Symmetric Cauchy stresses do not imply symmetric Biot strains in weak formulations of isotropic hyperelasticity with rotational degrees of freedom.

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Abstract

We show that symmetric Cauchy stresses do not imply symmetric Biot strains in weak formulations of finite isotropic hyperelasticity with exact rotational degrees of freedom. This is contrary to claims in the literature which are valid, however, in the linear isotropic case.

Key words: rotational degree of freedom, hyperelasticity, isotropy

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1 Introduction

This article is motivated by the numerous contributions which propose to introduce rotational degrees of freedom in a classical finite elasticity context in order to improve the numerical approximation of classical solutions [16, 5, 17, 20, 2, 3, 8, 19]. We refer to the introductions in [2, 8] for the historical development of this specific approach to the numerics of classical finite elasticity[1] and the relevance it has, e.g. in the numerical simulation of thin structures [6, 23]. The general idea underlying the approach, is to approximate the classical formulation by a weak formulation in which rotational degrees of freedom (also called drilling degrees of freedom) appear as a dedicated numerical intermediary device. Hence, no physical meaning is ascribed to them, as opposed to e.g. in a Cosserat theory.

The introduction of rotational degrees of freedom gives, in general, rise to a possible asymmetry of the relaxed Biot stretches. In an anisotropic setting, therefore, it is necessary to augment the energetic formulation with a term penalizing this possible asymmetry [3, Eq.(2.9)] in order to still approximate classical solutions with symmetric Biot stretch tensor $U = \sqrt{F^T F}$.

However, in [3, p.26] it is claimed that this penalization is unnecessary in the case of finite isotropic hyperelasticity similar to the case of isotropic linear elasticity with infinitesimal rotations. The argument supporting this claim is based on the "observation" that the isotropic formulation of the moment equilibrium equation enforces **automatically** the symmetry of the relaxed Biot stretch. Furthermore, it is this automatism which adds to the attractiveness of the numerical proposal [3, Rem.2].

In this note we clarify that, contrary to the above claim, **penalization is necessary even in the isotropic case**, in order to compute approximately symmetric Biot stretches, i.e., to recover the classical situation.

The paper is organized as follows. Firstly, we recall the isotropic hyperelastic formulation of elasticity in the classical symmetric Biot stretch and derive the corresponding Euler-Lagrange equations. Then, we introduce the formulation with rotational degrees of freedom and establish various connections between solutions of the different models. Moreover, we exhibit the well-known relation of the relaxed model to a finite-strain Cosserat model without curvature energy, see e.g., the pseudo-polar continuum in [8, p.158].

By way of a counterexample we show then that symmetry of relaxed Biot stretch may not be obtained without sufficient penalization. It is noted that in a linearized, isotropic setting the former cannot happen: satisfaction of moment equilibrium (symmetric Cauchy-stresses) implies symmetric infinitesimal stretch in the presence of infinitesimal skew-symmetric degrees of freedom for isotropy [5, 6].

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[1]In the literature this approach is also commonly referred to as a **relaxation** of classical finite elasticity.

2 The classical finite strain isotropic Biot model

2.1 The finite strain isotropic Biot model in variational form

For simplicity we restrict the exposition throughout to zero body forces. In a variational framework, the task is to find a deformation $\varphi : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ minimizing the energy functional I,

$$I(\varphi) = \int_\Omega W(\nabla\varphi) \, \mathrm{dV} \mapsto \min. \text{ w.r.t. } \varphi \,, \tag{2.1}$$

together with the Dirichlet boundary condition of place for the deformation φ on some part Γ of the boundary $\partial\Omega$: $\varphi_{|\Gamma} = g_\mathrm{d}$. In the Biot approach, the special constitutive assumptions are

$$W(F) = W^\sharp(U) \,. \tag{2.2}$$

The strain energy W depends on the deformation gradient $F = \nabla\varphi \in \mathrm{GL}^+(3)$ only through the **objective symmetric continuum Biot stretch tensor** $U = R^T F = \sqrt{F^T F}$: $T_x\Omega \mapsto T_x\Omega$, where $R = \mathrm{polar}(F) : T_x\Omega \mapsto T_{\varphi(x)}\varphi(\Omega)$ is the orthogonal part of the polar decomposition of F, i.e., the continuum rotation and U is positive definite symmetric. It is well known that every objective free energy, i.e., $\forall\, Q \in \mathrm{SO}(3) : W(Q\,F) = W(F)$, can be expressed in this way by a function $W^\sharp$ defined on the classical stretch U alone, see e.g.,[7].

In the case of material isotropy, the free energy W should be right-invariant under the group of special rotations $\mathrm{SO}(3)$, i.e.,

$$\begin{aligned}
\forall\, Q \in \mathrm{SO}(3) : &\quad W(F\,Q) = W(F) \;\Leftrightarrow \\
\forall\, Q \in \mathrm{SO}(3) : &\quad W^\sharp(Q^T\,U\,Q) = W^\sharp(U) \,.
\end{aligned} \tag{2.3}$$

For example, the most general isotropic quadratic energy in U with zero stresses in the reference configuration is given by

$$W^\sharp(U) = \mu \, \|U - \mathbb{1}\|^2 + \frac{\lambda}{2} \, \mathrm{tr}\,[U - \mathbb{1}]^2 \,, \tag{2.4}$$

where the parameters $\mu, \lambda > 0$ are the Lamé constants of classical isotropic elasticity.

2.2 The Euler-Lagrange equations of the finite Biot model

The following considerations are facilitated by using the representation $U(F) = R(F)^T F = \mathrm{polar}(F)^T F$. Moreover, let $v \in C_0^\infty(\Omega, \mathbb{R}^3)$. Taking free variations w.r.t. φ in the energy

leads to

$$\frac{\mathrm{d}}{\mathrm{dt}}_{|t=0} I(\varphi + t\,v) = \int_\Omega \langle D_F W(\nabla\varphi), \nabla v \rangle \,\mathrm{dV} = \int_\Omega \langle D_F[W^\sharp(U(F))], \nabla v \rangle \,\mathrm{dV}$$

$$= \int_\Omega \langle D_U W^\sharp(U), D_F U(F).\nabla v \rangle \,\mathrm{dV} = \int_\Omega \langle D_U W^\sharp(U), D_F[R(F)^T\,F].\nabla v \rangle \,\mathrm{dV}$$

$$= \int_\Omega \langle D_U W^\sharp(U), [D_F R(F).\nabla v]^T\,F + R(F)^T\,\nabla v \rangle \,\mathrm{dV} \tag{2.5}$$

$$= \int_\Omega \langle D_U W^\sharp(U), [\delta R(F, \nabla v)]^T\,R(F)\,R(F)^T\,F] + R(F)^T \nabla v \rangle \,\mathrm{dV}$$

$$= \int_\Omega \langle R(F)\,D_U W^\sharp(U), \nabla v \rangle + \langle D_U W^\sharp(U), [\delta R(F, \nabla v)]^T\,R(F)R(F)^T F] \rangle \,\mathrm{dV}$$

$$= \int_\Omega \langle R(F)\,D_U W^\sharp(U), \nabla v \rangle + \langle D_U W^\sharp(U)\,U^T, [\delta R(F, \nabla v)]^T\,R(F)] \rangle \,\mathrm{dV} .$$

Now, we use that on the one hand $D_U W^\sharp(U)\,U^T$ is symmetric for isotropic $W^\sharp$ and that on the other hand $[\delta R(F, \nabla v)]^T\,R(F)$ is always skew-symmetric. This implies that the product between them vanishes. Therefore, we obtain

$$0 = \frac{\mathrm{d}}{\mathrm{dt}}_{|t=0} I(\varphi + t\,v) = \int_\Omega \langle R(F)\,D_U W^\sharp(U), \nabla v \rangle \,\mathrm{dV}$$

$$= \int_\Omega \langle \mathrm{Div}[R(F)\,D_U W^\sharp(U)], v \rangle \,\mathrm{dV}, \quad \forall\, v \in C_0^\infty(\Omega, \mathbb{R}^3), \tag{2.6}$$

where we have used the divergence theorem. Thus, the strong form of equlibrium for the classical Biot model reads

$$\mathrm{Div}\,S_1(F) = \mathrm{Div}[R(F)\,D_U W^\sharp(U)] = 0, \tag{2.7}$$

with the first Piola-Kirchhoff tensor $S_1(F) = R(F)\,D_U W^\sharp(U)$. Since the second Piola-Kirchhoff tensor is defined as $S_2(F) = F^{-1}\,S_1(F)$ it holds

$$S_2(F) = F^{-1}\,S_1(F) = F^{-1}\,R(F)\,D_U W^\sharp(U) = U^{-1}\,D_U W^\sharp(U) \in \mathrm{Sym} \tag{2.8}$$

from the fact that for isotropic $W^\sharp$, the tensors $D_U W^\sharp$ and U^{-1} commute and are each symmetric. If we define the Biot stress tensor by $T = D_U W(U)$, then the following relation between the Biot stresses (living on the reference configuration) and the Cauchy-stresses in the actual configuration holds

$$\sigma = \frac{1}{\det[F]} R\,T\,F^T = \frac{1}{\det[F]} F\,S_2(F)\,F^T \in \mathrm{Sym} . \tag{2.9}$$

We note that the classical Biot model is not known to be well-posed when (2.4) is used. In this case Legendre-Hadamard ellipticity is lost [1].

Using the polar decomposition we may write equivalently

$$\mathrm{Div}[R\,D_U W^\sharp(R^T\,F)] = 0, \quad R = \mathrm{polar}(F). \tag{2.10}$$

A weaker formulation is obtained by replacing the constraint $R = \mathrm{polar}(F)$ in (2.10) into

$$\mathrm{Div}[R\,D_U W^\sharp(R^T\,F)] = 0\,, \quad R^T\,F \in \mathrm{Sym}\,. \tag{2.11}$$

The difference between (2.11) and (2.10) is that in (2.10) the stretch $U = R^T F$ is not only symmetric, but also positive definite symmetric. In fact it holds

$$\forall\, R \in \mathrm{SO}(3)\,, F \in \mathrm{GL}^+(3):\quad R^T\,F \in \mathrm{Sym} \Leftrightarrow R = Q_i\,\mathrm{polar}(F)\,, \tag{2.12}$$

for

$$Q_1 = \mathbb{1}\,, \quad Q_2 = \mathrm{diag}(1,-1,-1)\,, \quad Q_3 = \mathrm{diag}(-1,1,-1)\,, \quad Q_4 = \mathrm{diag}(-1,-1,1)\,.$$

Thus every solution to (2.10) is a solution to (2.11) but not vice versa. Despite the difference between the formulations (2.10) and (2.11) it is (2.11) which is sought to be approximated by a formulation with rotational degrees of freedom which we introduce presently.

3 The Biot model with rotational degrees of freedom

The Biot model with rotational degrees of freedom is obtained by formally relaxing the constraint on the rotations R in the previous approach to coincide either with the polar-decomposition or to make $U = R^T F$ symmetric. Instead, one introduces an independent rotation field $\overline{R} : \Omega \mapsto \mathrm{SO}(3)$ and writes, cf. [3, 2.3]

$$I^{\mathrm{rel}}(\varphi,\overline{R}) = \int_\Omega W^\sharp(\overline{R}^T\nabla\varphi)\,\mathrm{dV} \mapsto \min.\ \mathrm{w.r.t.}\ (\varphi,\overline{R})\,, \tag{3.1}$$

taking free variations w.r.t φ and $\overline{R}$. Let us abbreviate $\overline{U} = \overline{R}^T F$, which is in general non-symmetric. Repeating the same steps as before leads us to the balance of forces equation

$$0 = \frac{\mathrm{d}}{\mathrm{dt}}_{|t=0} I^{\mathrm{rel}}(\varphi + t\,v,\overline{R}) = \int_\Omega \langle D_F[W^\sharp(\overline{R}^T\nabla\varphi)], \nabla v\rangle\,\mathrm{dV} = \int_\Omega \langle D_{\overline{U}}W^\sharp(\overline{R}^T\nabla\varphi), \overline{R}^T\,\nabla v\rangle\,\mathrm{dV}$$

$$= \int_\Omega \langle \overline{R}\,D_{\overline{U}}W^\sharp(\overline{U}), \nabla v\rangle\,\mathrm{dV} = \int_\Omega \langle \mathrm{Div}[\overline{R}\,D_{\overline{U}}W^\sharp(\overline{U}), v\rangle\,\mathrm{dV}\,, \ \forall\,v \in C_0^\infty(\Omega,\mathbb{R}^3)\,. \tag{3.2}$$

Free variation w.r.t. to the independent rotations $\overline{R}$ leads to an algebraic side condition. Since $\overline{R}^T\overline{R} = \mathbb{1}$ we have $\delta\overline{R}^T\,\overline{R} = A \in \mathfrak{so}(3)$ for some arbitrary skew-symmetric matrix A. Thus for all variations $\delta\overline{R}$

$$0 = \langle D_{\overline{R}}[W^\sharp(\overline{R}^T\,F), \delta\overline{R}\rangle = \langle DW^\sharp(\overline{R}^T F), \delta\overline{R}^T\,F\rangle = \langle D_{\overline{U}}W^\sharp(\overline{R}^T\,F), \underbrace{\delta\overline{R}^T\,\overline{R}}_{A}\,\overline{R}^T\,F\rangle$$

$$= \langle D_{\overline{U}}W^\sharp(\overline{R}^T F), A\,\overline{U}\rangle = \langle D_{\overline{U}}W^\sharp(\overline{U})\,\overline{U}^T, A\rangle \tag{3.3}$$

and balance of angular momentum follows as

$$\forall\,A \in \mathfrak{so}(3):\quad 0 = \langle D_U W^\sharp(\overline{U})\,\overline{U}^T, A\rangle \Leftrightarrow D_{\overline{U}}W^\sharp(\overline{U})\,\overline{U}^T \in \mathrm{Sym}\,. \tag{3.4}$$

5

Gathering the Euler-Lagrange equations we have for the model with rotational degrees of freedom

$$0 = \mathrm{Div}[\overline{R}\, D_{\overline{U}} W^{\sharp}(\overline{U})]\,, \qquad D_{\overline{U}} W^{\sharp}(\overline{U})\, \overline{U}^{T} \in \mathrm{Sym}\,. \tag{3.5}$$

This proposed relaxed Biot model with independent rotations is a special case of a nonlinear Cosserat continuum. To see this let us continue by introducing a finite strain Cosserat model.

4 The finite strain Cosserat model in variational form

In [12, 13] a finite-strain, fully frame-indifferent, three-dimensional Cosserat micropolar model is introduced, cf. [22, 18]. The two-field problem has been posed in a variational setting. The task is to find a pair $(\varphi, \overline{R}) : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \times \mathrm{SO}(3)$ of deformation φ and **independent microrotation**[2] $\overline{R} \in \mathrm{SO}(3)$, minimizing the energy functional I,

$$I(\varphi, \overline{R}) = \int_{\Omega} W_{\mathrm{mp}}(\overline{R}^T \nabla \varphi) + W_{\mathrm{curv}}(\overline{R}^T \mathrm{D_x} \overline{R})\, \mathrm{dV} \mapsto \min. \text{ w.r.t. } (\varphi, \overline{R})\,, \tag{4.1}$$

together with the Dirichlet boundary condition of place for the deformation φ on Γ: $\varphi_{|\Gamma} = g_{\mathrm{d}}$ and Neumann conditions on the microrotations $\overline{R}$ everywhere on $\partial\Omega$. The constitutive assumptions are

$$\overline{R}(x) : T_x\Omega \mapsto T_{\varphi(x)}\varphi(\Omega)\,, \quad \overline{U}(x) := \overline{R}^T(x)F(x) : T_x\Omega \mapsto T_x\Omega\,,$$

$$W_{\mathrm{mp}}(\overline{U}) = \mu\,\|\,\mathrm{sym}(\overline{U} - 1\!\!1)\|^2 + \mu_c\,\|\,\mathrm{skew}(\overline{U} - 1\!\!1)\|^2$$

$$+ \frac{\lambda}{4}\left((\det[\overline{U}] - 1)^2 + (\frac{1}{\det[\overline{U}]} - 1)^2\right)\,,\ F = \nabla\varphi\,,$$

$$W_{\mathrm{curv}}(\mathrm{D_x}\overline{R}) = \mu\,L_c^q\,\|\,\mathrm{Curl}\,\overline{R}\|^q\,, \tag{4.2}$$

under the minimal requirement $q \geq 2$. The total elastically stored energy $W = W_{\mathrm{mp}} + W_{\mathrm{curv}}$ depends on the generalized stretch $\overline{U}$ and on the curvature measure $\mathrm{Curl}\,\overline{R}$ [15] which describe the interaction of the microstructure on the macroscale. The strain energy W_{mp} depends on the deformation gradient $F = \nabla\varphi$ and the microrotations $\overline{R} \in \mathrm{SO}(3)$, which do not necessarily coincide with the **continuum rotations** $R = \mathrm{polar}(F) : T_x\Omega \mapsto T_{\varphi(x)}\varphi(\Omega)$.[3] In general, the **micropolar stretch tensor** $\overline{U}$ is **not symmetric** and does not coincide with the **symmetric continuum stretch tensor** $U = R^T F = \sqrt{F^T F}$: $T_x\Omega \mapsto T_x\Omega$.

Here $\Gamma \subset \partial\Omega$ is that part of the boundary, where Dirichlet conditions g_{d} for deformations are prescribed. The parameters $\mu, \lambda > 0$ are again the Lamé constants of classical isotropic elasticity, the additional parameter $\mu_c \geq 0$ is called the **Cosserat couple modulus**. For $\mu_c > 0$ the elastic strain energy density $W_{\mathrm{mp}}(\overline{U})$ is **uniformly convex** in $\overline{U}$

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[2]The microrotation $\overline{R}$ "lives" on the macroscale.

[3]The continuum rotation and the microrotation rotate infinitesimal volumina and move base points.

and satisfies the **standard growth assumption**

$$\forall F \in \mathrm{GL}^+(3): \ W_{\mathrm{mp}}(\overline{U}) = W_{\mathrm{mp}}(\overline{R}^T F) \geq \min(\mu, \mu_c) \, \|\overline{R}^T F - \mathbb{1}\|^2 = \min(\mu, \mu_c) \, \|F - \overline{R}\|^2$$
$$\geq \min(\mu, \mu_c) \inf_{R \in \mathrm{O}(3)} \|F - R\|^2 = \min(\mu, \mu_c) \, \mathrm{dist}^2(F, \mathrm{O}(3))$$
$$= \min(\mu, \mu_c) \, \mathrm{dist}^2(F, \mathrm{SO}(3)) = \min(\mu, \mu_c) \, \|F - \mathrm{polar}(F)\|^2$$
$$= \min(\mu, \mu_c) \, \|U - \mathbb{1}\|^2 \,, \tag{4.3}$$

where $\mathrm{dist} : \mathbb{M}^{3\times 3} \mapsto \mathbb{R}$ is the euclidian distance function on second order tensors. In contrast, for the case $\mu_c = 0$ the strain energy density is **only convex** w.r.t. F and does not satisfy (4.3).[4]

The parameter $L_c > 0$ (with dimension length) introduces an **internal length** which is **characteristic** for the material, e.g., related to the grain size in a polycrystal. The internal length $L_c > 0$ is responsible for **size effects** in the sense that smaller samples are relatively stiffer than larger samples.

In the Cosserat model it is still possible to compute a tensor, formally taking on the role of the Cauchy-stresses:

$$\sigma = \frac{1}{\det[F]} \, S_1(F, \overline{R}) \, F^T = \frac{1}{\det[F]} \, D_F W(F, \overline{R}) \, F^T$$
$$= \frac{1}{\det[F]} \, \overline{R} \, D_{\overline{U}} W(\overline{U}) \, F^T = \frac{1}{\det[F]} \, \overline{R} \, T(\overline{U}) \, F^T \,. \tag{4.4}$$

It is of prime importance to realize that a linearization of this isotropic Cosserat bulk model with $\mu_c = 0$ for small displacement and small microrotations completely decouples the two fields of deformation φ and microrotations $\overline{R}$ and leads to the classical linear elasticity problem for the deformation. In [10] it is nevertheless shown that $\mu_c = 0$ is a reasonable choice.[5] For more details on the modelling of the three-dimensional Cosserat model we refer the reader to [12]. Extensions to a micromorphic model have been given in [14]. The Cosserat model is well-posed in the sense that the existence of minimizers is obtained for various combinations of constitutive parameters [9, 11], including $\mu_c = 0$, provided that L_c is strictly positive.

The Biot model with independent rotations is obtained from the Cosserat model by neglecting the curvature, i.e., setting $L_c = 0$. By a scaling argument it is easy to see that $L_c = 0$ corresponds to the limit of arbitrarily large samples. Therefore, the proposed Cosserat model can be seen as a regularization of the Biot model with independent rotations. For $L_c = 0$, balance of angular momentum is equivalently expressed as $\sigma \in \mathrm{Sym}$.

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[4] The condition $F \in \mathrm{GL}^+(3)$ is necessary, otherwise $\|F - \mathrm{polar}(F)\|^2 = \mathrm{dist}^2(F, \mathrm{O}(3)) < \mathrm{dist}^2(F, \mathrm{SO}(3))$, as can be easily seen for the reflection $F = \mathrm{diag}(1, -1, 1)$.

[5] Thinking in the context of an infinitesimal-displacement Cosserat theory one might believe that $\mu_c > 0$ is necessary also for a "true" finite-strain Cosserat theory.

5 Symmetry of stresses versus symmetry of stretches

Let us now return to the Euler-Lagrange equations for the Biot model with rotational degrees of freedom (3.5)

$$0 = \mathrm{Div}[\overline{R}\, D_{\overline{U}} W^{\sharp}(\overline{U})]\,, \qquad D_{\overline{U}} W^{\sharp}(\overline{U})\, \overline{U}^{T} \in \mathrm{Sym}\,. \tag{5.1}$$

Using the representation theorems for isotropic functions of non-symmetric tensor arguments [21] it is easy to see that for isotropic $W^{\sharp}$ and for symmetric $\overline{U}$ the balance of angular momentum (3.4) is automatically satisfied, see e.g., [19]. We observe with [3] that a (not necessarily unique) solution $(\varphi, \overline{R})$ of (3.5) cannot solve (2.11) unless $\overline{U}$ is symmetric.

However, in [3] the author proceeds and arrives correctly at

$$\gamma_1\,(\overline{U} - \overline{U}^{T}) + \gamma_2\,(\overline{U}^2 - \overline{U}^{T,2}) + \gamma_3\,(\overline{U}^3 - \overline{U}^{T,3}) = 0\,, \tag{[3] Eq.(2.25)}$$

for some scalar functions $\gamma_i = \gamma_i(\overline{U})$. He concludes: "The moment equilibrium ... leads to the symmetry condition $\overline{U} = \overline{U}^{T}$. This result, ..., can be explained as follows: For an isotropic material the stretch $\overline{U}$ and the stress r $(= D_{\overline{U}} W^{\sharp}(\overline{U})$ our addition) are coaxial; consequently the moment equilibrium in (2.19) is satisfied identically for every symmetrical $\overline{U}$. Vice versa the moment equilibrium condition enforces this symmetry under the assumption of an isotropic material as demonstrated in (2.25)."

This statement is only partially true: symmetric $\overline{U}$ satisfies, for isotropic $W^{\sharp}$, always moment equilibrium (3.4). The converse, is, however, not necessarily the case. To see this, choose e.g.

$$
\begin{aligned}
W^{\sharp}(\overline{U}) = {}& \mu\,\|\operatorname{sym}(\overline{U} - \mathbb{1})\|^2 + \mu_c\,\|\operatorname{skew}(\overline{U} - \mathbb{1})\|^2 \\
& + \frac{\lambda}{4}\left((\det[\overline{U}] - 1)^2 + (\frac{1}{\det[\overline{U}]} - 1)^2\right),
\end{aligned} \tag{5.2}
$$

as in the Cosserat model (4.2). Clearly, $W^{\sharp}$ is an isotropic scalar valued function of the non-symmetric tensor argument $\overline{U}$. Since the volumetric term is independent of $\overline{R}$ we can concentrate for balance of angular momentum on

$$W_{\mu,\mu_c}(F, \overline{R}) := \mu\,\|\operatorname{sym}(\overline{U} - \mathbb{1})\|^2 + \mu_c\,\|\operatorname{skew}(\overline{U} - \mathbb{1})\|^2\,. \tag{5.3}$$

Balance of angular momentum reads now

$$
\begin{aligned}
D_{\overline{U}} W^{\sharp}(\overline{U})\, \overline{U}^{T} \in \mathrm{Sym} \;&\Leftrightarrow\; D_{\overline{U}} W_{\mu,\mu_c}(\overline{U})\, \overline{U}^{T} \in \mathrm{Sym} \;\Leftrightarrow\; \\
\left[2\mu\,(\operatorname{sym}(\overline{U} - \mathbb{1})) + 2\mu_c\,\operatorname{skew}\overline{U}\right] \overline{U}^{T} &\in \mathrm{Sym} \;\Leftrightarrow\; \\
\left[\mu\,(\overline{U} + \overline{U}^{T} - 2\,\mathbb{1}) + \mu_c(\overline{U} - \overline{U}^{T})\right] \overline{U}^{T} &\in \mathrm{Sym} \;\Leftrightarrow\; \\
(\mu - \mu_c)\,\overline{U}\,\overline{U} - 2\mu\,\overline{U} &\in \mathrm{Sym} \;\Leftrightarrow\; \\
(\mu - \mu_c)\,[\overline{U}^2 - \overline{U}^{T,2}] - 2\mu\,[\overline{U} - \overline{U}^{T}] &= 0\,.
\end{aligned} \tag{5.4}
$$

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Obviously, for $\mu_c = \mu$ the symmetric solution is unique. Therefore, assume presently that $0 \le \mu_c < \mu$ but we note that the choice $\mu_c = 0$ is not necessary. Define $\rho := \frac{2\mu}{\mu - \mu_c}$ and set

$$F = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_1 + \lambda_2 > \rho,$$

$$\overline{R} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha = \arccos\left(\frac{\rho}{\lambda_1 + \lambda_2}\right) \in \left[0, \frac{\pi}{2}\right]. \tag{5.5}$$

This yields the explicit forms

$$\overline{R} = \begin{pmatrix} \frac{\rho}{\lambda_1+\lambda_2} & -\sqrt{1 - \frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 \\ \sqrt{1 - \frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & \frac{\rho}{\lambda_1+\lambda_2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\overline{U} = \overline{R}^T F = \begin{pmatrix} \frac{\rho\,\lambda_1}{\lambda_1+\lambda_2} & \lambda_2\sqrt{1 - \frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 \\ -\lambda_1\sqrt{1 - \frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & \frac{\rho\,\lambda_2}{\lambda_1+\lambda_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.6}$$

as can be easily seen from a straight forward calculation. It is obvious, that $\overline{U}$ **is in general not symmetric**. On the other hand, we obtain

$$\overline{U} - \overline{U}^T = \begin{pmatrix} 0 & (\lambda_1+\lambda_2)\sqrt{1 - \frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 \\ -(\lambda_1+\lambda_2)\sqrt{1 - \frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\overline{U}^2 - \overline{U}^{T,2} = \begin{pmatrix} 0 & \rho\,(\lambda_1+\lambda_2)\sqrt{1 - \frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 \\ -\rho\,(\lambda_1+\lambda_2)\sqrt{1 - \frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.7}$$

Thus, $\overline{U}^2 - \overline{U}^{T,2} = \frac{2\mu}{\mu - \mu_c}(\overline{U} - \overline{U}^T)$ from which we readily infer

$$(\mu - \mu_c)[\overline{U}^2 - \overline{U}^{T,2}] - 2\mu[\overline{U} - \overline{U}^T] = 0.$$

Hence, $\overline{U} = \overline{R}^T F \notin \mathrm{Sym}$, but satisfies the equilibrium equation of angular momentum.

Furthermore, it can be shown, that the given rotation $\overline{R}$ is not only a solution of the balance of momentum equation $D_{\overline{U}} W^\sharp(\overline{U})\,\overline{U}^T \in \mathrm{Sym}$ but realizes indeed the global minimum w.r.t. $\overline{R}$ of the energy $W^\sharp$ at given deformation gradient F. Moreover, if $|\lambda_1 - \lambda_2| < \rho$, then $\langle \xi, \overline{U}.\xi \rangle_{\mathbb{R}^3}$ defines a positive definite quadratic form. This is shown in the Diploma-Thesis of A. Fischle [4]. There, an exhaustive discussion of the structure of optimal rotations for the energy (5.2) is povided. It should also be noted that the always

possible symmetric solution $\overline{R} = \text{polar}(F)$ need not even be a local minimizer. Stability considerations do not speak in favour of the polar rotation!

Interesting enough, if in the former, we choose $\mu_c \geq \mu$ (a specific kind of penalty) then the only solution of balance of momentum is indeed a symmetric $\overline{U}$ [4]. Here, the penalty term enforces exactly the symmetry and not only approximately.

6 Conclusion

Summarizing the situation, we can say: the Cosserat model turns into a Biot model with independent rotations whenever the internal length scale is absent, i.e., $L_c = 0$. Even in the case of isotropy the equilibrium solutions of the model (3.5) are not necessarily equilibrium solutions of the weak Biot model (2.11). If it is intended to approximate classical solutions by the model with independent rotations, then a sufficiently large penalty term $\mu_c \| \text{skew } \overline{U} \|^2$ needs to be added. In the investigated isotropic case (5.2), a finite penalty parameter $\mu_c \geq \mu$ is sufficient to enforce symmetry of the relaxed Biot stretches exactly. If the penalty parameter μ_c is small or absent the relation of the relaxed Biot model with independent rotations to the classical isotropic Biot model is lost. Since in the classical Biot model, the stretches are not only symmetric but positive definite, the solutions of the relaxed Biot model with penalty need to be checked w.r.t. positive definiteness in order to maintain their physical relevance.

It remains to find a sufficiently large class of isotropic free energies such that moment equilibrium in the relaxed formulation implies automatically the symmetry of the relaxed Biot stretch $\overline{U}$.

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Notation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with non-vanishing 2-dimensional Hausdorff measure. We denote by $\mathbb{M}^{3\times3}$ the set of real 3×3 second order tensors, written in capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times3}$ is given by $\langle X, Y \rangle_{\mathbb{M}^{3\times3}} = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{M}^{3\times3}}$ (we use these symbols indifferently for tensors and vectors). The identity tensor on $\mathbb{M}^{3\times3}$ will be denoted by $\mathbb{1}$, so that $\operatorname{tr}[X] = \langle X, \mathbb{1} \rangle$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie-theory, i.e. $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3\times3} \,|\, X^T = -X\}$ are skew symmetric second order tensors and $\mathfrak{sl}(3) := \{X \in \mathbb{M}^{3\times3} \,|\, \operatorname{tr}[X] = 0\}$ are traceless tensors. We set $\operatorname{sym}(X) = \frac{1}{2}(X^T + X)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \operatorname{sym}(X) + \operatorname{skew}(X)$.

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