A Chart Preserving the Normal Vector and Extensions of Normal Derivatives in Weighted Function Spaces

Katrin Schumacher*

Given a domain Ω of class $C^{k,1}$, $k \in \mathbb{N}$ we construct a chart that maps normals to the boundary of the half space to normals to the boundary of Ω in the sense that $\frac{\partial}{\partial x_n}\alpha(x',0) = -N(x')$ and that still is of class $C^{k,1}$. As an application we prove the existence of a continuous extension operator for all normal derivatives of order 0 to k on domains of class $C^{k,1}$. The construction of this operator is performed in weighted function spaces where the weight function is taken from the class of Muckenhoupt weights.

Key Words and Phrases: Chart, coordinate transformation, normal vector, normal derivative, extension theorem
2000 Mathematics Subject Classification Numbers: 35A99, 46E35, 47A20

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a $C^{k,1}$-domain, i.e., the boundary $\partial\Omega$ can locally be expressed as the graph of a $C^{k,1}$-function

$$a : V \cap (\mathbb{R}^{n-1} \times \{0\}) \to \mathbb{R}$$

with an appropriate open set $V \subset \mathbb{R}^n$; here $k \in \mathbb{N}$. Then we are looking for a chart $\alpha : V \to U \subset \mathbb{R}^n$ of regularity as high as possible such that

$$\frac{\partial}{\partial x_n}\alpha(x',0) = -N(x') \quad \text{whenever} \quad (x',0) \in V, \tag{1.1}$$

where $N(x')$ denotes the unit outer normal vector at $(x', a(x')) \in \partial\Omega$. This means that normals to the boundary of the half space are mapped to normals to $\partial\Omega$. The natural mapping with this property is

$$x = (x', x_n) \mapsto \begin{pmatrix} x' \\ a(x') \end{pmatrix} - x_n \cdot N(x'). \tag{1.2}$$

*Department of Mathematics, Darmstadt University of Technology, Schlossgartenstr. 7, 64289 Darmstadt, Germany, e-mail: schumacher@mathematik.tu-darmstadt.de

However, if a is a $C^{k,1}$-function, then, since it includes the outer normal N, the chart (1.2) is only of class $C^{k-1,1}$.

For this reason we introduce a different chart which conserves the $C^{k,1}$-regularity and still has the property (1.1).

Coordinate transforms as in (1.2) are used e.g. by Nečas [13], to prove extension theorems of normal derivatives, see also Chapter 4 of this paper. Moreover, in [12, Chapter 4.1] Giga uses such a coordinate transformation to obtain symbols of pseudo-differential operators of a particular form. In a similar context, according to Abels, the proof of results in [1] can be significantly simplified by the use of a chart with the property (1.1) but which preserves the regularity of a.

In the second part of this paper we present an application of the chart mentioned above. We prove the existence of a continuous operator extending functions defined on the boundary in the following way. Given functions $g_1, ..., g_m$ on the boundary we find a function u defined on Ω such that

$$\frac{\partial^j u}{\partial N^j} = g_j \quad \text{on} \quad \partial\Omega, \;\; 0 \leq j \leq k.$$

In the context of classical Sobolev spaces this result can be found in [13].

The result of [13] is generalized in two aspects. First, using the particular chart constructed in the first part of this paper, one can deal with more general domains. More precisely, one can permit domains with a boundary regularity that is of one order lower than in the former results. Using this it is possible to show that the results on very weak solutions to the Navier-Stokes equations by Galdi, Simader and Sohr in [10] and by Farwig, Galdi and Sohr in [5] hold not only in $C^{2,1}$-domains but, more generally, in $C^{1,1}$-domains. This can be seen in [14] where a weighted approach to this problem is given.

Second, we consider the problem in weighted function spaces. This means, we consider weighted Lebesgue spaces $L^q_w(\Omega)$ and Sobolev spaces $W^{k,q}_w(\Omega)$ which means that we integrate with respect to the measure $w\,dx$ for an appropriate weight function w, see Section 3 below for the exact definition of these spaces.

All weight functions, that we use are contained in the Muckenhoupt class A_q. This is the class of nonnegative and locally integrable weight functions, for which the expression

$$A_q(w) := \sup_Q \left(\frac{1}{|Q|} \int_Q w\,dx \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{q-1}} dx \right)^{q-1}$$

is finite, where the supremum is taken over all cubes Q in $\mathbb{R}^n$. As shown in [6] examples of Muckenhoupt weights are $w(x) = (1+|x|)^\alpha$, with $-n < \alpha < n(q-1)$ or $\operatorname{dist}(x, M)^\alpha$, $-(n-k) < \alpha < (n-k)(q-1)$, where M is a compact k-dimensional Lipschitzian manifold.

One reason why the class of Muckenhoupt weights is appropriate for analysis is that the maximal operator is continuous in weighted L^q-spaces, if and only if the weight function is a Muckenhoupt weight. Thus the powerful tools of harmonic analysis may be applied, cf. García-Cuerva and Rubio de Francia [11] and Stein [16]. Moreover, there is a powerful extrapolation theorem by Curbera, García-Cuerva, Martell and Pérez [3] that guarantees estimates in very general Banach function spaces provided that the estimates in weighted function spaces are known for all weights from the Muckenhoupt class A_q.

2 Construction of the Chart

Let $\Omega \subset \mathbb{R}^n$ be a $C^{k,1}$-domain, $k \in \mathbb{N}$. This means that for every $x_0 \in \partial\Omega$ we can rotate and shift the coordinate system such that its origin is x_0 and such that in a neighborhood $U(x_0)$ of x_0 one has

$$\partial\Omega \cap U(x_0) = \{(x', a(x')) \mid x' \in V(0)\}, \tag{2.1}$$

where $V(0)$ is an appropriate ($(n-1)$-dimensional) neighborhood of 0 and $a : V(0) \to \mathbb{R}$ is a $C^{k,1}$-function.

Lemma 2.1. *For $k \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^n$ be a $C^{k,1}$-domain. Then for every $x_0 \in \partial\Omega$ there exists a neighborhood $U \subset \mathbb{R}^n$ of x_0, a neighborhood $V \subset \mathbb{R}^n$ of 0 and a bijective map $\alpha : V \to U$ such that*

$$\alpha(0) = x_0, \quad \alpha(V \cap (\mathbb{R}^{n-1} \times \{0\})) = U \cap \partial\Omega, \qquad \alpha(V \cap \mathbb{R}^n_+) = U \cap \Omega$$

and with the following properties:

(1) $\alpha \in C^{k,1}(V, U)$,

(2) $\frac{\partial}{\partial x_n}\alpha(x', 0) = -N(x')$ *and* $\left(\frac{\partial}{\partial x_n}\right)^j \alpha(x', 0) = 0$ *for* $j \geq 2$ *even.*

(3) With the notation of (2.1) one has

- *a)* $\|\alpha\|_{C^{k,1}(V,U)}$ *can be estimated by* $\|a\|_{C^{k,1}(V \cap (\mathbb{R}^{n-1} \times \{0\}))}$.
- *b) There exists some $r > 0$ which only depends on the sets $U(x_0)$, $V(x_0)$ and the size of $\|a\|_{C^{k,1}(V \cap (\mathbb{R}^{n-1} \times \{0\}))}$ such that $B_r(x_0) \subset U$.*

Proof. We use the notation $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$ and $\partial^\gamma = \partial^{\gamma_1}_{x_1} \cdot \ldots \cdot \partial^{\gamma_1}_{x_n}$ for $\gamma \in \mathbb{N}^n$.

After rotating and shifting the coordinate system we may assume that $x_0 = 0$, $(0, a(0)) = 0$ and $\nabla a(0) = 0$.

Let $0 \leq \rho \in C_0^\infty(\mathbb{R}^{n-1})$ be radially symmetric such that

$$\operatorname{supp} \rho \subset B_1(0) \quad \text{and} \quad \int_{\mathbb{R}^{n-1}} \rho = 1.$$

For $t \neq 0$ we set $\rho_t(x') = \frac{1}{|t|^{n-1}}\rho(\frac{x'}{t})$. We define the function α as follows:

$$\alpha(x', x_n) = \begin{cases} \binom{x'}{a(x')} - (x_n \rho_{x_n} * N)(x') & \text{if } x_n \neq 0 \\ \binom{x'}{a(x')} & \text{if } x_n = 0, \end{cases}$$

where the convolution takes place in $\mathbb{R}^{n-1}$.

Obviously, if $V \subset \mathbb{R}^n$ is small enough, then $\alpha(x', 0) \in \partial\Omega$ for $(x', 0) \in V$. Moreover, since $\partial\Omega$ is at least of class $C^{1,1}$ it follows easily from the construction of α that $\alpha(x', x_n) \in \Omega$ and $\alpha(x', -x_n) \notin \Omega$, if $x_n > 0$ is small.

Next we show (1) For every multi-index $\gamma = (\gamma', \gamma_n) \in \mathbb{N}_0$, with $|\gamma| \leq k$ and $|\gamma'| < k$ one has for $x_n \neq 0$

$$\begin{aligned}\partial^\gamma (x_n \rho_{x_n} * N)(x') =& \gamma_n (-1)^{\gamma_n - 1} \int \rho(\xi) \nabla^{\gamma_n - 1} \partial^{\gamma'} N(x' - x_n \xi) \underbrace{(\xi, ..., \xi)}_{\gamma_n - 1} d\xi \\ &+ x_n \frac{\partial}{\partial x_n} \left((-1)^{\gamma_n - 1} \int \rho(\xi) \nabla^{\gamma_n - 1} \partial^{\gamma'} N(x' - x_n \xi) \underbrace{(\xi, ..., \xi)}_{\gamma_n - 1} d\xi \right).\end{aligned}$$

Then using change of variables an the fact that the map $\nabla^{\gamma_n - 1} \partial^{\gamma'} N(x' - \xi)$ is $(\gamma_n - 1)$-linear, the second summand is equal to

$$\begin{aligned}&(-1)^{\gamma_n - 1} x_n \frac{\partial}{\partial x_n} \int \frac{1}{|x_n|^{n + \gamma_n - 2}} \rho\left(\frac{\xi}{x_n}\right) \nabla^{\gamma_n - 1} \partial^{\gamma'} N(x' - \xi)(\xi, ..., \xi) d\xi \\ &= (-1)^{\gamma_n - 1} \int ((-n - \gamma_n + 2)\rho(\xi) - \nabla\rho(\xi) \cdot \xi)\, \nabla^{\gamma_n - 1} \partial^{\gamma'} N(x' - \xi x_n) \underbrace{(\xi, ..., \xi)}_{\gamma_n - 1} d\xi.\end{aligned}$$

Hence

$$\begin{aligned}\partial^\gamma (x_n \rho_{x_n} * N)(x') =& (-1)^{\gamma_n - 1} \int ((-n + 2)\rho(\xi) - \nabla\rho(\xi) \cdot \xi) \\ &\cdot \nabla^{\gamma_n - 1} \partial^{\gamma'} N(x' - \xi x_n) \underbrace{(\xi, ..., \xi)}_{\gamma_n - 1} d\xi.\end{aligned} \tag{2.2}$$

Still we have to consider the case $|\gamma'| = k$ in which the situation is easier. Integration by parts yields

$$\partial^\gamma (x_n \rho_{x_n} * N)(x') = \int \partial^{\beta_1} \rho(\xi) \partial^{\beta_2} N(x' - x_n \xi) d\xi, \tag{2.3}$$

where $\gamma = \beta_1 + \beta_2$ and $|\beta_1| = 1$.

The map $x \mapsto \binom{x'}{a(x')}$ is of type $C^{k,1}$ because a is. It remains to show that $\partial^\gamma (x_n \rho_{x_n} * N(x'))$ is Lipschitz continuous for every $\gamma \in \mathbb{N}^n$, $|\gamma| \leq k$. This is an easy consequence of the representations (2.2) and (2.3) and of $N \in C^{k-1,1}$, e.g.,

$$\begin{aligned}&|\partial^\gamma (x_n \rho_{x_n} * N(x')) - \partial^\gamma (y_n \rho_{y_n} * N(y'))| \\ &\leq \int_{B_1(0)} |c_n \rho(\xi) - \nabla\rho(\xi) \cdot \xi| \, |\nabla^{\gamma_n - 1} \partial^{\gamma'} N(x' - \xi x_n) - \nabla^{\gamma_n - 1} \partial^{\gamma'} N(y' - \xi y_n)| d\xi \\ &\leq c L_\gamma |x - y|.\end{aligned}$$

A similar calculation shows that the right-hand side of (2.3) is Lipschitz continuous.

It remains to show (2) From (2.2) we have for $j > 1$ even

$$\begin{aligned}\left(\frac{\partial}{\partial x_n}\right)^j \alpha(x',0) &= (-1)^{j-1}\int((-n+2)\rho(\xi)-\nabla\rho(\xi)\cdot\xi)\nabla^{j-1}N(x')(\xi,...,\xi)d\xi \\ &=(-1)^{j-1}\left(\int(-n+2)\rho(\xi)\nabla^{j-1}N(x')(\xi,...,\xi)d\xi\right. \\ &\quad\left.+\sum_{k=1}^{n-1}\int\rho(\xi)\left[\nabla^{j-1}N(x')(\xi,...,\xi)+(j-1)\xi_k\nabla^{j-1}N(x')(e_k,\xi,...,\xi)\right]d\xi\right) \\ &=(-1)^{j-1}j\int\rho(\xi)\nabla^{j-1}N(x')(\xi,...,\xi)d\xi = 0,\end{aligned}$$

since ρ is presumed to be rotationally symmetric and $\xi \mapsto \nabla^{j-1}N(x')(\xi,...,\xi)$ is an odd function for $j-1$ odd. Similarly,

$$\frac{\partial}{\partial x_n}\alpha(x',0) \;=\; -N(x')\left((2-n)\int\rho(\xi)d\xi-\sum_{i=1}^{n-1}\int\partial_i\rho(\xi)\xi_i d\xi\right) = -N(x').$$

It remains to show (3) b).
By (2.2) and (2.3) one has, since $\nabla a(0)=0$ and $N(0)=-e_n$,

$$\nabla\alpha(0) \;=\; \nabla\begin{pmatrix} x' \\ a(x')\end{pmatrix} - \nabla(x_n\rho_{x_n} * N(x'))\big|_{x=0} = \mathrm{id}\,.$$

Since $\nabla\alpha$ is Lipschitz continuous with constant K, we get for $x,y\in\overline{B_r(0)}$, $r<\frac{1}{2K}$, that

$$\begin{aligned}|\alpha(x)-\alpha(y)| &= \sup_{|v|=1}|v\cdot\nabla\alpha(\xi_v)(x-y)|, \qquad \xi_v\in\{(1-t)x+ty\,|\,t\in(0,1)\} \\ &\geq \inf_{\xi\in B_r(0)}\left(\frac{|x-y|^2}{|x-y|}-\left|(x-y)(\nabla\alpha(\xi)-\nabla\alpha(0))\frac{x-y}{|x-y|}\right|\right) > \frac{1}{2}|x-y|.\end{aligned}$$

From this inequality it immediately follows that α is injective on $B_r(0)$.

Moreover, it is easily seen that $B_{\frac{r}{2}}(x_0)\subset\alpha B_r(0)$. Indeed for $x\in\partial B_r(0)$ one has $|\alpha(x)-x_0|>\frac{1}{2}|x-0|$. Since $\nabla\alpha(x)$ is invertible for every $x\in B_r(0)$ it follows from the Inverse Function Theorem that $\alpha(B_r(0))$ is open. Together with the continuity of α we obtain

$$B_{\frac{r}{2}}(x_0)\cap\partial\alpha(B_r(0)) = B_{\frac{r}{2}}(x_0)\cap\alpha(\partial B_r(0)) = \emptyset.$$

Assume now that $y\in B_{\frac{r}{2}}(x_0)\setminus\alpha(B_r(0))$. Then the straight line from y to x_0 intersects $\partial\alpha(B_r(0))$. Thus this intersection point is contained in the intersection which we have shown to be empty. This is a contradiction.

This argument finishes the proof. □

3 Weighted Function Spaces

In Section 4 we want to prove an extenstion theorem that prescribes a low boundary regularity. Since this is done in weighted function spaces in this section we collect the basic definitions of weight functions and function spaces, which are needed in this below.

Definition 3.1. *Let A_q, $1 < q < \infty$, the set of Muckenhoupt weights, be given by all $0 \le w \in L^1_{loc}(\mathbb{R}^n)$ for which*

$$A_q(w) := \sup_Q \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{q-1}}dx\right)^{q-1} < \infty. \tag{3.1}$$

The supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $|Q|$ denotes the Lebesgue measure od Q. To avoid trivial cases, we exclude the case where w vanishes almost everywhere.

We introduce some function spaces. First by $C_0^\infty(\Omega)$ we denote the space of smooth functions with compact support in Ω. For $1 < q < \infty$, $w \in A_q$ and an open set Ω we define the weighted Lebesgue space by

$$L^q_w(\Omega) := \left\{ f \in L^1_{loc}(\overline{\Omega}) \mid \|f\|_{q,w} := \left(\int_\Omega |f|^q w\,dx\right)^{\frac{1}{q}} < \infty\right\}.$$

For $k \in \mathbb{N}_0$, the set of nonnegative integers, the weighted Sobolev spaces are defined by

$$W^{k,q}_w(\Omega) = \left\{ u \in L^q_w(\Omega) \;\middle|\; \|u\|_{k,q,w} := \sum_{|\alpha|\le k} \|D^\alpha u\|_{q,w} < \infty\right\}.$$

Finally for $k \in \mathbb{N}$ we define the space $T^{k,q}_w(\partial\Omega) := (W^{k,q}_w(\Omega))|_{\partial\Omega}$ equipped with the norm $\|\cdot\|_{T^{k,q}_w} = \|\cdot\|_{T^{k,q}_w(\partial\Omega)}$ of the factor space, i.e.,

$$\|g\|_{T^{k,q}_w(\partial\Omega)} := \inf\left\{\|u\|_{W^{k,q}_w(\Omega)} \;\middle|\; u \in W^{k,q}_w(\Omega) \text{ and } u|_{\partial\Omega} = g\right\}.$$

By [7], [9] and [2] the spaces $L^q_w(\Omega)$, $W^{k,q}_w(\Omega)$ and $T^{k,q}_w(\partial\Omega)$ are reflexive Banach spaces in which $C_0^\infty(\overline{\Omega})$ or $C_0^\infty(\Omega)$ or $C_0^\infty(\overline{\Omega})|_{\partial\Omega}$, respectively, are dense.

Note that by Slobodeckiĭ [15] and Nečas [13, Chapitre 2, §5] in the unweighted case one has

$$T^{k,q}_1(\partial\Omega) = W^{k-\frac{1}{q},q}(\partial\Omega).$$

However, in the setting of Muckenhoupt weights such a characterization of the spaces by an intrinsic norm is known only for few examples of weight functions.

For weighted function spaces change of variables is possible in the following sense.

Lemma 3.2. *Let Ω and $\mathcal{O}$ be two domains in $\mathbb{R}^n$ and*

$$\alpha : \overline{\mathcal{O}} \to \overline{\Omega}$$

a $C^{k-1,1}$-diffeomorphism, $k \ge 1$.

1. *The operator*
$$T : u \mapsto u \circ \alpha : \quad W^{k,q}_w(\Omega) \to W^{k,q}_{w\circ\alpha}(\mathcal{O})$$
is continuous.

2. *The same is true for the operator*
$$S : g \mapsto g \circ \alpha : \quad T^{k,q}_w(\partial\Omega) \to T^{k,q}_{w\circ\alpha}(\partial\mathcal{O}).$$

Proof. The first assertion follows immediately from the change of variables formula, the second follows from the first using the definition of $T_w^{k,q}(\partial\Omega)$. $\square$

By [8] the following weighted analogue of the Poincaré inequality holds: there exists a constant $c = c(q, w) > 0$ such that

$$\|u\|_{q,w} \leq c\|\nabla u\|_{q,w} \quad \text{for every} \quad u \in W_{w,0}^{1,q}(\Omega). \tag{3.2}$$

Moreover the solvability of the following Laplace resolvent problem continues to hold in weighted function spaces.

Theorem 3.3. (Regularity of the Dirichlet Problem)
Let $1 < q < \infty$, $k \in \mathbb{Z}$, $k \geq -1$ and let $f \in W_w^{k,q}(\mathbb{R}_+^n)$. Then there exists a unique weak solution $u \in W_w^{k+2,q}(\mathbb{R}_+^n)$ to the Dirichlet Problem

$$(1 - \Delta)u = f \quad \textit{and} \quad u|_{\mathbb{R}^{n-1}} = 0.$$

It fulfills the estimate $\|u\|_{k+2,q,w} \leq c\|f\|_{k,q,w}$, where $c = c(k, q, w)$.

The same is true for the solution u of $(1 - \Delta)u = 0$, $u|_{\mathbb{R}^{n-1}} = g$, if $g \in T_w^{k+2,q}(\mathbb{R}^{n-1})$, i.e., it fulfills the estimate

$$\|u\|_{k+2,q,w} \leq c\|g\|_{T_w^{k+2,q}}.$$

Proof. For $k = -1$ the first assertion has been proved by Fröhlich in [7]. Using this, one obtains the regularity of this boudary value problem as in the classical unweighted case which can be found e.g. in Evans [4].

For the second assertion let $v \in W_w^{k+2,q}(\mathbb{R}_+^n)$ be an extension of g. Then we find a unique $u \in W_w^{k+2,q}(\mathbb{R}_+^n)$ with $(\mathrm{id} - \Delta)u = (\mathrm{id} - \Delta)v$ and $u|_{\mathbb{R}^{n-1}} = 0$. Thus $v - u$ solves the problem and by the first assertion it fulfills the estimate. $\square$

4 Extensions of Functions on the Boundary

Our next objective is to construct a linear extension operator that maps functions defined on the boundary $\partial\Omega$ to a function defined on the domain Ω whose boundary values or normal derivatives are the given preimages.

We start with the half space.

Theorem 4.1. *Let $1 < q < \infty$, $w \in A_q$ and $k \in \mathbb{N}$. Then there exists a continuous linear operator*

$$T : \prod_{j=0}^{k-1} T_w^{k-j,q}(\mathbb{R}^{n-1}) \to W_w^{k,q}(\mathbb{R}_+^n)$$

such that $\frac{\partial^j}{\partial x_n^j} T(g_0, ..., g_{k-1})|_{x_n=0} = g_j$, $j = 0, ..., k-1$.

Proof. It suffices to show that for every $g \in T_w^{k-j,q}(\mathbb{R}^{n-1})$, $j = 0, ..., k-1$, there exists $u \in W_w^{k,q}(\mathbb{R}_+^n)$ depending continuously and linearly on g such that $\frac{\partial^j}{\partial x_n^j} u = g$ and $\frac{\partial^i}{\partial x_n^i} u =$

0 for every $i = 0, ..., j-1$. To see this assume that for every $j = 0, ..., k-1$ there exists a continuous linear operator

$$T_j : T_w^{k-j,q}(\mathbb{R}^{n-1}) \to W_w^{k,q}(\mathbb{R}^n_+), \quad \frac{\partial^i}{\partial x_n^i} T_j(h)\big|_{x_n=0} = \begin{cases} 0, & \text{if } i < j \\ h, & \text{if } i = j \end{cases}.$$

For $g = (g_0, ..., g_{k-1})$ we can define $S_0(g) := T_0(g)$ and

$$S_{j+1}(g) := S_j(g) + T_{j+1}\left(g_{j+1} - \frac{\partial^{j+1}}{\partial x_n^{j+1}} S_j(g)\right).$$

Then $T = S_{k-1}$ solves our problem.

Next we show the weaker assertion. For $g \in T_w^{k-j,q}(\mathbb{R}^{n-1})$ let $v \in W_w^{k-j,q}(\mathbb{R}^n_+)$ with $(1-\Delta)v = 0$ and $v|_{\mathbb{R}^{n-1}} = g$ which is uniquely defined by Theorem 3.3. Let $\zeta \in C^\infty(\mathbb{R}_+)$ be a cut-off function with $\zeta(t) = 1$ for $t < 1$ and $\zeta(t) = 0$ for $t > 2$. We set

$$\phi(x) = \phi(x_n) = \frac{1}{j!} x_n^j \cdot \zeta(x_n) \quad \text{and} \quad u(x) = \phi(x) v(x). \tag{4.1}$$

We show that ϕu solves the problem. More precisely we prove the following claim:

If $\phi \in C^\infty(\overline{\mathbb{R}^n_+})$ with $\phi(x) = \phi(x_n)$, $\operatorname{supp}\phi \subset \mathbb{R}^{n-1} \times [0,2]$ and $(\frac{\partial}{\partial x_n})^m \phi|_{x_n=0} = 0$ for $m = 0, ..., l$ and $v \in W_w^{k,q}(\mathbb{R}^n_+)$ with $(1-\Delta)v = 0$ then $\phi v \in W_w^{k+l,q}(\mathbb{R}^n_+)$ with $\|\phi v\|_{k+l,q,w} \le c\|v\|_{k,q,w}$.

To prove this we use mathematical induction with respect to l and assume that we already know the assertion is true for $l-1$, $l-2$ and all k.

Since $(1-\Delta)v = 0$ we obtain

$$(1-\Delta)(\phi v) = -\Delta\phi v - 2\nabla v \cdot \nabla\phi. \tag{4.2}$$

As $(\frac{\partial}{\partial x_n})^m \Delta\phi|_{x_n=0} = 0$ for $m = 0, ..., l-2$, $(\frac{\partial}{\partial x_n})^m \nabla\phi|_{x_n=0} = 0$ for $m = 0, ..., l-1$ and $(1-\Delta)\nabla v = 0$, (4.2) and the induction hypothesis yield $(1-\Delta)(\phi v) \in W_w^{k+l-2,q}(\mathbb{R}^n_+)$. Thus and since $\phi v|_{\mathbb{R}^{n-1}} = 0$, one has $\phi v \in W_w^{k+l,q}(\mathbb{R}^n_+)$ by the regularity of the Laplace resolvent problem. Moreover

$$\|\phi v\|_{k+l,q,w} \le c\|(\Delta\phi)v + 2\nabla v \nabla\phi\|_{k+l-2,q,w} \le c(\|v\|_{k,q,w} + \|\nabla v\|_{k-1,q,w}) \le c\|v\|_{k,q,w}.$$

For the start of induction we need the cases $l = 0$ and $l = 1$. The case $l = 0$ is trivial, the case $l = 1$ is proved in the same way as the induction step.

If one applies the above claim to $u\phi v$ given by (4.1) we get $u \in W_w^{k,q}(\Omega)$. Moreover,

$$\frac{\partial^l}{\partial x_n^l} u(x', 0) = \sum_{\nu=0}^{l} \binom{l}{\nu} \frac{\partial^\nu}{\partial x_n^\nu} v \frac{\partial^{l-\nu}}{\partial x_n^{l-\nu}} \phi(x', 0) = \begin{cases} 0 & \text{if } l < j \\ g(x') & \text{if } l = j. \end{cases}$$

This shows the assertion about the boundary values. □

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{k-1,1}$-domain, $k \ge 1$. Then there exists a continuous linear operator*

$$L : \prod_{j=0}^{k-1} T_w^{k-j,q}(\partial\Omega) \to W_w^{k,q}(\Omega)$$

such that $\frac{\partial^j}{\partial N^j} L(g)|_{\partial\Omega} = (-1)^j g_j$, $0 \le j \le k-1$, where $g = (g_0, ..., g_{k-1})$.

Proof. As in the proof of Theorem 4.1 we construct an operator

$$L_j : T_w^{k-j,q}(\partial\Omega) \to W_w^{k,q}(\Omega), \quad \frac{\partial^k}{\partial N^j} L_j(g) = \begin{cases} (-1)^j g & \text{if } k = j \\ 0 & \text{if } k < j. \end{cases}$$

Then the general case follows as in the proof of Theorem 4.1.

We choose the collection of charts $(\alpha_i, V_i, U_i)_{i=1}^m$ according to Lemma 2.1 and a decomposition of unity $(\phi_i)_{i=1}^m$ subordinate to the covering $\{U_i\}$.

To simplify the notation we fix i and set $\gamma = \alpha_i$, $U = U_i$, $V = V_i$ and $\phi = \phi_i$. Moreover, for $g \in T_w^{k-j,q}(\partial\Omega)$ we set $\tilde{g} = (g \cdot \phi) \circ \gamma$. By Lemma 3.2 we know $\tilde{g}_j \in T_{w\circ\gamma}^{k-j,q}(\mathbb{R}^{n-1})$. Thus we may apply the operator T from Theorem 4.1 and set

$$v := v_i := L_{i,j}(g) := (\psi_i T(0, ..., 0, \tilde{g}, 0, ..., 0)) \circ \gamma^{-1},$$

meaning, that the j'th component of $(0, ..., 0, \tilde{g}, 0, ..., 0)$ is $\tilde{g}$.

Moreover, $(\psi_i)_i \subset C_0^\infty(\overline{\mathbb{R}}_+^n)$ with $\psi_i = 1$ in a neighborhood of $\operatorname{supp} \tilde{g}$ and $\operatorname{supp} \psi_i \subset V_i$. Here ψ_i can be chosen such that $\frac{\partial^k}{\partial x_n^k}\psi_i(x', 0) = 0$ for every $k \in \mathbb{N}$.

Then we have by the choice of γ according to Lemma 2.1 for every $k \leq j$.

$$\begin{aligned}
(-1)^k \delta_{j,k} \tilde{g}(x') &= \frac{\partial^k}{\partial x_n^k} T(...0, \tilde{g}, 0...)(x', 0) = \left(\frac{\partial^k}{\partial x_n^k}(v \circ \gamma)\right)(x', 0) \\
&= (\nabla^k v \circ \gamma) \cdot (\partial_n \gamma, ..., \partial_n \gamma)(x', 0) + \text{terms containing } \nabla^i v \circ \gamma(x', 0),\ i < j \\
&= (\nabla^k v(\gamma(x', 0)))\underbrace{(-N(x'), ..., -N(x'))}_{k} = (-1)^k \left(\frac{\partial^k}{\partial^k N} v\right)(\gamma(x', 0)).
\end{aligned}$$

The terms containing $\nabla^i v \circ \gamma(x', 0)$ vanish for $i < j$, since

$$\nabla^i (v \circ \gamma)(x', 0) = \nabla^i (\psi_i T(0, ..., 0, \tilde{g}, 0, ..., 0))(x', 0) = 0$$

for $i = 1, ..., j-1$ by the definition of T.

Finally we set $L_j(g) = \sum_{i=1}^m L_{i,j}(g)$, and obtain

$$\frac{\partial^k}{\partial N^k} L_j(g)|_{\partial\Omega} = \sum_{i=1}^m \frac{\partial^k}{\partial N^k} L_{i,j}(g)|_{\partial\Omega} = \begin{cases} g & \text{if } k = j \\ 0 & \text{if } k < j. \end{cases}$$

The continuity of L_j follows from Lemma 3.2 and the continuity of T in Theorem 4.1. □

References

[1] H. Abels. Reduced and generalized Stokes resolvent equations in asymptotically flat layers. II: H_∞-calculus. *J. Math. Fluid Mech.*, 7:223–260, 2005.

[2] S.-K. Chua. Extension theorems on weighted Sobolev spaces. *Indiana Univ. Math. J.*, 41:1027–1076, 1992.

[3] G. P. Curbera, J. García-Cuerva, J. M. Martell, C. Pérez. Extrapolation with weights, rearrangement-invariant function spaces, modular inequalities and applications to singular integrals. *Adv. Math.*, 203:256–318, 2006.

[4] L. C. Evans. *Partial Differential Equations.* American Mathematical Society, Providence, 1998.

[5] R. Farwig, G. P. Galdi, H. Sohr. Very weak solutions of stationary and instationary Navier-Stokes equations with nonhomogeneous data. *Nonlinear Elliptic and Parabolic Problems, Progr. Nonlinear Differential Equations Appl., Birkhäuser*, 64:113–136, 2005.

[6] R. Farwig, H. Sohr. Weighted L^q-theory for the Stokes resolvent in exterior domains. *J. Math. Soc. Japan*, 49:251–288, 1997.

[7] A. Fröhlich. *Stokes- und Navier-Stokes-Gleichungen in gewichteten Funktionenräumen.* Shaker Verlag, Aachen, 2001.

[8] A. Fröhlich. The Stokes operator in weighted L^q-spaces II: Weighted resolvent estimates and maximal L^p-regularity. *Preprint No. 2173, FB Mathematik, TU Darmstadt*, 2001, to appear in Math. Ann.

[9] A. Fröhlich. The Stokes operator in weighted L^q-spaces I: Weighted estimates for the Stokes resolvent problem in a half space. *J. Math. Fluid Mech.*, 5:166–199, 2003.

[10] G.P. Galdi, C.G. Simader, H. Sohr. A class of solutions to stationary Stokes and Navier-Stokes equations with boundary data in $W^{-\frac{1}{q},q}$. *Math. Ann.*, 331:41–74, 2005.

[11] J. García-Cuerva, J. L. Rubio de Francia. *Weighted norm inequalities and related topics.* North Holland, Amsterdam, 1985.

[12] Y. Giga. Analyticity of the semigroup generated by the Stokes operator in L_r-spaces. *Math. Z.*, 178:297–329, 1981.

[13] J. Nečas. *Les Méthodes Directes en Théorie des Équations Elliptiques.* Academia, Prague, 1967.

[14] K. Schumacher. The Stokes probelm with low-regularity data in weighted function spaces. *In preparation.*

[15] L. N. Slobodeckiĭ. Generalized Sobolev spaces and their application to boundary problems for partial differential equations. *Leningrad. Gos. Ped. Inst. Učen. Zap.*, 197:54–112., 1958.

[16] E. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton Mathematical Series. 43, Princeton University Press, Princeton, N.J, 1993.