Regularity up to the boundary for nonlinear elliptic systems arising in time-incremental infinitesimal elasto-plasticity

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Abstract

In this note we investigate the question of higher regularity up to the boundary for quasilinear elliptic systems which origin from the time-discretization of models from infinitesimal elasto-plasticity. Our main focus lies on an elasto-plastic Cosserat model. More specifically we show that the time discretization renders H^2-regularity of the displacement and H^1-regularity for the symmetric plastic strain ε_p up to the boundary provided the plastic strain of the previous time step is in H^1, as well. This result contrasts with classical Hencky and Prandtl-Reuss formulations where it is known not to hold due to the occurrence of slip lines and shear bands. Similar regularity statements are obtained for other regularizations of ideal plasticity like viscosity or isotropic hardening.

In the first part we recall the time continuous Cosserat elasto-plasticity problem, provide the update functional for one time step and show various preliminary results for the update functional (Legendre-Hadamard/monotonicity). Using non standard difference quotient techniques we are able to show the higher global regularity. Higher regularity is crucial for qualitative statements of finite element convergence. As a result we may obtain estimates linear in the mesh-width h in error estimates.

Key words: polar materials, perfect plasticity, higher global regularity, quasilinear elliptic systems, error estimates, time-increments

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Contents

1 Introduction

1.1 Plasticity and Cosserat models

This article addresses the regularity question for time-incremental formulations of *geometrically linear* elasto-plasticity. As a representative model problem we consider generalized continua of *Cosserat-micropolar* type.

The basic difference of a Cosserat model as compared with classical continuum models is the appearance of a nonsymmetric stress tensor which is augmented by a generalized balance of angular momentum equation allowing to model interaction of particles not only by surface forces (classical Cauchy continuum) but also through surface couples (Cosserat continuum). General continuum models involving *independent rotations* as additional degrees of freedom have been first introduced by the Cosserat brothers [15]. For an introduction to the theory of Cosserat and micropolar models we refer to the introduction in [49, 43, 45, 44, 48], see also [22, 9].

There are a great many proposals for extensions of the elastic Cosserat framework to infinitesimal elasto-plasticity. We mention only [17, 19, 31, 55]. Recently the finite-strain formulation has been put into focus, see, e.g., [56, 62, 23] and references therein.

The first author has also proposed an elasto-plastic Cosserat model [45, 44] in a finite strain framework. A geometrical linearization of this model has been investigated in [46, 48] and is shown to be well-posed also in the rate-independent limit for both quasistatic and dynamic processes.

When it comes to numerically solving problems in elasto-plasticity, then it is common practice to discretize the time-evolution in the flow-rule for the plastic variable with a backward Euler method and to consider a sequence of discrete-in-time problems [50]. Provided that the elasto-plastic model has certain variational features (hyperelasticity of the elastic response, associative flow rule) it is possible to recast the problem for one time-step (called the update problem in the following) itself into a variational framework: the updated displacement is obtained as a minimizer of some update functional, see e.g., [61, 60, 2, 66, 67]. This line of thought can be nicely extended to finite-strain multiplicative plasticity, see [52, 37, 36, 38] and references therein. In the geometrically linear setting the resulting variational update problem usually has the form of a quasilinear elliptic system whose corresponding energy has only linear growth (in case of perfect plasticity).

For qualitative statements on the rate of convergence of finite element methods it is necessary to know precisely the regularity of the function to be approximated. This then is the question on the regularity of the solution of the quasilinear elliptic system constituting the update problem.

As far as classical rate-independent (perfect) elasto-plasticity is concerned we remark that global existence for the displacement has been shown only in a very weak, measure-valued sense, while the stresses could be shown to remain in $L^2(\Omega)$, provided a safe-load condition is assumed. For these results we refer for example to [3, 13, 64]. If hardening or viscosity is added, then global H^1-displacement solutions are found see e.g. [1, 12, 11], already without safe-load assumption. A complete theory for the classical rate-independent case remains, however, elusive, see also the remarks in [13].

Since classical perfect plasticity is, therefore, notoriously ill-posed (the updated displacements have derivatives only in a measure-valued sense) we focus in our investigation of higher regularity on certain modified update functionals which might allow for more regular updates. The Cosserat elasto-plastic model in [46] is our basic candidate. Based on this time-continuous model we investigate the time-incremental formulation and study the global regularity of minimizers of the corresponding update functional. In [49] this time-incremental formulation is the basis of a finite-element approximation.

Our focus on Cosserat models is justified by the fact that the Cosserat type models are today increasingly advocated as a means to regularize the pathological mesh size dependence of localization computations where shear failure mechanisms [14, 40, 4] play a dominant role, for applications in plasticity, see the non-exhaustive list [31, 19, 55, 17].

1.2 Outline of this contribution

Our contribution is organized as follows: first, we recall the time-continuous geometrically linear elasto-plastic Cosserat model as introduced in [45, 44] and investigated mathematically in [46, 48, 47].

Referring to the development in [49] we provide in section 2 the corresponding time-discretized formulation based on a fully implicit backward Euler discretization of the plastic flow rule in time. It is shown in [49] that at each time step t_n the updated displacement field u^n and the updated "Cosserat–microrotation–matrix" A^n can equivalently be obtained from a convex minimization problem which involves only data from the previous time step. The plastic strain ε_p^n is then derived from A^n and u^n via a simple update formula. Furthermore, in [49] it has been shown that the update problem admits unique minimizers $u^n \in H^1(\Omega, \mathbb{R}^3)$, $A^n \in H^1(\Omega, \mathfrak{so}(3))$ and $\varepsilon_p^n \in L^2(\Omega, \mathrm{Sym}(3))$ provided that the data coming from the previous time step are smooth enough. In order to quantify the rate of convergence of corresponding finite element methods for the update problem we investigate the regularity of the displacements u^n by studying the corresponding weak Euler–Lagrange equations. These equations form a quasilinear elliptic system of partial differential equations. The main result of this paper is Theorem 5.2 in section 5, where we formulate a global regularity result for weak solutions of a rather general class of quasilinear elliptic systems of second order. The time-incremental Cosserat plasticity formulation satisfies all the necessary assumptions of the regularity result which allows us to show higher regularity to the extent that $\forall n \in \mathbb{N}: \quad u^n \in H^2(\Omega, \mathbb{R}^3), A^n \in H^2(\Omega, \mathfrak{so}(3)), \varepsilon_p^n \in H^1(\Omega, \mathrm{Sym}(3))$ if pure Dirichlet data are assumed. Let us remark that it remains an open problem whether a similar regularity result is also valid for the time-continuous Cosserat model or other regularized time-continuous plasticity formulations.

The general quasilinear elliptic systems, which we study in section 5, are of the following type: Find $u \in H_0^1(\Omega)$ such that for every $v \in H_0^1(\Omega)$

$$\int_\Omega \langle \mathcal{M}(x, \nabla u(x), z(x)), \nabla v(x) \rangle \,\mathrm{dx} = \int_\Omega \langle f, v \rangle \,\mathrm{dx}.$$

Here, $z \in L^2(\Omega, \mathbb{R}^N)$ and $f \in L^2(\mathbb{R}^3)$ are given data. For the Cosserat model, z is identified with (ε_p^n, A^n), the explicit structure of $\mathcal{M} = \mathcal{M}_C$ is given in section 2.4. It is shown that $\mathcal{M}_C$ is rank–one monotone in ∇u and Lipschitz continuous but not differentiable. Consequently, we assume in the general case that the function $\mathcal{M} : \Omega \times \mathbb{M}^{m\times d} \times \mathbb{R}^N \to \mathbb{M}^{m\times d}$ is Lipschitz continuous, rank–one monotone in ∇u and induces a Gårding inequality. The precise conditions on $\mathcal{M}$ are formulated as **R1**–**R3** in section 5.1. Our main result is theorem 5.2, where we prove for smooth domains that $u \in H^2(\Omega)$ provided that $z \in H^1(\Omega)$ and $f \in L^2(\Omega)$. We emphasize that we do not need the differentiability of $\mathcal{M}$ and that we require $\mathcal{M}$ to be rank-one monotone, only, instead of uniformly or strongly monotone. A further new aspect compared to systems studied in the literature is the presence of the function z in the definition of the differential operator.

Let us give a short overview on global regularity results for quasilinear second order systems. Systems with quadratic growth or, more general, with p–growth are studied by several authors. We mention here the books [42, 39, 6], and the paper [53] where global regularity results for systems of the type

$$\mathrm{Div}\, \mathcal{M}(x, \nabla u(x)) + f(x) = 0, \quad u\big|_{\partial\Omega} = g_{\mathrm{D}}\,,$$

are shown for smooth domains assuming that $\mathcal{M}$ is differentiable and strongly monotone. Further results for Lipschitz domains were obtained in [21, 20, 57] again assuming that $\mathcal{M}$ is strongly monotone (or uniformly monotone, if $p \neq 2$), differentiable and that there is a function W such that $\mathcal{M} = DW$. These results are proved with a difference quotient technique which relies on the standard finite differences $\delta_h u(x) := u(x+h) - u(x)$.

In Daněček [16] the authors study systems, where $\mathcal{M}(x, u, \nabla u) = B(x)\nabla u + h(x, u, \nabla u)$. The main assumption in [16] is that B is uniformly positive definite, h is Hölder-continuous with respect to ∇u and $h(x, u, \cdot)$ is uniformly monotone in zero. They prove that the gradient of solutions belongs locally to certain Campanato-Spanne spaces. With our main result we can treat the case, where h does not depend on u, is Lipschitz continuous and monotone but not necessarily uniformly monotone and where B induces a rank-one positive quadratic form. We obtain $u \in H^2(\Omega)$ globally.

In [58] a nonlinear elliptic system is studied which is more related to our Cosserat-model. There, $\mathcal{M}$ is chosen as $\mathcal{M}(\nabla u) = \frac{h(|\varepsilon(u)|)}{|\varepsilon(u)|}\,\varepsilon(u)$, where $\varepsilon(u)$ is the linearized strain tensor, and it is assumed that h is differentiable except for a finite number of points and that h is strongly monotone. It is shown for smooth domains that $u \in H^2(\Omega)$ by investigating the regularity of functions u_δ with $\mathrm{Div}(\delta\varepsilon(u_\delta) + \mathcal{M}(\varepsilon(u_\delta))) + f = 0$ for $\delta \searrow 0$. The results for u_δ are obtained with standard finite differences. Further results for related models where obtained in [54, 7]. Let us remark that the quasilinear system we are interested in contains the above described systems as special cases (if $p = 2$) and that our main result is not covered by the above references. The local and global regularity of the stress fields of a class of degenerated quasilinear elliptic systems is investigated in the papers [10, 33].

Note that higher regularity is not known to hold for the displacements of the classical limit of our formulation, which is the classical time-incremental Prandtl-Reuss model. In the first update step this model in turn is nothing else than the total deformation Hencky plasticity model. The Hencky model does not allow for regular displacements. Here, it is known that the displacement $u \in L^{\frac{3}{2}}(\Omega, \mathbb{R}^3)$ (see, e.g., [6, p.423]) while the classical symmetric stresses satisfy $\sigma \in H^1_{\mathrm{loc}}(\Omega, \mathrm{Sym}(3)) \cap H^{\frac{1}{2}-\delta}(\Omega)$ for every $\delta > 0$ if the data are sufficiently regular and if Ω is a Lipschitz domain. See [59, 24, 5, 51, 18] for the local and [32] for the global result.

The proof of our own regularity result is split into the three classical steps. In the first step we investigate the tangential regularity of weak solutions in the case where Ω is a cube. Since we assumed rank–one monotonicity, only, we cannot apply the standard difference quotient technique in this step. Instead, we use finite differences which are based on inner variations: $\triangle_h u(x) = u(\tau_h(x)) - u(x)$, where $\tau_h(x) = x + \varphi^2(x)h$ for $h \in \mathbb{R}^d$ and a cut–off function φ. This will be explained in more detail in remark 5.5. Let us note that these nonstandard differences where recently applied by Nesenenko [51] in order to obtain higher local regularity for models from elasto-plasticity with linear hardening. In the second step we prove higher regularity in directions normal to the boundary. Due to the lack of differentiability of $\mathcal{M}$ we cannot apply the usual arguments (i.e. solving the equation for the normal derivatives) to obtain the differentiability of ∇u in the normal direction. Instead, we exploit the rank–one monotonicity of $\mathcal{M}$ in order to get more information on the missing derivative. In the final step we prove the result for arbitrary bounded $C^{1,1}$-smooth domains by the usual localization procedure. The notation is found in the appendix.

2 The infinitesimal elasto-plastic Cosserat model

In this section we recall the specific isotropic infinitesimal elasto-plastic Cosserat model which has been proposed in a finite-strain setting in [44] and which was analyzed in [46]. Moreover, we derive a discrete formulation. This section does not contain new results; it serves for the clear definition of the problem and for the introduction of some of the notation.

2.1 Time continuous infinitesimal elasto-plastic Cosserat model

The geometrically linear time continuous system in variational form with non-dissipative Cosserat effects reads: for given body forces $f(t) \in L^2(\Omega, \mathbb{R}^3)$ and given Dirichlet data find the **displacement** $u(t) \in H^1(\Omega, \mathbb{R}^3)$, the **skew-symmetric microrotation** $A(t) \in H^1(\Omega, \mathfrak{so}(3))$ and the **symmetric plastic strain** $\varepsilon_p(t) \in L^2(\Omega, \mathrm{Sym}(3))$ with

$$\begin{aligned}
&\int_\Omega W(\nabla u, A, \varepsilon_p(t)) - \langle f(t), u\rangle \,\mathrm{dx} \mapsto \min. \quad \text{w.r.t. } (u, A) \text{ at fixed } \varepsilon_p(t)\,,\\
&W(\nabla u, A, \varepsilon_p) = \mu \,\|\mathrm{sym}\,\nabla u - \varepsilon_p\|^2 \\
&\qquad\qquad + \mu_c \,\|\mathrm{skew}(\nabla u - A)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\,[\nabla u]^2 + 2\mu\, L_c^2\, \|\nabla\,\mathrm{axl}(A)\|^2\,,\\
&\dot{\varepsilon_p}(t) \in \partial\chi(T_E(t)), \qquad T_E = 2\mu\,(\varepsilon - \varepsilon_p)\,, \quad \varepsilon_p \in \mathrm{Sym}(3) \cap \mathfrak{sl}(3), \quad \varepsilon_p(0) = \varepsilon_p^0\,, \qquad (2.1)\\
&u_{|\Gamma_D} = g_{\mathrm{D}}(t,x) - x, \quad A_{|\Gamma_D} = \mathrm{skew}(\nabla g_{\mathrm{D}}(t,x))_{|\Gamma_D}\,.
\end{aligned}$$

Here, $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain and $\Gamma_D \subset \partial\Omega$ is that part of the boundary where Dirichlet data are prescribed. The parameters $\mu, \lambda > 0$ are the Lamé constants of isotropic linear elasticity, $\mu_c > 0$ is the Cosserat couple modulus and $L_c > 0$ is an internal length parameter.[1] The classical symmetric elastic strain $\mathrm{sym}\,\nabla u$ is denoted by ε. The linear operator $\mathrm{axl} : \mathfrak{so}(3) \to \mathbb{R}^3$ provides the canonical identification between the Lie-algebra $\mathfrak{so}(3)$ of skew-symmetric matrices and vectors in $\mathbb{R}^3$. The Lie-algebra of trace free matrices is denoted by $\mathfrak{sl}(3)$ and $\mathrm{dev} : \mathbb{M}^{3\times3} \to \mathfrak{sl}(3)$, $\mathrm{dev}\,X = X - \frac{1}{3}\mathbb{1}$ is the orthogonal projection onto $\mathfrak{sl}(3)$. As regards the plastic flow rule, $\partial\chi$ is the subdifferential of a convex flow potential $\chi : \mathbb{M}^{3\times3} \to \mathbb{R}^+$ acting on the generalized conjugate forces, i.e., the Eshelby-stress tensor $T_E = -\partial_{\varepsilon_p} W(\nabla u, A, \varepsilon_p)$, where W is the free energy used in (2.1).[2]

The corresponding system of partial differential equations coupled with the flow rule is given by (note that $\|A\|^2_{\mathbb{M}^{3\times3}} = 2\,\|\mathrm{axl}(A)\|^2_{\mathbb{R}^3}$ for $A \in \mathfrak{so}(3, \mathbb{R})$)

$$\begin{aligned}
&\mathrm{Div}\,\sigma = -f, \quad x \in \Omega\,, \quad \text{balance of forces}\,,\\
&\sigma = 2\mu\,(\varepsilon - \varepsilon_p) + 2\,\mu_c\,(\mathrm{skew}(\nabla u) - A) + \lambda\,\mathrm{tr}\,[\varepsilon]\cdot\mathbb{1}\,, \qquad (2.2)
\end{aligned}$$

[1]Observe that for $\mu_c = 0$ or $L_c = 0$ one recovers the classical Prandtl-Reuss formulation for the displacement u.

[2]The specification $\chi = I_K$ as indicatorfunction of some elastic domain is not necessary at this point.

$$
\begin{aligned}
-\mu\, L_c^2\, \Delta\, \mathrm{axl}(A) &= \mu_c\ \mathrm{axl}(\mathrm{skew}(\nabla u) - A)\,, \quad \text{balance of angular momentum}\,,\\
\dot{\varepsilon_p}(t) &\in \partial\chi(T_E), \quad T_E = 2\mu\,(\varepsilon - \varepsilon_p)\,,\\
u_{|\Gamma_D}(t,x) &= g_{\mathrm{D}}(t,x) - x\,, \quad A_{|\Gamma_D} = \mathrm{skew}(\nabla g_{\mathrm{D}}(t,x))_{|\Gamma_D}\,,\\
\sigma.\vec{n}|_{\partial\Omega\setminus\Gamma_D}(t,x) &= 0\,, \quad \mu\, L_c^2 \nabla\, \mathrm{axl}(A).\vec{n}|_{\partial\Omega\setminus\Gamma_D}(t,x) = 0\,,\\
\varepsilon_p(0) &\in \mathrm{Sym}(3) \cap \mathfrak{sl}(3)\,.
\end{aligned}
$$

Note that in this model the force stresses σ need not be symmetric and that the Cosserat effects, active through the microrotations A, only appear in the balance equations but not in the plastic flow rule since T_E does not depend on A. It is worth noting that this model is intrinsically thermodynamically correct. If $\Gamma_D = \partial\Omega$ then the model admits global weak solutions with the regularity [46]:

$$
\begin{aligned}
&u \in L^\infty([0,T], H^1(\Omega, \mathbb{R}^3))\,, \quad A \in L^\infty([0,T], H^1(\Omega, \mathfrak{so}(3)))\,,\\
&\varepsilon_p \in L^\infty([0,T], L^2(\Omega, \mathrm{Sym}(3) \cap \mathfrak{sl}(3)))\,.
\end{aligned} \tag{2.3}
$$

2.2 Backward Euler time discretization of the flow rule

For a numerical treatment we consider the time discretization of the flow rule with the fully implicit backward Euler scheme. Let $0 = t_0 < t_1 < \ldots < t_N = T$ be a subdivision of the time interval $[0,T]$ with $t_j - t_{j-1} = \vartriangle t$. Let $f^n(x) = f(x, t_n)$ and assume that at time t_{n-1} a sufficiently regular plastic strain field $\varepsilon_p^{n-1} \in \mathrm{Sym}(3) \cap \mathfrak{sl}(3)$ is given. We want to determine the **updated displacement** $u^n \in H^1(\Omega, \mathbb{R}^3)$, the **updated skew-symmetric microrotation** $A^n \in H^1(\Omega, \mathfrak{so}(3))$ and the **updated symmetric plastic strain** $\varepsilon_p^n \in L^2(\Omega, \mathrm{Sym}(3) \cap \mathfrak{sl}(3))$ satisfying

$$
\begin{aligned}
\mathrm{Div}\, \sigma^n &= -f^n, \quad x \in \Omega\,,\\
\sigma^n &= 2\mu\,(\varepsilon^n - \varepsilon_p^n) + 2\,\mu_c\,(\mathrm{skew}(\nabla u^n) - A^n) + \lambda\, \mathrm{tr}\,[\varepsilon^n] \cdot 1\!\!1\,,\\
-\mu\, L_c^2\, \Delta\, \mathrm{axl}(A^n) &= \mu_c\ \mathrm{axl}(\mathrm{skew}(\nabla u^n) - A^n)\,,\\
\frac{\varepsilon_p^n - \varepsilon_p^{n-1}}{\vartriangle t} &\in \partial\chi(T_E^n), \quad T_E^n = 2\mu\,(\varepsilon^n - \varepsilon_p^n)\,,\\
u^n_{|\Gamma_D}(x) &= g^n_{\mathrm{D}}(x) - x\,, \quad A^n_{|\Gamma_D} = \mathrm{skew}(\nabla g^n_{\mathrm{D}}(x))\,,\\
\sigma^n.\vec{n}|_{\partial\Omega\setminus\Gamma_D}(x) &= 0\,, \quad \mu\, L_c^2 \nabla\, \mathrm{axl}(A)^n.\vec{n}|_{\partial\Omega\setminus\Gamma_D}(x) = 0\,,\\
\varepsilon_p^{n-1} &\in L^2(\Omega, \mathrm{Sym}(3) \cap \mathfrak{sl}(3))\,.
\end{aligned} \tag{2.4}
$$

It is possible to explicitly solve the discretized flow rule $(2.4)_4$ for ε_p^n in terms of ε_p^{n-1}, ε^n and $\vartriangle t$. To see this, consider

$$
\frac{\varepsilon_p^n - \varepsilon_p^{n-1}}{\vartriangle t} \in \partial\chi(2\mu\,(\varepsilon^n - \varepsilon_p^n)) \;\Leftrightarrow\; 0 \in \partial\chi(2\mu\,(\varepsilon^n - \varepsilon_p^n)) - \frac{\varepsilon_p^n - \varepsilon_p^{n-1}}{\vartriangle t} \;\Leftrightarrow \tag{2.5}
$$

$$0 \in \partial_{\varepsilon_p^n} \left(\mu \left\| \varepsilon_p^n - \varepsilon_p^{n-1} \right\|^2 + \triangle t \, \mathcal{X}(2\mu \, (\varepsilon^n - \varepsilon_p^n)) \right) .$$

Thus we can define the local potential for the local flow rule

$$V^{\text{time}}(\varepsilon^n, \varepsilon_p^n, \varepsilon_p^{n-1}, \triangle t) := \mu \left\| \varepsilon_p^n - \varepsilon_p^{n-1} \right\|^2 + \triangle t \, \mathcal{X}(2\mu \, (\varepsilon^n - \varepsilon_p^n)) . \tag{2.6}$$

It is easy to see that V^{time} is strictly convex in ε_p^n, thus V^{time} admits a unique minimizer satisfying $(2.5)_3$. Moreover, we have

$$\begin{aligned} V^{\text{time}}(\varepsilon^n, \varepsilon_p^n, \varepsilon_p^{n-1}, \triangle t) &= \mu \left\| \varepsilon_p^n - \varepsilon_p^{n-1} \right\|^2 + \triangle t \, \mathcal{X}(2\mu \, (\varepsilon^n - \varepsilon_p^n)) \\ &= \frac{1}{4\mu} \left\| 2\mu(\varepsilon_p^n - \varepsilon^n + \varepsilon^n - \varepsilon_p^{n-1}) \right\|^2 + \triangle t \, \mathcal{X}(2\mu \, (\varepsilon^n - \varepsilon_p^n)) \\ &= \frac{1}{4\mu} \left\| \Sigma^n - \Sigma^n_{\text{trial}} \right\|^2 + \triangle t \, \mathcal{X}(\Sigma^n) = \tilde{V}(\Sigma^n, \Sigma^n_{\text{trial}}) , \end{aligned} \tag{2.7}$$

where $\Sigma^n = 2\mu \, (\varepsilon^n - \varepsilon_p^n)$ and the so-called trial stresses $\Sigma^n_{\text{trial}} = 2\mu \, (\varepsilon^n - \varepsilon_p^{n-1})$. Minimizing V^{time} w.r.t. ε_p^n is equivalent to minimizing $\tilde{V}$ w.r.t. Σ^n. Proceeding further, we specialize $\mathcal{X}$. Let us define the elastic domain in stress-space

$$K := \{ \Sigma \in \mathbb{M}^{3\times 3} \mid \|\text{dev}\, \Sigma\| \leq \sigma_{\text{y}} \} , \tag{2.8}$$

with initial yield stress σ_{y}, $[\sigma_{\text{y}}] = [\text{MPa}]$, and corresponding indicator function

$$I_K(\Sigma) = \begin{cases} 0 & \|\text{dev}\, \Sigma\| \leq \sigma_{\text{y}} \\ \infty & \|\text{dev}\, \Sigma\| > \sigma_{\text{y}} , \end{cases} \tag{2.9}$$

and let $\mathcal{X} = I_K$. We have therefore $\partial \mathcal{X} = \partial I_K$ in the sense of the subdifferential. With this choice, the unique minimizer of $\tilde{V}$ is simply characterized by

$$\inf_{\Sigma^n \in K} \left\| \Sigma^n - \Sigma^n_{\text{trial}} \right\|^2 , \tag{2.10}$$

independent of $\triangle t$. The solution is the orthogonal projection of Σ^n_{trial} onto the convex set K, denoted by

$$\Sigma^n = P_K(\Sigma^n_{\text{trial}}) \; \Rightarrow 2\mu \, (\varepsilon^n - \varepsilon_p^n)) = P_K(2\mu \, (\varepsilon^n - \varepsilon_p^{n-1})) . \tag{2.11}$$

Reintroducing the last result into the balance of forces equation $(2.4)_1$ delivers

$$\begin{aligned} \text{Div}\, \sigma^n &= -f^n, \quad x \in \Omega , \\ \sigma^n &= P_K(2\mu \, (\varepsilon^n - \varepsilon_p^{n-1})) + 2\, \mu_c \, (\text{skew}(\nabla u^n) - A^n) + \lambda \, \text{tr}\, [\varepsilon^n] \cdot 1\!\!1 . \end{aligned} \tag{2.12}$$

This step is called **return-mapping** [61, 60] in an engineering context of classical plasticity. At given plastic strain of the previous time step ε_p^{n-1} this equation is the strong form of the update problem for the force-balance equation.

Gathering the previous development the formal problem for the update consists in determining $u^n \in H^1(\Omega, \mathbb{R}^3)$, $A^n \in H^1(\Omega, \mathfrak{so}(3))$ and $\varepsilon_p^n \in L^2(\Omega, \mathrm{Sym}(3) \cap \mathfrak{sl}(3))$ satisfying

$$\begin{aligned} \operatorname{Div} \sigma^n &= -f^n, \quad x \in \Omega\,, \\ \sigma^n &= P_K(2\mu\,(\varepsilon^n - \varepsilon_p^{n-1})) + 2\,\mu_c\,(\operatorname{skew}(\nabla u^n) - A^n) + \lambda\,\operatorname{tr}[\varepsilon^n]\cdot 1\!\!1\,, \\ -\mu\, L_c^2\,\Delta\,\operatorname{axl}(A^n) &= \mu_c\ \operatorname{axl}(\operatorname{skew}(\nabla u^n) - A^n)\,. \end{aligned} \tag{2.13}$$

The updated plastic strain field is then given by

$$\varepsilon_p^n = \varepsilon^n - \frac{1}{2\mu}\,P_K(2\mu\,(\varepsilon^n - \varepsilon_p^{n-1}))\,. \tag{2.14}$$

For the precise formulation of this system we need the projection operator onto the yield surface which we recall in the following.

2.3 The projection onto the yield surface

Let K be a convex domain in stress space defined as

$$K := \left\{\Sigma \in \mathbb{M}^{3\times 3} \mid \|\operatorname{dev}\Sigma\| \le \sigma_{\mathrm{y}}\right\}\,. \tag{2.15}$$

The orthogonal projection $P_K : \mathbb{M}^{3\times 3} \to K$ onto this set is uniquely given by (see, e.g.,[29, 30])

$$\begin{aligned} P_K(\Sigma) &= \begin{cases} \Sigma & \Sigma \in K \\ \Sigma - (\|\operatorname{dev}\Sigma\| - \sigma_{\mathrm{y}})\,\frac{\operatorname{dev}\Sigma}{\|\operatorname{dev}\Sigma\|} & \Sigma \notin K \end{cases} \\ &= \begin{cases} \Sigma & \|\operatorname{dev}\Sigma\| \le \sigma_{\mathrm{y}} \\ \frac{1}{3}\operatorname{tr}[\Sigma]\,1\!\!1 + \frac{\sigma_{\mathrm{y}}}{\|\operatorname{dev}\Sigma\|}\,\operatorname{dev}\Sigma & \|\operatorname{dev}\Sigma\| > \sigma_{\mathrm{y}}\,. \end{cases} \end{aligned} \tag{2.16}$$

It is easy to see that P_K is Lipschitz continuous but not differentiable at Σ with $\|\operatorname{dev}\Sigma\| = \sigma_{\mathrm{y}}$.[3] From convex analysis it is clear that P_K represents a monotone operator which is non-expansive. Therefore, P_K has Lipschitz constant 1. Observe also that

$$P_K(\Sigma) = \frac{1}{3}\operatorname{tr}[\Sigma]\,1\!\!1 + P_K(\operatorname{dev}\Sigma)\,. \tag{2.18}$$

For future reference we calculate also

$$\begin{aligned} \Sigma - P_K(\Sigma) &= \begin{cases} 0 & \|\operatorname{dev}\Sigma\| \le \sigma_{\mathrm{y}} \\ \operatorname{dev}\Sigma\,\left(1 - \frac{\sigma_{\mathrm{y}}}{\|\operatorname{dev}\Sigma\|}\right) & \|\operatorname{dev}\Sigma\| > \sigma_{\mathrm{y}} \end{cases} \\ &= [\|\operatorname{dev}\Sigma\| - \sigma_{\mathrm{y}}]_+\,\frac{\operatorname{dev}\Sigma}{\|\operatorname{dev}\Sigma\|}\,, \\ \|\Sigma - P_K(\Sigma)\|^2 &= [\|\operatorname{dev}\Sigma\| - \sigma_{\mathrm{y}}]_+^2\,, \end{aligned} \tag{2.19}$$

where $[x]_+ := \max\{0, x\}$.

[3] Consider the simple example $p : \mathbb{R} \to \mathbb{R}$,

$$p(x) = \begin{cases} x & |x| \le \sigma_{\mathrm{y}} \\ \sigma_{\mathrm{y}}\,\frac{x}{|x|} & |x| > \sigma_{\mathrm{y}} \end{cases} \tag{2.17}$$

2.4 Weak form of the reduced update problem

From now onwards we take $\Gamma_D = \partial\Omega$ and assume $g_{\mathrm{D}} = x$, i.e. the body is fixed everywhere on its boundary and subject only to body forces. This assumption allows us to confine attention to the simpler setting in $H_0^1(\Omega)$. We introduce the nonlinear mapping

$$\begin{aligned} &\mathcal{M}_C : \mathbb{M}^{3\times3} \times \mathrm{Sym}(3) \times \mathfrak{so}(3) \to \mathbb{M}^{3\times3}\,, \\ &\mathcal{M}_C(X, \varepsilon_p, A) := P_K(2\mu(\mathrm{sym}\, X - \varepsilon_p)) + \lambda\, \mathrm{tr}\,[X]\, 1\!\!1 + 2\mu_c(\mathrm{skew}(X) - A)\,. \end{aligned} \tag{2.20}$$

The weak form of the update problem (2.13) now reads as follows: for given $f^n \in L^2(\Omega, \mathbb{R}^3)$ and $\varepsilon_p^{n-1} \in L^2(\Omega, \mathrm{Sym}(3) \cap \mathfrak{sl}(3))$ find $(u^n, A^n) \in H_0^1(\Omega, \mathbb{R}^3) \times H_0^1(\Omega, \mathfrak{so}(3))$ solving

$$\int_\Omega \langle \mathcal{M}_C(\nabla u^n, \varepsilon_p^n, A^n), \nabla v\rangle\, \mathrm{dx} = \int_\Omega \langle f^n, v\rangle\, \mathrm{dx} \qquad \forall v \in H_0^1(\Omega, \mathbb{R}^3)\,, \tag{2.21}$$

$$\mu\, L_c^2 \int_\Omega \langle DA^n, DB\rangle\, \mathrm{dx} = \mu_c \int_\Omega \langle \mathrm{skew}\, \nabla u^n - A^n, B\rangle\, \mathrm{dx}\,, \quad \forall\, B \in H_0^1(\Omega, \mathfrak{so}(3))\,. \tag{2.22}$$

The updated plastic strain field ε_p^n is then obtained by (2.14). It is shown in [49] that for every n the system (2.21)–(2.22) admits a unique weak solution $u^n \in H_0^1(\Omega, \mathbb{R}^3)$ and $A^n \in H_0^1(\Omega, \mathfrak{so}(3))$. Equation (2.21) represents the quasilinear elliptic system for determining u^n which will be discussed with respect to regularity. Together with $\varepsilon_p^{n-1}, \varepsilon^n \in H^1(\Omega, \mathrm{Sym}(3))$, which we will obtain from the regularity result to be proven below, using (2.14) we see that $\varepsilon_p^n \in H^1(\Omega, \mathrm{Sym}(3))$.

Lemma 2.1 (Strong Legendre-Hadamard ellipticity)
Let $\mu > 0$, $2\mu + 3\lambda > 0$ and $0 < \mu_c$. Then the matrix valued function $\mathcal{M}_C$ is strongly rank–one monotone, i.e., there exists a constant $c_{LH}^+ > 0$ such that for every $X \in \mathbb{M}^{3\times3}$, $\varepsilon_p \in \mathrm{Sym}(3)$, $A \in \mathfrak{so}(3)$ and for all ξ, $\eta \in \mathbb{R}^3$ we have

$$\langle \mathcal{M}_C(X + \xi\otimes\eta, \varepsilon_p, A) - \mathcal{M}_C(X, \varepsilon_p, A), \xi\otimes\eta\rangle \geq c_{LH}^+\, \|\xi\|^2\, \|\eta\|^2\,. \tag{2.23}$$

Proof. The projection P_K itself is monotone and for $\mu > 0$ there is no sign-change. Thus the map $X \to P_K(2\mu(\mathrm{sym}\, X - \varepsilon_p))$ is also monotone in X. Since (2.18) holds we have

$$\langle P_K(2\mu(\mathrm{sym}\, X + \xi\otimes\eta - \varepsilon_p)) - P_K(2\mu(\mathrm{sym}\, X - \varepsilon_p)), \xi\otimes\eta\rangle \geq \frac{2\mu}{3} \mathrm{tr}\,[\xi\otimes\eta]^2\,.$$

For the remaining linear contribution we have

$$\begin{aligned} &\langle \lambda\, \mathrm{tr}\,[X + \xi\otimes\eta]\, 1\!\!1 + 2\mu_c\, \mathrm{skew}(X + \xi\otimes\eta - A) - [\lambda\, \mathrm{tr}\,[X]\, 1\!\!1 + 2\mu_c\, \mathrm{skew}(X - A)]\,, \xi\otimes\eta\rangle \\ &= \lambda\, \mathrm{tr}\,[\xi\otimes\eta]^2 + 2\mu_c\, \|\mathrm{skew}(\xi\otimes\eta)\|^2\,. \end{aligned} \tag{2.24}$$

Thus

$$
\begin{aligned}
&\langle \mathcal{M}_C(X+\xi\otimes\eta,\varepsilon_p,A)-\mathcal{M}_C(X,\varepsilon_p,A),\xi\otimes\eta\rangle\\
&\quad\geq \frac{2\mu+3\lambda}{3}\mathrm{tr}\,[\xi\otimes\eta]^2+2\mu_c\,\|\mathrm{skew}(\xi\otimes\eta)\|^2=\frac{2\mu+3\lambda}{3}\langle\xi,\eta\rangle^2+\mu_c\,\left(\|\xi\|^2\,\|\eta\|^2-\langle\xi,\eta\rangle^2\right)\\
&\quad\text{split } \mu_c^1+\mu_c^2=\mu_c\\
&\quad=\left(\frac{2\mu+3\lambda}{3}-\mu_c^1\right)\langle\xi,\eta\rangle^2+\mu_c^1\,\|\xi\|^2\,\|\eta\|^2+\mu_c^2\,\underbrace{\left(\|\xi\|^2\,\|\eta\|^2-\langle\xi,\eta\rangle^2\right)}_{\geq 0}\\
&\quad\geq\left(\frac{2\mu+3\lambda}{3}-\mu_c^1\right)\langle\xi,\eta\rangle^2+\mu_c^1\,\|\xi\|^2\,\|\eta\|^2\geq\mu_c^1\,\|\xi\|^2\,\|\eta\|^2\,, \qquad (2.25)
\end{aligned}
$$

if $0<\mu_c^1<\frac{3\lambda+2\mu}{3}$. Thus $\mathcal{M}_C$ generates a strongly Legendre-Hadamard elliptic operator with ellipticity constant $c_{LH}^+=\min(\mu_c,\frac{2\mu+3\lambda}{3})$. ■

Obviously, $\mathcal{M}$ is Lipschitz continuous: for every $X_i\in\mathbb{M}^{3\times3}$, $P_i\in\mathrm{Sym}(3)$, $A_i\in\mathfrak{so}(3)$ we have

$$
\|\mathcal{M}_C(X_1,P_1,A_1)-\mathcal{M}_C(X_2,P_2,A_2)\|\leq L_{\mathcal{M}_C}\,\left(\|X_1-X_2\|+\|P_1-P_2\|+\|A_1-A_2\|\right)\,. \qquad (2.26)
$$

Lemma 2.2
Let $\mu>0, 2\mu+3\lambda>0$ and $\mu_c>0$. The operator $\mathcal{M}_C$ generates a strongly monotone operator on $H_0^1(\Omega,\mathbb{R}^3)$, that is, there exists a constant $c_{\mathcal{M}_C}>0$ such that for every $v_1,v_2\in H_0^1(\Omega,\mathbb{R}^3)$ and for all $\varepsilon_p\in L^2(\Omega,\mathrm{Sym}(3))$ and $A\in L^2(\Omega,\mathfrak{so}(3))$ we have

$$
\int_\Omega\langle\mathcal{M}_C(\nabla v_1,\varepsilon_p,A)-\mathcal{M}_C(\nabla v_2,\varepsilon_p,A),\nabla v_1-\nabla v_2\rangle\,\mathrm{dx}\geq c_{\mathcal{M}_C}\,\|v_1-v_2\|^2_{H_0^1(\Omega,\mathbb{R}^3)}\,. \qquad (2.27)
$$

Proof. The same calculation as in the proof of Lemma 2.1 yields the estimate

$$
\begin{aligned}
&\langle\mathcal{M}_C(\nabla v_1,\varepsilon_p,A)-\mathcal{M}_C(\nabla v_2,\varepsilon_p,A),\nabla v_1-\nabla v_2\rangle\\
&\quad\geq\frac{2\mu+3\lambda}{3}\,\mathrm{tr}\,[\nabla v_1-\nabla v_2]^2+2\mu_c\,\|\mathrm{skew}(\nabla v_1-\nabla v_2)\|^2\,. \qquad (2.28)
\end{aligned}
$$

Set $u=v_1-v_2$ and consider

$$
\frac{2\mu+3\lambda}{3}\,\mathrm{tr}\,[\nabla u]^2+2\mu_c\,\|\mathrm{skew}\,\nabla u\|^2=\frac{2\mu+3\lambda}{3}\,|\mathrm{Div}\,u|^2+\mu_c\,\|\mathrm{curl}\,u\|^2\,. \qquad (2.29)
$$

The Curl/Div inequality on the space $H_0^1(\Omega)$ guarantees that there exists $C^+>0$ such that

$$
\forall\,u\in H_0^1(\Omega,\mathbb{R}^3):\quad\int_\Omega|\mathrm{Div}\,u|^2+\|\mathrm{curl}\,u\|^2\,\mathrm{dx}\geq C^+\,\|u\|^2_{H_0^1(\Omega,\mathbb{R}^3)}\,, \qquad (2.30)
$$

see for example [28]. Applying this inequality to (2.29) implies finally (2.27). ■

It is instructive to realize that although the quadratic form (2.29) is formally positive in the sense of Nečas [41] and strongly Legendre-Hadamard elliptic with constant coefficients it is impossible to extend the analysis to Dirichlet boundary conditions given only on a part of the boundary $\partial\Omega$.

We observe that

$$\left\| \sqrt{\mu_c}\,\mathrm{skew}\,X + \sqrt{\frac{\lambda}{2\cdot 3}}\mathrm{tr}\,[X]\,1\!\!1 \right\|^2 = \frac{\lambda}{2}\,\mathrm{tr}\,[X]^2 + \mu_c\,\|\mathrm{skew}\,X\|^2\,. \tag{2.31}$$

Let $\widehat{\mathcal{A}}$ be the constant-coefficients first order differential operator

$$\widehat{\mathcal{A}}.\nabla u = \sqrt{\mu_c}\,\mathrm{skew}(\nabla u) + \sqrt{\frac{\lambda}{2\cdot 3}}\mathrm{tr}\,[\nabla u]\,1\!\!1\,. \tag{2.32}$$

The corresponding Fourier-symbol is given as a linear operator $\mathcal{A}(\xi) : \mathbb{C}^3 \to \mathbb{C}^{3\times 3}$ with

$$\mathcal{A}(\xi).\hat{u} := \sqrt{\mu_c}\,\mathrm{skew}(\xi\otimes\hat{u}) + \sqrt{\frac{\lambda}{2\cdot 3}}\mathrm{tr}\,[\xi\otimes\hat{u}]\,1\!\!1\,. \tag{2.33}$$

From (2.31) it follows

$$\|\mathcal{A}(\xi).\hat{u}\|^2 = \frac{\lambda}{2}\,\mathrm{tr}\,[\xi\otimes\hat{u}]^2 + \mu_c\,\|\mathrm{skew}\,\xi\otimes\hat{u}\|^2\,. \tag{2.34}$$

By algebraic completeness of the symbol $\mathcal{A}(\xi) : \mathbb{C}^3 \to \mathbb{C}^{3\times 3}$ it is meant

$$\forall\,\xi\in\mathbb{C}^3,\ \xi\neq 0: \quad \mathcal{A}(\xi).\hat{u} = 0_{\mathbb{C}^{3\times 3}} \ \Rightarrow \hat{u} = 0_{\mathbb{C}^3}\,. \tag{2.35}$$

Recall that the corresponding statement for real ξ, i.e.,

$$\forall\,\xi\in\mathbb{R}^3,\ \xi\neq 0: \quad \mathcal{A}(\xi).\hat{u} = 0_{\mathbb{R}^{3\times 3}} \ \Rightarrow \hat{u} = 0_{\mathbb{R}^3}\,, \tag{2.36}$$

is a consequence of strict Legendre-Hadamard ellipticity of $\widehat{\mathcal{A}}$. If the symbol is algebraically complete, then, using the result in Nečas [41] the induced quadratic form

$$\int_\Omega \left\|\widehat{\mathcal{A}}.\nabla u\right\|^2 + \|u\|^2\,\mathrm{dx} \tag{2.37}$$

is an equivalent norm on $H^1(\Omega,\mathbb{R}^3)$. However, we proceed to show that $\mathcal{A}$ as defined in (2.33) corresponding to our quadratic form (2.29) is not algebraically complete.

Proof. To this end we write

$$\mathcal{A}(\xi).\hat{u} = 0 \ \Rightarrow\ \mathrm{tr}\,[\xi\otimes\hat{u}] = 0,\ \text{and}\ \mathrm{skew}(\xi\otimes\hat{u}) = 0 \ \Rightarrow\ \xi = \hat{u}\,, \quad \mathrm{tr}\,[\xi\otimes\xi] = 0\,. \tag{2.38}$$

Consider for simplicity the $2D$-case:

$$\xi = \begin{pmatrix}\alpha_1 + i\,\beta_1\\ \alpha_2 + i\,\beta_2\end{pmatrix}, \quad \xi\otimes\xi = \begin{pmatrix}\xi_1\,\xi_1 & \xi_1\,\xi_2\\ \xi_2\,\xi_1 & \xi_2\,\xi_2\end{pmatrix},$$
$$\mathrm{tr}\,[\xi\otimes\xi] = \xi_1\xi_1 + \xi_2\xi_2 = \alpha_1^2 + \alpha_2^2 - (\beta_1^2+\beta_2^2) + 2i(\alpha_1\beta_1 + \alpha_2\beta_2) = 0\,. \tag{2.39}$$

Choosing $\xi = (i, 1)^T$ shows that $\operatorname{tr}[\xi \otimes \xi] = 0$, which proves the claim. ■

Thence, the quadratic form is not algebraically complete and this excludes the treatment of mixed boundary conditions on u in the following: we are forced to assume $\Gamma_D = \partial\Omega$. However, inhomogeneous Dirichlet conditions may be prescribed as far as the use of the Div / Curl estimate is concerned.

2.5 Variational form of the update problem

Due to the underlying variational formulation, the weak form (2.21) of the time-incremental Cosserat problem still has a variational structure. In [49] it is shown that solving (2.21)–(2.22) is equivalent to the following minimization problem: find $(u^n, A^n) \in H_0^1(\Omega, \mathbb{R}^3) \times H_0^1(\Omega, \mathfrak{so}(3))$ which minimize the functional

$$I_{\text{incr}}^n(u, A) = \mathcal{E}_{\text{incr}}(u, A, \varepsilon_p^{n-1}) - \int_\Omega \langle f^n, u\rangle \,\mathrm{dx} \tag{2.40}$$

in $H_0^1(\Omega, \mathbb{R}^3) \times H_0^1(\Omega, \mathfrak{so}(3))$. Here $\mathcal{E}_{\text{incr}}$ denotes the free energy of the incremental problem defined by

$$\begin{aligned}\mathcal{E}_{\text{incr}}(u, A, \varepsilon_p) = \frac{1}{2\mu}\int_\Omega \Psi\big(2\mu(\operatorname{sym}(\nabla u) - \varepsilon_p)\big)\,\mathrm{dx} + \frac{\lambda}{2}\int_\Omega \operatorname{tr}[\nabla u]^2\,\mathrm{dx} \\ + \mu_c\int_\Omega \|\operatorname{skew}(\nabla u) - A\|^2\,\mathrm{dx} + \mu L_c^2\int_\Omega \|DA\|^2\,\mathrm{dx},\end{aligned} \tag{2.41}$$

with a potential function $\Psi : \mathbb{M}^{3\times3} \to \mathbb{R}^+$ of the form

$$\begin{aligned}\Psi(X) :=& \begin{cases} \frac{1}{2}\|X\|^2 & \|\operatorname{dev} X\| \le \sigma_{\mathrm{y}} \\ \frac{1}{2}\left(\frac{1}{3}\operatorname{tr}[X]^2 + 2\,\sigma_{\mathrm{y}}\,\|\operatorname{dev} X\| - \sigma_{\mathrm{y}}{}^2\right) & \|\operatorname{dev} X\| > \sigma_{\mathrm{y}} \end{cases} \\ =& \frac{1}{2}\|X\|^2 - \frac{1}{2}\left[\|\operatorname{dev} X\| - \sigma_{\mathrm{y}}\right]_+^2 .\end{aligned} \tag{2.42}$$

Clearly, Ψ is convex but not strongly convex outside the yield surface. Moreover, it has only linear growth outside the yield surface. Note that for the first time step $n = 1$ and $\varepsilon_p^0 = 0$, $\mu_c = 0$, $L_c = 0$ the functional $I_{\text{incr}}^1(u, 0)$ reduces to the primal plastic functional of static perfect plasticity (Hencky-plasticity) [35, 63, 24, 25, 6].

Calculating the subdifferential of the convex potential shows that

$$\begin{aligned}\partial\Psi(\Sigma).H =& \begin{cases} \langle \Sigma, H\rangle & \|\operatorname{dev}\Sigma\| \le \sigma_{\mathrm{y}} \\ \frac{1}{3}\operatorname{tr}[\Sigma]\operatorname{tr}[H] + \frac{\sigma_{\mathrm{y}}}{\|\operatorname{dev}\Sigma\|}\langle \operatorname{dev}\Sigma, \operatorname{dev} H\rangle & \|\operatorname{dev}\Sigma\| > \sigma_{\mathrm{y}} \end{cases} \\ =& \langle P_K(\Sigma), H\rangle .\end{aligned} \tag{2.43}$$

Hence $\partial\Psi(\Sigma) = P_K(\Sigma)$ motivating the variational structure.

The following relationship between the potential Ψ and the projection P_K is also valid

$$\Psi(X) = \frac{1}{2}\|X\|^2 - \frac{1}{2}\|X - P_K(X)\|^2 . \tag{2.44}$$

For future reference the second differential of the potential Ψ can be calculated in those points where the potential is differentiable. It holds

$$D^2_X\Psi(X).(H,H) = \begin{cases} \|H\|^2 & \|\operatorname{dev} X\| < \sigma_{\mathrm{y}} \\ \text{does not exist} & \|\operatorname{dev} X\| = \sigma_{\mathrm{y}} \\ \frac{1}{3}\operatorname{tr}[H]^2 + \sigma_{\mathrm{y}}\left(\frac{\|\operatorname{dev} H\|^2}{\|\operatorname{dev} X\|} - \frac{\langle \operatorname{dev} X, H\rangle^2}{\|\operatorname{dev} X\|^3}\right) & \|\operatorname{dev} X\| > \sigma_{\mathrm{y}}\ . \end{cases} \tag{2.45}$$

The potential Ψ is not strictly rank-one convex in X, since, taking $H = \xi \otimes \eta$ with $\langle \xi, \eta\rangle = 0$ yields

$$D^2_X\Psi(X).(\xi\otimes\eta,\xi\otimes\eta) = \begin{cases} \|\xi\|^2\,\|\eta\|^2 & \|\operatorname{dev} X\| \le \sigma_{\mathrm{y}} \\ \sigma_{\mathrm{y}}\left(\frac{\|\operatorname{dev} \xi\otimes\eta\|^2}{\|\operatorname{dev} X\|} - \frac{\langle \operatorname{dev} X, \xi\otimes\eta\rangle^2}{\|\operatorname{dev} X\|^3}\right) & \|\operatorname{dev} X\| > \sigma_{\mathrm{y}} \end{cases} \tag{2.46}$$

Taking $X = \xi \otimes \eta$ shows finally

$$D^2_X\Psi(X).(\xi\otimes\eta,\xi\otimes\eta) = \begin{cases} \|\xi\|^2\,\|\eta\|^2 & \|\operatorname{dev} X\| \le \sigma_{\mathrm{y}} \\ 0 & \|\operatorname{dev} X\| > \sigma_{\mathrm{y}}\ . \end{cases} \tag{2.47}$$

3 Improved error estimates for Cosserat plasticity

Let $h > 0$ be the mesh-size of a finite element method and let $V_h \subset H^1_0(\Omega,\mathbb{R}^3)$ be a corresponding discrete finite-element space. Let us concentrate on the displacement approximation only. In [49, Th.8] the following error estimate for the discrete solution $u_h^{\mu_c,n} \in V_h$ of the Galerkin-approximation of (2.41) in V_h has been shown:

$$\|u^{\mu_c,n} - u_h^{\mu_c,n}\|_{H^1_0(\Omega)} \le \frac{C_1}{\mu_c} \inf_{v_h\in V_h} \|u^{\mu_c,n} - v_h\|_{H^1_0(\Omega)}\ , \tag{3.1}$$

with a constant $C_1 > 0$. Here, $u^{\mu_c,n} = u^n$ is the exact solution of (2.21).

Using our regularity result from section 5, i.e., $u^{\mu_c,n} \in H^2(\Omega,\mathbb{R}^3)$, the right hand side can be estimated qualitatively. If V_h is chosen to be the space of piecewise linear finite elements, then it holds [8, p.107]

$$\|u^{\mu_c,n} - u_h^{\mu_c,n}\|_{H^1_0(\Omega)} \le \frac{C_2}{\mu_c}\, h\, \|u^{\mu_c,n}\|_{H^2(\Omega)}\ . \tag{3.2}$$

In [49] it has also been shown that for $\mu_c \to 0$ the classical Prandtl-Reuss symmetric Cauchy stresses σ^0 are approximated by the sequence of non-symmetric stresses σ^{μ_c} whenever a safe load condition is satisfied. The estimate (3.2) strongly suggests therefore to balance h against μ_c to obtain optimal rates of convergence to the classical solution as in [54], where hardening type approximations have been considered.

4 Higher regularity for alternative regularized update potentials

Our regularity result can also be applied to many other problems arising in the context of infinitesimal plasticity. There exist several other possibilities to regularize the classical update problem for the Prandtl–Reuss model. We recall the classical update problem: find a minimizer $u^n \in BD(\Omega, \mathbb{R}^3)$ of the functional

$$I^{\text{class}}_{\text{incr}}(u) = \mathcal{E}^{\text{class}}_{\text{incr}}(u, \varepsilon_p^{n-1}) - \int_\Omega \langle f^n, u\rangle \,\mathrm{dx}\,, \tag{4.1}$$

where $\mathcal{E}^{\text{class}}_{\text{incr}}$ denotes the free energy of the classical incremental problem defined by

$$\mathcal{E}^{\text{class}}_{\text{incr}}(u, \varepsilon_p) = \frac{1}{2\mu}\int_\Omega \Psi\big(2\mu(\text{sym}(\nabla u) - \varepsilon_p)\big)\,\mathrm{dx} + \int_\Omega \frac{\lambda}{2}\,\text{tr}\,[\nabla u]^2\,\mathrm{dx}\,, \tag{4.2}$$

with the potential Ψ as in (2.42). There is a vast literature on this Prandtl-Reuss update problem, mostly for the first time step $n = 1$, in which case it is the classical Hencky-problem of total deformation plasticity [63, 54, 24, 25]. In this case, the plastic strain field ε_p is a symmetric bounded measure [63, 6]. The classical symmetric Cauchy stresses $\sigma = 2\mu\,(\text{sym}\,\nabla u - \varepsilon_p) + \lambda\,\text{tr}\,[\nabla u]\,\mathbb{1}$ satisfy $\sigma \in L^2(\Omega, \text{Sym}(3))$, indeed higher regularity for the stresses can be shown in the sense that $\sigma \in H^1_{\text{loc}}(\Omega, \text{Sym}(3)) \cap H^{\frac{1}{2}-\delta}(\Omega)$.

For regularization purposes the following proposals are usually made:

$$\mathcal{E}^{\text{reg}}_{\text{incr}}(u, \varepsilon_p) = \frac{1}{2\mu}\int_\Omega \Psi\big(2\mu(\text{sym}(\nabla u) - \varepsilon_p)\big)\,\mathrm{dx} + \int_\Omega \frac{\lambda}{2}\,\text{tr}\,[\nabla u]^2 + \text{Reg}(\nabla u, \varepsilon_p)\,\mathrm{dx}\,, \tag{4.3}$$

with the function Reg in the form

$$\begin{aligned}
\text{Reg}(\nabla u, \varepsilon_p) &= \frac{\mu\,\delta}{2}\,\|\text{dev}\,\text{sym}\,\nabla u - \varepsilon_p\|^2\,, \quad \text{Fuchs/Seregin [24, p.60]}\,,\\
\text{Reg}(\nabla u, \varepsilon_p) &= \frac{1}{2\,\mu\,(1+\frac{\Delta t}{\eta})}\,[\|\mu(\text{dev}\,\text{sym}\,\nabla u - \varepsilon_p)\| - \sigma_{\text{y}}]_+^2\,, \quad \text{linear viscosity } \eta\,,\\
\text{Reg}(\nabla u, \varepsilon_p) &= \frac{\mu\,\delta}{2}\,\|\nabla u - \varepsilon_p\|^2\,, \quad \text{locally strictly convex in } \nabla u\,.
\end{aligned} \tag{4.4}$$

In each case, for $\delta > 0$ the density of the update problem is then uniformly convex in the symmetric strain $\varepsilon = \text{sym}\,\nabla u$. Moreover,

$$\text{Reg}(\nabla u, \varepsilon_p) + \frac{\lambda}{2}\,\text{tr}\,[\nabla u]^2 \geq c^+\,\|\varepsilon - \varepsilon_p\|^2\,, \tag{4.5}$$

and Korn's first inequality establishes quadratic growth and we have uniform convexity for the regularized problem. Our main regularity result applies therefore also to these models.

In the case with linear hardening it is simpler to write the update potential directly. We consider as an example isotropic hardening with the hardening variable $\alpha \geq 0$ (a measure for the accumulated plastic strain in the previous time step). Here, the energy $\mathcal{E}_{\text{incr}}$ can be expressed as (cf. [60, p.124])

$$\mathcal{E}^{\text{hard}}_{\text{incr}}(u, \varepsilon_p, \alpha) = \frac{1}{2\mu}\int_\Omega \Psi_{\text{hard}}\big(2\mu(\text{sym}(\nabla u) - \varepsilon_p), \alpha\big)\,\mathrm{dx} + \int_\Omega \frac{\lambda}{2}\,\text{tr}\,[\nabla u]^2\,\mathrm{dx}\,, \tag{4.6}$$

with (cf. (2.42))

$$\Psi_{\text{hard}}(X,\alpha) = \begin{cases} \frac{1}{2}\,\|X\|^2 & \|\text{dev}\,X\| \leq \sigma_{\text{y}} + H\,\alpha \\ \frac{1}{2\,(1+\frac{H}{1[\text{MPa}]})}\Big(\frac{H}{1[\text{MPa}]}\,\|X\|^2 + \frac{1}{3}\,\text{tr}\,[X]^2 & \\ \quad +2\,(\sigma_{\text{y}}+H\,\alpha)\,\|\text{dev}\,X\| - (\sigma_{\text{y}}+H\,\alpha)^2\Big) & \|\text{dev}\,X\| > \sigma_{\text{y}} + H\,\alpha \end{cases}$$

$$= \frac{1}{2}\,\|X\|^2 - \frac{1}{2\,(1+\frac{H}{1[\text{MPa}]})}\,\big[\,\|\text{dev}\,X\| - (\sigma_{\text{y}}+H\,\alpha)\big]_+^2\,, \tag{4.7}$$

whose second derivative coincides with the consistent tangent method introduced already in [61]. The constant $H > 0$ is the hardening modulus with dimension [MPa]. In this form it is easy to see that for positive hardening modulus $H > 0$ the isotropic hardening update potential is uniformly convex in $\text{sym}\,\nabla u$ with quadratic growth and has a Lipschitz continuous derivative. Therefore, our main regularity result applies also to this functional.[4] The relative merits of each individual regularization scheme depend on their ability to balance regularization and approximation. Linear viscosity and hardening can be justified on physical grounds, but the (small) viscosity parameter $\eta > 0$ is difficult to estimate, as is the linear hardening modulus $H > 0$. The physically motivated regularization terms have the property to only control the symmetric part of the displacement gradient. The regularization $(4.4)_3$, however, does not satisfy the linearized frame-indifference condition.

All alternative regularization procedures thus establish local coercivity in the strains. In contrast, the Cosserat regularization is weaker in the sense that only strong Legendre-Hadamard ellipticity is reestablished, which, provided displacement boundary data are prescribed, suffice for existence, uniqueness and higher regularity. Thus the Cosserat approach appears as the weakest regularization among the considered ones.

5 The regularity theorem

We know already that (2.41) has solutions $u^n \in H^1(\Omega,\mathbb{R}^3)$. Looking at the system for the microrotations A^n at given $\nabla u^n \in L^2(\Omega,\mathbb{M}^{3\times3})$ we realize at once that the linearity in A^n together with the Laplacian structure allows to use standard elliptic regularity results for linear systems which yields higher regularity for the microrotations: $A^n \in H^2(\Omega,\mathfrak{so}(3))$. In this section we study the regularity of the displacement field u^n, which is determined through equation (2.21).

5.1 Higher regularity for a quasilinear elliptic system

The quasilinear elliptic system introduced in section 2.4 is a special case of the systems which we define here below. For $d, m, N \geq 1$ and $\Omega \subset \mathbb{R}^d$ let $\mathcal{M} : \Omega\times\mathbb{M}^{m\times d}\times\mathbb{R}^N \to \mathbb{M}^{m\times d}$ be a matrix valued function with the following properties:

[4] Repin [54, eq.(2.3)] calls $(4.4)_2$ linear hardening and shows the regularity $u^\delta \in H^2_{\text{loc}}(\Omega,\mathbb{R}^3)$ while for the planar case $n = 2$ he obtains $u^\delta \in H^2(\Omega,\mathbb{R}^2)$ if $\Gamma = \partial\Omega$ is smooth.

R1 The mapping $\mathcal{M} : \Omega \times \mathbb{M}^{m\times d} \times \mathbb{R}^N \to \mathbb{M}^{m\times d}$ is a Carathéodory function which is Lipschitz continuous in the following sense: there exist constants $L_1, L_2 > 0$ such that for every $x, x_i \in \Omega$, $a, a_i \in \mathbb{M}^{m\times d}$ and $z, z_i \in \mathbb{R}^N$ we have

$$\|\mathcal{M}(x_1, a, z) - \mathcal{M}(x_2, a, z)\| \le L_1(\|a\| + \|z\|)\, \|x_1 - x_2\| \,, \tag{5.1}$$
$$\|\mathcal{M}(x, a_1, z_1) - \mathcal{M}(x, a_2, z_2)\| \le L_2(\|a_1 - a_2\| + \|z_1 - z_2\|), \tag{5.2}$$
$$\mathcal{M}(x, 0, 0) = 0. \tag{5.3}$$

Assumption **R1** implies the useful estimate

$$\begin{aligned}&\|\mathcal{M}(x_1, a_1, z_1) - \mathcal{M}(x_2, a_2, z_2)\| \\ &\qquad \le L_1(\|a_1\| + \|z_1\|)\, \|x_1 - x_2\| + L_2(\|a_1 - a_2\| + \|z_1 - z_2\|).\end{aligned} \tag{5.4}$$

R2 The mapping $\mathcal{M}$ is strongly rank–one monotone. That means that there exists a constant $c_{LH} > 0$ such that for every $x \in \overline{\Omega}$, $a \in \mathbb{M}^{m\times d}$, $z \in \mathbb{R}^N$, $\xi \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^d$ we have

$$\langle \mathcal{M}(x, a + \xi \otimes \eta, z) - \mathcal{M}(x, a, z), \xi \otimes \eta \rangle \ge c_{LH} \|\xi\|^2 \|\eta\|^2 . \tag{5.5}$$

R3 The Gårding inequality shall be satisfied: there exist constants $C_G > 0$, $c_G \in \mathbb{R}$ such that for every $u_1, u_2 \in H^1(\Omega)$ with $u_1 - u_2 \in H_0^1(\Omega)$ and for every $z \in L^2(\Omega)$ the following inequality is valid:

$$\begin{aligned}&\int_\Omega \langle \mathcal{M}(x, \nabla u_1, z) - \mathcal{M}(x, \nabla u_2, z), \nabla(u_1 - u_2)\rangle \,\mathrm{dx} \\ &\qquad\qquad \ge C_G \|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)} - c_G \|u_1 - u_2\|^2_{L^2(\Omega)} .\end{aligned}$$

Remark 5.1
If $\mathcal{M}$ is differentiable, then the Gårding inequality already implies that $\mathcal{M}$ is rank–one monotone, see for example [65, Th.6.1].

We investigate the regularity properties of weak solutions to the following quasilinear elliptic boundary value problem. For given $g \in H^{\frac{1}{2}}(\partial\Omega)$, $z \in L^2(\Omega, \mathbb{R}^N)$ and $f \in L^2(\Omega, \mathbb{R}^m)$ find $u \in H^1(\Omega, \mathbb{R}^m)$ with $u\big|_{\partial\Omega} = g$ such that for every $v \in H_0^1(\Omega, \mathbb{R}^m)$ we have:

$$\int_\Omega \langle \mathcal{M}(x, \nabla u(x), z(x)), \nabla v(x)\rangle \,\mathrm{dx} = \int_\Omega \langle f, v\rangle \,\mathrm{dx}\,. \tag{5.6}$$

Theorem 5.2
Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^{1,1}$–smooth domain, $m \ge 1$, $N \ge 1$, and assume that $\mathcal{M} : \Omega \times \mathbb{M}^{m\times d} \times \mathbb{R}^N \to \mathbb{M}^{m\times d}$ satisfies **R1**–**R3**. *Let furthermore $g \in H^{\frac{3}{2}}(\partial\Omega)$, $z \in H^1(\Omega)$ and $f \in L^2(\Omega)$. Every weak solution $u \in H^1(\Omega)$ of (5.6) with $u\big|_{\partial\Omega} = g$ is an element of $H^2(\Omega)$ and satisfies*

$$\|u\|_{H^2(\Omega)} \le c\,\big(\|g\|_{H^{\frac{3}{2}}(\partial\Omega)} + \|z\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \big).$$

Before we prove theorem 5.2, we apply it to the situation described in section 2.4. There, $m = d = 3$ and $\mathbb{R}^N$ is identified with $\mathrm{Sym}(3) \times \mathfrak{so}(3)$ so that $z = (\varepsilon_p, A)$. Moreover,

$$
\begin{aligned}
\mathcal{M}(x, \nabla u, z) &= \mathcal{M}_C(\nabla u, \varepsilon_p, A) \\
&= P_K(2\mu(\operatorname{sym} \nabla u - \varepsilon_p)) + \lambda(\operatorname{tr}[\nabla u])\mathbb{1} + 2\mu_c(\operatorname{skew}(\nabla u) - A).
\end{aligned}
$$

Since P_K is a Lipschitz continuous mapping, we see immediately, that assumption **R1** is satisfied. **R2** is proved in lemma 2.1 and the Gårding inequality is satisfied since $\mathcal{M}_C$ generates a strongly monotone operator on $H_0^1(\Omega)$, see lemma 2.2. Therefore, we have the following result for the reduced update problem (2.13):

Theorem 5.3
Let Ω be $C^{1,1}$–smooth, $f^n \in L^2(\Omega)$ and $\varepsilon_p^{n-1} \in H^1(\Omega)$. Then $u_n \in H^2(\Omega)$, $A^n \in H^2(\Omega)$ and $\varepsilon_p^n \in H^1(\Omega)$.

The proof of theorem 5.2 is carried out with a difference quotient technique. We cover the boundary of Ω with a finite number of domains and map each of these domains with a $C^{1,1}$–diffeomorphism onto the unit cube in such a way that the image of the boundary of Ω lies on the midplane of the unit cube. We first prove higher regularity in directions tangential to the midplane by estimating difference quotients. The regularity in normal direction is then obtained on the basis of the tangential regularity and by using the differential equation together with the rank–one monotonicity of $\mathcal{M}$.

Since $\mathcal{M}$ is nonlinear and since we assumed rank–one monotonicity in stead of strong monotonicity, we cannot use as test functions the usual finite differences of the type $h^{-1}\varphi^2(x)(u(x+h) - u(x))$, where φ is a cut–off function. Instead, we use differences which are based on inner variations. We begin the proof of theorem 5.2 by studying a model problem on a half cube.

5.2 A model problem on a half cube

Let $C_r = \{\, x \in \mathbb{R}^d \,;\, |x_i| < r,\ 1 \le i \le d \,\}$ be a cube with side length $2r$, $C_r^\pm$ the upper and lower half–cube, respectively, and $M_r = \{\, x \in C_r \,;\, x_d = 0 \,\}$ the mid plane.

Lemma 5.4
Let $\Omega = C_1^-$, $f \in L^2(C_1^-)$, $z \in H^1(C_1^-)$ and assume that $u \in H^1(C_1^-)$ with $u\big|_{M_1} = 0$ satisfies (5.6). Then for every $r \in (0,1)$ and for $1 \le i \le d-1$ we have $\partial_i u \in H^1(C_r^-)$. Moreover, there is a constant $c_r > 0$ such that

$$
\|\partial_i u\|_{H^1(C_r^-)} \le c_r\big(\|u\|_{H^1(C_1^-)} + \|z\|_{H^1(C_1^-)} + \|f\|_{L^2(C_1^-)} \big). \tag{5.7}
$$

Proof. Let $r \in (0,1)$ and $\varphi \in \mathcal{C}_0^\infty(C_1)$ with $\varphi(x) = 1$ on C_r. For $h \in \mathbb{R}^d$ we introduce the mapping

$$
\tau_h : C_1 \to \mathbb{R}^d : \, x \to \tau_h(x) = x + \varphi(x)h.
$$

Let $h_0 = \|\varphi\|_{W^{1,\infty}(C_1)}^{-1} \min\big\{1, \operatorname{dist}(\operatorname{supp}\varphi, \partial C_1)\big\}$. For every $h \in \mathbb{R}^d$ with $|h| < h_0$ and h parallel to the plane M_1, the mapping τ_h is a diffeomorphism from C_1 onto itself with

$\tau_h(C_1^\pm) = C_1^\pm$, $\tau_h(M_1) = M_1$ and $\tau_h(x) = x$ for every $x \in \partial C_1$, see e.g. [26]. Moreover, τ_h has the following properties (if $|h| < h_0$):

$$\nabla\tau_h(x) = \big(\mathbb{1} + h \otimes \nabla\varphi(x)\big), \quad \det[\nabla\tau_h(x)] = 1 + \langle h, \nabla\varphi(x)\rangle,$$
$$\nabla_y\tau_h^{-1}(y) = \big(\mathbb{1} + h \otimes \nabla\varphi\big)^{-1}\big|_{\tau_h^{-1}(y)} = \mathbb{1} - \big((1 + \langle h, \nabla\varphi\rangle)^{-1} h \otimes \nabla\varphi\big)\big|_{\tau_h^{-1}(y)}.$$

For a function $v : C_1^- \to \mathbb{R}^s$ we introduce

$$\triangle_h v = v\circ\tau_h - v, \qquad \triangle^h v = v - v\circ\tau_h^{-1}.$$

For $f, g \in L^2(C_1^-)$, $|h| < h_0$ and $h \parallel M_1$ the following product rule is valid:

$$\int_{C_1^-} f \,\triangle^h g \,\mathrm{dx} = -\int_{C_1^-} g \,\triangle_h f \,\mathrm{dx} - \int_{C_1^-} (f\circ\tau_h\, g)\,\langle h, \nabla\varphi\rangle \,\mathrm{dx}. \tag{5.8}$$

This identity can be shown by a transformation of coordinates $y = \tau_h(x)$ in the term $(g\circ\tau_h^{-1})f$. Let $u \in H_0^1(C_1^-)$ be a solution of (5.6). We define for $h \in \mathbb{R}^d$ with $|h| < h_0$ and $h \parallel M_1$

$$v_h(x) = \triangle^h(\triangle_h u(x)).$$

In view of the assumptions on φ, h_0 and h it follows that $v_h \in H_0^1(C_1^-)$. Inserting v_h into (5.6) yields

$$\int_{C_1^-} \langle \mathcal{M}(x, \nabla u, z), \nabla v_h\rangle \,\mathrm{dx} = \int_{C_1^-} \langle f, v_h\rangle \,\mathrm{dx}. \tag{5.9}$$

Note that $\nabla v_h = \triangle^h\nabla(\triangle_h u) + [(\det[\nabla\tau_h])^{-1}(\nabla\triangle_h u)\, h\otimes\nabla\varphi]\circ\tau_h^{-1}$ and therefore, (5.9) is equivalent to

$$\begin{aligned}
&\int_{C_1^-} \langle \mathcal{M}(x, \nabla u, z), \triangle^h\nabla(\triangle_h u)\rangle \,\mathrm{dx} \\
&\qquad = -\int_{C_1^-} \langle \mathcal{M}(x, \nabla u, z), \big(\det[\nabla\tau_h]^{-1}(\nabla\triangle_h u)\, h \otimes \nabla\varphi\big)\circ\tau_h^{-1}\rangle \,\mathrm{dx} \\
&\qquad\quad + \int_{C_1^-} \langle f, \triangle^h\triangle_h u\rangle \,\mathrm{dx}.
\end{aligned}$$

Furthermore, the product rule (5.8) entails

$$\begin{aligned}
&\int_{C_1^-} \langle \triangle_h\mathcal{M}(x, \nabla u, z)\,,\, \nabla\triangle_h u\rangle \,\mathrm{dx} \\
&\qquad = -\int_{C_1^-} \langle \mathcal{M}(x, \nabla u, z)\circ\tau_h, \nabla\triangle_h u\rangle\langle h, \nabla\varphi\rangle \,\mathrm{dx} \\
&\qquad\quad + \int_{C_1^-} \langle \mathcal{M}(x, \nabla u, z), \big((\det[\nabla\tau_h])^{-1}(\nabla\triangle_h u)\, h \otimes \nabla\varphi\big)\circ\tau_h^{-1}\rangle \,\mathrm{dx} \\
&\qquad\quad + \int_{C_1^-} \langle f, \triangle^h\triangle_h u\rangle \,\mathrm{dx} \\
&\qquad =: S_1 + S_2 + S_3
\end{aligned} \tag{5.10}$$

Finally we have

$$
\begin{aligned}
\int_{C_1^-} & \langle \mathcal{M}(x, \nabla(u\circ\tau_h), z) - \mathcal{M}(x, \nabla u, z)\,,\ \nabla\triangle_h u\rangle \,\mathrm{dx} \\
&= \int_{C_1^-} \langle \triangle_h \mathcal{M}(x, \nabla u, z), \nabla\triangle_h u\rangle \,\mathrm{dx} \\
&\quad + \int_{C_1^-} \langle \mathcal{M}(x, \nabla(u\circ\tau_h), z) - \mathcal{M}(x, \nabla u, z)\circ\tau_h\big), \nabla\triangle_h u\rangle \,\mathrm{dx} \\
&\overset{(5.10)}{=} S_1 + S_2 + S_3 \\
&\quad + \int_{C_1^-} \langle \mathcal{M}(x, \nabla(u\circ\tau_h), z) - \mathcal{M}(x, \nabla u, z)\circ\tau_h, \nabla\triangle_h u\rangle \,\mathrm{dx} \\
&= S_1 + \ldots + S_4. \qquad (5.11)
\end{aligned}
$$

The next task is to show that there is a constant $c > 0$, which does not depend on h, such that

$$
|S_1 + \ldots + S_4| \leq c\,|h|\,\big(\,\|u\|_{H^1(C_1^-)} + \|z\|_{H^1(C_1^-)} + \|f\|_{L^2(C_1^-)}\big)\,\|\triangle_h u\|_{H^1(C_1^-)}\,. \qquad (5.12)
$$

Due to the Lipschitz assumptions on $\mathcal{M}$ we have

$$
\begin{aligned}
|S_1| + |S_2| &\leq c\,|h|\,\|\mathcal{M}(\cdot, \nabla u, z)\|_{L^2(C_1^-)}\,\|\triangle_h u\|_{H^1(C_1^-)} \\
&\leq c\,|h|\,\big(\,\|u\|_{H^1(C_1^-)} + \|z\|_{L^2(C_1^-)}\big)\,\|\triangle_h u\|_{H^1(C_1^-)}\,.
\end{aligned}
$$

Moreover, since $f \in L^2(C_1^-)$, the term S_3 can be estimated as

$$
|S_3| \leq c\,|h|\,\|f\|_{L^2(C_1^-)}\,\|\triangle_h u\|_{H^1(C_1^-)}\,.
$$

By inequality (5.4) we see that

$$
\begin{aligned}
|S_4| &\leq cL_1\,|h|\,\big(\,\|u\|_{H^1(C_1^-)} + \|z\|_{H^1(C_1^-)}\big)\,\|\triangle_h u\|_{H^1(C_1^-)} \\
&\quad + cL_2\big(\,\|\nabla(u\circ\tau_h) - (\nabla u)\circ\tau_h\|_{L^2(C_1^-)} + c\,|h|\,\|z\|_{H^1(C_1^-)}\big)\,\|\triangle_h u\|_{H^1(C_1^-)}\,.
\end{aligned}
$$

The identity $\nabla(u\circ\tau_h) - (\nabla u)\circ\tau_h = (\nabla u)\circ\tau_h\,(h\otimes\nabla\varphi)$ leads to

$$
|S_4| \leq c\,|h|\,\big(\,\|u\|_{H^1(C_1^-)} + \|z\|_{H^1(C_1^-)}\big)\,\|\triangle_h u\|_{H^1(C_1^-)}\,.
$$

Collecting all the above estimates we finally arrive at inequality (5.12). Gårding's inequality (see **R3**) and Poincaré's inequality imply that

$$
\begin{aligned}
\int_{C_1^-} & \langle \mathcal{M}(x, \nabla(u\circ\tau_h), z) - \mathcal{M}(x, \nabla u, z)\,,\ \nabla\triangle_h u\rangle \,\mathrm{dx} \\
&\geq C_G\,\|\nabla\triangle_h u\|^2_{L^2(C_1^-)} - c_G\,\|\triangle_h u\|^2_{L^2(C_1^-)} \\
&\geq c\big(\,\|\triangle_h u\|^2_{H^1(C_1^-)} - |h|^2\,\|u\|^2_{H^1(C_1^-)}\big).
\end{aligned}
$$

Combining the above estimates with (5.11) and (5.12) results finally in

$$\begin{aligned}\|\triangle_h u\|^2_{H^1(C_1^-)} &\le c\,|h|\,\big(\,\|u\|_{H^1(C_1^-)} + \|z\|_{H^1(C_1^-)} + \|f\|_{L^2(C_1^-)}\big)\,\|\triangle_h u\|_{H^1(C_1^-)}\\ &\quad + c\,|h|^2\,\|u\|^2_{H^1(C_1^-)}\end{aligned}$$

and the constant c is independent of h. From Young's inequality we obtain

$$|h|^{-1}\,\|\triangle_h u\|_{H^1(C_1^-)} \le c\big(\,\|u\|_{H^1(C_1^-)} + \|z\|_{H^1(C_1^-)} + \|f\|_{L^2(C_1^-)}\big). \tag{5.13}$$

It follows from this inequality that $\partial_i u \in H^1(C_r^-)$ for $1 \le i \le d-1$ and that $\|\partial_i u\|_{H^1(C_r^-)}$ is bounded by the right hand side in (5.13), see e.g. [34]. ■

Remark 5.5
If we choose the usual finite differences as test functions, i.e. $\tilde v_h(x) = \delta_{-h}(\varphi^2\delta_h u)$, *where* $\delta_h u = u(x+h) - u(x)$, *then similar calculations as those for* v_h *lead to the estimate*

$$\int_{C_1^-} \varphi^2(x)\langle \mathcal{M}(x,\nabla u(x+h),z(x)) - \mathcal{M}(x,\nabla u(x),z(x)),\delta_h\nabla u\rangle\,\mathrm{dx} \le c\,|h|\,\big\|\varphi^2\delta_h u\big\|_{H^1(C_1^-)}, \tag{5.14}$$

compare also (5.11) *and* (5.12). *But now neither* **R2** *nor* **R3** *help us to find a lower bound for the left hand side of* (5.14) *in terms of* $\|\varphi^2\delta_h\nabla u\|^2_{L^2(C_1^-)}$, *since in general* $\delta_h\nabla u$ *is not a rank–one matrix, and since we cannot interchange* φ *and* $\mathcal{M}$ *due to the nonlinearity of* $\mathcal{M}$.

Lemma 5.6 (Regularity in normal direction)
With the same assumptions as in lemma 5.4 it follows for every $r \in (0,1)$ *that* $\partial_d u \in H^1(C_r^-)$. *Furthermore, there exists a constant* $c_r > 0$ *such that*

$$\|u\|_{H^2(C_r^-)} \le c_r\big(\,\|z\|_{H^1(C_1^-)} + \|f\|_{L^2(C_1^-)} + \|u\|_{H^1(C_1^-)}\big). \tag{5.15}$$

Proof. Let $r \in (0,1)$. Equation (5.6) implies that

$$\operatorname{Div}\mathcal{M}(x,\nabla u(x),z(x)) + f(x) = 0 \tag{5.16}$$

for almost every $x \in C_1^-$. Let $\mathcal{M}_i$ denote the columns of the matrix valued function $\mathcal{M}$, i.e. $\mathcal{M}_i(x,a,z) = (\mathcal{M}_i^\alpha(x,a,z))_{1\le\alpha\le m} \in \mathbb{R}^m$ for $1 \le i \le d$. The Lipschitz continuity of $\mathcal{M}$ and the tangential regularity proved in lemma 5.4 guarantee that $\partial_i\mathcal{M}_i(\cdot,\nabla u,z) \in L^2(C_r^-)$ for $1 \le i \le d-1$ and is bounded by the right hand side in (5.7). Together with (5.16) we obtain therefore

$$\partial_d\mathcal{M}_d(\cdot,\nabla u,z) = -f - \sum_{i=1}^{d-1}\partial_i\mathcal{M}_i(\cdot,\nabla u,z) \quad \in L^2(C_r^-).$$

By Lemma 7.23 in [27] the derivative ∂_d can be replaced with a finite difference in the following way: For every $\Omega' \subset\subset C_r^-$ and every $h \in \mathbb{R}^d$ with $|h| < \operatorname{dist}(\Omega',\partial C_r^-)$ and $h \perp M_1$

we have

$$\|\delta_h \mathcal{M}_d(\cdot, \nabla u, z)\|_{L^2(\Omega')} \le \big(\|f\|_{L^2(C_r^-)} + \sum_{i=1}^{d-1} \|\partial_i \mathcal{M}_i(\cdot, \nabla u, z)\|_{L^2(C_r^-)} \big) |h|$$
$$=: c_0 |h| . \tag{5.17}$$

Here, $\delta_h v(x) := v(x+h) - v(x)$ for $h \in \mathbb{R}^d$. Thus, for every $h \perp M_1$ with $|h| < \operatorname{dist}(\Omega', \partial C_r^-)$ we have

$$\int_{\Omega'} \langle \delta_h \mathcal{M}_d(x, \nabla u, z), \delta_h \partial_d u \rangle \,\mathrm{dx} \le c_0 |h| \, \|\delta_h \partial_d u\|_{L^2(\Omega')} , \tag{5.18}$$

where c_0 is the constant from (5.17). We split now the left hand side into a term which can be estimated from below due to the rank-one monotonicity of $\mathcal{M}$ and into terms which may be estimated from above using the Lipschitz continuity of $\mathcal{M}$ and the regularity results from lemma 5.4. For functions $v : C_1^- \to \mathbb{R}^m$ we define $\tilde{\nabla} v(x) = (\partial_1 v(x), \ldots, \partial_{d-1} v(x), 0) \in \mathbb{M}^{m \times d}$. Furthermore, $v_h(x) := v(x+h)$ and $e_d = (0, \ldots, 0, 1)^\top \in \mathbb{R}^d$. With these notations we have

$$\begin{aligned} &\int_{\Omega'} \langle \mathcal{M}_d(x, \tilde{\nabla} u + \partial_d u_h \otimes e_d, z) - \mathcal{M}_d(x, \nabla u, z), \delta_h \partial_d u \rangle \,\mathrm{dx} \\ &\quad = \int_{\Omega'} \langle \delta_h \mathcal{M}_d(x, \nabla u, z), \delta_h \partial_d u \rangle \,\mathrm{dx} \\ &\qquad + \int_{\Omega'} \langle \mathcal{M}_d(x, \tilde{\nabla} u + \partial_d u_h \otimes e_d, z) - \mathcal{M}_d(x+h, \nabla u_h, z_h), \delta_h \partial_d u \rangle \,\mathrm{dx} \\ &\quad = S_1 + S_2. \end{aligned} \tag{5.19}$$

The term S_1 is already estimated in (5.18). From the Lipschitz continuity of $\mathcal{M}$ (see (5.4)) and the regularity results of lemma 5.4 we obtain by straightforward calculations

$$\begin{aligned} |S_2| &\le c \, \|\delta_h \partial_d u\|_{L^2(\Omega')} \big((\|\tilde{\nabla} u + \partial_d u_h \otimes e_d\|_{L^2(\Omega')} + \|z\|_{H^1(C_1^-)}) |h| + \|\delta_h \tilde{\nabla} u\|_{L^2(\Omega')} \big) \\ &\le c \, |h| \, \big(\|u\|_{H^1(C_1^-)} + \|z\|_{H^1(C_1^-)} + \|\partial_d \tilde{\nabla} u\|_{L^2(C_r^-)} \big) \|\delta_h \partial_d u\|_{L^2(\Omega')} \end{aligned} \tag{5.20}$$

and the constant c is independent of Ω' and h. Moreover, choosing $\xi = \partial_d u_h$ and $\eta = e_d$ in (5.5), we obtain for the left hand side in (5.19) from the rank-one monotonicity of $\mathcal{M}$ that

$$\int_{\Omega'} \langle \mathcal{M}_d(x, \tilde{\nabla} u + \partial_d u_h \otimes e_d, z) - \mathcal{M}_d(x, \nabla u, z), \ \delta_h \partial_d u \rangle \,\mathrm{dx} \ge c_{LH} \, \|\delta_h \partial_d u\|^2_{L^2(\Omega')} . \tag{5.21}$$

Estimates (5.18)–(5.21) together with Young's inequality finally imply that

$$|h|^{-1} \, \|\delta_h \partial_d u\|_{L^2(\Omega')} \le c \big(\|u\|_{H^1(C_1^-)} + \|z\|_{H^1(C_1^-)} + \|\partial_d \tilde{\nabla} u\|_{L^2(C_r^-)} \big) \tag{5.22}$$

for every $h \perp M_1$ and the constant c is independent of h and $\Omega' \Subset C_r^-$. This implies that $\partial_d^2 u \in L^2(C_r^-)$ and $\|\partial_d^2 u\|_{L^2(c_r^-)}$ is bounded by the right hand side in (5.22). Estimate (5.15) is a combination of (5.22) and (5.7). ■

5.3 Proof of theorem 5.2

Let the assumptions of theorem 5.2 be valid and assume that $g = 0$. Choose $x_0 \in \partial\Omega$ and let U_{x_0} be a neighborhood of x_0 such that there exists a $\mathcal{C}^{1,1}$-diffeomorphism $\Phi_{x_0} : U_{x_0} \to C_1$, where C_1 is the unit cube in $\mathbb{R}^d$, with the following properties (we omit the index x_0): $\Phi(U) = C_1$, $\Phi(U \cap \Omega) = C_1^-$, $\Phi(U \backslash \overline{\Omega}) = C_1^+$, $\Phi(U \cap \partial\Omega) = M_1$ and $\Phi(x_0) = 0$. Let $u \in H_0^1(\Omega)$ be a solution for (5.6) with the data $f \in L^2(\Omega)$ and $z \in H^1(\Omega)$. It follows that

$$\int_{U\cap\Omega} \langle \mathcal{M}(x, \nabla u, z), \nabla v\rangle \,\mathrm{dx} = \int_{U\cap\Omega} \langle f, v\rangle \,\mathrm{dx}$$

for every $v \in H_0^1(\Omega \cap U)$. After a transformation of coordinates with $y = \Phi(x)$ and $\Psi := \Phi^{-1}$, the previous equation can be written as follows: Let $\tilde{u}(y) = u(\Psi(y))$. For every $v \in H_0^1(C_1^-)$ we have

$$\int_{C_1^-} \langle \tilde{\mathcal{M}}(y, \nabla\tilde{u}, \tilde{z}), \nabla v\rangle \,\mathrm{dy} = \int_{C_1^-} \langle \tilde{f}, v\rangle \,\mathrm{dy}.$$

Here, we use the abbreviations

$$\tilde{\mathcal{M}}(y, a, \zeta) = |\det[\nabla\Psi(y)]| \; \mathcal{M}(\Psi(y), a(\nabla\Psi(y))^{-1}, \zeta)(\nabla\Psi(y))^{-\top}, \tag{5.23}$$

$$\tilde{f}(y) = |\det[\nabla\Psi(y)]| \; f(\Psi(y)), \tag{5.24}$$

$$\tilde{z}(y) = z(\Psi(y)) \tag{5.25}$$

for $y \in C_1^-$, $a \in \mathbb{M}^{m\times d}$ and $\zeta \in \mathbb{R}^N$. It follows immediately from the properties of the diffeomorphism Φ and from those of $\mathcal{M}$ that $\tilde{\mathcal{M}}$ satisfies **R1**–**R3** with respect to C_1^-. Furthermore, $\tilde{f}$ and $\tilde{z}$ have the smoothness required in lemma 5.4. Thus, lemmata 5.4 and 5.6 guarantee that $\tilde{u} \in H^2(C_r^-)$ for every $r < 1$ and that estimate (5.15) is valid. After applying the inverse transformation $\Psi : C_1^- \to U \cap \Omega$, we have finally shown the following: For every $x_0 \in \overline{\Omega}$ exists an open neighborhood $\tilde{U}_{x_0}$ such that $u\big|_{\tilde{U}_{x_0}\cap\Omega} \in H^2(\tilde{U}_{x_0} \cap \Omega)$ and estimate (5.15) is valid with respect to $\tilde{U}_{x_0} \cap \Omega$. The constants may depend on x_0. Since Ω is assumed to be bounded, we can cover $\overline{\Omega}$ by a finite number of the domains $\tilde{U}_{x_0}$ and obtain finally that $u \in H^2(\Omega)$ with

$$\|u\|_{H^2(\Omega)} \le c\big(\|z\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\big). \tag{5.26}$$

This proves theorem 5.2 for the case of vanishing Dirichlet conditions. The general case can be seen as follows. There exists a linear and continuous extension operator $F : H^{\frac{3}{2}}(\partial\Omega) \to H^2(\Omega)$ with $(F(g))\big|_{\partial\Omega} = g$ for every $g \in H^{\frac{3}{2}}(\Omega)$, see for example [68]. Then $u \in H^1(\Omega)$ with $u\big|_{\partial\Omega} = g$ for some $g \in H^{\frac{3}{2}}(\partial\Omega)$ is a solution to (5.6) if and only if there exists an element $\tilde{u} \in H_0^1(\Omega)$ with $u = \tilde{u} + F(g)$ and for every $v \in H_0^1(\Omega)$, $\tilde{u}$ satisfies

$$\int_{\Omega} \langle \hat{\mathcal{M}}(x, \nabla\tilde{u}, \tilde{z}), \nabla v\rangle \,\mathrm{dx} = \int_{\Omega} \langle f, v\rangle \,\mathrm{dx},$$

where $\tilde{z} = (F(g), z)$ and $\hat{\mathcal{M}}(x, a, \tilde{z}) = \mathcal{M}(x, a + F(g)(x), z)$. Clearly, $\hat{\mathcal{M}}$ satisfies **R1**–**R3** as well and by the first part of this proof it follows that $\tilde{u} \in H^2(\Omega)$. This finishes the proof of theorem 5.2.

6 Discussion

We have shown that the time-incremental Cosserat elasto-plasticity problem admits $H^1(\Omega)$-regular updates of the symmetric plastic strain ε_p^n provided that the previous plastic strain ε_p^{n-1} is in $H^1(\Omega)$ and the domain and data are suitably regular. Altogether, the time-incremental problem allows the regularity $\forall\, n \in \mathbb{N}:\ u^n \in H^2(\Omega, \mathbb{R}^3)$, $\varepsilon_p^n \in H^1(\Omega, \mathrm{Sym}(3))$ and $A^n \in H^2(\Omega, \mathfrak{so}(3))$. Uniform bounds in time are missing and it is an open question whether a similar result holds for the time continuous problem.

The presented method of proof for higher regularity uses a difference quotient method which is based on inner variations and can be extended to more general problems. This will be the subject of further investigations.

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Notation

We denote by $\mathbb{M}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X,Y\rangle_{\mathbb{M}^{3\times 3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X,X\rangle_{\mathbb{M}^{3\times 3}}$ (we use these symbols indifferently for tensors and vectors). The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $1\!\!1$, so that $\operatorname{tr}[X] = \langle X, 1\!\!1\rangle$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-algebra theory, i.e. $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times 3} \,|X^T = -X\}$ are skew symmetric second order tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times 3} \,|\operatorname{tr}[X] = 0\}$ are traceless tensors. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$. For $X \in \mathbb{M}^{3\times 3}$ we set for the deviatoric part $\operatorname{dev} X = X - \frac{1}{3}\operatorname{tr}[X]\, 1\!\!1 \in \mathfrak{sl}(3)$.

For a second order tensor X we let $X.e_i$ be the application of the tensor X to the column vector e_i. The first and second differential of a scalar valued function $W(F)$ are written $D_F W(F).H$ and $D_F^2 W(F).(H,H)$, respectively. Sometimes we use also $\partial_X W(X)$ to denote the first derivative of W with respect to X. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H_\circ^{1,2}(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions.