_2^2 \, \|u\|_2^{s-2} \, d\tau, \end{aligned} \tag{3.14}$$

where $C_1 = C_1(q) > 0$ means the constant in (3.1).

Let $C_0 = C_0(\Omega, q, s) > 0$ be the constant in (2.4). Then we use $\frac{2}{s} = \frac{3}{2} - \frac{3}{q}$, $\frac{2}{s_*} = 1 - \frac{3}{q}$, $\frac{s}{4} = 1 - \frac{s}{s_*}$, and obtain from (3.14) that the condition (2.4) is satisfied when $C = C_0 C_1^{-1}$ in (2.6). This proves Lemma 2.5 in the case (i).

Using the energy inequality (1.6) with $t_0 = 0$ we obtain $\|u(\tau)\|_2 \le \|u_0\|_2$ for $t - \delta \le \tau \le t$, and (3.14) leads to

$$\int_{t-\delta}^{t} \|u\|_q^s \, d\tau \le C_1 \|u_0\|_2^{s-2} \int_{t-\delta}^{t} \|\nabla u\|_2^2 \, d\tau. \tag{3.15}$$

Therefore, if (2.7) holds with $C = C_0 C_1^{-1}$, then (2.4) is satisfied. ■

Proof of Theorem 1.2 Let Ω, a, b, T, u, u_0 be as in this theorem. Using the strong energy inequality (1.6) for almost all $t_0 = t' \in (a,b)$, and all $t = t' + \delta$, $t \in (t', b)$, we obtain that

$$\int_{t-\delta}^{t} \|\nabla u\|_2^2 \, d\tau \le \frac{1}{2}\|u(t')\|_2^2 - \frac{1}{2}\|u(t)\|_2^2. \tag{3.16}$$

Now setting $\alpha = \frac{s}{4} + \varepsilon \in (\frac{1}{2}, 1)$, $\varepsilon > 0$, (3.16) implies that

$$\frac{1}{\delta^{s/4}} \int_{t-\delta}^{t} \|\nabla u\|_2^2 \, d\tau \le \frac{\left|\frac{1}{2}\|u(t)\|_2^2 - \frac{1}{2}\|u(t')\|_2^2\right|}{|t-t'|^{\alpha}} \, \delta^{\varepsilon} \le C_1 \delta^{\varepsilon}, \tag{3.17}$$

where C_1 denotes the left-side side of (1.9). Therefore, for each $t \in (a,b)$ there exist $\delta > 0$, $s \in (2,4)$, $\varepsilon > 0$ as above such that (3.17) is satisfied. Choosing this δ sufficiently small we see that (2.7) is satisfied. Thus by Lemma 2.5 (ii) each $t \in (a,b)$ is regular, and, therefore, u is regular in (a,b). ■

Proof of Theorem 1.4 Let Ω, a, b, T, u, u_0 be as in this theorem, let $t \in (a,b)$, $\alpha \in (\frac{1}{2}, 1)$, and assume that (1.14) is satisfied. Then there is a sequence $(\delta_j)_{j \in \mathbb{N}} \subset (0,t)$ with $\lim_{j\to\infty} \delta_j = 0$ such that

$$C_1 := \lim_{j\to\infty} \frac{1}{\delta_j^{\alpha}} \int_{t-\delta_j}^{t} \|\nabla u\|_2^2 \, d\tau < \infty. \tag{3.18}$$

Next we choose some $\varepsilon > 0$ and $s \in (2,4)$ such that $\frac{s}{4} = \alpha - \varepsilon$. Using $\lim_{j\to\infty} \delta_j = 0$ and (3.18) find $j_0 \in \mathbb{N}$ such that

$$\frac{1}{\delta_{j_0}^{\alpha}} \int_{t-\delta_{j_0}}^{t} \|\nabla u\|_2^2 \, d\tau \le C_1 + \varepsilon.$$

This yields – for j_0 sufficiently large –

$$\frac{1}{\delta_{j_0}^{s/4}} \int_{t-\delta_{j_0}}^{t} \|\nabla u\|_2^2 \, d\tau \le \delta_{j_0}^{\varepsilon}(C_1 + \varepsilon) \le \frac{C}{\|u_0\|_2^{s-2}}\,.$$

Then Lemma 2.5, (ii), shows that u is regular in some interval $(t-\delta', t+\delta') \subseteq (0,T)$ in the sense of (2.8). We can choose $\delta' > 0$ small enough such that $(t-\delta', t+\delta') \subseteq (a,b)$. Since $t \in (a,b)$ was arbitrary, we obtain the result of Theorem 1.4. This completes the proof. ■

4 A local space-time regularity result

The results in Sections 1 and 2 rest on the idea to identify the given weak solution u locally in time via a unique very weak solution with a regular one. For this identification we use Serrin's uniqueness argument, and therefore the strong energy inequality (1.6). In principle we can apply the same method in both space and time direction. We mention a result in this context in the following Theorem 4.1.

In this section $\Omega \subseteq \mathbb{R}^3$ means a completely general domain, i.e. a connected open subset, with boundary $\partial\Omega$ uniformly of class $C^{2,1}$. Further we need a special weak solution, the so-called suitable weak solution introduced in [2]:

A weak solution u defined in $[0,T) \times \Omega$, $0 < T \leq \infty$, as in Definition 1.1 is called a *suitable weak solution* of (1.1) with initial value $u_0 \in L^2_\sigma(\Omega)$ if additionally the following conditions are satisfied:

(i) The associated pressure term ∇p, defined in the sense of distributions by (1.5), satisfies

$$\nabla p \in L^{5/4}_{\mathrm{loc}}\big((0,T) \times \Omega\big). \tag{4.1}$$

(ii) It holds the local energy inequality in the form

$$\begin{aligned} \frac{1}{2}\|\varphi u(t)\|_2^2 + \int_{t_0}^t \|\varphi \nabla u\|_2^2 \, d\tau \quad \leq \quad & \frac{1}{2}\|\varphi u(t_0)\|_2^2 + \frac{1}{2}\int_{t_0}^t \langle \nabla\varphi^2, \nabla|u|^2\rangle \, d\tau \\ & - \int_{t_0}^t \langle p + \frac{1}{2}|u|^2, u \cdot \nabla\varphi^2\rangle \, d\tau \end{aligned} \tag{4.2}$$

for almost all $t_0 > 0$, all $t \in [t_0, T)$, and all test functions $\varphi \in C_0^\infty(\Omega)$.

The existence of such a suitable weak solution has been shown in [5]. The reason that we need (4.2) is again the local identification procedure, now in space-time, with Serrin's uniqueness argument. However, this local energy inequality contains the associated pressure p which should satisfy (4.1); the exponent $5/4$ comes from the nonlinear term in (1.1). In the following we use the parabolic cylinder

$$Q_r = Q_r(t_0, x_0) = (t_0 - r^2, t_0) \times B_r(x_0) \subseteq (0,T) \times \Omega \tag{4.3}$$

with $r > 0$, $t_0 \in (0,T)$, $x_0 \in \Omega$, $B_r(x_0) = \{x \in \mathbb{R}^3; |x - x_0| < r\}$, such that $\overline{B_r(x_0)} \subseteq \Omega$, cf. [2].

Theorem 4.1 *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain with boundary $\partial\Omega$ uniformly of class $C^{2,1}$, and let $0 < T \leq \infty$. Consider a parabolic cylinder $Q_r(t_0, x_0)$ with r, t_0, x_0 as in (4.3), and a suitable weak solution u in $[0,T) \times \Omega$ defined by Definition 1.1 and by (4.1), (4.2). Then there is an absolute constant $C > 0$ with the following property: If*

$$\|u\|_{L^s(t_0-r^2,t_0;L^q(B_r(x_0)))} \leq C\, r^{\frac{2}{s}+\frac{3}{q}-1} \tag{4.4}$$

with $2 < s \leq q < \infty$, $1 \leq \frac{2}{s} + \frac{3}{q} \leq 1 + \frac{1}{q}$, then u is regular in $Q_{\frac{r}{2}}(t_0, x_0)$ in the sense that the local Serrin condition

$$u \in L^{s^*}\big(t_0 - (r/2)^2; t_0; L^{q^*}\big(B_{\frac{r}{2}}(x_0)\big)\big) \tag{4.5}$$

is satisfied with exponents $2 < s^* < \infty$, $3 < q^* < \infty$ satisfying $\frac{s}{s^*} + \frac{3}{q^*} = 1$.

From the local Serrin condition (4.5) we are not able to prove the same smoothness property (1.8) for $Q_{\frac{r}{2}}(t_0, x_0)$ as from the global Serrin condition (1.7). However, see [2], p. 780, and [16], p. 440, p. 453, (4.5) implies the following property: In each subdomain $D \subseteq Q_{r/2}$ with $\overline{D} \subseteq Q_{r/2}$ there exists spatial derivatives of arbitrary order which are essentially bounded in D.

In the special case $\frac{2}{s}+\frac{3}{q} = 1$, (4.4) means the well-known local Serrin condition while in the case $1 < \frac{2}{s} + \frac{3}{q} \leq 1 + \frac{1}{q}$ we obtain a new regularity criterion beyond Serrin's condition.

Sketch of the proof. In the first step we reduce the general case $1 \leq \frac{2}{s} + \frac{3}{q} \leq 1 + \frac{1}{q}$ to the case $\frac{2}{s} + \frac{3}{q} = 1 + \frac{1}{q}$. Indeed, if we choose s_0 with $2 < s_0 \leq s$ such that $\frac{2}{s_0} + \frac{3}{q} = 1 + \frac{1}{q}$, then we see using Hölder's inequality that if (4.4) is satisfied with the given s, q, then (4.4) holds with the same constant for s_0, q also. Thus we may assume in the following that $\frac{2}{s} + \frac{3}{q} = 1 + \frac{1}{q}$.

In the next step we reduce the given r, x_0 to the case $r = 1$, $x_0 = 0$. For this purpose we use the scaling transform similarly as in [2], and define, with $\lambda > 0$, the functions $\tilde{u}, \tilde{p}, \tilde{u}_0$ in the variables $\tau = \lambda^{-2}t$, $y = \lambda^{-1}(x - x_0)$ by setting $\tilde{u}(\tau, y) = \lambda u(t, x)$, $\tilde{p}(\tau, y) = \lambda^2 p(t, x)$, $\tilde{u}_0 = \lambda u_0$. Then $(t, x) \in Q_r(t_0, x_0)$ if and only if $(\tau, y) \in Q_{r/\lambda}(\tau_0, 0)$. Setting $\lambda = r$ we conclude that (4.4) is satisfied for $\tilde{u}$ with $r = 1$, $x_0 = 0$ if (4.4) holds with given u, r, x_0. This shows that we may assume in the following that $r = 1$, $x_0 = 0$.

In the third step we choose a certain cylinder $Q' = [t_0', t_0) \times B_{r'}$, $B_{r'} = B_{r'}(0)$, with $Q_{1/2}(t_0, 0) \subset Q' \subset Q_1(t_0, 0)$, $t_0' \in (t_0 - 1, t_0 - \frac{1}{2})$, $r' \in (\frac{1}{2}, 1)$, in such a way that

$$u(t_0') \in L^q(B_{r'})\,, \quad u\big|_{(t_0', t_0)\times\partial B_{r'}} \in L^s\big(t_0 - 1, t_0; L^q(\partial B_{r'})\big) \tag{4.6}$$

are well-defined and satisfy the estimates

$$\begin{aligned} \|u(t_0')\|_{L^q(B_{r'})} &\leq C_1 \|u\|_{L^s(t_0-1,t_0;L^q(B_1))}, & (4.7)\\ \|u\|_{L^s(t_0',t_0;\partial B_{r'})} &\leq C_1 \|u\|_{L^s(t_0-1,t_0;L^q(B_1))} & (4.8) \end{aligned}$$

with $C_1 = C_1(r') > 0$. To prove these properties we use the mean value argument in the same way as in (3.4), and the traces in (4.6) are well-defined in the sense of Lebesgue points, see [14], p. 341.

The properties (4.6) – (4.8) enable us to construct a very weak solution $v \in L^{s^*}\big(t'_0, t_0; L^{q^*}(B_{r'})\big)$, $2 < s^* < \infty$, $3 < q^* < \infty$, $\frac{2}{s^*} + \frac{3}{q^*} = 1$, of the local system

$$\begin{aligned} &v_t - \Delta v + v \cdot \nabla v + \nabla h = 0, \qquad \operatorname{div} v = 0 \\ &v\big|_{(t'_0,t_0)\times\partial B_{r'}} = u\big|_{(t'_0,t_0)\times\partial B_{r'}}, \qquad v\big|_{t=0} = u(t'_0)\big|_{B_{r'}}, \end{aligned} \tag{4.9}$$

see [1] and [4], Theorem 1, if the smallness condition (4.4) is satisfied (with $r = 1$, $x_0 = 0$).

In the last step we have to identify $u = v$ in Q'. For this purpose we use the local energy inequality (4.2) in a similar way as in Serrin's uniqueness criterion, see [13], [14], V, Theorem 1.5.1. This leads to the estimate

$$\frac{1}{2}\|u(t) - v(t)\|_2^2 + \int_{t'_0}^t \|\nabla(u - v)\|_2^2 \, d\tau \le \int_{t'_0}^t \big|\langle (u - v) \cdot \nabla(u - v), v\rangle_{B_{r'}}\big| \, d\tau.$$

Using a standard estimate of the right-hand side, and the well-known absorption argument, we obtain in a finite number of steps with $t'_0 < t_1 < \cdots < t_n = t_0$, $n \in \mathbb{N}$, that $u = v$ in Q'. This completes the proof. ■

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