On the Trace Space of a Sobolev Space with a Radial Weight

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Abstract

Our concern in this paper lies with trace spaces for weighted Sobolev spaces, when the weight is a power of the distance to a point at the boundary. For a large range of powers we give a full description of the trace space.

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1 Introduction and main result

We consider integer order weighted Sobolev spaces with weights equal to a power of the distance to a point of the boundary and more general weights modelled upon such weights. Our concern in this paper lies with a characterization of trace spaces of these weighted Sobolev spaces. Rather surprisingly there are not too many trace theorems for weighted Sobolev spaces even though traces belong to the fundamental concepts both in the theory and applications, and they have been studied for a very long time. One of the major reasons is that there are no straightforward analogs of methods known from the non-weighted theory, which allow a description of values on manifolds of lower dimensions. Note in passing that the study of traces has been closely connected with extension of integer order spaces to spaces with non-integer derivatives, and it was one of the motivation for establishing the general theory of Besov spaces.

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The non-weighted theory for the W_p^k was studied in many papers and it can be found in a number of well-known monographs. We shall make no attempt to make an account of that; let us collect just some of the important references. The pioneering works by Aronszajn [4] and Slobodetskii [25] for the Hilbert case and the papers by Gagliardo [11] and Stein [26] should be mentioned. The theory for $p = 2$ based on abstract methods can be found in Lions and Magenes' monograph [18]. The case of general p is treated for instance in monographs by Nečas [20], Adams [2], Kufner, John and Fučík [16], Bergh and Löfström [5], Triebel [28]. An immense work has been done by the Soviet school (Lizorkin, Besov, Nikol'skii, Il'in, Uspenskii, and many others). We refer to [28] for a large list of references.

Spaces with weights which equal to a power of the distance to the boundary appeared in many papers; let us refer at least to [14] and [15]. A standard approach consists in taking the trace space as a factor space (modulo equality on the boundary). Nikol'skii in his monograph [21] (especially its second edition) established a trace theorem for these Sobolev weighted spaces: For a suitable range of parameters and under assumption on the regularity on $\partial\Omega$, the boundary of Ω, he identified the trace space with an unweighted Besov space with a modified smoothness parameter—the effect of the weight on the domain (Hardy's inequality behind the scenes).

Let us recall the very basic setting of the trace problem. For simplicity we shall consider spaces on $\mathbb{R}^n$ and traces on $\mathbb{R}^{n-1}$, that is, on $\partial\mathbb{R}^n_+$, the boundary of $\mathbb{R}^n_+$. By virtue of extension theorems the Sobolev space on $\mathbb{R}^n_+$ equals (up to equivalence of norms) to the restriction of the corresponding Sobolev space on $\mathbb{R}^n$, equipped with the factornorm (modulo equality on $\mathbb{R}^n_+$). This can be transferred to spaces on a smooth domain Ω and its boundary $\partial\Omega$ in a standard way—using resolution of unity and local coordinates. Let $s > 0$ be a non-integer and denote by $[s]$ the integer part of s. Let $1 < p \leq \infty$. Then the *Sobolev-Slobodetskii space* $W_p^s = W_p^s(\mathbb{R}^n)$ is defined as the linear space of all functions $f \in L_p(\mathbb{R}^n)$ with

$$\begin{aligned}\|f|W_p^s(\mathbb{R}^n)\| &= \|f\|_{L_p(\mathbb{R}^n)} \\ &\quad + \sum_{|\alpha|=[s]} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{(s-[s])p+N}} \, dx dy \right)^{1/p} < \infty.\end{aligned}$$

(Note that this is a special case of a general Besov space $B^s_{p,q}(\mathbb{R}^n)$ for $p = q$.) Here and in the following we shall use the notation $\|f|X\|$ instead of $\|f\|_X$ whenever it might improve legibility of the text. Recall that $C(\overline{\mathbb{R}^n_+}) \cap W_p^1(\mathbb{R}^n_+)$ is dense in $W_p^1(\mathbb{R}^n_+)$. One can prove that there is a bounded linear operator

$$\mathrm{tr} : W_p^1(\mathbb{R}^n) \to W_p^{1-1/p}(\partial\mathbb{R}^n_+)$$

such that $\operatorname{tr} f(y') = f(y')$ for every $y' \in \partial\mathbb{R}^n_+$ and every $f \in C(\overline{\mathbb{R}^n_+})$. This gives a natural meaning to values of a general $f \in W^1_p(\mathbb{R}^n_+)$ on $\partial\mathbb{R}^n_+$. Moreover, it is well known that there exists a bounded linear operator

$$\operatorname{ext} : W_p^{1-1/p}(\partial\mathbb{R}^n_+) \to W^1_p(\mathbb{R}^n_+)$$

such that $\operatorname{ext} \circ \operatorname{tr} = \operatorname{id}$ on $W_p^{1-1/p}(\partial\mathbb{R}^n_+)$. Theorems of this kind are now well-known in a general setting of Besov and Lizorkin-Triebel spaces; we refer to [28].

Now let w be a *weight function* (shortly a *weight*) in $\mathbb{R}^n$, that is, $w \in L_{1,\mathrm{loc}}$ and $w > 0$ a.e. in $\mathbb{R}^n$. Let $W^k_p(w) = W^k_p(\mathbb{R}^n, w)$ be the *weighted Sobolev space*, i.e. the space of all functions f, which together with their generalized derivatives $D^\alpha f$ up to the order k belong to

$$L_p(w) = L_p(\mathbb{R}^n; w) = \left\{ f : \|f\|^p_{L_p(w)} = \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx < \infty \right\},$$

with the norm

$$\|f|W^k_p(w)\| = \sum_{|\alpha| \le k} \|D^\alpha f | L_p(w)\|.$$

Only special weights (of type $(1 + |x|^2)^{r/2}$ and their generalizations) and rather sophisticated methods permit to conclude that a function f belongs to $W^k_p(w)$ if and only if $fw^{1/p} \in W^k_p(\chi_\Omega) = W^k_p$, see [24] and [7] for the so called W^n classes (one has to assume that the weighted space in question can be extended to the whole of $\mathbb{R}^n$, too). In particular, the class W^n excludes singularities so that another approach must be used for weights vanishing or blowing-up at the boundary. The situation is now well understood for weights, which equal to a power of the distance to the boundary. (Note also that such weights can be used to characterize zero traces, even in case of a quite general boundary; see e.g. [13].) The trace theorem for such weights was proved by Nikol'skii in [21] with help of real analysis methods. Let us recall Nikol'skii's result. Assume that Ω is a domain with a sufficiently smooth boundary Γ (as to the required smoothness we refer to [21] for details) and let

$$\varrho(x) = \operatorname{dist}(x, \Gamma), \qquad x \in \Omega.$$

For $k \in \mathbb{N}$, $1 \le p \le \infty$, and $\gamma \in \mathbb{R}$, denote by $W^k_{p,\gamma}$ the weighted Sobolev space with the norm

$$\|f|W^k_{p,\gamma}\| = \|f\|_{L_p(\Omega)} + \sum_{|\alpha|=k} \|(D^\alpha f)\varrho^{-\gamma}\|_{L_p(\Omega)}.$$

Suppose that

$$0 < k + \gamma - 1/p < k.$$

Then

$$W^k_{p,\gamma}(\Omega) \hookrightarrow W^{k+\gamma-1/p}_p(\Gamma)$$

and, moreover, there exists a bounded extension operator

$$\mathrm{ext}_\gamma : W^{k+\gamma-1/p}_p(\Gamma) \to W^k_{p,\gamma}(\Omega).$$

A by far more general setting—spaces on fractals with this type of weights—was recently considered by Piotrowska in [22].

In the following we shall make use of a Fourier analytic approach to Sobolev spaces and their weighted generalizations, therefore we recall the most important definitions and fix the notation.

Let $\{\varphi_j\}_{j=0}^\infty$ is the *smooth (dyadic) decomposition of unity* (see [28], [5]): $\operatorname{supp}\varphi_j \subseteq \{2^{j-1} \le |\xi| \le 2^{j+1}\}$ for $j \in \mathbb{N}_0$ and $\operatorname{supp}\varphi_0 \subseteq \overline{B_1(0)}$ and $\varphi_j(\xi) = \varphi_1(2^{-j+1}\xi)$ for $j \in \mathbb{N}$.

For $1 \le p \le \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}^1$ the *Besov space* $B^s_{p\,q} = B^s_{p\,q}(\mathbb{R}^n)$ is the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ with the finite norm

$$\|f|B^s_{p\,q}\| = \left(\sum_{k=0}^\infty 2^{ksq}\|\mathcal{F}^{-1}(\varphi_k\widehat{f})\|^q_{L_p}\right)^{1/q} \tag{1.1}$$

if $q < \infty$ and with the finite norm

$$\|f|B^s_{p\,\infty}\| = \sup_k 2^{ks}\|\mathcal{F}^{-1}(\varphi_k\widehat{f})\|_{L_p} \tag{1.2}$$

if $q = \infty$. Replacing the L_p space in the above definitions by $L_p(w)$ we get a formal definition of the *weighted Besov space* $B^s_{p\,q}(\mathbb{R}^n; w)$. Here $\mathcal{S}(\mathbb{R}^n)$ denotes the space of smooth rapidly decreasing functions $f\colon \mathbb{R}^n \to \mathbb{C}$ and $\mathcal{S}'(\mathbb{R}^n) = (\mathcal{S}(\mathbb{R}^n))'$ its dual.

We shall also use the *Bessel potential spaces* $H^s_p = H^s_p(\mathbb{R}^n)$ and their weighted clones: For s real and $1 < p < \infty$,

$$H^s_p(\mathbb{R}^n) = \left\{f \in \mathcal{S}'(\mathbb{R}^n) : \left\|\mathcal{F}^{-1}\left[(1+|\xi|^2)^{s/2}\mathcal{F}f\right]\right\|_{L_p} < \infty\right\},$$
$$H^s_p(\mathbb{R}^n; w) = \left\{f \in \mathcal{S}'(\mathbb{R}^n) : \left\|\mathcal{F}^{-1}\left[(1+|\xi|^2)^{s/2}\mathcal{F}f\right]\right\|_{L_p(w)} < \infty\right\}$$

normed in the obvious way.

For Lipschitz domains there exists a universal extension operator working on Sobolev, Besov and Bessel potential spaces (and also on the Lizorkin-Triebel spaces, even for all real s, see Rychkov [23]); this means that many relevant properties of spaces on Lipschitz domains follow from the claims on the whole of $\mathbb{R}^n$. That is, one can work either with a formal definition of spaces on domains as factorspaces of spaces on $\mathbb{R}^n$ modulo equality on the domain in question or with a space on the domain with a usual intrinsic norm (if it is available). This can be partly extended to weighted spaces with the Muckenhoupt weights. Recall that a weight w belongs to the Muckenhoupt class $A_p(\mathbb{R}^n)$ $(1 < p < \infty)$ if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{1/(p-1)}\, dx \right)^{p-1} < \infty, \tag{1.3}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with edges parallel to the coordinate axes. We shall write simply A_p if no misunderstanding can occur. Note in passing that $x' \mapsto |x'|^\alpha$ belongs to A_q in $\mathbb{R}^{n-1}$ if and only if $-(n-1) < \alpha < (q-1)(n-1)$ (see e.g. [8]).

We also refer to Chua [6] for an extension theorem for Sobolev spaces on domains and to Rychkov [23] as to the formulae for the norm in Sobolev spaces with A_p weights in terms of a weighted Littlewood-Paley decomposition. Specifically, for a positive integer k, $1 < p < \infty$, and $w \in A_p$,

$$\|f|W_p^k(\mathbb{R}^n; w)\| \sim \left\| \left(\sum_{k=0}^{\infty} 2^{2jk} |\mathcal{F}^{-1}(\varphi_k \widehat{f})(x)|^2 \right)^{1/2} \right\|_{L_p(w)}.$$

This holds even for a bigger class of the so called local A_p weights (see [23]) (one requires the condition (1.3) only for small cubes).

In Section 4 we also make use of weighted Sobolev spaces of negative order. It well-known that for $1 < p < \infty$ and $w \in A_p$, the dual space of $L_p(w)$ is given by $L_{p'}(w')$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and $w' = w^{-\frac{1}{p-1}} \in A_{p'}$. Accordingly, for a positive integer k we define

$$W_q^{-k}(\mathbb{R}^n; w) := \left(W_{q'}^k(\mathbb{R}^n; w') \right)'.$$

For more details about weighted spaces of negative order we refer to [23].

To avoid technicalities we shall not deal with the case of Lipschitz domains and we will concentrate on the basic case of a Sobolev space on $\mathbb{R}^n$ and a trace on the boundary of a half-space $\mathbb{R}^n_+$.

Our main result is:

THEOREM 1.1 *Let $\alpha \in (-(n-1), (q-1)(n-1))$. Then*

$$\mathrm{tr}_{\mathbb{R}^{n-1}} W_q^1(\mathbb{R}^n_+; |x|^\alpha) = B_{qq}^{1-\frac{1}{q}}(\mathbb{R}^{n-1}; |x'|^\alpha).$$

For the precise definition of the function spaces we refer to Section 2 below.

The structure of the paper is as follows: In Section 2 we prove some preliminary results concerning weighted spaces. Then in Section 3 the proof of the Theorem 1.1 for $\alpha > 0$ is given, based on a suitable estimate of the solution operator to a Dirichlet boundary value problem. Finally, in Section 4 the case $\alpha < 0$ is proved by a duality argument.

2 Preliminary results on weighted function spaces

By Garcia-Cuerva and Rubio de Francia [12], Theorem 3.9. the following weighted version of the Hörmander-Mikhlin multiplier theorem holds.

THEOREM 2.1 *Let $m \in C^n(\mathbb{R}^n \setminus \{0\})$ fulfill the property*

$$|\partial^\alpha m(\xi)| \leq K|\xi|^{-|\alpha|}, \quad \textit{for every} \ \ \xi \in \mathbb{R}^n \setminus \{0\}, \quad |\alpha| = 0, 1, ..., n,$$

for some constant $K > 0$. Then T defined by

$$\widehat{Tf} = m\widehat{f} \qquad \textit{for} \ \ f \in \mathcal{S}(\mathbb{R}^n)$$

extends to a continuous operator on $L^q_w(\mathbb{R}^n)$ for every $q \in (1,\infty)$ and $w \in A_q$.

In [12] this theorem is stated for an even larger class of multipliers m. The assertion on the operator norm is not mentioned explicitly, but it follows from the same proof.

Moreover, recall that a smooth function $p\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is in the pseudodifferential symbol class $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, if and only if for every $\alpha, \beta \in \mathbb{N}_0^n$ there is a constant $C_{\alpha,\beta}$ such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}$$

uniformly in $x, \xi \in \mathbb{R}^n$, where $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$. Moreover, $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is a Fréchet space e.g. with respect to the semi-norms

$$|p|^{(N)}_{S^m_{1,0}} = \sup_{|\alpha|+|\beta|\leq N} \sup_{x,\xi\in\mathbb{R}^n} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)|\langle\xi\rangle^{-m+|\alpha|},$$

cf. e.g. [17, 27]. It is well-known that $\langle\xi\rangle^m \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, i.e., for every $\alpha \in \mathbb{N}_0^n$

$$|\partial_\xi^\alpha \langle\xi\rangle^m| \leq C_\alpha \langle\xi\rangle^{m-|\alpha|} \tag{2.1}$$

uniformly in $\xi \in \mathbb{R}^n$. This can e.g. be proved by using the fact that $f(a,\xi) := |(a,\xi)|^m$, $(a,\xi) \in \mathbb{R}^{n+1} \setminus \{0\}$, is a smooth and homogeneous function of degree m.

For $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ the associated pseuododifferential operator is defined by

$$p(x,D_x)f = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\widehat{f}(\xi)\frac{d\xi}{(2\pi)^n}, \qquad f \in \mathcal{S}(\mathbb{R}^n), \tag{2.2}$$

where $\widehat{f} = \mathcal{F}[f](\xi)$ and $D_x = \frac{1}{i}\partial_x$. Then $p(x,D_x)$ can be extended to a bounded operator on weighted Bessel potential spaces by the following result due to Marschall [19, Theorem 1]:

THEOREM 2.2 *Let $1 < q < \infty$, $s \in \mathbb{R}$, $w \in A_q$, and let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$. Then $p(x,D_x)$ defined as above extends to a bounded linear operator $p(x,D_x)\colon H^{s+m}_q(\mathbb{R}^n;w) \to H^s_q(\mathbb{R}^n;w)$. Moreover, there exists $N = N(s,m,n,q) \in \mathbb{N}_0$ and $C = C(s,m,n,q) > 0$ such that*

$$\|p(x,D_x)|\mathcal{L}(H^{s+m}_q(\mathbb{R}^n;w), H^s_q(\mathbb{R}^n;w))\| \leq C|p|^{(N)}_{S^m_{1,0}}$$

uniformly in $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$.

Proof: The first part follows directly from [19, Theorem 1]. The second part follows easily from the linearity of the mapping $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \ni p \mapsto p(x,D_x) \in \mathcal{L}(H^{s+m}_q(\mathbb{R}^n;w), H^s_q(\mathbb{R}^n;w))$ and the fact that the mapping is bounded, which can be easily checked by observing that all constants in the proof of [19, Theorem 1] only depend on some semi-norm $|p|^{(N)}_{S^m_{1,0}}$ with a sufficiently large $N \in \mathbb{N}_0$. ■

Let $\omega \in A_q(\mathbb{R}^n)$, let φ_j, $j \in \mathbb{N}_0$, be a dyadic decomposition of unity as in the introduction and let $s \in \mathbb{R}$, $1 \leq p,q \leq \infty$. Note that φ_j, $j \in \mathbb{N}_0$, can be chosen such that $\varphi_j(\xi) = (\varphi_1(2^{-j+1}\xi)$ for all $j \geq 1$. In particular, this implies

$$|\partial^\alpha_\xi \varphi_j(\xi)| \leq C_\alpha 2^{-j|\alpha|} \tag{2.3}$$

uniformly in $j \in \mathbb{N}_0$ and for all $\alpha \in \mathbb{N}_0$.

With the notation as in (2.2) we can define weighted Besov space by

$$B^s_{pq}(\mathbb{R}^n;\omega) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^s_{pq}(\mathbb{R}^n;\omega)} < \infty \right\},$$

$$\|f\|_{B^s_{pq}(\mathbb{R}^n;\omega)} = \left(\sum_{j=0}^\infty 2^{sqj} \|\varphi_j(D_x)f\|^q_{L^p(\mathbb{R}^n;\omega)} \right)^{1/q}$$

with the obvious modifications if $q=\infty$. We note that $B^s_{pq}(\mathbb{R}^n;\omega)$ is a retract of $\ell^s_q(\mathbb{N}_0;L^p(\mathbb{R}^n;\omega))$, where

$$\begin{aligned}
\ell^s_q(\mathbb{N}_0;X) &= \left\{(a_j)_{j\in\mathbb{N}_0}\in X^{\mathbb{N}_0} : \|(a_j)_{j\in\mathbb{N}_0}\|_{\ell^s_q(\mathbb{N}_0;X)}<\infty\right\},\\
\|(a_j)_{j\in\mathbb{N}_0}\|_{\ell^s_q(\mathbb{N}_0;X)} &= \left(\sum_{j=0}^\infty 2^{sjq}\|a_k\|_X^q\right)^{\frac{1}{q}} \quad \text{if } q<\infty,\\
\|(a_j)_{j\in\mathbb{N}_0}\|_{\ell^s_\infty(\mathbb{N}_0;X)} &= \sup_{j\in\mathbb{N}_0} 2^{sj}\|a_j\|_X.
\end{aligned}$$

More precisely, the retractions and coretractions are given by

$$\begin{aligned}
R\colon \ell^s_q(\mathbb{N}_0;L^p(\mathbb{R}^n;\omega)) &\to B^s_{pq}(\mathbb{R}^n;\omega), & R((a_j)_{j\in\mathbb{N}_0}) &= \sum_{j=0}^\infty \psi_j(D_x)a_j,\\
S\colon B^s_{pq}(\mathbb{R}^n;\omega) &\to \ell^s_q(\mathbb{N}_0;L^p(\mathbb{R}^n;\omega)), & Sf &= (\varphi_j(D_x)f)_{j\in\mathbb{N}_0},
\end{aligned}$$

where $\psi_j(\xi)=\varphi_{j-1}(\xi)+\varphi_j(\xi)+\varphi_{j+1}(\xi)$, $j\in\mathbb{N}_0$, and $\varphi_{-1}(\xi)\equiv 0$.

2.1 Interpolation of weighted Besov spaces

Lemma 2.3 *Let $1<q<\infty$, $s\in\mathbb{R}$, and let $\omega\in A_q(\mathbb{R}^n)$. Then*

$$B^s_{q1}(\mathbb{R}^n;\omega)\hookrightarrow H^s_q(\mathbb{R}^n;\omega)\hookrightarrow B^s_{q\infty}(\mathbb{R}^n;\omega).$$

Proof: First of all,

$$|\partial^\alpha_\xi(\langle\xi\rangle^s\varphi_j(\xi))|\le C_{\alpha,s}2^{sj}\langle\xi\rangle^{-|\alpha|}$$

for all $\alpha\in\mathbb{N}_0^n$, $s\in\mathbb{R}$, because of (2.1), (2.3), and since $c2^j\le|\xi|\le C2^j$ on $\operatorname{supp}\varphi_j$. Hence

$$\|\langle D_x\rangle^s\varphi_j(D_x)f\|_{L^q(\mathbb{R}^n;\omega)}\le C_s2^{sj}\|f\|_{L^q(\mathbb{R}^n;\omega)} \tag{2.4}$$

by the Mikhlin multiplier theorem for weighted L^q-spaces, Theorem 2.1, or Theorem 2.2 with C_s independent of $j\in\mathbb{N}_0$. Since $\varphi_j(D_x)f=\varphi_j(D_x)(\varphi_{j-1}(D_x)f+\varphi_j(D_x)f+\varphi_{j+1}(D_x)f)$,

$$\|\langle D_x\rangle^s\varphi_j(D_x)f\|_{L^q(\mathbb{R}^n;\omega)}\le C_s2^{sj}\|\varphi_{j-1}(D_x)f+\varphi_j(D_x)f+\varphi_{j+1}(D_x)f\|_{L^q(\mathbb{R}^n;\omega)}$$

Therefore

$$\begin{aligned}
\|f\|_{H^s_q(\mathbb{R}^n;\omega)} &\le \sum_{j=0}^\infty \|\langle D_x\rangle^s\varphi_j(D_x)f\|_{L^q(\mathbb{R}^n;\omega)}\\
&\le C\sum_{j=0}^\infty 2^{sj}\|\varphi_{j-1}(D_x)f+\varphi_j(D_x)f+\varphi_{j+1}(D_x)f\|_{L^q(\mathbb{R}^n;\omega)}\\
&\le C\|f\|_{B^s_{q1}(\mathbb{R}^n;\omega)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}\|f\|_{B^s_{q\infty}(\mathbb{R}^n;\omega)} &= \sup_{j\in\mathbb{N}_0} 2^{sj}\|\langle D_x\rangle^{-s}\varphi_j(D_x)\langle D_x\rangle^s f\|_{L^q(\mathbb{R}^n;\omega)}\\ &\le C\|\langle D_x\rangle^s f\|_{L^q(\mathbb{R}^n;\omega)} = C\|f\|_{H^s_q(\mathbb{R}^n;\omega)}\end{aligned}$$

by (2.4), which finishes the proof. ■

Lemma 2.4 *Let $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, $1 < p < \infty$, $1 \le q, q_0, q_1 \le \infty$, $\theta \in (0,1)$, and let $s = (1-\theta)s_0 + \theta s_1$. Then*

$$(B^{s_0}_{pq_0}(\mathbb{R}^n;\omega), B^{s_1}_{pq_1}(\mathbb{R}^n;\omega))_{\theta,q} = B^s_{pq}(\mathbb{R}^n;\omega)$$

for any weight function $\omega \in A_p$.

Proof: Use that $B^{s_j}_{pq_j}(\mathbb{R}^n;\omega)$ is a retract of $\ell^{s_j}_{q_j}(\mathbb{N}_0;L^p(\mathbb{R}^n;\omega))$ and apply [5, Theorem 5.6.1]. ■

Corollary 2.5 *Let $1 < q < \infty$, $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, $\theta \in (0,1)$ and let $s = (1-\theta)s_0 + \theta s_1$ and let $\omega \in A_q(\mathbb{R}^n)$. Then*

$$(H^{s_0}_q(\mathbb{R}^n;\omega), H^{s_1}_q(\mathbb{R}^n;\omega))_{\theta,q} = B^s_{qq}(\mathbb{R}^n;\omega).$$

Proof: The corollary follows directly from Lemma 2.3 and Lemma 2.4. ■

Corollary 2.6 *Let $1 < q < \infty$ and let $\omega = \omega(x') \in A_q(\mathbb{R}^{n-1})$. Then*

$$\operatorname{tr}_{\mathbb{R}^{n-1}} W^1_q(\mathbb{R}^n_+;\omega) = (L^q(\mathbb{R}^{n-1};\omega), W^1_q(\mathbb{R}^{n-1};\omega))_{1-\frac{1}{q},q} = B^{1-\frac{1}{q}}_{qq}(\mathbb{R}^{n-1};\omega)$$

Proof: The first equality follows from

$$W^1_q(\mathbb{R}^n_+;\omega) = L^q(\mathbb{R}_+;W^1_q(\mathbb{R}^{n-1};\omega)) \cap W^1_q(\mathbb{R}_+;L^q(\mathbb{R}^{n-1};\omega))$$

and Lions' trace method for real interpolation, cf. [5, Corollary 3.12.3] or apply [3, Chapter III, Corollary 4.10.2]. The second equality follows from the previous corollary and the fact that $W^1_q(\mathbb{R}^n;\omega) = H^1_q(\mathbb{R}^n;\omega) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \langle D_x\rangle f \in L^q(\mathbb{R}^n;\omega)\}$, cf. Fröhlich [10, Lemma 3.1] or [9]. ■

2.2 An Embedding for $L^q(\mathbb{R}^n; |x|^\alpha)$

Let $(M, \mathcal{B}, \mu)$ be a measure space and let $L^{p,\infty}$, $1 \leq p < \infty$, be the corresponding *weak L^p-space* (the *Marcinkiewicz space*) as e.g. defined in [5, Section 1.3].

Lemma 2.7 *Let $1 \leq p, p_1, p_2 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then there exists $C > 0$ such that*

$$\|fg\|_{L^{p,\infty}(M,\mu)} \leq C\|f\|_{L^{p_1,\infty}(M,\mu)}\|g\|_{L^{p_2,\infty}(M,\mu)}.$$

Proof: Since the mapping $(f,g) \mapsto fg$ is bilinear, it is sufficient to consider the case $\|f\|_{L^{p_1,\infty}(M,\mu)}, \|g\|_{L^{p_2,\infty}(M,\mu)} \leq 1$. Let $\alpha = \frac{p_1}{p_2}$. Then we either have $|f(x)| \geq |g(x)|^\alpha$ or $|f(x)| < |g(x)|^\alpha$. Hence

$$\begin{aligned}\mu\left(\{x : |f(x)g(x)| \geq \lambda\}\right) &\leq \mu\left(\{x : |f(x)|^{1+\alpha} \geq \lambda\}\right) + \mu\left(\{|g(x)|^{1+\frac{1}{\alpha}} \geq \lambda\}\right)\\ &\leq \lambda^{-\frac{p_1}{1+\alpha}} + \lambda^{-\frac{\alpha p_2}{1+\alpha}} = 2\lambda^{-p}\end{aligned}$$

for every $\lambda > 0$, which finishes the proof. ■

Corollary 2.8 *Let $1 < q < \infty$ and let $0 \leq \alpha < (q-1)n$. Then*

$$L^q(\mathbb{R}^n; |x|^\alpha) \hookrightarrow L^{r,\infty}(\mathbb{R}^n) \qquad \textit{where } \frac{1}{r} = \frac{1}{q} + \frac{\alpha}{qn}. \tag{2.5}$$

Proof: Let $p = \frac{qn}{\alpha}$. Then $|x|^{-\frac{\alpha}{q}} \in L^{p,\infty}(\mathbb{R}^n)$ and therefore

$$\|f\|_{L^{r,\infty}(\mathbb{R}^n)} \leq \||x|^{-\frac{\alpha}{q}}\|_{L^{p,\infty}(\mathbb{R}^n)}\||x|^{\frac{\alpha}{q}}\|_{L^{q,\infty}(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n;|x|^\alpha)}.$$

■

For the following we denote

$$B^s_{pq,(r)}(\mathbb{R}^n, \omega) = \left\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B^s_{pq,(r)}(\mathbb{R}^n;\omega)\| < \infty\right\},$$

$$\|f|B^s_{pq,(r)}(\mathbb{R}^n;\omega)\| = \left(\sum_{j=0}^\infty \|\varphi_j(D_x)f|L^{p,r}(\mathbb{R}^n;\omega)\|^q\right)^{1/q}$$

with the obvious modification if $q = \infty$, where $1 \leq p, q, r \leq \infty$ and $s \in \mathbb{R}$, cf. [28, Section 2.4.1]. We need the following simple lemma.

Lemma 2.9 *Let $s_0, s_1 \in \mathbb{R}$, $1 \le q_0, q_1, r_0, r_1 \le \infty$, $q_0 \ne q_1$, $\theta \in (0,1)$, and let $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then*

$$(B^{s_0}_{q_0q_0,(r_0)}(\mathbb{R}^n;\omega), B^{s_1}_{q_1q_1,(r_1)}(\mathbb{R}^n;\omega))_{\theta,q} = B^s_{qq}(\mathbb{R}^n;\omega).$$

Proof: First of all, we note that $B^{s_j}_{q_jq_j,(r_j)}(\mathbb{R}^n;\omega)$ and $B^s_{qq}(\mathbb{R}^n;\omega)$ are retracts of $\ell^{s_j}_{q_j}(\mathbb{N}_0;L^{q_j,r_j}(\mathbb{R}^n;\omega))$, $\ell^s_q(\mathbb{N}_0;L^q(\mathbb{R}^n;\omega))$, resp., with respect to the same retraction mappings. Hence the statement follows from

$$\begin{aligned} &\left(\ell^{s_0}_{q_0}(\mathbb{N}_0;L^{q_0,r_0}(\mathbb{R}^n;\omega)), \ell^{s_1}_{q_1}(\mathbb{N}_0;L^{q_1,r_1}(\mathbb{R}^n;\omega))\right)_{\theta,q} \\ = \;& \ell^s_q(\mathbb{N}_0;(L^{q_0,r_0}(\mathbb{R}^n;\omega), L^{q_1,r_1}(\mathbb{R}^n;\omega))_{\theta,q}) = \ell^s_q(\mathbb{N}_0;L^q(\mathbb{R}^n;\omega)) \end{aligned}$$

where we have used [5, Theorem 5.6.2] and [5, Theorem 5.3.1]. ∎

The following theorem is a key result for the proof of Theorem 1.1.

THEOREM 2.10 *Let $s \in \mathbb{R}$, $1 < q < \infty$, and let $0 < \alpha < (q-1)n$. Then*

$$B^{s+\frac{\alpha}{q}}_{qq}(\mathbb{R}^n;|x|^\alpha) \hookrightarrow B^s_{qq}(\mathbb{R}^n) \cap H^s_q(\mathbb{R}^n). \tag{2.6}$$

Proof: By Corollary 2.8 $L^q(\mathbb{R}^n;|x|^\alpha) \hookrightarrow L^{r,\infty}(\mathbb{R}^n)$ for all $0 < \alpha < (q-1)n$ and $\frac{1}{r} = \frac{1}{q} + \frac{\alpha}{qn}$. Using the generalized Marcinkiewicz interpolation theorem, cf. [5, Theorem 5.3.2] for different values of q yields

$$L^{q,r}(\mathbb{R}^n;|x|^\alpha) \hookrightarrow L^r(\mathbb{R}^n) \qquad \text{where } \frac{1}{r} = \frac{1}{q} + \frac{\alpha}{qn} \tag{2.7}$$

for all $0 < \alpha < (q-1)n$. Hence for all $0 < \alpha < (q-1)n$ and $\frac{1}{r} = \frac{1}{q} + \frac{\alpha}{qn}$

$$B^{s+\frac{\alpha}{q}}_{qq,(r)}(\mathbb{R}^n;|x|^\alpha) \hookrightarrow B^{s+\frac{\alpha}{q}}_{rq}(\mathbb{R}^n) \hookrightarrow B^s_{qq}(\mathbb{R}^n) \cap H^s_q(\mathbb{R}^n)$$

due to [28, Section 2.8.2, Equation (2) and (18)]. Hence using Lemma 2.9 for $B^{s+\frac{\alpha}{q}}_{qq,(r)}(\mathbb{R}^n;|x|^\alpha)$ with different values of q together with

$$(B^s_{q_0q_0}(\mathbb{R}^n), B^s_{q_1q_1}(\mathbb{R}^n))_{\theta,q} = B^s_{qq}(\mathbb{R}^n),\;\; (H^s_{q_0}(\mathbb{R}^n), H^s_{q_1}(\mathbb{R}^n))_{\theta,q} = H^s_q(\mathbb{R}^n), \tag{2.8}$$

where $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, cf. [28, Theorem 2.4.1] and [5, Theorem 6.4.5], we conclude (2.6). ∎

3 Proof for positive α

If $0 < \alpha < (q-1)(n-1)$, then $|x|^\alpha \geq |x'|^\alpha$ and therefore

$$\operatorname{tr} W^1_q(\mathbb{R}^n_+; |x|^\alpha) \subseteq \operatorname{tr} W^1_q(\mathbb{R}^n_+; |x'|^\alpha) = B^{1-\frac{1}{q}}_{qq}(\mathbb{R}^{n-1}; |x'|^\alpha) \tag{3.1}$$

by Corollary 2.6. Hence it remains to prove the converse inclusion. To this end we use the following extension operator: We denote by $u = K_D a$, $a \in \mathcal{S}(\mathbb{R}^{n-1})$, the solution of

$$\begin{aligned} (1-\Delta)u &= 0 \qquad \text{in } \mathbb{R}^n_+\,, \\ u|_{\partial\mathbb{R}^n_+} &= a \qquad \text{on } \mathbb{R}^{n-1}. \end{aligned}$$

Using partial Fourier transformation $\tilde{a}(\xi') = \mathcal{F}_{x'\mapsto\xi'}[a](\xi')$ the solution $u = K_D a$ can be easily calculated as

$$u(x', x_n) = K_D a = \mathcal{F}^{-1}_{\xi'\mapsto x'}\left[e^{-\langle\xi'\rangle x_n}\tilde{a}(\xi')\right], \qquad x = (x', x_n) \in \mathbb{R}^n_+.$$

Note that $\langle\xi'\rangle = (1+|\xi'|^2)^{\frac{1}{2}}$ as above. It is well known that the *symbol-kernel* $\tilde{k}(\xi', x_n) := e^{-\langle\xi'\rangle x_n}$ satisfies the following estimate

$$\sup_{x_n\geq 0} |x_n^s \partial^l_{x_n} \partial^\alpha_{\xi'} \tilde{k}(\xi', x_n)| \leq C_{\alpha,s,l} \langle\xi'\rangle^{l-s-|\alpha|} \tag{3.2}$$

uniformly in $\xi' \in \mathbb{R}^{n-1}$ and for all $\alpha \in \mathbb{N}_0^{n-1}$, $s \geq 0$, $l \in \mathbb{N}_0$, see e.g. [1, Lemma 2.9]. Using the latter estimate we show

Lemma 3.1 *Let $1 < q < \infty$, let $s \geq 0$, and let $\omega \in A_q(\mathbb{R}^{n-1})$. Then $x_n^s\,(\nabla K_D, K_D)$ extends to a bounded operator*

$$x_n^s \begin{pmatrix} \nabla K_D \\ K_D \end{pmatrix} : B^{1-\frac{1}{q}}_{qq}(\mathbb{R}^{n-1}; \omega) \to L^q(\mathbb{R}_+; B^s_{qq}(\mathbb{R}^{n-1}; \omega) \cap H^s_q(\mathbb{R}^{n-1}; \omega)).$$

Proof: First of all,

$$\begin{pmatrix} \nabla K_D \\ K_D \end{pmatrix} a = \mathcal{F}^{-1}_{\xi'\mapsto x'}\left[\begin{pmatrix} i\xi' \\ -\langle\xi'\rangle \\ 1 \end{pmatrix} e^{-\langle\xi'\rangle x_n}\tilde{a}(\xi')\right] \equiv \mathcal{F}^{-1}_{\xi'\mapsto x'}\left[\tilde{k}'(\xi', x_n)\tilde{a}(\xi')\right].$$

Here $\tilde{k}'(\xi', x_n)$ satisfies

$$|\partial^\alpha_{\xi'} \tilde{k}'(\xi', x_n)| \leq C_{\alpha,s,l} \langle\xi'\rangle^{1-s-|\alpha|} |x_n|^{-s} \tag{3.3}$$

uniformly in $\xi' \in \mathbb{R}^{n-1}$, $x_n > 0$, and for all $\alpha \in \mathbb{N}_0^{n-1}$, $s \geq 0$, $l \in \mathbb{N}_0$, by virtue of (3.2), (2.1), and the product rule. Hence for every $x_n > 0$ $\tilde{k}'(\xi', x_n) \in S^{1-s}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ is a pseudodifferential symbol with seminorms bounded by $C|x_n|^{-s}$. Hence

$$\|\nabla K_D a(., x_n)\|_{H_q^{s-\frac{1}{p}}(\mathbb{R}^{n-1};\omega)} \leq C|x_n|^{-s}\|a\|_{H_q^{1-\frac{1}{p}}(\mathbb{R}^{n-1};\omega)}$$

by Theorem 2.2. Replacing s by $s + \frac{1}{p}$ we conclude

$$\|x_n^s \nabla K_D a(., x_n)\|_{H_q^s(\mathbb{R}^{n-1};\omega)} \leq C|x_n|^{-\frac{1}{p}}\|a\|_{H_q^{1-\frac{1}{p}}(\mathbb{R}^{n-1};\omega)}.$$

Since $\|f\|_{L^{p,\infty}(\mathbb{R}_+)} \leq \|t^{-\frac{1}{p}}\|_{L^{p,\infty}}\|f\|_{L^\infty} \leq C\sup_{t>0} t^{\frac{1}{p}}|f(t)|$, we get

$$\|x_n^s \nabla K_D a(., x_n)\|_{L^{p,\infty}(\mathbb{R}_+;H_q^s(\mathbb{R}^{n-1};\omega))} \leq C\|a\|_{H_q^{1-\frac{1}{p}}(\mathbb{R}^{n-1};\omega)}.$$

Using real interpolation for different values of p and setting $p = q$ afterwards, we conclude

$$x_n^s \nabla K_D \colon B_{qq}^{1-\frac{1}{q}}(\mathbb{R}^{n-1};\omega) \to L^q(\mathbb{R}_+; H_q^s(\mathbb{R}^{n-1};\omega)),$$

where we have used Corollary 2.5, [28, Section 1.18.6, Theorem 2], and (2.8). One more real interpolation with different values of q yields

$$x_n^s \nabla K_D \colon B_{qq}^{1-\frac{1}{q}}(\mathbb{R}^{n-1};\omega) \to L^q(\mathbb{R}_+; B_{qq}^s(\mathbb{R}^{n-1};\omega)),$$

which completes the proof. ■

Proof of Theorem 1.1, case $\alpha > 0$: Using Lemma 3.1 with $s = 0$, we conclude $K_D \colon B_{qq}^{1-\frac{1}{q}}(\mathbb{R}^{n-1}, |x'|^\alpha) \to W_q^1(\mathbb{R}^n_+; |x'|^\alpha)$. Moreover, applying Lemma 3.1 with $s = \frac{\alpha}{q}$, we conclude

$$x_n^{\frac{\alpha}{q}} \begin{pmatrix} \nabla K_D \\ K_D \end{pmatrix} : B_{qq}^{1-\frac{1}{q}}(\mathbb{R}^{n-1}, |x'|^\alpha) \to L^q(\mathbb{R}_+; B_{qq}^{\frac{\alpha}{q}}(\mathbb{R}^{n-1}; |x'|^\alpha)) \hookrightarrow L^q(\mathbb{R}^n_+),$$

where we have also used Theorem 2.10. Hence $K_D \colon B_{qq}^{1-\frac{1}{q}}(\mathbb{R}^{n-1}, |x'|^\alpha) \to W_q^1(\mathbb{R}^n_+; x_n^\alpha)$. Since $|x|^\alpha \leq C_\alpha\left(|x'|^\alpha + x_n^\alpha\right)$, the result for $\alpha > 0$ follows. ■

4 The result for negative α

The case $\alpha < 0$ is derived from the case $\alpha \geq 0$ by a duality argument. More precisely, we use the following abstract lemma.

Lemma 4.1 *Let $\omega_1, \omega_2 \in A_q$ be given such that one has $\operatorname{tr}_{\mathbb{R}^{n-1}} W_q^1(\mathbb{R}^n_+;\omega_1) = \operatorname{tr}_{\mathbb{R}^{n-1}} W_q^1(\mathbb{R}^n_+;\omega_2)$ with equivalent norms. Then*

$$\operatorname{tr}_{\mathbb{R}^{n-1}} W_{q'}^1(\mathbb{R}^n_+;\omega_1') = \operatorname{tr}_{\mathbb{R}^{n-1}} W_{q'}^1(\mathbb{R}^n_+;\omega_2') \tag{4.1}$$

with equivalent norms, where $\frac{1}{q}+\frac{1}{q'}$ and $\omega_j' = \omega_j^{-\frac{1}{q-1}}$.

Proof: Let $g \in \left(\operatorname{tr}_{\mathbb{R}^{n-1}} W_{q'}^1(\mathbb{R}^n_+;\omega_1')\right)'$ then

$$G := [\phi \mapsto \langle g, \phi\rangle_{\mathbb{R}^{n-1}}] \in \left(W_{q'}^1(\mathbb{R}^n;\omega_1')\right)' = W_q^{-1}(\mathbb{R}^n;\omega_1)$$

with $\|g\|_{\left(\operatorname{tr}_{\mathbb{R}^{n-1}} W_{q'}^1(\mathbb{R}^n_+;\omega_1')\right)'} = \|G\|_{W_q^{-1}(\mathbb{R}^n;\omega_1)}$.

By [9] one has that $(1-\Delta) : W_q^1(\mathbb{R}^n;\omega_j) \to W_p^{-1}(\mathbb{R}^n;\omega_j)$, $j = 1,2$ is an isomorphism. Since G has its support in $\mathbb{R}^{n-1}$, it follows that $(1-\Delta)^{-1}G$ is a weak solution to the boundary value problem

$$(1-\Delta)(1-\Delta)^{-1}G = 0 \quad \text{on } \Omega \quad \text{and} \quad (1-\Delta)^{-1}G \in \operatorname{tr}_{\mathbb{R}^{n-1}} W_q^1(\mathbb{R}^n_+;\omega_1),$$

where $\Omega = \mathbb{R}^n_+$ or $\Omega = \mathbb{R}^n_-$. By the a priori estimate of this boundary value problem in [9] one has

$$\|(1-\Delta)^{-1}G\|_{W_q^1(\mathbb{R}^n;\omega_1)} \leq c\|\operatorname{tr}_{\mathbb{R}^{n-1}}(1-\Delta)^{-1}G\|_{\operatorname{tr}_{\mathbb{R}^{n-1}} W_q^1(\mathbb{R}^n_+;\omega_1)}.$$

Thus we may estimate using the assumption (4.1)

$$\begin{aligned}
\|g\|_{\left(\operatorname{tr}_{\mathbb{R}^{n-1}} W_{q'}^1(\mathbb{R}^n_+;\omega_1')\right)'} &= \|G\|_{W_q^{-1}(\mathbb{R}^n;\omega_1)} \leq c\|(1-\Delta)^{-1}G\|_{W_q^1(\mathbb{R}^n;\omega_1)} \\
&\leq c\|\operatorname{tr}_{\mathbb{R}^{n-1}}(1-\Delta)^{-1}G\|_{\operatorname{tr}_{\mathbb{R}^{n-1}} W_q^1(\mathbb{R}^n;\omega_1)} \\
&\leq c\|\operatorname{tr}_{\mathbb{R}^{n-1}}(1-\Delta)^{-1}G\|_{\operatorname{tr}_{\mathbb{R}^{n-1}} W_q^1(\mathbb{R}^n;\omega_2)} \\
&\leq c\|(1-\Delta)^{-1}G\|_{W_q^1(\mathbb{R}^n;\omega_2)} \\
&\leq c\|G\|_{W_q^{-1}(\mathbb{R}^n;\omega_2)} = c\|g\|_{\left(\operatorname{tr}_{\mathbb{R}^{n-1}} W_{q'}^1(\mathbb{R}^n_+;\omega_2')\right)'}.
\end{aligned}$$

Interchanging the roles of ω_1 and ω_2, we obtain the reverse estimate

$$\|g\|_{\left(\operatorname{tr}_{\mathbb{R}^{n-1}} W_{q'}^1(\mathbb{R}^n_+;\omega_2')\right)'} \leq c\|g\|_{\left(\operatorname{tr}_{\mathbb{R}^{n-1}} W_{q'}^1(\mathbb{R}^n_+;\omega_1')\right)'}$$

for some $c > 0$. We have shown that

$$\left(\operatorname{tr}_{\mathbb{R}^{n-1}} W^1_{q'}(\mathbb{R}^n_+;\omega_2')\right)' = \left(\operatorname{tr}_{\mathbb{R}^{n-1}} W^1_{q'}(\mathbb{R}^n_+;\omega_1')\right)'.$$

Thus, since $\operatorname{tr}_{\mathbb{R}^{n-1}} W^1_{q'}(\mathbb{R}^n_+;\omega_j')$, $j = 1,2$, is a factor space of the reflexive Banach space $W^1_{q'}(\mathbb{R}^n_+;\omega_2')$ with respect to a closed subspace, it is reflexive and we obtain

$$\operatorname{tr}_{\mathbb{R}^{n-1}} W^1_{q'}(\mathbb{R}^n_+;\omega_1') = \operatorname{tr}_{\mathbb{R}^{n-1}} W^1_{q'}(\mathbb{R}^n_+;\omega_2')$$

as asserted. ■

Corollary 4.2 *Let $-(n-1) < \alpha < 0$. Then*

$$\operatorname{tr}_{\mathbb{R}^{n-1}} W^1_q(\mathbb{R}^n_+;|x|^\alpha) = B^{1-\frac{1}{q}}_{qq}(\mathbb{R}^{n-1};|x'|^\alpha).$$

Proof: Let $\beta := -\frac{\alpha}{q-1} = -(q'-1)\alpha$. Then $0 < \beta < (n-1)(q'-1)$. Thus we are in the range of indices that have already been considered and one obtains from the results of Section 3 that

$$\operatorname{tr}_{\mathbb{R}^{n-1}} W^1_{q'}(\mathbb{R}^n_+;|x|^\beta) = B^{1-\frac{1}{q'}}_{q'q'}(\mathbb{R}^{n-1};|x'|^\beta) = \operatorname{tr}_{\mathbb{R}^{n-1}} W^1_{q'}(\mathbb{R}^n_+;|x'|^\beta),$$

using Corollary 2.6 and $|x'|^\beta \in A_{q'}(\mathbb{R}^{n-1})$. Thus by Lemma 4.1 we obtain

$$\operatorname{tr}_{\mathbb{R}^{n-1}} W^1_q(\mathbb{R}^n_+;|x|^\alpha) = \operatorname{tr}_{\mathbb{R}^{n-1}} W^1_q(\mathbb{R}^n_+;|x'|^\alpha) = B^{1-\frac{1}{q}}_{qq}(\mathbb{R}^{n-1};|x'|^\beta),$$

where we again applied Corollary 2.6. This finishes the proof of Theorem 1.1. ■

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