STABILITY ESTIMATES FOR THE INVERSE CONDUCTIVITY PROBLEM FOR LESS REGULAR CONDUCTIVITIES

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Abstract. We prove a log-type stability estimate for the inverse conductivity problem in space dimension $n \geq 3$, if the conductivity has $C^{3/2+\varepsilon}$ regularity.

1. Introduction

Since Calderón's pioneering work [Cal80] the inverse conductivity problem has been investigated very intensively. The conductivity equation can be formulated as follows.

Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $\gamma \in L^\infty(\Omega)$ such that $\gamma \geq \tau > 0$ a.e. for some positive constant τ. Then the equation

$$\begin{cases} \operatorname{div}(\gamma \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

models the electrical potential u in the interior without sinks or sources for given voltage f.

For $f \in C^\infty(\partial\Omega)$ the Dirichlet-to-Neumann map Λ_γ associated to this problem is defined by

$$\Lambda_\gamma f := \gamma \partial_\nu u|_{\partial\Omega}, \tag{1.2}$$

where u is the weak solution of (1.1) corresponding to the Dirichlet data f. It is well known that Λ_γ can be extended to a bounded operator from $H^{1/2}(\Omega)$ to $H^{-1/2}(\Omega)$. In this case the equality (1.2) has to be exchanged by

$$\langle \Lambda_\gamma f, \overline{v|_{\partial\Omega}} \rangle = \int_\Omega \gamma \nabla u \nabla v \, \mathrm{d}x \qquad \forall v \in H^1(\Omega).$$

We set $\| \cdot \|_* := \| \cdot \|_{H^{1/2} \to H^{-1/2}}$.

The inverse conductivity problem can be formulated as follows. *What kind of information on γ can we get from the knowledge of Λ_γ?*

Calderón attacked this problem by using harmonic functions in order to get uniqueness of γ if it is analytic and close to a constant. In the seminal work of Sylvester and Uhlmann [SU87] they succeeded in proving uniqueness for $\gamma \in C^2(\overline{\Omega})$ by using special solutions of (1.1) instead of harmonic functions — so called complex geometrical optics solutions. This method has become a great tool for investigating many kinds of inverse problems. For an overview of this method and its application see e.g. Uhlmann's survey paper [Uhl99] and the references therein.

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It is reasonable to ask which regularity condition for γ is needed such that it is uniquely identified by Λ_γ. Recently, Astala and Päivärinta answered this question in space dimension $n = 2$. It turned out to be true for $\gamma \in L^\infty$, see [AP06].

For space dimension $n \geq 3$ it seems that the lowest regularity condition has not been found yet. Brown solved the uniqueness question for $\gamma \in C^{3/2+\varepsilon}(\overline{\Omega})$ in [Bro96]. In [PPU03] Panchenko, Päivärinta and Uhlmann constructed complex geometrical optics solutions for Lipschitz continuous conductivities. With the help of these functions they succeeded in proving uniqueness for $\gamma \in C^{3/2}(\overline{\Omega})$. Brown and Torres treated in [BT03] the case of $\gamma \in W^{3/2,p}(\Omega)$, where $p > 2n$.

In view of the continuous dependence of Λ_γ on γ Alessandrini proved in [Ale88] a log-type stability estimate for $\gamma \in C^2(\overline{\Omega})$ having a uniform bound in the Bessel potential space $H^{n/2+2}(\Omega)$. On the other hand, it was shown by Mandache [Man01] that the log-type estimate is optimal. Recently Wang and the author proved in [HW06] a log-log-type stability estimate for the local Dirichlet-to-Neumann map if $\gamma \in C^2(\overline{\Omega})$. In this work we aim to get a log-type stability estimate assuming $\gamma \in C^{3/2+\varepsilon}(\overline{\Omega})$. More precisely, the main result of this work reads as follows.

Theorem 1.1. *Let $n \geq 3$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let γ_1, $\gamma_2 \in C^{3/2+\varepsilon}(\overline{\Omega})$ with $0 < \varepsilon < \frac{1}{2}$ be such that $\gamma_i \geq \alpha > 0$ in $\overline{\Omega}$ and*

$$\|\gamma_i\|_{H^{n/2+\varepsilon,2}(\Omega)} + \|\gamma_i\|_{C^{3/2+\varepsilon}(\overline{\Omega})} \leq M \tag{1.3}$$

for $i = 1, 2$ and some constants α, $M > 0$. Then

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C(\|\Lambda_1 - \Lambda_2\|_*^{1/24} + |\log \|\Lambda_1 - \Lambda_2\|_*|^{-\theta_2})^{\theta_1} \tag{1.4}$$

for $\theta_1 = \frac{\varepsilon}{n-2+2\varepsilon}$, $\theta_2 = \frac{\varepsilon}{8(n+1)-4\varepsilon}$ and some constant $C > 0$.

Here we used the notation $\Lambda_i := \Lambda_{\gamma_i}$.

Remark 1.2. *Note that for $n = 3$ the $H^{n/2+\varepsilon,2}(\Omega)$-norm can be estimated by the $C^{3/2+\varepsilon}(\overline{\Omega})$-norm such that in this case no apriori information for a higher Sobolev regularity is needed.*

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2. Preliminaries

In this section we collect the basic statements which we will use in the proof of the main result. First of all we like to mention an embedding result for the product of two functions in the Bessel potential spaces $H^{s,p}(\Omega)$. This result is well known and can be found e.g. in the monograph of Runst and Sickel [RS96, p. 176 and p.150]. For the definition of $H^{s,p}(\Omega)$ and properties of these spaces see also [Tri78].

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be open and $1 < p, q < \infty$, $s, t \in \mathbb{R}$. Let $f \in H^{s,p}(\Omega)$ and $g \in H^{t,q}(\Omega)$, where the parameters s, t, p, q satisfy $\frac{1}{p} + \frac{1}{q} \leq 1$, $0 < s < t < \frac{n}{q}$ and $s \leq \frac{n}{p}$.*

Then for $\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - \frac{s}{n}$ it holds that

$$\|f \cdot g\|_{H^{s,r}(\Omega)} \leq C\|f\|_{H^{s,p}(\Omega)}\|g\|_{H^{t,q}(\Omega)}.$$

In the following we will often use the shortcut $H^s(\Omega) = H^{s,2}(\Omega)$. We will also use appropriate weighted Sobolev spaces. For $\delta \in \mathbb{R}$ we define the weighted Lebesgue space as

$$L^2_\delta(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{C} : \|f\|_{L^2_\delta} = \int |f(x)|^2 (1+|x|^2)^\delta \,\mathrm{d}x < \infty \right\}.$$

The weighted Sobolev space of order $k \in \mathbb{N}$ is given by

$$H^k_\delta(\mathbb{R}^n) := \{ f \in L^2_\delta(\mathbb{R}^n) : D^\alpha f \in L^2_\delta(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{N}^n,\ |\alpha| \leq k \}.$$

For arbitrary $s > 0$ we define the Sobolev space $H^s_\delta(\mathbb{R}^n)$ of fractional order as usual by using complex interpolation.

In order to get suitable estimates for the complex geometrical optics solutions for the conductivity equation with conductivity γ we first take a smooth approximation of $\log \gamma$. We denote this function by $\gamma^\#_\tau$. More precisely let $\phi \in C^\infty_c(\mathbb{R}^n)$ with supp $\phi \in B(0,1)$, $\phi \geq 0$ and $\int \phi = 1$. Then we set $\phi_\tau(x) := \tau^{-n}\phi(\frac{x}{\tau})$ and define $\gamma^\#_\tau := \phi_\tau * (\log \gamma)$.

The next lemma collects a few simple estimates for the approximations $\gamma^\#_\tau$. The proof is straight forward and can be found in [Knu06] or [Sal04] for example.

Lemma 2.2. *Suppose that $\gamma \in C^{1+s}(\mathbb{R}^n)$ for some $0 \leq s \leq 1$ such that $\gamma(x) = 1$ if $|x| > R$. Then*

$$\|\Delta \gamma^\#_\tau\|_{L^2} \leq C\tau^{s-1}$$

$$\|\partial^\alpha(\log\gamma - \gamma^\#_\tau)\|_{L^2} \leq C\tau^{s+1-|\alpha|} \qquad \text{for } |\alpha| \leq 1$$

and

$$\|\partial^\alpha(\gamma^{-\frac{1}{2}} - e^{-\frac{1}{2}\gamma^\#_\tau})\|_{L^\infty} \leq C\tau^{s+1-|\alpha|} \qquad \text{for } |\alpha| \leq 1.$$

Hence, we also get

$$\|(\gamma^{-\frac{1}{2}} - e^{-\frac{1}{2}\gamma^\#_\tau})\|_{H^1} \leq C\tau^s.$$

The constant C depends on n, R and the C^{1+s}-norm of γ.

We will also need the stable determination of the conductivity at points on the boundary of Ω. This result has been proven in [SU88] as well as [Ale88].

Proposition 2.3. *Let $\gamma_1, \gamma_2 \in C(\overline{\Omega})$ such that for some $\alpha > 0$*

$$0 < \alpha < \gamma_i(x) < \alpha^{-1}$$

holds for all $x \in \overline{\Omega}$. Then there exists a constant $C > 0$ such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_1 - \Lambda_2\|_*.$$

For our proof of the stability estimate we will intensively use complex geometrical optics solutions as they are constructed in [PPU03] and their properties. In order to fix notation we initially will describe the form and properties we need.

At first, the existence of complex geometrical optics solutions we are going to use is ensured by the following proposition.

Proposition 2.4. *Let $\rho \in \mathbb{C}^n$ such that $\rho \cdot \rho = \sum_{i=1}^n \rho_i^2 = 0$. Let $\gamma \in W^{1,\infty}(\mathbb{R}^n)$ such that $\gamma > \alpha$ a.e. for some constant $\alpha > 0$. Additionally assume that $\gamma = 1$ outside a large ball. Then there exists $\rho_0 > 0$ such that for $|\rho| > \rho_0$ the equation*

$$\operatorname{div}(\gamma \nabla u) = 0 \qquad \text{in } \mathbb{R}^n$$

has a unique solution $u = e^{x\cdot\rho}(e^{-\frac{1}{2}\gamma^\#_\tau} + w) = e^{x\cdot\rho}\gamma^{-1/2}(1+\psi)$ with $\psi \in L^2_\delta(\mathbb{R}^n)$.

During the construction of complex geometrical optics solutions in [PPU03] the operators $\Delta_\rho := e^{-\rho x}\Delta(e^{\rho x}\cdot) = \Delta + 2\rho\cdot\nabla$ and $\nabla_\rho := e^{-\rho x}\nabla(e^{\rho x}\cdot) = \nabla + \rho$ were used. The next proposition states the mapping properties of these operators between weighted Sobolev spaces which we will need in the future.

Proposition 2.5. *Let $n \geq 2$, $-1 < \delta < 0$ and $\rho \in \mathbb{C}$ such that $\rho\cdot\rho = 0$.*

(a) *For $s \geq 0$ the operator $\Delta_\rho : H^s_\delta(\mathbb{R}^n) \to H^s_{\delta+1}(\mathbb{R}^n)$ is invertible and Δ_ρ^{-1} satisfies*

$$\|\Delta_\rho^{-1}\|_{H^s_{\delta+1}\to H^s_\delta} \leq \frac{C(s,\delta,n)}{|\rho|}.$$

Moreover $\Delta_\rho^{-1} : L^2_{\delta+1}(\mathbb{R}^n) \to H^k_\delta(\mathbb{R}^n)$ is bounded for $k = 1,2$ with

$$\|\Delta_\rho^{-1}\|_{L^2_{\delta+1}\to H^k_\delta} \leq \frac{C(k,\delta,n)}{|\rho|^{1-k}}.$$

(b) *Suppose that $g \in C^{1+\varepsilon}(\mathbb{R}^n)$ for some $0 < \varepsilon < 1$ and $\nabla g(x) = 0$ for $|x| > R$. Then there exist constants $C_1 = C_1(\delta,n,g,R)$ and $C_2 = C_2(\delta,n,g,R)$ such that the operator*

$$T_\rho = I + \nabla g\cdot\nabla_\rho\Delta_\rho^{-1} : L^2_{\delta+1}(\mathbb{R}^n) \to L^2_{\delta+1}(\mathbb{R}^n)$$

is invertible with $\|T_\rho^{-1}\| \leq C_2$ if $|\rho| > C_1$.

Proofs of these facts can be found in [PPU03].

Remark 2.6. *Exploring the estimates of the approximation of g from Lemma 2.2 it follows that the constants C_1 and C_2 depend on δ, n, R and the $C^{1+\varepsilon}$-norm of g.*

Proof. The proof given here follows the arguments from [Sal04, Proposition 2.1]. In fact, writing $g = g^\# + g^b$, where $g^\# := \phi_\tau * g$ is a smooth approximation of g, we get $T_\rho = A - B$ where $A := e^{-g^\#/2}\Delta_\rho e^{g^\#/2}\Delta_\rho^{-1}$ and $B := (\frac{1}{2}\Delta g^\# + \frac{1}{4}|\nabla g^\#|^2 - \nabla g^b\cdot\nabla_\rho)\Delta_\rho^{-1}$. Note, that $(\Delta_\rho + \nabla g^\#\cdot\nabla_\rho)e^{-g^\#/2} = e^{-g^\#/2}(\Delta_\rho - \frac{1}{2}\Delta g^\# + \frac{1}{4}|\nabla g^\#|^2)$. Furthermore, we have $A^{-1} = \Delta_\rho e^{-g^\#/2}\Delta_\rho^{-1}e^{g^\#/2}$. Since $T_\rho = A(1 + A^{-1}B)$ it is enough to investigate $\|A^{-1}B\|$ for the existence of T_ρ^{-1}.

We start with

$$\begin{aligned} A^{-1} &= (\Delta_\rho e^{-g^\#/2})\Delta_\rho^{-1}e^{g^\#/2} + 2\nabla(e^{-g^\#/2})\cdot\nabla\Delta_\rho^{-1}e^{g^\#/2} + e^{-g^\#/2}\Delta_\rho\Delta_\rho^{-1}e^{g^\#/2} \\ &= I + (-\frac{1}{2}\Delta g^\# + \frac{1}{4}|\nabla g^\#|^2)e^{-g^\#/2}\Delta_\rho^{-1}e^{g^\#/2} - (\rho\cdot\nabla g^\#)e^{-g^\#/2}\Delta_\rho^{-1}e^{g^\#/2} \\ &\quad - e^{-g^\#/2}\nabla g^\#\cdot\nabla\Delta_\rho^{-1}e^{g^\#/2}. \end{aligned}$$

Therefore it follows that

$$\|A^{-1}\| \leq C(1 + |\rho|^{-1}(\|\Delta g^\#\|_\infty + \|\nabla g^\#\|_\infty^2) + \|\nabla g^\#\|_\infty) \leq C.$$

For B we calculate

$$\|B\| \leq C(\|\frac{1}{2}\Delta g^\# + \frac{1}{4}|\nabla g^\#|^2\|_\infty)\|\Delta_\rho^{-1}\| + C\|\nabla g^b\|_\infty\|\nabla_\rho\Delta_\rho^{-1}\| \leq C|\rho|^{-\varepsilon}$$

using $\tau = |\rho|^{-1}$ and the approximation estimates from Lemma 2.2. The constant C depends on n, δ and R. This now implies $\|A^{-1}B\| < 1$ for $|\rho| > C_1$, hence $\|T_\rho^{-1}\| \leq C_2$. □

Since we are assuming $\gamma \in C^{3/2+\varepsilon}$ we are able to use the "ε" in order to get a convenient form of the stability estimate. We believe that stability estimates could be derived for $\gamma \in C^{3/2}$ with a uniform modulus of continuity. But the stability estimate would depend heavily on this modulus. Here we will restrict ourselves on moduli of continuity of the form $|x|^\varepsilon$. In order to use this additional ε for the stability estimate we need the following refinement of Proposition 2.3 from [Sal04].

Proposition 2.7. *Suppose $\gamma \in C^{3/2+\varepsilon}(\overline{\Omega})$, $0 < \varepsilon \le \frac{1}{2}$ and $-1 < \delta < 0$. If $|\rho| > \rho_0$ then the function w from Proposition 2.4 satisfies*

$$\|w\|_{H^\eta_\delta} \le C|\rho|^{-(\frac{1}{2}+\varepsilon)+\eta} \qquad \textit{for } \eta \in [0,2]. \tag{2.1}$$

Proof. First note that by interpolation we get from Proposition 2.5 that

$$\|\Delta_\rho^{-1}\|_{L^2_{\delta+1}\to H^s_\delta} \le C|\rho|^{s-1}.$$

Next we extend γ to a function in $C^{3/2+\varepsilon}(\mathbb{R}^n)$, which we also denote by γ, such that $\gamma(x) = 1$ for $|x| > R$. By the construction of the complex geometrical optics solutions, the function w in Proposition 2.4 satisfies $(\Delta_\rho + \nabla(\log\gamma)\cdot\nabla_\rho)w = f_\tau$ where f_τ is given by

$$\begin{aligned} f_\tau &:= -(\Delta_\rho + \nabla(\log\gamma)\cdot\nabla_\rho)e^{-\frac{1}{2}\gamma_\tau^\#} \\ &= -(\Delta_\rho + \nabla(\gamma_\tau^\#)\cdot\nabla_\rho)e^{-\frac{1}{2}\gamma_\tau^\#} - \nabla(\log\gamma - \gamma_\tau^\#)\cdot\nabla_\rho e^{-\frac{1}{2}\gamma_\tau^\#}. \end{aligned}$$

(See also [PPU03, Equation (2.11)].) Using Lemma 2.2 and choosing $\tau = |\rho|^{-1}$ we get

$$\begin{aligned} &\|f_\tau\|_{L^2_{\delta+1}} \\ &\le Ce^{\frac{1}{2}\|\gamma_\tau^\#\|_{L^\infty}}(\|\Delta\gamma_\tau^\#\|_{L^2} + \|\nabla\gamma_\tau^\#\|^2_{L^\infty} + \|\nabla(\log\gamma - \gamma_\tau^\#)\|^2_{L^2}(\|\nabla\gamma_\tau^\#)\|_{L^\infty} + |\rho|)) \\ &\le C(\tau^{\varepsilon-\frac{1}{2}} + |\rho|\tau^{\frac{1}{2}+\varepsilon}) \\ &\le C|\rho|^{\frac{1}{2}-\varepsilon} \end{aligned}$$

where C depends on R, δ and $\|\gamma\|_{C^{3/2+\varepsilon}}$. The norm bounds of Δ_ρ^{-1} and T_ρ^{-1} from Proposition 2.5 finally give

$$\begin{aligned} \|w\|_{H^\eta_\delta} &= \|\Delta_\rho^{-1}T_\rho^{-1}f_\tau\|_{H^\eta_\delta} \\ &\le \|\Delta_\rho^{-1}\|_{L^2_{\delta+1}\to H^\eta_\delta}\|T_\rho^{-1}\|_{L^2_{\delta+1}}\|f_\tau\|_{L^2_{\delta+1}} \\ &\le C|\rho|^{-\frac{1}{2}-\varepsilon+\eta} \end{aligned}$$

for $|\rho| > \rho_0$, where ρ_0 depends on n, δ, R and the $C^{3/2+\varepsilon}$-norm of γ (cf. Remark 2.6). □

The next lemma contains the fundamental identity which will be the starting point of our proof of the stability estimate. A proof of this identity can be found in [PPU03, Lemma 3.1].

Lemma 2.8. *Let $\gamma \in W^{1,\infty}(\Omega)$, γ strictly positive on $\overline{\Omega}$. We set $a := \sqrt{\gamma}$. Let $u \in H^1(\Omega)$ be a solution of $\operatorname{div}(\gamma\nabla u) = 0$ in Ω. Then*

$$\int_\Omega (\nabla a \cdot \nabla(uv) - \nabla(au)\cdot\nabla v)\,\mathrm{d}x = \langle \frac{1}{a}\Lambda_\gamma u|_{\partial\Omega}, \overline{v|_{\partial\Omega}}\rangle \tag{2.2}$$

for any $v \in H^1(\Omega)$.

3. The stability estimate

In this section we give a proof of Theorem 1.1. Let $k, \ell, m \in \mathbb{R}^n$ be mutually orthogonal with $|m|^2 = |\ell|^2 - \frac{|k|^2}{4}$. We choose $\rho_1 := \ell + i(-\frac{k}{2} + m)$ and $\rho_2 := -\ell + i(-\frac{k}{2} - m)$. Note that $\rho_i \cdot \rho_i = 0$, $\rho_1 + \rho_2 = -ik$ and that for fixed k we can choose $|\rho_i|$ as large as we like. It is clear that such a construction is not possible for $n = 2$.

We take the complex geometrical optics solutions $u_1 := e^{x\rho_1}(e^{-\frac{1}{2}\gamma^{\#}_{1,\tau}} + w_1) = e^{x\rho_1}a_1^{-1}(1 + \psi_1)$ and $u_2 := e^{x\rho_2}(e^{-\frac{1}{2}\gamma^{\#}_{2,\tau}} + w_2) = e^{x\rho_2}a_2^{-1}(1 + \psi_2)$ as guaranteed by Proposition 2.4. Moreover, we extend the functions γ_i to all of $\mathbb{R}^n$ such that $\gamma_i \in C^{3/2+\varepsilon}(\mathbb{R}^n)$ and $\gamma_i = 1$ outside a ball of radius $R > 0$ with center at the origin. Since $\psi_i = e^{-\frac{1}{2}\gamma^{\#}_{i,\tau}} - a_i^{-1} + w_i$ using (2.1) and Lemma 2.2 it is clear that ψ_i, $i = 1, 2$, satisfies the inequality

$$\|\psi_i\|_{H^\eta_\delta} \le C|\rho|^{-(\frac{1}{2}+\varepsilon)+\eta} \qquad \text{for } \eta \in [0,1].$$

Next we define $v_i := e^{x\rho_i}(1 + \psi_i)$ for $i = 1, 2$ and use (2.2) once for $a = a_1$, $u = u_1$, $v = v_2$ and once for $a = a_2$, $u = u_2$, $v = v_1$. With this definitions we have $a_i u_i = v_i$, for $i = 1, 2$. Subtracting the resulting equalities we get

$$\int_\Omega \nabla a_1 \cdot \nabla(u_1 v_2) - \nabla a_2 \cdot \nabla(u_2 v_1) = -\left(\langle \frac{1}{a_1}\Lambda_1 u_1, \overline{v_2}\rangle - \langle \frac{1}{a_2}\Lambda_2 u_2, \overline{v_1}\rangle\right).$$

Setting in the form of the geometrical optics solutions we arrive at

$$\begin{aligned}
&\int_\Omega \nabla a_1 \nabla(a_1^{-1}e^{-ixk}) - \nabla a_2 \nabla(a_2^{-1}e^{-ixk})\,\mathrm{d}x \\
&= -\int_\Omega \nabla a_1 \nabla\left(e^{-ixk}a_1^{-1}W\right) - \nabla a_2 \nabla\left(e^{-ixk}a_2^{-1}W\right)\,\mathrm{d}x \\
&\quad - \left(\langle \frac{1}{a_1}\Lambda_1 u_1, \overline{v_2}\rangle - \langle \frac{1}{a_2}\Lambda_2 u_2, \overline{v_1}\rangle\right)
\end{aligned}$$

where we used $W := \psi_1 + \psi_2 + \psi_1\psi_2$ for short.

Since

$$\begin{aligned}
\left\langle \frac{1}{a_1}\Lambda_1 u_1, \overline{v_2}\right\rangle - \left\langle \frac{1}{a_2}\Lambda_2 u_2, \overline{v_1}\right\rangle &= \left\langle \Lambda_1 u_1, \frac{1}{a_1}\overline{v_2}\right\rangle - \left\langle \Lambda_2 u_2, \frac{1}{a_2}\overline{v_1}\right\rangle \\
&= \left\langle \Lambda_1 u_1, \left(\frac{1}{a_1} - \frac{1}{a_2}\right)\overline{v_2}\right\rangle + \langle \Lambda_1 u_1, \overline{u_2}\rangle \\
&\quad - \left\langle \Lambda_2 u_2, \frac{1}{a_2}\overline{v_1}\right\rangle \\
&= \left\langle \Lambda_1 u_1, \left(\frac{1}{a_1} - \frac{1}{a_2}\right)\overline{v_2}\right\rangle + \langle \Lambda_1 u_2, \overline{u_1}\rangle \\
&\quad - \left\langle \Lambda_1 u_2, \frac{1}{a_2}\overline{v_1}\right\rangle + \left\langle (\Lambda_1 - \Lambda_2)\, u_2, \frac{1}{a_2}\overline{v_1}\right\rangle \\
&= \left\langle \Lambda_1 u_1, \left(\frac{1}{a_1} - \frac{1}{a_2}\right)\overline{v_2}\right\rangle + \left\langle (\Lambda_1 - \Lambda_2)\, u_2, \frac{1}{a_2}\overline{v_1}\right\rangle \\
&\quad + \left\langle \Lambda_1 u_2, \left(\frac{1}{a_1} - \frac{1}{a_2}\right)\overline{v_1}\right\rangle
\end{aligned}$$

we next aim to get the following inequalities for the terms containing ψ_i or the boundary integrals. Note that as in the proof of Proposition 2.7 we necessarily make the choice $\tau = |\rho|^{-1}$.

$$\left|\int_\Omega e^{-ikx}(-ik)(\nabla a_i)a_i^{-1} W \,\mathrm{d}x\right| \leq C|k|(|\rho|^{-(1/2+\varepsilon)} + |\rho|^{-(1+2\varepsilon)}) \tag{3.1}$$

$$\left|\int_\Omega e^{-ikx}(\nabla a_i)(\nabla a_1^{-1}) W \,\mathrm{d}x\right| \leq C(|\rho|^{-(1/2+\varepsilon)} + |\rho|^{-(1+2\varepsilon)}) \tag{3.2}$$

$$\left|\int_\Omega e^{-ikx}(\nabla a_i)a_1^{-1}\nabla W \,\mathrm{d}x\right| \leq C|k|^{1/2-\varepsilon/2}|\rho|^{-\varepsilon/2} \tag{3.3}$$

$$\left|\langle \Lambda_1 u_1, (\frac{1}{a_1} - \frac{1}{a_2})v_2\rangle\right| \leq Ce^{c|\rho|}\|\Lambda_1 - \Lambda_2\|_*^{1/6} \tag{3.4}$$

$$\left|\langle \Lambda_1 u_2, (\frac{1}{a_1} - \frac{1}{a_2})v_1\rangle\right| \leq Ce^{c|\rho|}\|\Lambda_1 - \Lambda_2\|_*^{1/6} \tag{3.5}$$

$$\left|\langle (\Lambda_1 - \Lambda_2)u_2, \frac{v_1}{a_2}\rangle\right| \leq Ce^{c|\rho|}\|\Lambda_1 - \Lambda_2\|_* \tag{3.6}$$

The constant C in these inequalities depends on n, Ω, M and the constant c depends on the diameter of Ω. The proofs of these inequalities use estimates of the complex geometrical optics solutions given in the Propositions 2.4 and 2.7 as well as the stability result on the boundary (Proposition 2.3) and Lemma 2.2. We further assume that $|\rho| > \rho_0$ whereρ_0 is the constant from Proposition 2.4.

Inequality (3.1) follows from

$$\begin{aligned}\left|\int_\Omega e^{-ikx}(-ik)a_i^{-1}\nabla a_i W \,\mathrm{d}x\right| &\leq |k|\|e^{-ixk}a_i^{-1}\nabla a_i\|_{L^\infty(\Omega)}\|W\|_{L^1(\Omega)}\\ &\leq C|k|(\|\psi_1\|_{L^2(\Omega)} + \|\psi_2\|_{L^2(\Omega)} + \|\psi_1\psi_2\|_{L^1(\Omega)})\\ &\leq C|k|(|\rho|^{-1/2-\varepsilon} + |\rho|^{-1-2\varepsilon}).\end{aligned}$$

The second inequality follows by using similar arguments.

During the proof of inequality (3.3) we will use Proposition 2.1 in order to estimate the product of two functions. Here we will use the choice of parameters as follows: $p := \frac{2n}{1-\varepsilon}$, $q := r := 2$, $s := \frac{1-\varepsilon}{2}$, $t := \frac{2-\varepsilon}{4}$. Note that for $-\frac{1}{2} < s < \frac{1}{2}$ it holds that $(H^s(\Omega))' = H^{-s}(\Omega)$, see e.g. [Tri78, 4.8.2]. Hence using the fact that ∇a_i has compact support yields

$$\begin{aligned}&\left|\int_\Omega e^{-ikx}(\nabla a_i)a_i^{-1}\nabla W \,\mathrm{d}x\right|\\ &\leq C\|e^{-ixk}(\nabla \log a_i)\|_{H^{(1-\varepsilon)/2}(\Omega)}(\|\nabla\psi_1\|_{H^{-(1-\varepsilon)/2}(\Omega)} + \|\nabla\psi_2\|_{H^{-(1-\varepsilon)/2}(\Omega)})\\ &\quad + \left|\int_\Omega e^{-ixk}(\nabla \log a_i)(\psi_1\nabla\psi_2 + \psi_2\nabla\psi_1)\,\mathrm{d}x\right|\\ &\leq C(|k|^{(1-\varepsilon)/2}|\rho|^{-\varepsilon/2} + \|e^{-ixk}(\nabla \log a_i)\psi_1\|_{H^{(1-\varepsilon)/2}(\Omega)}\|\nabla\psi_2\|_{H^{-(1-\varepsilon)/2}(\Omega)}\\ &\quad + \|e^{-ixk}(\nabla \log a_i)\psi_2\|_{H^{(1-\varepsilon)/2}(\Omega)}\|\nabla\psi_1\|_{H^{-(1-\varepsilon)/2}(\Omega)})\\ &\leq C|k|^{(1-\varepsilon)/2}(|\rho|^{-\varepsilon/2} + \|(\nabla \log a_i)\|_{H^{(1-\varepsilon)/2,p}(\Omega)}\|\psi_1\|_{H^{(2-\varepsilon)/4}(\Omega)}\|\nabla\psi_2\|_{H^{-(1-\varepsilon)/2}(\Omega)}\\ &\quad + \|(\nabla \log a_i)\|_{H^{(1-\varepsilon)/2,p}(\Omega)}\|\psi_2\|_{H^{(1-\varepsilon)/4}(\Omega)}\|\nabla\psi_1\|_{H^{-(1-\varepsilon)/2}(\Omega)})\\ &\leq CM|k|^{(1-\varepsilon)/2}|\rho|^{-\varepsilon/2}.\end{aligned}$$

The boundary term (3.4) is estimated by

$$\begin{aligned}\left|\left\langle \Lambda_1 u_1, \overline{(\frac{1}{a_1}-\frac{1}{a_2})v_2}\right\rangle\right| &\le \|\Lambda_1\|_*\|u_1\|_{H^{1/2}(\partial\Omega)}\|(\frac{1}{a_1}-\frac{1}{a_2})v_2\|_{H^{1/2}(\partial\Omega)}\\ &\le Ce^{c|\rho|}\|\frac{1}{a_1}-\frac{1}{a_2}\|_{C^{1+1/4}(\partial\Omega)}\|v_2\|_{H^1(\Omega)}\\ &\le Ce^{c|\rho|}\|\frac{1}{a_1}-\frac{1}{a_2}\|^{1/6}_{L^\infty(\partial\Omega)}\|\frac{1}{a_1}-\frac{1}{a_2}\|^{1-1/6}_{C^{3/2}(\partial\Omega)}\\ &\le Ce^{c|\rho|}\|\Lambda_1-\Lambda_2\|_*^{1/6}\end{aligned}$$

using the stability at the boundary and interpolation. Inequality (3.5) follows by the same arguments.

Finally (3.6) follows from

$$\begin{aligned}\left|\langle(\Lambda_1-\Lambda_2)u_2, \overline{\frac{v_1}{a_2}}\rangle\right| &\le \|\Lambda_1-\Lambda_2\|_*\|u_2\|_{H^{1/2}(\partial\Omega)}\|\frac{1}{a_2}v_1\|_{H^{1/2}(\partial\Omega)}\\ &\le \|\frac{1}{a_2}\|_{C^1(\overline{\Omega})}\|u_2\|_{H^1(\Omega)}\|v_1\|_{H^1(\Omega)}\|\Lambda_1-\Lambda_2\|_*\\ &\le Ce^{c|\rho|}\|\Lambda_1-\Lambda_2\|_*.\end{aligned}$$

Putting all these results together we have proven that

$$\begin{aligned}&\left|\int_\Omega \nabla a_1\nabla(a_1^{-1}e^{-ixk})-\nabla a_2\nabla(a_2^{-1}e^{-ixk})\,\mathrm{d}x\right|\\ &=\left|\int_\Omega e^{-ixk}(ik+\nabla(\log a_1+\log a_2))\nabla(\log a_1-\log a_2)\,\mathrm{d}x\right|\\ &\le Ce^{c|\rho|}\left(\|\Lambda_1-\Lambda_2\|_*+\|\Lambda_1-\Lambda_2\|_*^{1/6}\right)\\ &\quad+C\left((|k|+1)|\rho|^{(-1/2-\varepsilon)}+|k|^{1/2-\varepsilon/2}|\rho|^{-\varepsilon/2}\right).\end{aligned}\tag{3.7}$$

Next we consider the function $v := \log a_1 - \log a_2 \in H^1(\Omega)$. This function is a weak solution of

$$\begin{aligned}\Delta v-\nabla(\log a_1+\log a_2)\nabla v&=F &&\text{in }\Omega\\ v|_{\partial\Omega}&=(\log a_1-\log a_2)|_{\partial\Omega}\end{aligned}$$

with $F\in H^{-1}(\Omega)$.

Since v is also a weak solution of the elliptic equation $\operatorname{div}(a_1a_2)\nabla v=(a_1a_2)\cdot F$ in Ω, we get the estimate

$$\|v\|_{H^1(\Omega)}\le C\left(\|F\|_{H^{-1}(\Omega)}+\|v\|_{H^{1/2}(\partial\Omega)}\right)$$

for some constant $C>0$. Hence using the mean value theorem, Sobolev embedding and interpolation theory we get

$$\begin{aligned}\|\gamma_1-\gamma_2\|_{L^\infty(\Omega)} &\le C\|\log a_1-\log a_2\|_{L^\infty(\Omega)}\\ &\le C\|v\|^\theta_{H^1(\Omega)}\|v\|^{1-\theta}_{H^{n/2+\varepsilon}(\Omega)}\\ &\le CM^{1-\theta}(\|F\|_{H^{-1}(\Omega)}+\|v\|_{H^{1/2}(\partial\Omega)})^\theta\\ &\le CM^{1-\theta}(\|F\|_{H^{-1}(\Omega)}+\|v\|^{1/2}_{L^2(\partial\Omega)}\|v\|^{1/2}_{H^1(\partial\Omega)})^\theta\\ &\le CM^{1-\theta}(\|F\|_{H^{-1}(\Omega)}+CM^{1/2}\|\Lambda_1-\Lambda_2\|_*^{1/2})^\theta\end{aligned}\tag{3.8}$$

where $\theta = \frac{\varepsilon}{n-2+2\varepsilon}$. For the last inequality we used the stability estimate on the boundary and the apriori information on γ_i.

The stability estimate will now follow after treating $\|F\|_{H^{-1}(\Omega)}$. In order to begin with this we write $g := \nabla(\log a_1 + \log a_2)$ and denote by $\tilde{f}$ the extension of $f \in L^2(\Omega)$ by zero to $\mathbb{R}^n$. Then for $\varphi \in H_0^1(\Omega)$ we have

$$\begin{aligned}\langle F, \varphi\rangle &= \int_\Omega -\nabla v \nabla\varphi + (g\nabla v)\varphi \,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} -\widetilde{\nabla v}\nabla\tilde{\varphi} + (g\widetilde{\nabla v})\tilde{\varphi}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \Big((ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\Big)\,\mathcal{F}\tilde{\varphi}\,\mathrm{d}k.\end{aligned}$$

Hence

$$|\langle F,\varphi\rangle| \le \left(\int_{\mathbb{R}^n}\Big|(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\Big|^2 (1+|k|^2)^{-1}\,\mathrm{d}k\right)^{1/2} \|\varphi\|_{H^1(\mathbb{R}^n)} \tag{3.9}$$

where $\mathcal{F}$ denotes the Fourier transform. For the time being we will assume that $\|\Lambda_1 - \Lambda_2\|_* < 1$ and we choose $R > 1$. Its size will be chosen suitably later. The integral on the right hand side of (3.9) can be estimated as follows using (3.7) in order to bound $(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})$.

$$\begin{aligned}&\int_{\mathbb{R}^n}\Big|(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\Big|^2 (1+|k|^2)^{-1}\,\mathrm{d}k \\ &\le \int_{|k|\le R}\Big|(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\Big|^2 (1+|k|^2)^{-1}\,\mathrm{d}k \\ &\quad + \int_{|k|>R}\Big|(ik)\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\Big|^2 (1+|k|^2)^{-1}\,\mathrm{d}k \\ &\le C\bigg(R^n\|k\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\|^2_{L^\infty(B(0,R))} + \frac{1}{R^2}\|g\widetilde{\nabla v}\|^2_{L^2(\mathbb{R}^n)} \\ &\quad + \int_{|k|>R}(1+|k|^2)^{1/4}|\mathcal{F}\widetilde{\nabla v}|^2(1+|k|^2)^{-1/4}\bigg) \\ &\le C\left(R^n\|k\mathcal{F}\widetilde{\nabla v} + \mathcal{F}(g\widetilde{\nabla v})\|^2_{L^\infty(B(0,R))} + \frac{C}{R^2}\|v\|^2_{H^1(\mathbb{R}^n)} + \frac{1}{\sqrt{R}}\|v\|^2_{H^{1+1/4}(\Omega)}\right) \\ &\le CR^n e^{c|\rho|}\|\Lambda_1-\Lambda_2\|_*^{1/6} + CR^{n+1}|\rho|^{-1/2-\varepsilon} + CR^{n+1/2-\varepsilon}|\rho|^{-\varepsilon/2} + \frac{C}{\sqrt{R}}.\end{aligned}$$

Next we like to choose ρ such that the second and the third summand on the right hand side of the last inequality decay like $R^{-1/2}$. Hence we set $|\rho| := R^\tau$ where $\tau := \max\{\frac{2n+3}{1+2\varepsilon}, \frac{2(n+1)-\varepsilon}{\varepsilon}\} = \frac{2(n+1)-\varepsilon}{\varepsilon}$ in our situation ($n \ge 3$, $\varepsilon < \frac{1}{2}$). In order to have the complex geometrical optics solutions available we require $|\rho| > \rho_0$. Therefore we choose $R > 1$ such that $R^\tau > \rho_0$.

Next we set R in dependence of $\|\Lambda_1 - \Lambda_2\|_*$, i.e. $R = \frac{1}{c}|\log(\|\Lambda_1-\Lambda_2\|_*^{1/12})|^{1/\tau}$. It is clear that R satisfies the size condition $R^\tau > \rho_0$ if $\|\Lambda_1 - \Lambda_2\|_* < \kappa$ with κ small enough. So we first assume the required smallness of $\|\Lambda_1 - \Lambda_2\|_*$.

This gives now the estimate for F.

$$\|F\|_{H^{-1}(\Omega)} \le C(\|\Lambda_1-\Lambda_2\|_*^{1/12} + |\log\|\Lambda_1-\Lambda_2\|_*|^{-1/(2\tau)})^{1/2}$$

Setting in this inequality into (3.8) we end up with

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \le C(\|\Lambda_1 - \Lambda_2\|_*^{1/24} + |\log \|\Lambda_1 - \Lambda_2\|_*|^{-1/(4\tau)})^\theta.$$

Now let $\|\Lambda_1 - \Lambda_2\|_* \ge \kappa$ then

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \le \frac{2M}{\kappa^{\theta/24}}\kappa^{\theta/24} \le \|\Lambda_1 - \Lambda_2\|_*^{\theta/24}$$

hence (1.4) holds in this case, too and the proof is complete. □

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