

An implicit function theorem for Banach spaces and some applications

Jinpeng An* and Karl-Hermann Neeb†

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Abstract

We prove a generalized implicit function theorem for Banach spaces, without the usual assumption that the subspaces involved being complemented. Then we apply it to the problem of parametrization of fibers of differentiable maps, the Lie subgroup problem for Banach–Lie groups, as well as Weil’s local rigidity for homomorphisms from finitely generated groups to Banach–Lie groups.

Keywords: implicit function theorem, Banach manifold, Banach–Lie group

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1 Introduction

Motivated by the problem of local rigidity of homomorphisms from finitely generated groups to (finite-dimensional) Lie groups, Weil [We64] proved the following theorem.

Theorem 1.1 (Weil) *Let L, M and N be finite-dimensional C^1 -manifolds, $\varphi : L \rightarrow M$ and $\psi : M \rightarrow N$ be C^1 -maps. Let $x \in U$, $y = \varphi(x)$. Suppose*
(1) $\psi \circ \varphi \equiv z \in N$;

*ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland, jinpeng.an@math.ethz.ch

†Technische Universität Darmstadt, Schlossgartenstrasse 7, D-64289 Darmstadt, Deutschland, neeb@mathematik.tu-darmstadt.de

(2) $\text{im}(d\varphi(x)) = \ker(d\psi(y))$.

Then there exists a neighborhood U of y in M such that

$$\psi^{-1}(z) \cap U = \varphi(L) \cap U.$$

Weil's proof relies heavily on the Implicit Function Theorem. If L, M and N in Theorem 1.1 are Banach manifolds, and $\ker(d\varphi(x))$, $\ker(d\psi(y))$ and $\text{im}(d\psi(y))$ are closed complemented subspaces in the corresponding tangent spaces of the manifolds respectively, then Weil's proof still works, based on the Implicit Function Theorem for Banach spaces. Using the Nash-Moser Inverse Function Theorem, Hamilton [Ha77] proved a similar result in the setting of tame Fréchet spaces under more splitting assumptions (see also [Ha82]). Such kind of results are useful in problems of deformation rigidity (or local rigidity) of certain geometric structures (see e.g. [Ha82],[Be00], [Fi05], [AW05]). But such complementation or splitting conditions are not always satisfied and hard to verify in many concrete situations.

The first goal of this paper is to prove the Banach version of Theorem 1.1 without the assumption that the subspaces involved are complemented. The only additional assumption we have to make, comparing with the finite-dimensional case, is that $\text{im}(d\psi(y))$ is closed. More precisely, we will prove the following theorem. For convenience, we state its local form.

Theorem 1.2 *Let X, Y and Z be Banach spaces, $U \subset X$, $V \subset Y$ be open 0-neighborhoods. Let $\varphi : U \rightarrow V$ and $\psi : V \rightarrow Z$ be C^1 -maps. Suppose*

- (1) $\varphi(0) = 0$;
- (2) $\psi \circ \varphi \equiv 0$;
- (3) $\text{im}(d\varphi(0)) = \ker(d\psi(0))$;
- (4) $\text{im}(d\psi(0))$ is closed.

Then there exists a 0-neighborhood $W \subset V$ such that

$$\psi^{-1}(0) \cap W = \varphi(U) \cap W.$$

Informally, we may think Theorem 1.2 as saying that φ determines some “implicit function” from $\psi^{-1}(0) \cap W$ to U (although we do not even know a priori if it is continuous). So we call Theorem 1.2 a “generalized implicit function theorem”.

We will give three applications of the generalized implicit function theorem.

(1) **Parametrization of fibers of differentiable maps.** Suppose M, N are finite-dimensional C^k -manifolds and $\psi : M \rightarrow N$ is a C^k -map. If $z \in N$ is a regular value of ψ in the sense that for each $y \in \psi^{-1}(z)$, the differential $d\psi(y) : T_yM \rightarrow T_zN$ is surjective, then the Implicit Function Theorem implies that $\psi^{-1}(z)$ is a C^k -submanifold of M . This fact can be generalized directly to Banach manifolds under the additional assumption that $\ker(d\psi(y))$ is complemented in T_yM . Without the complementation assumption, we can not conclude that $\psi^{-1}(z)$ is a submanifold in general. However, using the generalized implicit function theorem, we can prove that under the assumption of the existence of the map φ as in Theorem 1.2, $\psi^{-1}(z)$ can be parametrized as a C^k -manifold in such a way that the manifold structure is compatible with the subspace topology, and the inclusion map is C^k .

(2) **Lie subgroups of Banach–Lie groups.** If G is a Banach Lie group, we call a closed subgroup H of G a Lie subgroup if there is a Banach–Lie group structure on H compatible with the subspace topology (there is at most one such structure). It is well-known that closed subgroups of finite-dimensional Lie groups are Lie subgroups but this is false for Banach–Lie groups (cf. [Hof75]; [Ne06], Rem. IV.3.17). Using the generalized implicit function theorem and a theorem of Hofmann [Hof75], we will prove that certain isotropy groups are Lie subgroups. In particular, we will prove that if the coset space G/H carries a Banach manifold structure for which G acts smoothly and the quotient map $q : G \rightarrow G/H, g \mapsto gH$ has surjective differential in some point $g \in G$, then H is a Banach–Lie subgroup of G .

(3) **Local rigidity of homomorphisms.** Let G be a Banach–Lie group, Γ a finitely generated group. A homomorphism $r : \Gamma \rightarrow G$ is locally rigid if any homomorphism from Γ to G sufficiently close to r is conjugate to r . Using Theorem 1.1, Weil [We64] proved that if $H^1(\Gamma, \mathbf{L}(G)) = \{0\}$, then r is locally rigid. We will generalize Weil’s result to the Banach setting under the assumption that a certain linear operator has closed range.

Theorem 1.2 will be proved in Section 2. The aforementioned three applications will be derived in Sections 3–5, respectively.

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2 A generalized implicit function theorem for Banach spaces

In this section we prove the generalized implicit function theorem.

Theorem 2.1 *Let X, Y and Z be Banach spaces, $U \subset X$, $V \subset Y$ be open 0-neighborhoods. Let $\varphi : U \rightarrow V$ and $\psi : V \rightarrow Z$ be C^1 -maps. Suppose*

- (1) $\varphi(0) = 0$, $\psi(0) = 0$;
- (2) $\psi \circ \varphi \equiv 0$;
- (3) $\text{im}(d\varphi(0)) = \ker(d\psi(0))$;
- (4) $\text{im}(d\psi(0))$ is closed.

Then there exists a 0-neighborhood $W \subset V$ such that

$$\psi^{-1}(0) \cap W = \varphi(U) \cap W.$$

Proof. By the Open Mapping Theorem, there exist positive constants C_1, C_2 such that for any $y \in \text{im}(d\varphi(0))$ and $z \in \text{im}(d\psi(0))$, there are $\sigma(y) \in X$ and $\tau(z) \in Y$ with $d\varphi(0)(\sigma(y)) = y$, $d\psi(0)(\tau(z)) = z$, and such that

$$\|\sigma(y)\| \leq C_1\|y\|, \quad \|\tau(z)\| \leq C_2\|z\|.$$

Let

$$C = \max\{C_1, C_2, \|d\varphi(0)\|, \|d\psi(0)\|, 1\}.$$

Choose a $\delta > 0$ with $B_X(0, \delta) \subset U$, $B_Y(0, \delta) \subset V$ such that

$$x \in B_X(0, \delta) \Rightarrow \|d\varphi(x) - d\varphi(0)\| < \frac{1}{6C^3},$$

$$y \in B_Y(0, \delta) \Rightarrow \|d\psi(y) - d\psi(0)\| < \frac{1}{6C^3},$$

where $B_X(0, \delta)$, $B_Y(0, \delta)$ are the open balls in X and Y with centers 0 and radius δ . We claim that the 0-neighborhood

$$W = B_Y\left(0, \frac{\delta}{9C^3}\right)$$

satisfies the request of the theorem. In fact, we prove below that for $y \in B_Y(0, \frac{\delta}{9C^3})$, if $\psi(y) = 0$, then there exists $x \in B_X(0, \frac{\delta}{2})$ such that $\varphi(x) = y$.

Suppose $y \in B_Y(0, \frac{\delta}{9C^3})$ such that $\psi(y) = 0$. Choose a point $x_0 \in B_X(0, \frac{\delta}{162C^4})$, and define $u_n \in Y$, $v_n, x_{n+1} \in X$ inductively by

$$\begin{cases} u_n = \varphi(x_n) - y & (\text{whenever } x_n \in U), \\ v_n = \sigma(u_n - \tau(d\psi(0)(u_n))), \\ x_{n+1} = x_n - v_n. \end{cases}$$

Here we use that $u_n - \tau(d\psi(0)(u_n)) \in \ker d\varphi(0) = \text{im}(d\psi(0))$. We prove inductively that

$$\begin{cases} \|x_n\| < \frac{\delta}{162} + \frac{20\delta}{81} \sum_{i=0}^{n-1} \frac{1}{2^i} & (\text{hence } x_n \in B_X(0, \frac{\delta}{2}) \subset U); \\ \|u_n\| < \frac{10\delta}{81C^3} \frac{1}{2^n}; \\ \|v_n\| < \frac{20\delta}{81} \frac{1}{2^n} \leq \frac{10}{81} \delta < \frac{\delta}{2}. \end{cases}$$

Since

$$\|x_0\| < \frac{\delta}{162C^4} \leq \frac{\delta}{162},$$

$$\begin{aligned} \|u_0\| &= \|\varphi(x_0) - y\| \leq \|\varphi(x_0) - \varphi(0)\| + \|y\| \\ &< \left\| \int_0^1 d\varphi(tx_0)(x_0) dt \right\| + \frac{\delta}{9C^3} \leq \int_0^1 \|d\varphi(tx_0)\| \|x_0\| dt + \frac{\delta}{9C^3} \\ &\leq \|x_0\| \int_0^1 (\|d\varphi(0)\| + \|d\varphi(tx_0) - d\varphi(0)\|) dt + \frac{\delta}{9C^3} \\ &\leq \frac{\delta}{162C^4} \left(C + \frac{1}{6C^3} \right) + \frac{\delta}{9C^3} < \frac{\delta}{81C^3} + \frac{\delta}{9C^3} = \frac{10\delta}{81C^3}, \end{aligned}$$

and

$$\begin{aligned} \|v_n\| &= \|\sigma(u_n - \tau(d\psi(0)(u_n)))\| \\ &\leq C \|u_n - \tau(d\psi(0)(u_n))\| \leq C(\|u_n\| + \|\tau(d\psi(0)(u_n))\|) \\ &\leq C(\|u_n\| + C\|d\psi(0)(u_n)\|) \leq C(\|u_n\| + C^2\|u_n\|) \leq 2C^3\|u_n\|, \end{aligned}$$

which implies that

$$\|v_0\| \leq 2C^3\|u_0\| < \frac{20\delta}{81},$$

the inequalities hold for $n = 0$. Suppose that the inequalities hold for some $n \geq 0$. We first have

$$\|x_{n+1}\| = \|x_n - v_n\| \leq \|x_n\| + \|v_n\| < \frac{\delta}{162} + \frac{20\delta}{81} \sum_{i=0}^n \frac{1}{2^i}.$$

Next we note that

$$\begin{aligned} u_{n+1} &= \varphi(x_{n+1}) - y = \varphi(x_n - v_n) - y = \varphi(x_n - v_n) - \varphi(x_n) + u_n \\ &= \varphi(x_n - v_n) - \varphi(x_n) + d\varphi(0)(v_n) + \tau(d\psi(0)(u_n)), \end{aligned}$$

and that $x_n, x_n - v_n \in B_X(0, \delta)$ implies that $x_n - tv_n \in B_X(0, \delta)$ holds for $0 \leq t \leq 1$. We thus obtain

$$\begin{aligned} \|u_{n+1}\| &\leq \|\varphi(x_n - v_n) - \varphi(x_n) + d\varphi(0)(v_n)\| + \|\tau(d\psi(0)(u_n))\| \\ &\leq \left\| \int_0^1 (d\varphi(x_n - tv_n) - d\varphi(0))(v_n) dt \right\| + C\|d\psi(0)(u_n)\| \\ &\leq \int_0^1 \|d\varphi(x_n - tv_n) - d\varphi(0)\| \|v_n\| dt \\ &\quad + C\|\psi(\varphi(x_n)) - \psi(y) - d\psi(0)(\varphi(x_n) - y)\| \\ &\leq \frac{1}{6C^3}\|v_n\| + C\left\| \int_0^1 (d\psi(y + tu_n) - d\psi(0))(u_n) dt \right\| \\ &\leq \frac{1}{3}\|u_n\| + C\int_0^1 \|d\psi(y + tu_n) - d\psi(0)\| \|u_n\| dt \\ &\leq \frac{1}{3}\|u_n\| + C\frac{1}{6C^3}\|u_n\| \leq \frac{1}{2}\|u_n\| < \frac{10\delta}{81C^3} \frac{1}{2^{n+1}}. \end{aligned}$$

So we also have

$$\|v_{n+1}\| \leq 2C^3\|u_{n+1}\| < \frac{20\delta}{81} \frac{1}{2^{n+1}}.$$

This proves the inequalities.

By the definition of x_n and the inequalities, the sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, hence converges to some $x \in X$, and we have

$$\|x\| = \|x_0 - \sum_{i=0}^{\infty} v_i\| \leq \|x_0\| + \sum_{i=0}^{\infty} \|v_i\| < \frac{\delta}{162} + \frac{20\delta}{81} \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{\delta}{2}.$$

We also have $\lim_{n \rightarrow \infty} u_n = 0$. Hence

$$y = \lim_{n \rightarrow \infty} (\varphi(x_n) - u_n) = \varphi(x).$$

This proves the above claim, hence the theorem. ■

Remark 2.2 Theorem 2.1 does not hold without the assumption that $\text{im}(d\psi(0))$ being closed. Here is a counter-example. Let H be a Hilbert space with an orthonormal basis $\{e_n : n \in \mathbb{N}\}$, f be the C^1 -map from H to itself defined by $f(v) = \|v\|v$, A be the linear map from H to itself determined by $Ae_n = -\frac{1}{n}e_n$. Let $X = 0$, $Y = Z = H$, $\varphi \equiv 0$, $\psi = f + A$. Then $d\psi(0) = A$, which is injective. It is obvious that all the conditions in Theorem 2.1 hold except that $\text{im}(d\psi(0)) = \text{im}(A)$ is closed. $\frac{1}{n}e_n \in \psi^{-1}(0)$, but $\frac{1}{n}e_n$ does not belong to $\text{im}(\varphi) = \{0\}$ for each $n \in \mathbb{N}$.

Corollary 2.3 *Let Y and Z be Banach spaces, $V \subset Y$ an open 0-neighborhood, $\psi : V \rightarrow Z$ be a C^1 -map and $X := \ker(d\psi(0))$. Suppose $\psi(X \cap V) = \{0\}$ and that $\text{im}(d\psi(0))$ is closed. Then there exists a 0-neighborhood $W \subset V$ such that*

$$\psi^{-1}(0) \cap W = X \cap W.$$

3 Parametrization of fibers of differentiable maps

In this section we refine the Implicit Function Theorem 2.1 in such a way that it also provides a manifold structure on the 0-fiber of ψ . Clearly, this is not a submanifold in the sense of Bourbaki [Bou89] because its tangent space need not be complemented, but it nevertheless is a Banach manifold.

Proposition 3.1 *Let X , Y and Z be Banach spaces. Then the following assertions hold:*

- (1) *The set $L_s(X, Y)$ of all surjective linear operators from X to Y is open. If $f \in L(X, Y)$ and $B_Y(0, r) \subseteq f(B_X(0, 1))$, then $f + g \in L_s(X, Y)$ whenever $\|g\| < r$.*
- (2) *The set $L_e(X, Y)$ of all closed embeddings of X into Y is open. If $f \in L(X, Y)$ satisfies $\|f(x)\| \geq r\|x\|$ for some $r > 0$, then $f + g \in L_e(X, Y)$ whenever $\|g\| < r$.*
- (3) *The set of pairs (f, g) with $\text{im}(f) = \ker(g)$ is an open subset in*

$$\{(f, g) \in L_e(X, Y) \times L_s(Y, Z) : gf = 0\}.$$

Proof. (1) (This proof is due to H. Glöckner. Another argument is given in [Bro76], Lemma and Theorem 2.1 on pp.20/21.) Let $f: X \rightarrow Y$ be open. In view of the Open Mapping Theorem, there exists an $r > 0$ with $B_Y(0, r) \subseteq f(B_X(0, 1))$. We claim that $f + g$ is surjective for each $g \in L(X, Y)$ with $\|g\| < r$.

Let $v \in Y$ and $a := \frac{\|g\|}{r} < 1$. Then there exists some $w_0 \in X$ with $f(w_0) = v$ and $\|w_0\| \leq \frac{\|v\|}{r}$. Then we have $(f + g)(w_0) = v + g(w_0)$ with

$$\|g(w_0)\| \leq r^{-1}\|v\|\|g\| = a\|v\|.$$

Now we pick $w_1 \in X$ with $f(w_1) = -g(w_0)$ and $\|w_1\| \leq r^{-1}\|g(w_0)\| \leq a\|w_0\| \leq \frac{a}{r}\|v\|$ and obtain

$$(f + g)(w_0 + w_1) = v + g(w_1) \quad \text{with} \quad \|g(w_1)\| \leq a^2\|v\|.$$

Iterating this procedure, we obtain a sequence $(w_n)_{n \in \mathbb{N}}$ in X with $\|w_n\| \leq a^n \frac{\|v\|}{r}$, so that $w := \sum_{i=0}^{\infty} w_i$ converges, and $f(w) = \lim_{n \rightarrow \infty} v + g(w_n) = v$.

(2) The Open Mapping Theorem implies that $f \in L(X, Y)$ is a closed embedding if and only if there exists an $r > 0$ with $\|f(x)\| \geq r\|x\|$ for all $x \in X$. If this is the case and $g \in L(X, Y)$ satisfies $\|g\| < r$, then

$$\|(f + g)(x)\| \geq \|f(x)\| - \|g(x)\| \geq (r - \|g\|)\|x\|$$

implies that $f + g$ is a closed embedding.

(3) Let $(f, g) \in L_e(X, Y) \times L_s(Y, Z)$. Without loss of generality, we may assume that f is an isometric embedding and that $g: Y \rightarrow Z$ is a metric quotient map, i.e., $\|g(x)\| = \inf_{y \in x + \ker g} \|y\|$. We show that if $\delta < \frac{1}{10}$ then $\|f - \tilde{f}\|, \|g - \tilde{g}\| < \delta$ and $\tilde{g}\tilde{f} = 0$ imply that $\text{im}(\tilde{f}) = \ker(\tilde{g})$.

Let $v \in \ker \tilde{g}$. We put $v_1 := v$ and assume that we have already constructed a sequence v_1, \dots, v_n in $\ker \tilde{g}$ with

$$\|v_{j+1}\| \leq \frac{1}{2}\|v_j\| \quad \text{and} \quad v_{j+1} - v_j \in \text{im}(\tilde{f}), \quad j = 1, \dots, n-1.$$

To find v_{n+1} , we first observe that

$$\|g(v_n)\| = \|g(v_n) - \tilde{g}(v_n)\| \leq \|g - \tilde{g}\|\|v_n\| \leq \delta\|v_n\|.$$

Hence there exists $w_n \in Y$ with $\|w_n\| \leq 2\delta\|v_n\| \leq \|v_n\|$ and $g(w_n) = g(v_n)$. Then $v_n - w_n \in \ker g = \text{im } f$ and $x_n := f^{-1}(v_n - w_n)$ satisfies

$$\|x_n\| = \|v_n - w_n\| \leq 2\|v_n\|.$$

For $v_{n+1} := v_n - \tilde{f}(x_n)$ we now have $v_{n+1} \in \ker \tilde{g}$ and

$$\begin{aligned} \|v_{n+1}\| &= \|v_n - \tilde{f}(x_n)\| = \|v_n - f(x_n) + (f - \tilde{f})(x_n)\| \\ &= \|w_n + (f - \tilde{f})(x_n)\| \leq \|w_n\| + \delta\|x_n\| \\ &\leq 2\delta\|v_n\| + 2\delta\|v_n\| \leq \frac{1}{2}\|v_n\|. \end{aligned}$$

We thus obtain sequences (v_n) and (x_n) satisfying the above conditions. In particular, $x := \sum_{n=1}^{\infty} x_n$ converges in F and $\tilde{f}(x) = \sum_{n=1}^{\infty} v_n - v_{n+1} = v_1 = v$. ■

With the preceding proposition we can strengthen Theorem 2.1 under the assumption that $d\varphi(0)$ is injective and $d\psi(0)$ is surjective as follows.

Proposition 3.2 *Let X, Y and Z be Banach spaces, $U \subset X$, $V \subset Y$ be open 0-neighborhoods. Let $\varphi : U \rightarrow V$ and $\psi : V \rightarrow Z$ be C^1 -maps. Suppose*

- (1) $\varphi(0) = 0$, $\psi(0) = 0$;
- (2) $\psi \circ \varphi \equiv 0$;
- (3) $\text{im}(d\varphi(0)) = \ker(d\psi(0))$;
- (4) $d\varphi(0)$ is injective, $d\psi(0)$ is surjective.

Then there exists an open 0-neighborhood $U' \subseteq U$ such that $\varphi|_{U'}$ is an open map into $\psi^{-1}(0)$.

Proof. For each $u \in U$ the relation $\psi \circ \varphi = 0$ implies that

$$d\psi(\varphi(u))d\varphi(u) = 0,$$

so that Proposition 3.1 implies that

$$\ker(d\psi(\varphi(u))) = \text{im}(d\varphi(u))$$

holds for each u in some open 0-neighborhood $U' \subseteq U$. Applying Theorem 2.1 to the maps $\psi_u(y) := \psi(\varphi(u) + y)$ and $\varphi_u(x) := \varphi(u + x)$, it follows that $\varphi|_{U'}$ is an open map to $\psi^{-1}(0)$. ■

Lemma 3.3 *Let $U \subseteq X$ be an open subset containing 0, $f : U \rightarrow Y$ a C^1 -map for which $df(0)$ is a closed embedding. Then there exists an open 0-neighborhood $U' \subset U$ and constants $0 < c < C$ with*

$$c\|x - y\| \leq \|f(x) - f(y)\| \leq C\|x - y\| \quad \text{for } x, y \in U'.$$

Proof. Since $df(0)$ is a closed embedding, there exists some $r > 0$ with $\|df(0)x\| \geq r\|x\|$ for $x \in X$. Pick an open ball $U' \subset U$ around 0 with

$$\|df(p) - df(0)\| \leq \frac{r}{4} \quad \text{for } p \in U'$$

and note that this implies that

$$\|df(p)v\| \geq \frac{3r}{4}\|v\| \quad \text{for } v \in X.$$

We now obtain for $x, y \in U'$:

$$\begin{aligned} & \|f(y) - f(x) - df(x)(y - x)\| \\ = & \left\| \int_0^1 df(x + t(y - x))(y - x) dt - df(x)(y - x) \right\| \\ = & \left\| \int_0^1 (df(x + t(y - x)) - df(x))(y - x) dt \right\| \leq \frac{r}{2}\|y - x\|, \end{aligned}$$

which in turn leads to

$$\|f(y) - f(x)\| \geq \|df(x)(y - x)\| - \frac{r}{2}\|y - x\| \geq \frac{3r}{4}\|y - x\| - \frac{r}{2}\|y - x\| = \frac{r}{4}\|y - x\|.$$

With $c := \frac{r}{4}$ and $C := \|df(0)\| + \frac{r}{4}$ the assertion now follows from

$$\|f(y) - f(x)\| = \left\| \int_0^1 df(x + t(y - x))(y - x) dt \right\| \leq C\|y - x\|.$$

■

Let $\psi: V \rightarrow Z$ be a C^k -map from an open 0-neighborhood V in a Banach space Y into another Banach space Z with $\psi(0) = 0$ and $d\psi(0)$ is surjective. Assume that there is an open 0-neighborhood U in some Banach space X and a C^k -map $\varphi: U \rightarrow \psi^{-1}(0) \subseteq V$ such that $\varphi(0) = 0$, $d\varphi(0)$ is injective, and $\text{im}(d\varphi(0)) = \ker(d\psi(0))$. Then by Proposition 3.2 and Lemma 3.3, $\psi^{-1}(0)$ is locally a Banach C^k -manifold around 0, and a local C^k -chart is provided by φ . To get the global version of such result, we need more preparation.

Lemma 3.4 *The map*

$$\mu: \{(f, g) \in L_e(X, Y)^2: \text{im}(f) = \text{im}(g)\} \rightarrow \text{GL}(X), \quad (f, g) \mapsto f^{-1} \circ g$$

is continuous.

Proof. Fix $(f_0, g_0) \in L_e(X, Y)^2$ with $\text{im}(f_0) = \text{im}(g_0)$. We may w.l.o.g. assume that f_0 is an isometric embedding $X \hookrightarrow Y$. Then for each $h \in L(X, Y)$ with $\|h\| < 1$ the map $f_0 + h$ is a closed embedding with

$$\|(f_0 + h)v\| \geq (1 - \|h\|)\|v\| \quad \text{for } v \in X.$$

We conclude that for any $g \in L(X, Y)$ with $\text{im}(g) \subseteq \text{im}(f_0 + h)$ we have

$$\|(f_0 + h)^{-1}g\| \leq (1 - \|h\|)^{-1}\|g\|.$$

For any $h, g \in L(X, Y)$ with $\text{im}(f_0 + h) = \text{im}(g)$ and $x \in X$ we thus obtain

$$\begin{aligned} \|(f_0 + h)^{-1}g - f_0^{-1}g_0\| &= \|(f_0 + h)^{-1}(g - (f_0 + h)f_0^{-1}g_0)\| \\ &\leq (1 - \|h\|)^{-1}\|g - (f_0 + h)f_0^{-1}g_0\| \\ &\leq (1 - \|h\|)^{-1}(\|g - g_0\| + \|\mathbf{1} - (f_0 + h)f_0^{-1}\|\|g_0\|). \end{aligned}$$

Since f_0 is assumed to be isometric, the map

$$\mathbf{1} - (f_0 + h)f_0^{-1}: \text{im}(f_0) = \text{im}(g_0) \rightarrow Y$$

satisfies

$$\|\mathbf{1} - (f_0 + h)f_0^{-1}\| = \|f_0 - (f_0 + h)\| = \|h\|.$$

Therefore the estimate above implies the continuity of μ . ■

Theorem 3.5 *Let $V \subseteq Y$ be an open subset containing 0 and $\psi: V \rightarrow Z$ be a C^k -map, $k \in \mathbb{N} \cup \{\infty\}$, with $\psi(0) = 0$ for which $d\psi(0)$ is surjective. Further, let $U_1, U_2 \subseteq X$ be open subsets containing 0 and $\varphi_i: U_i \rightarrow S := \psi^{-1}(0)$ be C^k -maps with*

$$\varphi_i(0) = 0, \quad d\varphi_i(0) \text{ is injective, and } \text{im}(d\varphi_i(0)) = \ker d\psi(0) \quad \text{for } i = 1, 2.$$

Then there exist open 0-neighborhoods $U'_i \subseteq U_i$ such that $\varphi_i|_{U'_i}$ are homeomorphisms onto a 0-neighborhood in S and the map

$$\varphi_{12} := \varphi_1^{-1} \circ \varphi_2: U'_2 \cap \varphi_2^{-1}(\varphi_1(U'_1)) \rightarrow X$$

is C^k .

Proof. We prove the theorem by induction over $k \in \mathbb{N}$. If it holds for each $k \in \mathbb{N}$, it clearly holds for $k = \infty$.

We first prove the theorem for $k = 1$. Using Proposition 3.2, we may w.l.o.g. assume that φ_i is a homeomorphism onto an open 0-neighborhood in S and that

$$d\varphi_i(x) \text{ is injective, } d\psi(\varphi_i(x)) \text{ is surjective, and } \text{im}(d\varphi_i(x)) = \ker(d\psi(\varphi_i(x)))$$

hold for all $x \in U_i$. Next we use Lemma 3.3 to see that we may further assume that there exist constants $0 < c < C$ such that for $x, y \in U_i$ we have

$$c\|x - y\| \leq \|\varphi_i(x) - \varphi_i(y)\| \leq C\|x - y\|. \quad (1)$$

Now fix $p \in U'_2$ with $\varphi_2(p) \in \varphi_1(U_1)$. We have to show that φ_{12} is differentiable in p . Let $y := \varphi_2(p)$ and $q := \varphi_1^{-1}(y)$. Since $d\varphi_2(p)$ and $d\varphi_1(q)$ are closed embeddings, the linear map $A := d\varphi_1(q)^{-1} \circ d\varphi_2(p): X \rightarrow X$ is invertible.

We have

$$\varphi_2(p + h) = \varphi_2(p) + d\varphi_2(p)h + r_2(h) = y + d\varphi_2(p)h + r_2(h)$$

with $\lim_{h \rightarrow 0} \frac{r_2(h)}{\|h\|} = 0$ and

$$\varphi_1(q + h) = \varphi_1(q) + d\varphi_1(q)h + r_1(h) = y + d\varphi_1(q)h + r_1(h)$$

with $\lim_{h \rightarrow 0} \frac{r_1(h)}{\|h\|} = 0$. For $\nu(h) := \varphi_{12}(p + h) - q$ we then have

$$y + d\varphi_2(p)h + r_2(h) = \varphi_2(p + h) = \varphi_1(q + \nu(h)) = y + d\varphi_1(q)\nu(h) + r_1(\nu(h)),$$

so that

$$d\varphi_2(p)h + r_2(h) = d\varphi_1(q)\nu(h) + r_1(\nu(h)).$$

This implies that $r_2(h) - r_1(\nu(h)) \in \ker d\psi(y) = \text{im}(d\varphi_1(q))$ and that

$$\nu(h) = Ah + (d\varphi_1(q))^{-1}(r_2(h) - r_1(\nu(h))).$$

From (1) we derive

$$\frac{c}{C}\|h\| \leq \|\nu(h)\| \leq \frac{C}{c}\|h\|,$$

and hence

$$\lim_{h \rightarrow 0} \frac{r_1(\nu(h))}{\|h\|} = 0.$$

Therefore ν is differentiable in 0 with $d\nu(0) = A$.

We conclude that φ_{12} is differentiable in p with

$$d(\varphi_{12})(p) = (d\varphi_1(q))^{-1} \circ d\varphi_2(p).$$

The continuity of the differential $d(\varphi_{12})$ follows from Lemma 3.4. This finishes the proof of the theorem for $k = 1$.

Now let us assume that $k > 1$ and that the theorem holds for C^{k-1} -maps, showing that φ_{12} is a C^{k-1} -map. It remains to show that the tangent map $T\varphi_{12}$ is also C^{k-1} , which therefore proves that φ_{12} is C^k .

The tangent map

$$T\psi: TV \cong V \times Y \rightarrow TZ \cong Z \times Z, \quad (x, v) \mapsto (\psi(x), d\psi(x)v)$$

is a C^{k-1} -map with $T\psi(0, 0) = (0, 0)$ for which $d(T\psi)(0, 0) \cong d\psi(0) \times d\psi(0)$ is surjective. Further, the maps

$$T\varphi_i: TU_i \cong U_i \times X \rightarrow \tilde{S} := (T\psi)^{-1}(0, 0)$$

are C^{k-1} -maps with

$$T\varphi_i(0, 0) = (0, 0) \quad \text{and} \quad \text{im}(d(T\varphi_i)(0, 0)) = \ker d(T\psi)(0, 0), \quad i = 1, 2.$$

Our induction hypothesis implies that $T\varphi_{12} = T\varphi_1^{-1} \circ T\varphi_2$ is a C^{k-1} -map, and this completes the proof. \blacksquare

Theorem 3.6 (Regular Value Theorem; without complements) *Let $\psi: M \rightarrow N$ be a C^k -map between Banach C^k -manifolds, $q \in N$, $S := \psi^{-1}(q)$ and X a Banach space. Assume that*

- (1) q is a regular value in the sense that for each $p \in S$ the differential $d\psi(p): T_p(M) \rightarrow T_q(N)$ is surjective, and
- (2) for every $p \in S$, there exists an open 0-neighborhood $U_p \subseteq X$ and a C^k -map $\varphi_p: U_p \rightarrow S \subseteq M$ with $\varphi_p(0) = p$, $d\varphi_p(0)$ is injective, and $\text{im}(d\varphi_p(0)) = \ker(d\psi(p))$.

Then S carries the structure of a Banach C^k -manifold modeled on X .

Proof. In view of Proposition 3.2, the maps φ_p whose existence is assured by (2) yield local charts and Theorem 3.5 implies that the chart changes are also C^k , so that we obtain a C^k -atlas of S . \blacksquare

4 Applications to Lie subgroups

If G_1 and G_2 are Banach–Lie groups, then the existence of canonical coordinates (given by the exponential function) implies that each continuous homomorphism $\varphi: G_1 \rightarrow G_2$ is actually smooth, hence a morphism of Lie groups. This implies in particular that a topological group G carries at most one Banach–Lie group structure.

If G is a Banach–Lie group, then we call a closed subgroup $H \subseteq G$ a *Banach–Lie subgroup* if it is a Banach–Lie group with respect to the subspace topology. It is an important problem in infinite-dimensional Lie theory to find good criteria for a closed subgroup of a Banach–Lie group to be a Banach–Lie subgroup.

Regardless of any Lie group structure, we may define for each closed subgroup H of a Banach–Lie group its Lie algebra by

$$\mathbf{L}(H) := \{x \in \mathbf{L}(G) : \exp_G(\mathbb{R}x) \subseteq H\},$$

and the Product and Commutator Formula easily imply that $\mathbf{L}(H)$ is a closed Lie subalgebra of the Banach–Lie algebra $\mathbf{L}(G)$ (cf. [Ne06], Lemma IV.3.1). By restriction we thus obtain an exponential map

$$\exp_H: \mathbf{L}(H) \rightarrow H.$$

From [Ne06], Thm. IV.4.16 we know that there exists a Banach–Lie group H_L with Lie algebra $\mathbf{L}(H)$ and an injective morphism of Lie groups $i_H: H_L \hookrightarrow G$ for which $i_H(H_L) = \langle \exp_G \mathbf{L}(H) \rangle$ is the subgroup of H generated by the image of \exp_H . Since $\mathbf{L}(H)$ is a closed subspace of $\mathbf{L}(G)$, the map $\mathbf{L}(i_H): \mathbf{L}(H) \rightarrow \mathbf{L}(G)$ is a closed embedding, and H is a Banach–Lie subgroup if and only if $i_H: H_L \rightarrow H$ is an open map, which is equivalent to the existence of an open 0-neighborhood $U \subseteq \mathbf{L}(H)$ for which $\exp_H|_U: U \rightarrow H$ is an open embedding (cf. [Ne06], Thm. IV.3.3). This notion of a Banach–Lie subgroup is weaker than the concept used by Bourbaki ([Bou89]), where it is assumed that $\mathbf{L}(H)$ has a closed complement, a requirement that is not always satisfied and hard to verify in many concrete situations.

For closed normal subgroups H of Banach–Lie groups one has the nice characterization that H is a Banach–Lie subgroup if and only if the topological quotient group G/H is a Banach–Lie group ([GN03]). It would be nice to have a similar criterion for general subgroups:

Conjecture 4.1 *A closed subgroup H of a Banach–Lie group G is a Banach–Lie subgroup if and only if G/H carries the structure of a Banach manifold for which the quotient map $q: G \rightarrow G/H, g \mapsto gH$ has surjective differential in each point and G acts smoothly on G/H .*

The main difficulty arises from the possible absence of closed complements to the closed subspace $\mathbf{L}(H)$ in $\mathbf{L}(G)$. If H is a Banach–Lie subgroup of G and $\mathbf{L}(H)$ has a closed complement, then the classical Inverse Function Theorem provides natural charts on the quotient space G/H , hence a natural manifold structure with all nice properties (cf. [Bou89], Ch. 3). Without a closed complement for $\mathbf{L}(H)$ it is not known how to construct charts of G/H . However, the natural model space is the quotient space $\mathbf{L}(G)/\mathbf{L}(H)$.

With the aid of the “implicit function theorem”, we can now prove one half of the conjecture above:

Theorem 4.2 *Let G be a Banach–Lie group and $\sigma: G \times M \rightarrow M$ a smooth action of G on a Banach manifold M , $p \in M$. Suppose that for the derived action*

$$\dot{\sigma}: \mathbf{L}(G) \rightarrow \mathcal{V}(M), \quad \dot{\sigma}(x)(m) := -d\sigma(\mathbf{1}, m)(x, 0)$$

the subspace $\dot{\sigma}(\mathbf{L}(G))(p)$ of $T_p(M)$ is closed. Then the stabilizer

$$G_p := \{g \in G: g.p = p\}$$

is a Lie subgroup.

Proof. First we note that G_p is a closed subgroup of G with

$$\mathbf{L}(G_p) = \{x \in \mathbf{L}(G): \dot{\sigma}(x)(p) = 0\}.$$

We consider the smooth maps

$$\psi: G \rightarrow M, \quad g \mapsto g.p \quad \text{and} \quad \varphi: \mathbf{L}(G_p) \rightarrow G_p, \quad x \mapsto \exp_G(x).$$

Then $G_p = \psi^{-1}(p)$, $\varphi(0) = \mathbf{1}$,

$$\text{im}(d\varphi(0)) = \mathbf{L}(G_p) = \ker(d\psi(\mathbf{1})).$$

Since $\text{im}(d\varphi(0)) = \dot{\sigma}(\mathbf{L}(G))(p)$ is closed, Theorem 2.1 implies that there exists arbitrarily small open 0-neighborhood $U \subseteq \mathbf{L}(G_p)$ such that $\exp_G(U)$ is a $\mathbf{1}$ -neighborhood in G_p . By [Ne06], Thm. IV.3.3 (cf. also [Hof75], Prop. 3.4), G_p is a Banach–Lie subgroup of G . ■

Corollary 4.3 *Let G be a Banach–Lie group and $H \subseteq G$ a closed subgroup for which G/H carries the structure of a Banach manifold. Suppose the quotient map $q: G \rightarrow G/H, g \mapsto gH$ has surjective differential in some point $g_0 \in G$ and G acts smoothly on G/H . Then H is a Banach–Lie subgroup of G .*

Proof. The action of G on G/H is given by $\sigma(g, xH) = gxH = q(gx)$, so that

$$\dot{\sigma}(\mathbf{L}(G))(gH) = dq(g)(\mathbf{L}(G)) = T_{gH}(G/H)$$

and the stabilizer of $p := gH = q(g) \in G/H$ is gHg^{-1} . By Theorem 4.2, $g_0Hg_0^{-1}$ is a Banach–Lie subgroup. So H is a Banach–Lie subgroup. ■

Remark 4.4 The preceding theorem has a natural generalization to the following setting. Let H be a closed subgroup of a Banach–Lie group G . Suppose that there exists an open $\mathbf{1}$ -neighborhood $U_G \subseteq G$ and a C^1 -map $F: U_G \rightarrow M$ to some Banach manifold M such that $dF(\mathbf{1})$ is surjective and that for $m := F(\mathbf{1})$ we have

$$F^{-1}(m) \cap U_G = H \cap U_G.$$

Then H is a Banach–Lie subgroup of G .

Problem 4.5 Suppose that the Banach–Lie group G acts smoothly on the Banach manifold M and $p \in M$. Is it always true that the stabilizer G_p is a Lie subgroup of G ?

From the theory of algebraic Banach–Lie groups it follows that for each Banach space Y , for each $y \in Y$ and each closed subspace $X \subseteq Y$, the subgroups

$$\mathrm{GL}(Y)_y := \{g \in \mathrm{GL}(Y) : g(y) = y\}$$

and

$$\mathrm{GL}(Y)_X := \{g \in \mathrm{GL}(Y) : g(X) = X\}$$

are Banach–Lie subgroups. Since inverse images of Lie subgroups are Lie subgroups ([Ne04], Lemma IV.11) the problem has a positive solution for linear actions and for stabilizers of closed subspaces.

If X is not complemented in Y , then it is not clear how to turn the Grassmannian $\mathrm{Gr}_X(Y) := \{g.X : g \in \mathrm{GL}(Y)\}$ into a smooth manifold on which $\mathrm{GL}(Y)$ acts. We are therefore in a situation where we know that the

closed subgroup $\mathrm{GL}(X)_Y$ is a Banach–Lie subgroup, but nothing is known on the quotient $\mathrm{Gr}_X(Y) \cong \mathrm{GL}(Y)/\mathrm{GL}(Y)_X$. Therefore a solution of Conjecture 4.1 would in particular lead to a natural Banach manifold structure on the Grassmannian of all closed subspaces of Y .

5 Weil’s local rigidity for Banach–Lie groups

Let G be a topological group, Γ a finitely generated (discrete) group. Let $\mathrm{Hom}(\Gamma, G)$ be the set of all homomorphisms from Γ to G , endowed with the compact-open topology, which coincides with the topology of pointwise convergence. A homomorphism $r \in \mathrm{Hom}(\Gamma, G)$ is *locally rigid* if there is an open neighborhood U of r in $\mathrm{Hom}(\Gamma, G)$ such that for any $r' \in U$, there exists some $g \in G$ with $r'(\gamma) = gr(\gamma)g^{-1}$ for all $\gamma \in \Gamma$.

If, moreover, G is a Banach–Lie group, then every homomorphism $r \in \mathrm{Hom}(\Gamma, G)$ defines naturally a Γ -module structure on the Lie algebra $\mathbf{L}(G)$ of G , via $\gamma.v = \mathrm{Ad}(r(\gamma))(v)$, $\gamma \in \Gamma, v \in \mathbf{L}(G)$. For a fixed $r \in \mathrm{Hom}(\Gamma, G)$ and a map $f: \Gamma \rightarrow G$ the pointwise product $f \cdot r: \Gamma \rightarrow G$ is a homomorphism if and only if f satisfies the cocycle condition

$$f(\gamma_1\gamma_2) = f(\gamma_1) \cdot (r(\gamma_1)f(\gamma_2)r(\gamma_1)^{-1}).$$

For a smooth family of homomorphisms $r_t \in \mathrm{Hom}(\Gamma, G)$ ($-\varepsilon < t < \varepsilon$) with $r_0 = r$, this observation directly implies that the differential of $r_t r^{-1}$ at $t = 0$ is a cocycle of Γ in $\mathbf{L}(G)$, and if $r_t = g_t r g_t^{-1}$ for a smooth curve g_t in G with $g_0 = e$, then the above differential is a coboundary. So it is reasonable to expect some relation between the local rigidity of r and the vanishing of $H^1(\Gamma, \mathbf{L}(G))$.

The following classical result is due to Weil [We64].

Theorem 5.1 (Weil) *Let G be a finite-dimensional Lie group and $r \in \mathrm{Hom}(\Gamma, G)$. If $H^1(\Gamma, \mathbf{L}(G)) = \{0\}$, then r is locally rigid.*

The main ingredient in Weil’s proof of Theorem 5.1 is the finite-dimensional version of Theorem 2.1. Using Theorem 2.1, in this section we generalize Weil’s Theorem to Banach–Lie groups. To state our result, we first recall a definition of the first cohomology of groups, which is related to presentations of groups.

Let Γ be a group. Let

$$1 \rightarrow R \xrightarrow{\iota} F \xrightarrow{\pi} \Gamma \rightarrow 1$$

be a presentation of Γ , where F is a free group on a subset $S \subset F$, R is a free group on a subset $T \subset R$. For $t \in T$, $\iota(t)$ can be expressed as a reduced word

$$\iota(t) = s_{t,1}^{\epsilon_{t,1}} \cdots s_{t,m_t}^{\epsilon_{t,m_t}}$$

in S , where $s_{t,j} \in S$, $\epsilon_{t,j} = \pm 1$.

Let V be a Γ -module. For a set A , let $\text{Map}(A, V)$ denote the set of all maps from A to V . Then $\text{Map}(A, V)$ has a natural abelian group structure induced from that of V . We define the coboundary operators

$$\delta^0 : V \rightarrow \text{Map}(S, V) \quad \text{and} \quad \delta^1 : \text{Map}(S, V) \rightarrow \text{Map}(T, V)$$

as follows:

$$\delta^0(v)(s) = v - \pi(s).v, \quad \delta^1(c)(t) = \sum_{j=1}^{m_t} \epsilon_{t,j} \pi(s_{t,1}^{\epsilon_{t,1}} \cdots s_{t,j-1}^{\epsilon_{t,j-1}} s_{t,j}^{\epsilon'_{t,j}}).c(s_{t,j})$$

for $v \in V$ and $c \in \text{Map}(S, V)$, where

$$\epsilon'_{t,j} = \begin{cases} 0, & \epsilon_{t,j} = 1; \\ -1, & \epsilon_{t,j} = -1. \end{cases}$$

Then δ^0 and δ^1 are homomorphisms of abelian groups, $\delta^1 \circ \delta^0 = 0$. The first cohomology group of Γ in V is, by definition, $H^1(\Gamma, V) = \ker(\delta^1) / \text{im}(\delta^0)$.

Remark 5.2 This definition of $H^1(\Gamma, V)$ coincides with the usual one which is defined by $H^1(\Gamma, V) = Z^1(\Gamma, V) / B^1(\Gamma, V)$, where

$$Z^1(\Gamma, V) = \{f \in \text{Map}(\Gamma, V) \mid (\forall \gamma_1, \gamma_2 \in \Gamma) f(\gamma_1 \gamma_2) = f(\gamma_1) + \gamma_1.f(\gamma_2)\},$$

$$B^1(\Gamma, V) = \{f \in \text{Map}(\Gamma, V) \mid (\exists v \in V)(\forall \gamma \in \Gamma) f(\gamma) = v - \gamma.v\}.$$

A direct way to see this is to look at the bijection between $Z^1(\Gamma, V)$ and $\ker(\delta^1)$ which sends $f \in Z^1(\Gamma, V)$ to the element $s \mapsto f(\pi(s))$ in $\ker(\delta^1)$, under which $B^1(\Gamma, V)$ is mapped bijectively onto $\text{im}(\delta^0)$. A more intrinsic way to see the coincidence is explained in Remark 5.5 below.

Now suppose that Γ is a finitely generated group. Then the presentation of Γ may be chosen in such a way that F is finitely generated. So we may assume that S is finite, T is countable. Suppose moreover that V is a continuous Banach Γ -module. Then for a countable set A , $\text{Map}(A, V)$ a Fréchet space with respect to the semi-norms

$$p_a(f) = \|f(a)\|, \quad a \in A.$$

If, moreover, A is finite, then $\text{Map}(A, V)$ is a Banach space with respect to the norm

$$\|f\|_A = \sup_{a \in A} \|f(a)\|.$$

In particular, $(\text{Map}(S, V), \|\cdot\|_S)$ is a Banach space, $(\text{Map}(T, V), \{p_t(\cdot) | t \in T\})$ is a Fréchet space. It is easy to see that the coboundary operators δ^0 and δ^1 are continuous. We say that δ^1 has closed image if $\text{im}(\delta^1)$ is a closed subspace of $\text{Map}(T, V)$. Our generalization of Weil's Theorem can be stated as follows.

Theorem 5.3 *Let G be a Banach–Lie group, Γ be a finitely generated group and $r \in \text{Hom}(\Gamma, G)$. If $H^1(\Gamma, \mathbf{L}(G)) = \{0\}$ and δ^1 has closed image, then r is locally rigid.*

In general, the set of generators T of the group R is infinite. So $\text{Map}(T, V)$ is only a Fréchet space. The following lemma enable us to reduce the problem to the category of Banach spaces so as to we can apply Theorem 2.1.

Lemma 5.4 *Let Γ be a finitely generated group, V a continuous Banach Γ -module, and let the notation be as above. If δ^1 has closed image, then there exists a finite subset $T' \subset T$ such that the map $\delta_{T'}^1 : \text{Map}(S, V) \rightarrow \text{Map}(T', V)$ defined by $\delta_{T'}^1(c) = \delta^1(c)|_{T'}$ satisfies $\ker(\delta_{T'}^1) = \ker(\delta^1)$, and $\text{im}(\delta_{T'}^1)$ is closed in $(\text{Map}(T', V), \|\cdot\|_{T'})$.*

Proof. Since δ^1 is continuous, $\ker(\delta^1)$ is a closed subspace of $\text{Map}(S, V)$. So $X = \text{Map}(S, V)/\ker(\delta^1)$ is a Banach space with respect to the norm

$$\|[c]\|' = \inf_{z \in \ker(\delta^1)} \|c + z\|_S,$$

where $[c] = c + \ker(\delta^1)$, and the linear injection $\tilde{\delta}^1 : X \rightarrow \text{Map}(T, V)$ induced by δ^1 is continuous. Since $\text{im}(\tilde{\delta}^1) = \text{im}(\delta^1)$ is closed in $\text{Map}(S, V)$, $\text{im}(\tilde{\delta}^1)$

is a Fréchet space with respect to the semi-norms p_t , $t \in T$. By the Open Mapping Theorem for Fréchet spaces (see, for example, [Bou87], Section I.3, Corollary 1), $\widetilde{\delta}^1$ is an isomorphism of Fréchet spaces from X onto $\text{im}(\delta^1)$. By [Bou87], Section II.1, Corollary 1 to Proposition 5, there is a finite subset $T' \subset T$ such that

$$\|[c]\|' \leq C \sup_{t \in T'} p_t(\widetilde{\delta}^1([c])) = C \sup_{t \in T'} p_t(\delta^1(c)) = C \sup_{t \in T'} \|\delta_{T'}^1(c)(t)\| = C \|\delta_{T'}^1(c)\|_{T'},$$

for all $c \in \text{Map}(S, V)$, where $C > 0$ is a constant. If $c \in \ker(\delta_{T'}^1)$, the above inequality implies that $\|[c]\|' = 0$. So $c \in \ker(\delta^1)$. Then $\ker(\delta_{T'}^1) \subset \ker(\delta^1)$. The converse inclusion is obvious.

Let $\delta_{T'}^1 : X \rightarrow \text{Map}(T', V)$ be the continuous linear map induced by $\delta_{T'}^1$. The above inequality says nothing but $\|[c]\|' \leq C \|\widetilde{\delta}_{T'}^1([c])\|_{T'}$ for $[c] \in X$. So $\text{im}(\delta_{T'}^1) = \text{im}(\widetilde{\delta}_{T'}^1)$ is closed in $\text{Map}(T', V)$. This proves the lemma. \blacksquare

Proof of Theorem 5.3. Since the Lie algebra $\mathbf{L}(G)$ of G is a continuous Banach Γ -module, and we have assumed that δ^1 has closed image, by Lemma 5.4, there is a finite subset $T' \subset T$ such that the map $\delta_{T'}^1 : \text{Map}(S, \mathbf{L}(G)) \rightarrow \text{Map}(T', \mathbf{L}(G))$ defined by $\delta_{T'}^1(c) = \delta^1(c)|_{T'}$ satisfies $\ker(\delta_{T'}^1) = \ker(\delta^1)$, and $\text{im}(\delta_{T'}^1)$ is closed in $(\text{Map}(T', \mathbf{L}(G)), \|\cdot\|_{T'})$.

For a finite set A , the set $\text{Map}(A, G)$ of all maps from A to G is isomorphic to the product G^A of $|A|$ copies of G , hence carries a natural Banach manifold structure, and the tangent space of $\text{Map}(A, G)$ at the constant map $e_A : a \mapsto e$ may be identified with $\text{Map}(A, \mathbf{L}(G))$.

Define the maps $\varphi : G \rightarrow \text{Map}(S, G)$ and $\psi : \text{Map}(S, G) \rightarrow \text{Map}(T', G)$ by

$$\begin{aligned} \varphi(g)(s) &= gr(\pi(s))g^{-1}r(\pi(s))^{-1}, \\ \psi(\alpha)(t) &= (\alpha(s_{t,1})r(\pi(s_{t,1})))^{e_{t,1}} \cdots (\alpha(s_{t,m_t})r(\pi(s_{t,m_t})))^{e_{t,m_t}}. \end{aligned}$$

Then it is easy to see that $\varphi(e) = e_S$, $\psi(e_S) = e_{T'}$, and that $\psi \circ \varphi \equiv e_{T'}$. Now we compute the differentials of φ and ψ at e and e_S , respectively. We have

$$\begin{aligned} d\varphi(e)(v)(s) &= \left. \frac{d}{d\lambda} \right|_{\lambda=0} \varphi(\exp(\lambda v))(s) \\ &= \left. \frac{d}{d\lambda} \right|_{\lambda=0} \exp(\lambda v)r(\pi(s)) \exp(-\lambda v)r(\pi(s))^{-1} = v - \text{Ad}(r(\pi(s)))v = \delta^0(v)(s), \end{aligned}$$

where $v \in \mathbf{L}(G)$, $s \in S$. So

$$d\varphi(e) = \delta^0.$$

We also have

$$\begin{aligned} d\psi(e_S)(c)(t) &= \frac{d}{d\lambda} \Big|_{\lambda=0} \psi(\exp(\lambda c))(t) \\ &= \frac{d}{d\lambda} \Big|_{\lambda=0} (\exp(\lambda c(s_{t,1}))r(\pi(s_{t,1})))^{\epsilon_{t,1}} \cdots (\exp(\lambda c(s_{t,m_t}))r(\pi(s_{t,m_t})))^{\epsilon_{t,m_t}} \\ &= \frac{d}{d\lambda} \Big|_{\lambda=0} \prod_{j=1}^{m_t} r(\pi(s_{t,1}^{\epsilon_{t,1}} \cdots s_{t,j-1}^{\epsilon_{t,j-1}} s_{t,j}^{\epsilon'_{t,j}})) \exp(\epsilon_{t,j} \lambda c(s_{t,j})) r(\pi(s_{t,1}^{\epsilon_{t,1}} \cdots s_{t,j-1}^{\epsilon_{t,j-1}} s_{t,j}^{\epsilon'_{t,j}}))^{-1} \\ &= \sum_{j=1}^{m_t} \epsilon_{t,j} \text{Ad}(r(\pi(s_{t,1}^{\epsilon_{t,1}} \cdots s_{t,j-1}^{\epsilon_{t,j-1}} s_{t,j}^{\epsilon'_{t,j}}))) c(s_{t,j}) \\ &= \delta^1(c)(t), \end{aligned}$$

where $c \in \text{Map}(S, \mathbf{L}(G))$, $t \in T'$. So $d\psi(e_S) = \delta_{T'}^1$. Since $H^1(\Gamma, \mathbf{L}(G)) = \{0\}$, we have

$$\text{im}(d\varphi(e)) = \text{im}(\delta^0) = \ker(\delta^1) = \ker(\delta_{T'}^1) = \ker(d\psi(e_S)).$$

We also have that $\text{im}(d\psi(e_S)) = \text{im}(\delta_{T'}^1)$ is closed in $\text{Map}(T', \mathbf{L}(G))$. This verifies all the conditions of Theorem 2.1. So by Theorem 2.1, there exists a neighborhood $W \subset \text{Map}(S, G)$ of e_S such that

$$\psi^{-1}(e_{T'}) \cap W = \varphi(G) \cap W.$$

Let U be the neighborhood of r in $\text{Hom}(\Gamma, G)$ such that $r' \in U$ if and only if the map $\alpha_{r'} : S \rightarrow G$ defined by $\alpha_{r'}(s) = r'(\pi(s))r(\pi(s))^{-1}$ is in W . Then for any $r' \in U$, $\alpha_{r'} \in \psi^{-1}(e_{T'}) \cap W$. By what we have proved above, $\alpha_{r'} \in \varphi(G)$, which means that there exists $g \in G$ such that

$$gr(\pi(s))g^{-1}r(\pi(s))^{-1} = r'(\pi(s))r(\pi(s))^{-1}$$

for all $s \in S$, that is, $r'(\gamma) = gr(\gamma)g^{-1}$ for $\gamma \in \pi(S)$. But $\pi(S)$ generates Γ . So $r'(\gamma) = gr(\gamma)g^{-1}$ for any $\gamma \in \Gamma$. This shows that r is locally rigid. \blacksquare

Remark 5.5 The condition that δ^1 has closed image in Theorem 5.3 can be replaced by a more intrinsic condition. We recall the definition of group cohomology by projective resolutions. Let Γ be a group, let

$$\cdots \rightarrow P_2 \xrightarrow{\partial_1} P_1 \xrightarrow{\partial_0} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0$$

be a projective Γ -resolution of the trivial Γ -module \mathbb{Z} . Then for a Γ -module V , the cohomology $H^*(\Gamma, V)$ of Γ in V is, by definition, the cohomology of the complex

$$0 \rightarrow \mathrm{Hom}_\Gamma(P_0, V) \xrightarrow{\delta^0} \mathrm{Hom}_\Gamma(P_1, V) \xrightarrow{\delta^1} \mathrm{Hom}_\Gamma(P_2, V) \rightarrow \cdots,$$

and $H^*(G, V)$ is independent of the particular choice of the projective resolution. The most usual definition of group cohomology that appeared in Remark 5.2 corresponds to the standard resolution. If each P_j in the resolution is countable, and V is a continuous Banach Γ -module, then each $\mathrm{Hom}_\Gamma(P_j, V)$, viewed as a closed subspace of $\mathrm{Map}(P_j, V)$, is a Fréchet space, and the coboundary operators δ^j are continuous. For some $n \geq 0$, to say that δ^n has closed image is equivalent to say that $H^{n+1}(G, V)$ is Hausdorff (see [BW]). Using a standard argument of homological algebra, it can be shown that $H^{n+1}(G, V)$ being Hausdorff is independent of the particular choice of the countable projective resolution. The operator δ^1 that we used in Theorem 5.3 is just the first coboundary operator defined by the Gruenberg resolution of Γ associated with the presentation of Γ (see [Gr60]), and the modules P_j in the Gruenberg resolution are countable if the free group F in the presentation of Γ is countable. So the condition that δ^1 has closed image in Theorem 5.3 is equivalent to the condition that $H^2(\Gamma, \mathbf{L}(G))$ is Hausdorff, which is independent of the particular countable resolution that we use in the definition of the group cohomology. By [BW], Chapter IX, Proposition 3.5, if $H^2(\Gamma, \mathbf{L}(G))$ is finite-dimensional, then $H^2(\Gamma, \mathbf{L}(G))$ is Hausdorff.

In view of the above remark, the following corollary of Theorem 5.3 is obvious.

Corollary 5.6 *Let G be a Banach–Lie group, Γ be a finitely generated group and $r \in \mathrm{Hom}(\Gamma, G)$. If $H^1(\Gamma, \mathbf{L}(G)) = \{0\}$ and $H^2(\Gamma, \mathbf{L}(G))$ is finite-dimensional, then r is locally rigid.*

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