Stability of B-Splines on Bounded Domains

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Abstract

We construct a uniformly stable family of bases for tensor product spline approximation on bounded domains in $\mathbb{R}^d$. These bases are derived from the standard B-spline basis by normalization with respect to the L^p-norm and a selection process relying on refined estimates for the de Boor-Fix functionals.

1 Introduction

Uniform stability of tensor product B-spline bases on $\mathbb{R}^d$ is a well known fact [2] and one of the many favorable properties of this class of functions. However, when approximating functions on a bounded domain $\Omega \subset \mathbb{R}^d$, stability is typically lost because of B-splines with only small parts of their support lying inside the domain. This problem was observed in [5], and probably also much earlier, and taken for granted ever since.

In [8, 6], an extension procedure is suggested to stabilize B-spline bases. There, outer B-splines supported near the boundary of the domain are suitably coupled with inner ones, and it can be shown that the resulting basis combines stability with full approximation power, despite its reduced dimension.

In this paper, we revisit the stability problem and show that instability of the standard B-spline basis is mostly due to bad scaling, and not to an inherent closeness to linear dependence. For instance, B-splines with support trimmed by a hyperplane can always be stabilized by scaling, no matter how small their active part becomes.

In the next section, we briefly recall the standard estimates for stability on $\mathbb{R}^d$ and explain possible sources of instability on bounded domains. Then, in Section 3, we introduce the notion of (r, p)-stability of B-splines, normalized with respect to the L^p-norm, and show that the set of all such B-splines forms a uniformly stable basis. While the basic Definition 3.1 and Theorem 3.2 are geared to great generality, we also provide sufficient conditions which are easy to verify in practice. Finally, in Section 4, we briefly show that the set of (r, p)-stable B-splines is useful in the sense that it provides, for instance, full approximation power with respect to the L^2-norm if the domain is coordinate-wise convex and sufficiently smooth.

At this place, we want to remark that approximation properties even of complete spaces of B-splines on bounded domains are not sufficiently well understood, today. The classical results in [5] are based on quite restrictive conditions on the geometry of the domain. In particular, it is assumed that the domain is coordinate-wise convex. The results in [6], which are based on extending functions defined only on the domain to intervals, are limited by the fact that the existence of such an extension may rely on the smoothness of the boundary of the domain. Both results have in common that the constants in the error estimates depend on the aspect ratio of the knot grid, and thus can grow unboundedly if, for instance, the knot sequence in one coordinate direction is repeatedly refined, while the other ones remain fixed. Presumably, this contra-intuitive behavior is due to technical limitations in the proofs, and not to the actual nature of spline approximation on grids with largely differing knot spacings in the coordinate directions.

2 Preliminaries

For an open set $\Omega \subset \mathbb{R}^d$ and $1 \leq p \leq \infty$, let $B = (b_i)_{i \in I}$ be a sequence of functions $b_i \in L^p(\Omega)$. The vector space of functions spanned by B is denoted

$$V^p(B) := \overline{\operatorname{span}\{b_i : i \in I\}},$$

where the closure is understood with respect to the L^p-norm $\|\cdot\|_{p,\Omega}$. For functions $f \in V^p(B)$ we write briefly

$$f = \sum_{i \in I} f_i b_i =: FB, \qquad F = (f_i)_{i \in I}.$$

The space of coefficients with finite l^p-norm is

$$l^p(I) := \{F \in \mathbb{R}^I : \|F\|_{p,I} < \infty\}.$$

If the underlying set is clear from the context, we also write $\|f\|_p$ and $\|F\|_p$ instead of $\|f\|_{p,\Omega}$ and $\|F\|_{p,I}$, respectively. B is a Riesz basis of $V^p(B)$ with respect to the given norms, if there exist constants $\tilde{c}, \tilde{C} > 0$, such that $\tilde{c}\|F\|_p \leq \|FB\|_p \leq \tilde{C}\|F\|_p$ for all $F \in l^p(I)$. In this case, the optimal constants are

$$c := \inf\{\|FB\|_p : \|F\|_p = 1\}, \qquad C := \sup\{\|FB\|_p : \|F\|_p = 1\}, \tag{1}$$

and the ratio $\operatorname{cond}_p B := C/c$ is called the *condition number* of B with respect to the p-norm.

Let $T = \{\ldots \leq \tau_{-1} \leq \tau_0 \leq \tau_1 \leq \ldots\}$, $\tau_i \in \mathbb{R}$, be a bi-infinite knot-sequence for a spline space of degree n. The corresponding order is denoted $\overline{n} := n+1$. The B-splines $(b_{i,n})_{i\in\mathbb{Z}}$ have support

$$S_{i,n} := \operatorname{supp} b_{i,n} = [\tau_i, \tau_{i+\overline{n}}], \quad |S_{i,n}| := \tau_{i+\overline{n}} - \tau_i,$$

and satisfy the *Marsden-identity*

$$(t-\tau)^n = \sum_{i\in\mathbb{Z}} b_{i,n}(t)\psi_{i,n}(\tau), \quad \psi_{i,n}(\tau) := \prod_{j=1}^{n}(\tau_{i+j} - \tau) \tag{2}$$

for all $t, \tau \in \mathbb{R}$. The function $\psi_{i,n}$ is a polynomial of degree n with the inner knots $\tau_{i+1}, \ldots, \tau_{i+n}$ of the B-spline $b_{i,n}$ as zeros. For $0 \leq \nu \leq n$, the νth derivative of $\psi_{i,n}$ can be written as

$$D^\nu \psi_{i,n}(\tau) = (-1)^\nu \frac{n!}{(n-\nu)!} \prod_{j=1}^{n-\nu}(\tau_{i,j}^\nu - \tau). \tag{3}$$

By Rolle's theorem, the zeros $\tau_{i,j}^\nu$ are all real and lie in the interval $S_{i,n}$. Hence, for $\tau \in S_{i,n}$, we have

$$|D^\nu \psi_{i,n}(\tau)| \leq n!\,|S_{i,n}|^{n-\nu}, \quad 0 \leq \nu \leq n. \tag{4}$$

To prove stability properties of B-splines, we use the de Boor-Fix functionals [3] given by

$$\lambda_{i,n}(u) := \frac{1}{n!} \sum_{\nu=0}^{n} (-1)^{n-\nu} D^{n-\nu}\psi_{i,n}(\xi_i) D^\nu u(\xi_i), \tag{5}$$

where ξ_i is an arbitrary point with $b_{i,n}(\xi_i) > 0$. The basic duality property is

$$\lambda_{i,n}(b_{j,n}) = \delta_{ij}, \quad i, j \in \mathbb{Z}. \tag{6}$$

Points $p \in \mathbb{R}^d$ are understood as row-vectors, and their components are indexed by superscripts, $p = (p^1, \dots, p^d)$. The component-wise product of two points $p, q \in \mathbb{R}^d$ is denoted

$$p * q := (p^1 q^1, \dots, p^d q^d) \in \mathbb{R}^d.$$

If, component-wise, $p \le q$, then the two points define the closed interval

$$P := [p, q] := [p^1, q^1] \times \cdots \times [p^d, q^d] \subset \mathbb{R}^d.$$

The vector of edge lengths, also called the *size* of P, is denoted

$$|P| := q - p \in \mathbb{R}^d.$$

The univariate knot sequences $T^1, \dots, T^d$ define a multivariate *knot grid* $T := T^1 \times \cdots \times T^d$ with knots $\tau_i := (\tau^1_{i^1}, \dots, \tau^d_{i^d})$ and *grid cells* $T_i := [\tau_i, \tau_{i+1}]$, $i \in \mathbb{Z}^d$. The basis functions of the tensor product spline space of coordinate degree $n \in \mathbb{N}^d$ with knots T are just products of the univariate B-splines, i.e.,

$$b_{i,n}(x) := b_{i^1,n^1}(x^1) \cdots b_{i^d,n^d}(x^d), \quad i \in \mathbb{Z}^d.$$

Denoting the order of the spline space again by $\overline{n} := n + (1, \dots, 1)$, their support is the interval

$$S_{i,n} := \operatorname{supp} b_{i,n} = [\tau_i, \tau_{i+\overline{n}}].$$

With the usual multi-index notation, the multivariate de Boor-Fix functionals are given by

$$\lambda_{i,n}(u) = \frac{1}{n!} \sum_{\nu \le n} (-1)^{n-\nu} \partial^{n-\nu} \psi_{i,n}(\xi_i) \partial^\nu u(\xi_i), \tag{7}$$

where $\psi_{i,n}(\xi_i) := \psi_{i^1,n^1}(\xi^1_i) \cdots \psi_{i^d,n^d}(\xi^d_i)$ and ξ_i is chosen such that $b_{i,n}(\xi_i) > 0$. To simplify notation, we fix the degree $n \in \mathbb{N}^d_0$ and drop the corresponding subscript, throughout.

In the following, we study stability properties of B-spline bases B spanning function spaces $V^p(B)$ on a domain $\Omega \subset \mathbb{R}^d$. For a given knot grid T, a natural choice of B is to employ all B-splines which do not vanish on Ω, i.e.,

$$B = (b_{i|\Omega})_{i \in I}, \quad I := \{i \in \mathbb{Z}^d : S_i \cap \Omega \neq \emptyset\}. \tag{8}$$

Typically, the results to be derived later are invariant with respect to *axis-aligned affine maps* (or briefly *a^3-maps*) in $\mathbb{R}^d$. Such maps have the form

$$\mathcal{A}\colon\ \mathbb{R}^d \ni x \mapsto (a_0 * x + a_1)A \in \mathbb{R}^d, \tag{9}$$

where $A \in \{0,1\}^{d\times d}$ is a permutation matrix, $a_0 \in (\mathbb{R}_{\neq 0})^d$ is a scaling vector, and $a_1 \in \mathbb{R}^d$ is a shift vector. If, in particular $a_0 = (\pm 1, \ldots, \pm 1)$, then $\mathcal{A}$ is called an *axis-aligned isometric map.* We write

$$\tilde{x} = \mathcal{A}(x), \quad \tilde{\Omega} = \mathcal{A}(\Omega), \quad \tilde{T} = \mathcal{A}(T),$$

etc., and observe that the B-splines $\tilde{b}_i$ and the de Boor-Fix functionals $\tilde{\lambda}_i$ with respect to $\tilde{T}$ are *affine invariant* in the sense that

$$\tilde{b}_i(\tilde{x}) = b_i(x), \quad \tilde{\lambda}_i(\tilde{u}) = \lambda_i(u), \text{ where } \tilde{u} := u \circ \mathcal{A}^{-1}.$$

For $\Omega = \mathbb{R}^d$, the classical result on the uniform stability of B-splines [2] states that $\operatorname{cond}_\infty B$ is bounded by a constant M depending only on the degree n (and the dimension d), but not on the choice of knots. A similar result holds for p-norms, $1 \leq p < \infty$, if the B-splines are normalized in a suitable way. Notably, $\operatorname{cond}_\infty B$ can be arbitrarily large for general Ω.

Let us illustrate this phenomenon by a simple univariate example. For

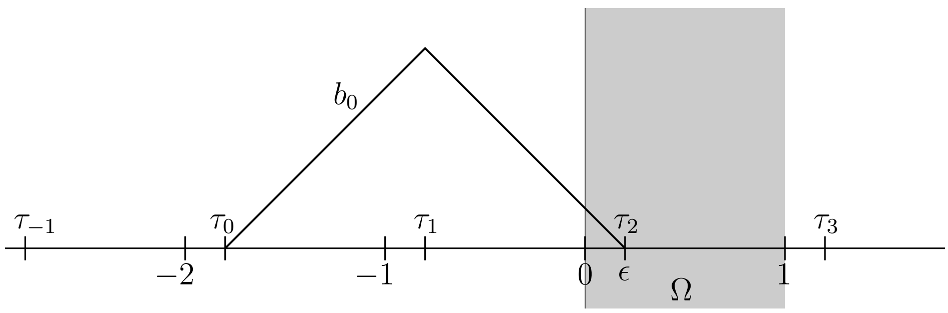

Figure 1: B-spline basis with large condition number.

$0 < \epsilon < 1$, we consider splines of degree $n = 1$ with knots $\tau_i = i - 2 + \epsilon$ on the domain $\Omega = (0, 1)$. Figure 1 shows the spline $FB = b_0$ corresponding to the coefficients $f_i = \delta_{i,0}$. From $\|b_0\|_{\infty,\Omega} = \epsilon$, we conclude that the lower bound in (1) is $c \le \epsilon$, while the upper bound is obviously $C = 1$. Hence, $\operatorname{cond}_\infty B \ge 1/\epsilon$ is not uniformly bounded.

To understand the differences between this case and the well known uniform stability on $\mathbb{R}$, let us briefly review the classical proof. For $\Omega = \mathbb{R}$ and $B = (b_i)_{i\in\mathbb{Z}}$, consider the estimate

$$c\|F\|_\infty \le \|FB\|_{\infty,\mathbb{R}} \le C\|F\|_\infty, \quad F \in l^\infty(\mathbb{Z}).$$

By partition of unity, the upper estimate holds with the optimal constant $C = 1$. For the lower estimate, a single coefficient f_i of the spline $f = FB$ is expressed with the help of the de Boor-Fix functional (5),

$$\begin{aligned} |f_i| = |\lambda_i(f)| &= \frac{1}{n!}\left|\sum_{\nu=0}^{n}(-1)^{n-\nu}D^{n-\nu}\psi_i(\xi_i)D^\nu f(\xi_i)\right| \\ &\le \frac{1}{n!}\max_{\nu=0}^{n}|D^\nu\psi_i(\xi_i)|\sum_{\nu=0}^{n}|D^\nu f(\xi_i)|. \end{aligned} \tag{10}$$

Affine invariance admits to assume $S_i = [0, 1]$ without loss of generality. In this case, by (4), $|D^\nu\psi_i(\xi)| \le n!$ for all ν. Further, there exists an interval $Q \subset S_i = [0, 1]$ of length $1/\overline{n}$ which does not contain a knot. Hence, $f_{|Q} \in \mathbb{P}_n(Q)$, i.e., the restriction of f to Q is a polynomial of degree $\le n$. With ξ_i the center of Q, the sum in the above estimate is a norm on the space $\mathbb{P}_n(Q)$. By equivalence of norms on finite-dimensional vector spaces, this norm is bounded from above by the L^∞-norm on Q with a constant C_n depending only on n. We obtain

$$|f_i| \le \sum_{\nu=0}^{n}|D^\nu f(\xi_i)| \le C_n\|f_{|Q}\|_{\infty,Q} \le C_n\|f\|_{\infty,\mathbb{R}}$$

for all $i \in \mathbb{Z}$ showing that $C_n^{-1}\|F\|_\infty \le \|FB\|_\infty$ for all $F \in l^\infty(\mathbb{Z})$. Hence, $\operatorname{cond}_\infty B \le M := C_n$ is bounded independent of the knot sequence.

For arbitrary domains $\Omega \subset \mathbb{R}$, the above argument can fail since it might be impossible to find an interval Q of length $1/\overline{n}$ in $S_i \cap \Omega$, if this set is small. The counterexample given above is based exactly on this observation. Of course, that problem is readily removed by adapting the knot sequence appropriately, for instance by setting $T = [0, 0, \epsilon, 1, 1]$. In this way, the instability is removed without changing the spline space on Ω. But unfortunately, this method does not work in general for domains in higher dimensions.

3 Stability

As shown above, the basis B according to (8) is not necessarily stable. A trivial way to circumvent this problem is to discard those B-splines for which a suitable interval Q does not exist. For instance, if the knot sequence is uniform, these are exactly those B-splines which do not have a complete grid cell of their support in Ω. Although in this way only relatively few B-splines near the boundary of Ω are ruled out, it is easily shown that the resulting spline space reveals a substantial loss of approximation power. A much more appropriate solution is based on the concept of *extension* as introduced in [8]. Here, the unstable outer B-splines are suitably attached to inner ones so that a uniformly stable basis with full approximation power is obtained.

In the following, we suggest an even simpler approach to the problem which is based on a natural normalization process. It turns out that most instabilities are not due to an inherent closeness to linear dependence, but merely to bad scaling. Our main result is essentially based on the following estimate for univariate B-splines:

Lemma 3.1 *Let b_i be a univariate B-spline of degree n with $\tau_i = 0$, and let $P := [0, 1]$. Then*

$$\|b_i\|_{\infty,P} \, \|D^\nu \psi_i\|_{\infty,P} \le n! \tag{11}$$

for all $\nu = 0, \ldots, n$.

Proof: The proof is by induction on ν, proceeding backwards from $\nu = n$. For $\nu = n$, the estimate follows immediately from $\|b_i\|_\infty \le 1$ and (4). Now, we assume that the estimate is true for $\nu + 1$. If $D^\nu \psi_i$ has a zero in $[0, 1]$, then, by the mean value theorem, $\|D^\nu \psi_i\|_{\infty,P} \le \|D^{\nu+1} \psi_i\|_{\infty,P}$. Otherwise, all zeros are ≥ 1 implying that $|D^\nu \psi_i|$ is monotone decreasing on $[0, 1]$. Hence,

$$\|b_i\|_{\infty,P} \|D^\nu \psi_i\|_{\infty,P} \le \|b_i\|_{\infty,P} |D^\nu \psi_i(0)|.$$

Since b_i and ψ_i depend only on the knots $\tau_i, \ldots, \tau_{i+n+1}$, we can assume $\tau_j = \tau_i = 0$ for all $j \le i$ without loss of generality. Differentiating (2) with respect to τ, we obtain for $\tau = 0$

$$\sum_{j\in\mathbb{Z}} b_j(t) D^\nu \psi_j(0) = (-1)^\nu \frac{n!}{(n-\nu)!}\, t^{n-\nu}.$$

By the special choice of knots, all zeros $\tau_{j,l}^\nu$ of $D^\nu\psi_j$ are non-negative. Thus, (3) yields $D^\nu\psi_j(0) = (-1)^\nu |D^\nu\psi_j(0)|$, and we obtain

$$b_i(t)|D^\nu\psi_i(0)| \le \sum_{j\in\mathbb{Z}} b_j(t)|D^\nu\psi_j(0)| = \frac{n!}{(n-\nu)!}\, t^{n-\nu}$$

for all $t \in P$. Hence,

$$\|b_i\|_{\infty,P}|D^\nu\psi_i(0)| \le \frac{n!}{(n-\nu)!} \le n!,$$

and the proof is complete. □

For a given domain $\Omega \subset \mathbb{R}^d$ and $1 \le p \le \infty$, we define the *normalized B-splines*

$$b_i^p := \frac{b_i}{\|b_i\|_{p,\Omega}}, \quad i \in I,$$

where I is the index set of relevant B-splines according to (8) so that the denominator is positive. The following definition characterizes a large class of normalized B-splines which lead to uniformly stable bases:

Definition 3.1 *Let T be a knot grid for a tensor product spline-space of degree n, and let $r = (r_1, r_2) \ge (1,1)$ be a pair of real parameters. Then the normalized B-spline b_i^p is called* (r,p)-stable *with respect to Ω and T, if there exist intervals $P_i, Q_i \subset S_i$ with the following properties:*

a) Q_i is contained in an interior grid cell, i.e., $Q_i \subset T_j \cap \overline{\Omega}$ for some $j \in \mathbb{Z}^d$.

b) The center ξ_i of Q_i is contained in P_i.

c) The sizes of Q_i and P_i are related by $r_1\big(\overline{n} * |Q_i|\big) = |P_i|$.

d) The intervals P_i and S_i have one corner in common.

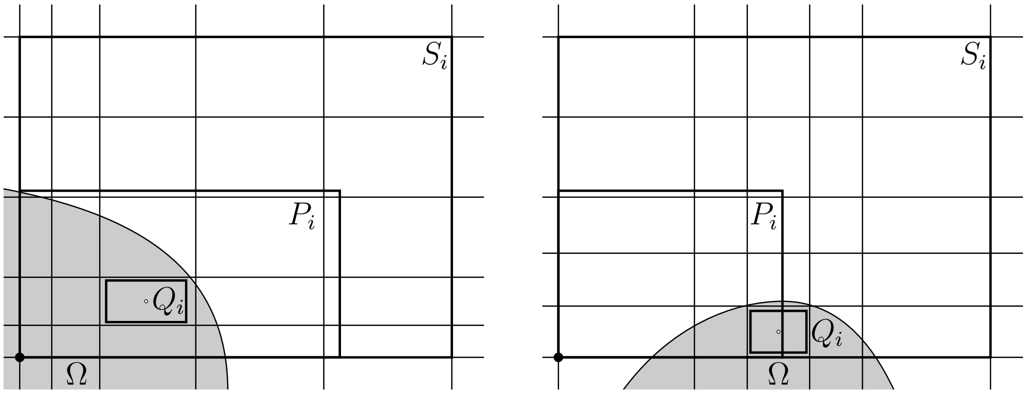

Figure 2: Illustration of Definition 3.1.

e) $\|b_i\|_{p,\Omega} \le r_2 \|b_i\|_{p,P_i}$.

The sequence of (r,p)-stable normalized B-splines is denoted

$$B_r^p := (b_i^p)_{i \in I_r^p}, \quad I_r^p := \{i \in I : b_i^p \textit{is } (r,p)\textit{-stable}\}.$$

Let us state some elementary facts about (r,p)-stability without proof.

- All B-splines with support completely contained in Ω admit the choice $S_i = P_i$ so that they are (r,p)-stable for all $r \ge (1,1)$ and $1 \le p \le \infty$.
- (r,p)-stability implies (r',p)-stability for $r' \ge r$.
- The notion of (r,p)-stability is affine invariant in the following sense: If $\mathcal{A}$ is an a^3-map according to (9) and b_i is (r,p)-stable with respect to Ω and T, then then $\tilde{b}_i$ is (r,p)-stable with respect to $\tilde{\Omega}$ and $\tilde{T}$.

Now, we will show that families of (r,p)-stable normalized B-splines form bases with a uniformly bounded condition number. To this end, the latter property will be used to map P_i to the unit interval, and to make the origin the common corner of P_i and S_i. The interval Q_i is a subset of Ω of fixed relative size on which splines are polynomial. The center ξ_i will be used for the de Boor-Fix functional. Condition d) concerning the common corner and the estimate e) are needed to satisfy the assumptions of Lemma 3.1.

Theorem 3.2 *The condition number of the basis B_r^p is bounded by*

$$\operatorname{cond}_p B_r^p \le M,$$

where the constant M depends on n, p, r, but neither on T nor on Ω.

Proof: We determine constants $c', C' > 0$ such that

$$c'\|F\|_p \leq \|FB_r^p\|_p \leq C'\|F\|_p$$

for all $F \in l^p(I_r^p)$. The upper bound follows from locality of the B-splines. On every grid cell T_j, there are $N := \overline{n}^1 \cdots \overline{n}^d$ non-vanishing basis functions. With $\|b_i^p\|_{p,T_j\cap\Omega} \leq 1$, standard arguments show that $C' := N^{1/q}$ is a valid constant, where p and q are related by $1/p + 1/q = 1$.

To prove the lower estimate, let P_i, Q_i be intervals according to Definition 3.1. Like the conditions specified in the definition, also the lower estimate $c'\|F\|_p \leq \|FB_r^p\|_p$ is affine invariant, i.e.,

$$\|FB_r^p\|_{p,\Omega} = \|F\tilde{B}_r^p\|_{p,\tilde{\Omega}}.$$

Hence, for fixed $i \in I_r^p$, we can assume without loss of generality that $P_i = [0,1]^d$, and that the common corner of P_i and S_i according to property d) is the origin. Applying the de Boor-Fix functional (7) to the spline $f = FB_r^p$, we obtain the estimate

$$\begin{aligned}\frac{|f_i|}{\|b_i\|_{p,\Omega}} = |\lambda_i(f)| &\leq \frac{1}{n!} \sum_{\nu\leq n} |\partial^{n-\nu}\psi_i(\xi_i)|\,|\partial^\nu f(\xi_i)| \\ &\leq \frac{1}{n!}\left(\max_{\nu\leq n} |\partial^\nu\psi_i(\xi_i)|\right) \sum_{\nu\leq n} |\partial^\nu f(\xi_i)|, \qquad (12)\end{aligned}$$

where ξ_i is the center of Q_i. Property a) implies that f restricted to Q_i is a polynomial. The sum in (12) is a norm on the space $\mathbb{P}^n(Q_i)$ of polynomials of degree $\leq n$. Therefore, by equivalence of norms on finite-dimensional vector spaces, this norm is bounded from above by the p-norm on Q_i times a constant $C_{n,p,r}$ which depends only on n, p, r because the size of Q_i is fixed by property c). Hence, since $\xi_i \in P_i$ by property b),

$$|f_i| \leq \frac{C_{n,p,r}}{n!}\,\|f\|_{p,Q_i}\,\|b_i\|_{p,\Omega}\,\max_{\nu\leq n}\|\partial^\nu\psi_i\|_{\infty,P_i}.$$

Now, by property e), $\|b_i\|_{p,\Omega} \leq r_2\,\|b_i\|_{p,P_i} \leq r_2\,\|b_i\|_{\infty,P_i}$. We apply Lemma 3.1 to all univariate factors of b_i and $\partial^\nu\psi_i$ and find

$$|f_i| \leq \frac{C_{n,p,r}}{n!}\,\|f\|_{p,Q_i}\,\|b_i\|_{\infty,P_i}\,\max_{\nu\leq n}\|\partial^\nu\psi_i\|_{\infty,P_i} \leq C_{n,p,r}\|f\|_{p,Q_i}.$$

Obeying the fact that at most $N = \overline{n}^1 \cdots \overline{n}^d$ of the intervals Q_i can overlap, a standard argument finally yields

$$\|F\|_p^p = \sum_{i \in I_r^p} |f_i|^p \leq C_{n,p,r}^p \sum_{i \in I_r^p} \|f\|_{p,Q_i}^p \leq N C_{n,p,r}^p \|f\|_{p,\Omega}^p,$$

and thus, $c' = (N^{1/p} C_{n,p,r})^{-1}$ is a valid constant for the lower estimate. Together, the condition number is bounded by

$$\operatorname{cond}_p B_r^p \leq M := \frac{C'}{c'} = C_{n,p,r} N^{1/p+1/q} = C_{n,p,r} N.$$

□

Next, we are going to specify sufficient conditions for (r,p)-stability which are easier to verify than those of Definition 3.1 but still yield sufficiently large subsets of B_r^p.

Lemma 3.3 *For all $p \in [1, \infty]$, the normalized B-spline b_i^p is (r,p)-stable with $r_2 = 1$ if there exist intervals P_i, R_i with the following properties:*

a) $R_i \subset S_i \cap \overline{\Omega} \subset P_i \subset S_i$

b) The sizes of R_i and P_i are related by $r_1 * |R_i| = |P_i|$.

c) The intervals P_i and S_i have one corner in common.

Proof: Being a subset of S_i, the interval R_i is partitioned into at most $\overline{n}^1 \times \cdots \times \overline{n}^d$ segments by the knot grid T. Hence, there exists a subinterval Q_i of R_i with $\overline{n} * |Q_i| = |R_i| = |P_i|/r_1$ which is completely contained in a grid cell. Also all other conditions of Definition 3.1 are obviously satisfied.□

With the help of this lemma, the univariate case $d = 1$ can be settled as follows:

Theorem 3.4 *Let T be a knot-sequence for an univariate spline-space of degree $n \geq 1$, and let $\Omega = (a, b) \subset \mathbb{R}$ be an open interval. Then, for every $1 \leq p \leq \infty$, the normalized B-spline basis $(b_i^p)_{i \in I}$ on Ω is $((1,1),p)$-stable if the number of knots in Ω is at least n, i.e.,*

$$\left|\{i \in \mathbb{Z} \colon a < \tau_i < b\}\right| \geq n.$$

Proof: We consider a B-spline b_i^p, $i \in I$, i.e., $S_i \cap \Omega \neq \emptyset$. Since Ω contains at least n knots, either τ_i or $\tau_{i+\bar{n}}$ must lie in Ω. Without loss of generality, we assume $\tau_i \in \Omega$. Defining

$$P_i := R_i := [\tau_i, \min(\tau_{i+\bar{n}}, b)]$$

we have $R_i = S_i \cap \overline{\Omega} = P_i \subset S_i$, $|P_i| = |R_i|$, and P_i and S_i have the knot τ_i in common. Thus, the conditions of Lemma 3.3 are satisfied with $r_1 = 1$. $\square$

We note that the required factor r_1 can become arbitrarily large if the conditions of the theorem are not satisfied. For instance, consider uniform knots $T = \mathbb{Z}$ and the domain $\Omega = (1/4 - \delta, 1/4 + \delta)$. Then, for sufficiently small δ, the normalized B-spline b_0^p is *not* (r, p)-stable for $r_1 < 1/(8\overline{n}\delta)$ and $r_2 = 1$. Indeed, the condition number of the normalized basis $(b_i^p)_{i \in I}$ is not uniformly bounded. In Figure 3, the condition number $\mathrm{cond}_2(b_i^2)_{i \in I}$ is plotted as a function of δ for degrees $n = 1, \ldots, 4$. The dot always indicates the point left of which the condition of the theorem is not satisfied.

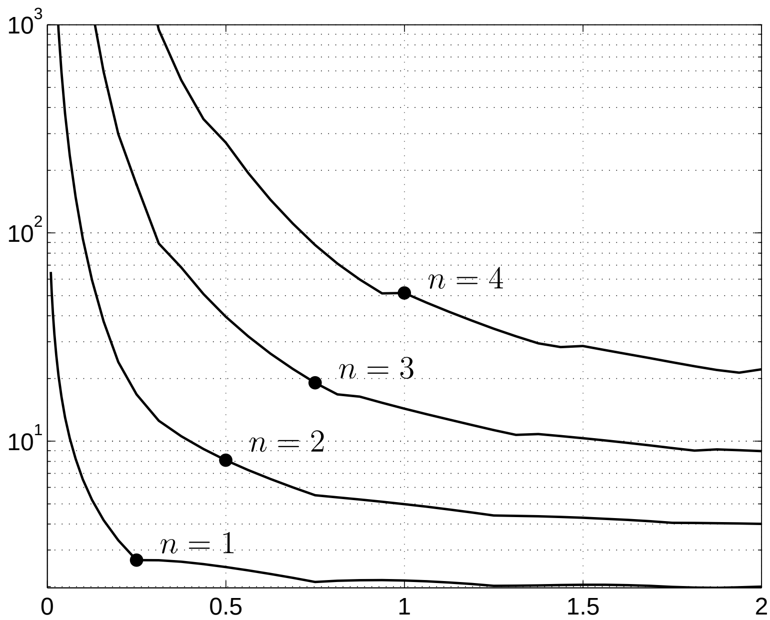

Figure 3: L^2-condition number of univariate B-splines.

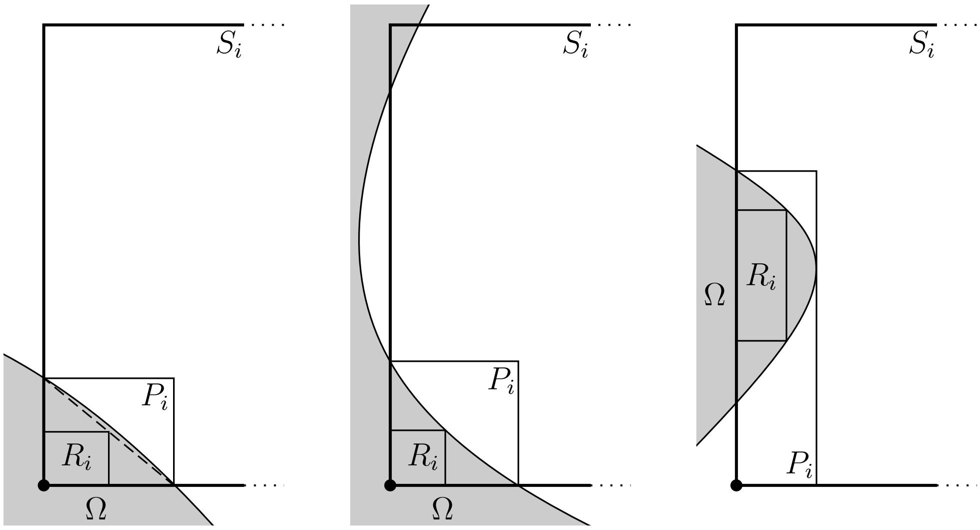

Figure 4: Examples for choices of the intervals P_i and R_i in Lemma 3.3.

Now, we focus on the less trivial multivariate case, and assume $d \geq 2$ for the remainder of this section without further notice. Figure 4 shows some typical cases in two dimensions:

- *(left)* $S_i \cap \Omega$ is convex, and b_i^p is (r, p) for $r = (2, 1)$ and all p, independent of the size of $S_i \cap \Omega$. If the intersection is not convex, then the required r_1 becomes larger, but typically remains bounded for small $S_i \cap \Omega$.
- *(middle)* $S_i \cap \Omega$ is not connected. We choose P_i to cover the larger part to find that b_i^p is (r, p) for $r = (3, 2)$ and all p.
- *(right)* Because P_i must contain one corner of S_i, the ratio between $|P_i|$ and $|R_i|$ can become arbitrarily large if $S_i \cap \Omega$ is small. This case is critical, but rare compared with the first one since it can occur only near points of the boundary where the tangent is parallel to one of the coordinate axes.

Already these three examples in 2 dimensions indicate that finding optimal constants r_1 and r_2 may be a non-trivial task. In the following, we provide a sufficient criterion which is easy to verify and applies to domains with smooth boundary. To this end, we introduce the following notational convention. The first $(d-1)$ components of a point p or an interval $P = [p, q]$ in $\mathbb{R}^d$ are denoted by a superscript star,

$$p = (p^*, p^d), \quad p^* := (p^1, \dots, p^{d-1}),$$
$$P = P^* \times P^d, \quad P^* = [p^*, q^*], \quad P^d = [p^d, q^d].$$

Definition 3.2 *Let $U := [0, u] \subset \mathbb{R}^d$ be an interval, and $\varphi \colon U^* \to \mathbb{R}$ a continuous function. We set*

$$v_\varphi := \min_{x^* \in U^*} \varphi(x^*), \qquad V_\varphi := \max_{x^* \in U^*} \varphi(x^*)$$

and, if φ is C^1,

$$g_\varphi := \min_{x^* \in U^*} \nabla\varphi(x^*), \quad G_\varphi := \max_{x^* \in U^*} \nabla\varphi(x^*),$$

where the extremal values of $\nabla\varphi$ are understood component-wise. We call

$$U_\varphi := \{x \in U : x^d < \varphi(x^*)\}$$

the restriction *of U by φ. Further, an open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is called a $C^{k,\alpha}$*-domain with resolution *$\delta \in \mathbb{R}^d$, if for every interval $S \subset \mathbb{R}^d$ of size $|S| \leq \delta$ there exists a $C^{k,\alpha}$-function φ, an axis-aligned isometry $\mathcal{A}$, and an interval $U = [0, u]$ such that*

$$S \cap \Omega = \mathcal{A}(U_\varphi).$$

The following theorem provides a sufficient condition for (r, p)-stability of a single normalized B-spline near the boundary of a C^1-domain.

Theorem 3.5 *Given a C^1-domain $\Omega \subset \mathbb{R}^d$ with resolution δ, consider the B-spline $b_i^p, i \in I$, with support $S := S_i$ of size $|S| \leq \delta$. Let $S \cap \Omega = \mathcal{A}(U_\varphi)$ according to the previous definition. Then, for all $p \in [1, \infty]$, the normalized B-spline b_i^p is (r, p)-stable with $r_2 = 1$ if either*

$$r_1 v_\varphi \geq u^d \quad \text{or} \quad \frac{r_1 - d}{d - 1} V_\varphi \geq U^* * (G_\varphi - g_\varphi). \tag{13}$$

Proof: The validity of both inequalities is invariant under application of an a³-map. Hence, we can assume without loss of generality that $S = U = [0, 1]^d$, i.e., $u^* = (1, \dots, 1), u^d = 1$.

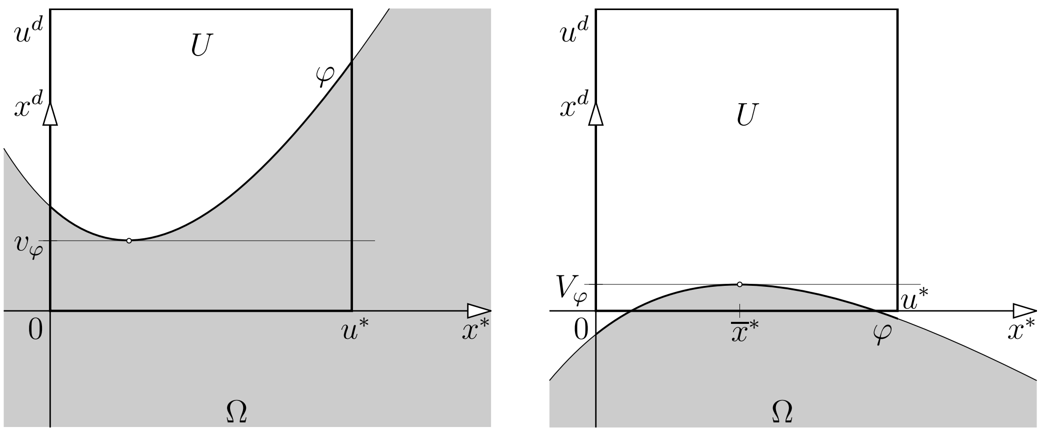

Figure 5: Illustration of Definition 3.5: Typical cases for condition $r_1 v_\varphi \geq u^d$ (left) and $(r_1 - d)V_\varphi/(d-1) \geq U^* * (G_\varphi - g_\varphi)$ (right).

To verify the first condition, we set $P := S$, $R := v_\varphi P$ and observe that the height v_φ of R is bounded from above by the values of φ so that $R \subset S \cap \overline{\Omega}$. Hence, the intervals $P = P_i$ and $R = R_i$ satisfy all assumptions of Lemma 3.3.

To verify the second condition, let $\overline{x}^* \in [0,1]^{d-1}$ denote a point, where φ attains its maximum,

$$V_\varphi = \varphi(\overline{x}^*).$$

We assume that the corner of S closest to $\overline{x}^*$ is the origin, i.e., $\overline{x}^* \in [0,1/2]^{d-1}$. Otherwise, S can be rotated in a suitable way by an axis-aligned isometry to obtain the desired configuration. Consequently, $\nabla\varphi(\overline{x}^*) \leq 0$ so that $g_\varphi \leq 0$. We define the interval $P := [0,p] \subset S$ by

$$p^k := \begin{cases} 1 & \text{if } k < d \text{ and } -G_\varphi^k \leq V_\varphi \\ -V_\varphi/G_\varphi^k & \text{if } k < d \text{ and } -G_\varphi^k > V_\varphi \\ \min\{1, V_\varphi\} & \text{if } k = d, \end{cases}$$

and the interval

$$R := (\overline{x}^*, 0) + P/r_1 = [\overline{x}^*, \overline{x}^* + p^*/r_1] \times [0, p^d/r_1].$$

Now, we show that $P = P_i$ and $R = R_i$ satisfy the assumptions of Lemma 3.3.

First, we have $r_1|R| = |P|$ by definition, and the origin is the common corner of S and P.

Second, we prove that $R \subset S \cap \overline{\Omega}$. To this end, let $x = (x^*, x^d) \in R$. The first component can be written in the form

$$x^* = \overline{x}^* + \eta * p^*/r_1, \quad \eta \in [0,1]^{d-1}.$$

We have $0 \leq \overline{x}^* \leq 1/2$ and, by (13), $r_1 \geq d \geq 2$. Hence, $p^*/r_1 \leq 1/2$ and $x^* \in [0,1]^{d-1}$ so that it remains to show that

$$p^d/r_1 \leq \varphi(\overline{x}^* + \eta * p^*/r_1) \quad \text{for all} \quad \eta \in [0,1]^{d-1},$$

i.e., that the upper face of R is bounded from above by the graph of φ. Obeying $g_\varphi \leq 0$, we estimate the function value at x^* by

$$\varphi(x^*) \geq \varphi(\overline{x}^*) + \sum_{k=1}^{d-1} g_\varphi^k \eta^k p^k/r_1 \geq V_\varphi + \frac{1}{r_1} \sum_{k=1}^{d-1} g_\varphi^k p^k. \tag{14}$$

To estimate $g_\varphi^k p^k$ we have to distinguish to following two cases: For $-G_\varphi^k \leq V_\varphi$ we have $p^k = 1$ and get from (13)

$$\frac{r_1 - d}{d-1} V_\varphi \geq G_\varphi^k - g_\varphi^k \geq -V_\varphi - g_\varphi^k,$$

hence

$$g_\varphi^k p^k = g_\varphi^k \geq \left(-\frac{r_1 - d}{d-1} - 1 \right) V_\varphi = \frac{1 - r_1}{d-1} V_\varphi.$$

Conversely, if $-G_\varphi^k > V_\varphi$, we have $p^k = -V_\varphi / G_\varphi^k$ and get from (13)

$$-g_\varphi^k \leq \frac{r_1 - d}{d-1} V_\varphi - G_\varphi^k \leq -\frac{r_1 - d}{d-1} G_\varphi^k - G_\varphi^k = \frac{1 - r_1}{d-1} G_\varphi^k.$$

Finally, using $G_\varphi^k \leq -V_\varphi \leq 0$, we find

$$g_\varphi^k p^k = -g_\varphi^k \frac{V_\varphi}{G_\varphi^k} \geq \frac{1 - r_1}{d-1} V_\varphi,$$

as in the first case. Applying this estimate to (14) we obtain

$$\varphi(x^*) \geq V_\varphi \left(1 + \frac{(1-r_1)(d-1)}{r_1(d-1)}\right) = \frac{V_\varphi}{r_1} \geq \frac{p^d}{r_1}.$$

Third, we prove that $S \cap \overline{\Omega} \subset P \subset S$. The second inclusion is trivial. To verify the first one, let $(x^*, x^d) \in S \cap \overline{\Omega}$. Then the last coordinate satisfies $0 \leq x^d \leq \min\{1, V_\varphi\} = p^d$, i.e.,

$$x^d \in [0, p^d].$$

It remains to show that $x^k \in [0, p^k]$ for all indices $k < d$. If $p^k = 1$, nothing has to be shown. Otherwise, if $-G_\varphi^k > V_\varphi$, we denote by e_k the kth unit vector to obtain

$$0 \leq \varphi(x^*) \leq \varphi(x^* - x^k e_k) + G_\varphi^k x^k \leq V_\varphi + G_\varphi^k x^k.$$

With $G_\varphi^k < 0$, we finally obtain the desired estimate

$$x^k \leq -V_\varphi / G_\varphi^k = p^k.$$

□

It follows immediately

Corollary 3.6 *Let Ω be a C^1-domain with resolution δ, and let T be a knot sequence with $|S_i| \leq \delta$ for all $i \in I$. Define*

$$B_r^* := (b_i^p)_{i \in I_r^*}, \quad I_r^* := \big\{ i \in I : b_i^p \text{ satisfies } (13) \big\},$$

then $I_r^* \subset I_r^p$ for all $p \in [1, \infty]$, and in particular

$$\operatorname{cond}_p B_r^* \leq M,$$

where the constant M depends on n, p, r, but neither on Ω nor on T.

4 Approximation

Although the focus of this paper is on stability issues, let us briefly discuss some aspects of approximation. More precisely, we want to estimate the additional L^p-error introduced by skipping parts of the full basis B with the goal to show that forming B-splines bases via *"scale and skip"* is a reasonable concept.

To keep things as simple as possible, let us assume that the degree of the spline space is constant, i.e., $n := n^1 = \cdots = n^d$. Further, we define the *global fineness* and the *global mesh-ratio* of the knot sequence T by

$$h := \max_{j=1}^{d} \max_{i} \left(\tau^j_{i^d+1} - \tau^j_{i^d} \right), \quad \varrho := h^{-1} \min_{j=1}^{d} \min_{i} \left(\tau^j_{i^d+1} - \tau^j_{i^d} \right),$$

respectively. To exclude multiple knots, we assume $\varrho > 0$.

The standard reference on approximation properties of full spline spaces on domains is [5]. Specializing the results of this remarkable paper to our setting, we recall the following: If the bounded domain Ω is coordinate-wise convex and satisfies several other technical conditions, then there exist uniformly bounded Hahn-Banach extensions

$$\Lambda_i : L^p(\Omega) \to \mathbb{R}, \quad \|\Lambda_i\|_{p,\Omega} \le C_{n,p,\varrho,\Omega},$$

of the suitably modified de Boor-Fix functionals, where here and in the following

$$C = C_{n,p,\varrho,\Omega}$$

is a generic constant depending on n, p, ϱ, Ω, which may attain different values at each occurrence, even at the same line. The error of the spline

$$\Lambda f := \sum_{i \in I} \Lambda_i(f) b_i$$

is bounded by

$$\|f - \Lambda f\|_{p,\Omega} \le C h^{\overline{n}} \sum_{k=1}^{d} \|\partial_k^{\overline{n}} f\|_{p,\Omega}. \tag{15}$$

Notably, only pure partial derivatives appear on the right hand side. However, we conjecture that the dependence of the constant C on the global mesh ratio ϱ is merely due to technical limitations of the known proofs, and does not indicate real problems when approximating with B-splines on grids with a bad aspect ratio.

Now, we consider approximation in the reduced space $V(B_r^*)$ on a $C^{1,1}$-domain Ω,

$$\Lambda_r^* f := \sum_{i \in I_r^*} \Lambda_i(f) b_i.$$

The additional error, introduced by skipping parts of the full basis, is bounded by

$$\|(\Lambda - \Lambda_r^*) f\|_{p,\Omega}^p \leq C \sum_{i \in I \setminus I_r^*} \|\Lambda_i(f) b_i\|_{p,\Omega}^p \leq C\, |I \setminus I_r^*| \max_{i \in I \setminus I_r^*} \|b_i\|_{p,\Omega}^p, \tag{16}$$

where the first estimate follows from the fact that at most $\overline{n}^d$ B-splines cover a given grid cell. The number of skipped B-splines is bounded by $\overline{n}^d$ times the number of grid cells intersecting the boundary. Hence, by smoothness and compactness of the boundary of Ω,

$$|I \setminus I_r^*| \leq C h^{1-d}. \tag{17}$$

It remains to estimate the norm of a B-spline b_i which does not satisfy (13). In particular, using the notation of Theorem 3.5, we have

$$\frac{r_1 - d}{d - 1} V_\varphi < U^* * (G_\varphi - g_\varphi).$$

By Lipschitz-continuity of $\nabla\varphi$ and compactness of the boundary, the maximal difference of gradients on $U^* \subset [0, h]^{n-1}$ is bounded by $G_\varphi - g_\varphi \leq Ch$. Hence, $V_\varphi < Ch^2$, where the constant C depends now also on r. Assuming without loss of generality that the axis-aligned isometry $\mathcal{A}$ according to Definition 3.2 is the identity, we obtain the inclusion

$$S_i \cap \Omega \subset [0, \overline{n}h]^{d-1} \times [0, Ch^2]$$

for the support of b_i. Hence,

$$\|b_i\|_{p,\Omega}^p \leq C h^{d-1} \int_0^{Ch^2} \big(b_{i^d}(t)\big)^p \, dt.$$

For $\tau = 0$, the univariate Marsden identity (2) for the dth coordinate yields

$$\psi_{i^d}(0) b_{i^d}(t) \leq \sum_{j \geq i^d} \psi_j(0) b_j(t) = t^n, \quad t \geq 0,$$

because $\psi_j(0)$ is non-negative for $j \geq i^d$. Since the mesh-ratio ϱ is assumed to be positive, we have $\psi_{i^d}(0) = \prod_{j=1}^n \tau_{i^d+j} \geq n!(\varrho h)^n$ and obtain $b_{i^d}(t) \leq C\,(t/h)^n$. Hence,

$$\|b_i\|_{p,\Omega}^p \leq C\,h^{d-1} \int_0^{Ch^2} (t/h)^{np}\,dt \leq C\,h^{np+d+1}.$$

Substituting this estimate and (17) into (16), we obtain

$$\|(\Lambda - \Lambda_r^*)f\|_{p,\Omega} \leq Ch^{n+2/p},$$

and finally

$$\|f - \Lambda_r^* f\|_{p,\Omega} \leq Ch^{\overline{n}} \left(h^{2/p-1} + \sum_{k=1}^{d} \|\partial_k^{\overline{n}} f\|_{p,\Omega} \right),$$

where the constant C depends on n, p, r, ϱ, and Ω. That is, the L^p-approximation order is optimal for $p \leq 2$, and reduced at most by 1 for $p > 2$.

We note that, typically, the number of omitted B-splines is substantially smaller than estimated here. For instance, if the boundary of Ω has a positive Gaussian curvature everywhere, one can show that $|I \backslash I_r^*| \leq Ch^{2-d}$. This leads to a similar error estimate as above, but with the term $h^{2/p-1}$ being replaced by $h^{3/p-1}$.

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