A geometrically exact thin membrane model - investigation of large deformations and wrinkling

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Abstract

We investigate a geometrically exact membrane model with respect to its capabilities in describing buckling and wrinkling. Contrary to more classical tension-field or relaxed approaches, our model is able to capture the detailed geometry of wrinkling while the balance of force equations remains elliptic throughout. This is achieved by introducing artficial viscosity related to the movement of an adjusted orthonormal frame (rotations) given by a local evolution equation. We discuss the consistent linearization of the model and investigate the efficiency of the local update of rotations. Numerical examples are presented that demonstrate the effectiveness of the new model for predicting wrinkles in membranes undergoing large deformation.

Key words: membranes, shells, thin films, energy minimization, viscoelasticity, transverse shear, tension-field theory, wrinkling, buckling.
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1 Introduction

Membrane-like structures are very diverse in nature but also in civil and mechanical engineering. In terms of structural mechanics membranes are gossamer, i.e., flexible and thin-walled areal structures with a high load bearing capacity and negligible bending resistance. In consequence, the equilibrium positions of membranes are characterized by a dominance of in-plane stresses and strains. Out-of-plane deformations are, of course, also observed in practice but they are usually understood as instabilities, e.g. folded, kinked or completely crashed (regions of) membranes.

Membrane-like structures are employed in many fields, just think of airbags, balloons, ultralight planes, solar sails of satellites or air-beams and air-cushions. Because of this huge range of applications there is a great request for a realistic description of the mechanical behavior of membranes. This can, in principle, be provided by means of classical shell theory. The different approaches toward elastic shell analysis are too numerous to list here, an illustrative review of today's state of the art provides [RW04], for a thorough mathematical analysis of the infinitesimal-displacement shell theory see [Cia98] and references therein. However, because shell theories are designed for thin but not gossamer structures the computation of membranes with shell finite elements invokes all the theoretical and numerical difficulties which are encountered for the limit of characteristic thickness $h \to 0$, [GDOC02].

Therefore, more suitable for such flexible applications is the membrane theory, see [Jen01] for an overview. Whereas a shell theory accounts for bending effects of the thin structures in classical membrane theory the bending stiffness is completely neglected. This a priori assumption renders the theory relatively simple, no (local) rotation of the membrane's mid-plane needs to be mapped. To develop their full load bearing capacity a membrane requires a tensional state of stress, this fact is considered by so called tension field theories [Ste90]. A traditional approach which goes back to the early 20th century is the introduction of a modified constitutive law to generate "no-compression material" models, [Wag29, Rei38]. In a similar manner relaxed strain energy densities may avoid compressive stresses, [Pip94, DR96]. More kinematically oriented approaches introduce a "corrected" deformation gradient, [Rod87, Rod91], what allows also to include anisotropic and irreversible material laws. All these membrane theories have in common that they determine the load bearing capacity of a membrane even in the presence of folds. Nonetheless, the detailed geometry, i.e., the actual position and amplitude of the folds remains undetermined.

However, instabilities like out-of-midplane deformation may also be of practical interest. An elastic membrane with no flexural rigidity will become wrinkled as soon as one of the principal stretches is non-positive. In general, three typical (local and in-plane) stress states may be distinguished. A membrane is *taut* when both principal stresses are tensile, a membrane is homogenously *wrinkled* when there is a uniaxial state of tensile stress, and, in the absence of tensile stresses the membrane is *slack*.

In this contribution we present results of a visco-elastic membrane model which allows for the computation of all three membrane states in a continuous way. It has been derived by dimensional descent from a bulk model, [Nef05a]. provides a geometrically exact finite deformation kinematic and is based on a modification of a visco-elastic strain energy density. Furthermore, the local well-posedness of the suggested model has been shown which sets it apart from practically all other geometrically exact membrane models, [Nef05b]. Typically, membrane models are either accompanied by a complete loss of ellipticity in a slack state [DR95a, Mia98, CSP95] or this loss of ellipticity is avoided by a quasiconvexification step making it impossible to describe the geometry of the wrinkles [DR95b, DR96]. We will show here by examples that such a stress state may be captured by our (elliptic) model, too. Moreover, the model predicts the detailed geometry of the deformation.

Remark on Notation: We work here in the context of nonlinear, finite visco-elasticity and consider a time period $t \in [0, T]$. In the following $\omega \subset \mathbb{R}^2$ denotes the flat referential domain of the membrane with smooth (Lipschitz continuous) boundary $\partial\omega$, the characteristic thickness is $h > 0$.
For vectors $a, b \in \mathbb{R}^3$ we denote with $\langle a, b\rangle$ the scalar product with associated vector norm $\|a\| = \langle a, a\rangle$. By $\mathbb{M}^{n\times m}$ the set of linear mappings $\mathbb{R}^n \mapsto \mathbb{R}^m$ is identified. In particular, $\mathbb{M}^{3\times 3}$ is the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3\times 3}$ is given by $\langle X, Y\rangle = \operatorname{tr}\left[XY^T\right]$, and thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle$. The identity tensor on $\mathbb{M}^{3\times 3}$ will be denoted by $\mathbb{1}$, so that $\operatorname{tr}[X] = \langle X, \mathbb{1}\rangle$. For $w \in \mathbb{M}^{2\times 3}$ and $X_3 \in \mathbb{R}^3$ we employ the notation $(w|X_3) \in \mathbb{M}^{3\times 3}$ to write the matrix composed of w and the (third) column vector of tensor X. Likewise $(x|y|z)$ is the matrix composed of the vectors $x, y, z \in \mathbb{R}^3$.
Moreover, we adopt here the usual abbreviations of Lie-group theory, i.e., $\mathrm{GL}(3) := \{X \in \mathbb{M}^{3\times 3} \,|\, \det X \neq 0\}$ is the general linear group and $\mathrm{SO}(3) := \{X \in \mathrm{GL}(3) \,|\, X^T X = \mathbb{1},\ \det X = 1\}$ is the subgroup of orthogonal tensors.

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2 The finite-strain-viscoelastic membrane model

The spatial deformation of a thin-walled structure $\phi_s : \omega \times (-\frac{h}{2}, \frac{h}{2}) \to \mathbb{R}^3$ can be viewed as being composed of the motion of the midsurface $m : \omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^2$ and of the motion of the director (initially) orthogonal to the midsurface. Presuming a plane initial state and with the coordinates indicated in Figure 1 we write for the displacement of the midsurface $u : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, such that $m(x, y) = (x, y, 0)^T + u(x, y)$. The membrane model presented here uses the polar decomposition of the deformation gradient into a continuum rotation R and a symmetric stretch tensor U. Let us write the polar decomposition in the form $R\,U = \operatorname{polar}(F)\,U$ with $R = \operatorname{polar}(F)$ being the orthogonal part of deformation gradient F, $R \in \mathrm{SO}(3)$. The out-of plane component of this continuum rotation, $R(x, y).e_3$, is the natural choice for the director of midsurface $m(x, y)$ consistent with small strains in the three-dimensional model. An additional variable $\varrho_m \in \mathbb{R}$ accounts for a varying thickness. In consequence, the spacial motion of an initially plane membrane can be written as

$$\phi_s(x, y, z) = m(x, y) + z\varrho_m(x, y)\, R(x, y).e_3\,. \tag{2.1}$$

The basic idea of our viscoelastic membrane model is the introduction of an *additional* field of independently evolving viscoelastic rotations $\overline{R} \in \mathrm{SO}(3)$. These rotations $\overline{R}$ are thought of as being physical meaningful but not exact continuum rotations R. With $R_3 \equiv \overline{R}(x, y).e_3$ denoting the corresponding out-of plane component the dimensional reduction of a three-dimensional continuum solid to a geometrically exact membrane model results in a deformation gradient of the form

$$F = (\nabla m|\varrho_m\,\overline{R}_3) \quad \text{with} \quad \varrho_m = 1 - \frac{\lambda}{2\mu + \lambda}\left[\langle(\nabla m|0), \overline{R}\rangle - 2\right], \tag{2.2}$$

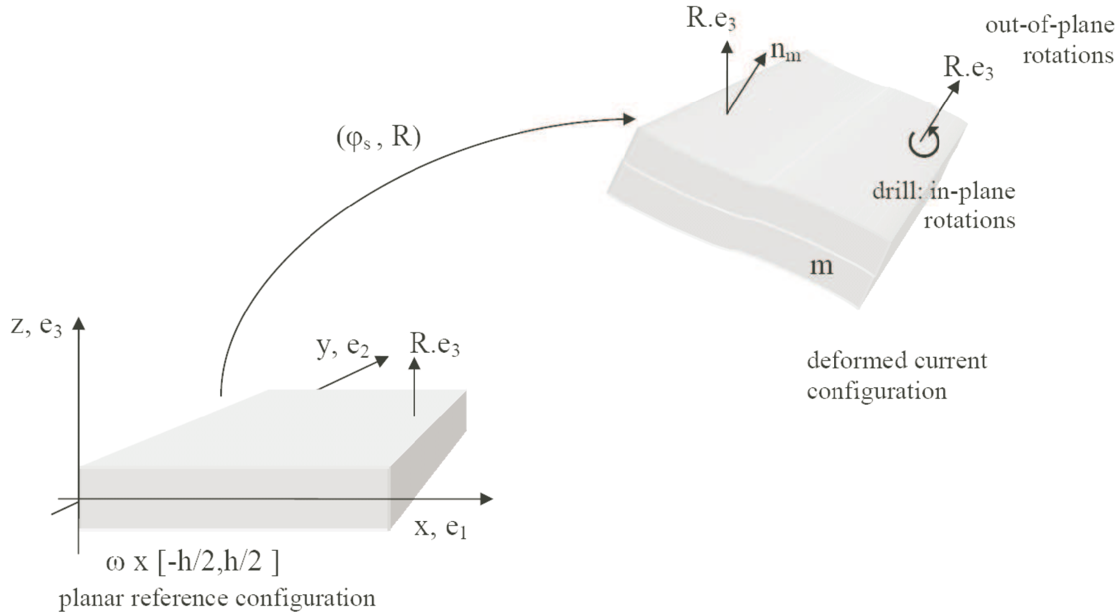


Figure 1: The three-dimensional membrane kinematics incorporating viscoelastic transverse shear ($\overline{R}_3 \neq \vec{n}_m$) and instantaneous thickness stretch ($\varrho_m \neq 1$).

where λ and μ are the Lame moduli; $\nabla m \in \mathbb{M}^{3\times 3}$ is the deformation gradient of the midsurface with $m_x = (m_{1,x}, m_{2,x}, m_{3,x})^T$, $m_y = (m_{1,y}, m_{2,y}, m_{3,y})^T$. The function $\varrho_m : \mathbb{M}^{3\times 2} \times \mathrm{SO}(3,\mathbb{R}) \mapsto \mathbb{R}$ accounts for a thickness stretch of the membrane, i.e., the thickness is decreasing for increasing membrane stretch, see remark below.

The membrane problem in a variational formulation then reads: find the deformation of the midsurface $m : [0,T] \times \omega \mapsto \mathbb{R}^3$ and the independent local viscoelastic rotation $\overline{R} : [0,T] \times \omega \mapsto \mathrm{SO}(3,\mathbb{R})$ such that

$$\int_\omega h\, W(F, \overline{R})\, \mathrm{d}\omega - \int_\omega \langle f_b, m\rangle\, \mathrm{d}\omega - \int_{\gamma_s} \langle f_s, m\rangle\, \mathrm{ds} \mapsto \min ., \tag{2.3}$$

w.r.t. m at fixed rotation $\overline{R}$. The strain energy density $W(F, \overline{R})$ in (2.3) is of the form

$$W(F, \overline{R}) = \frac{\mu}{4}\, \|F^T \overline{R} + \overline{R}^T F - 2\mathbb{1}\|^2 + \frac{\lambda}{8}\, \mathrm{tr}\left[F^T \overline{R} + \overline{R}^T F - 2\mathbb{1}\right]^2 . \tag{2.4}$$

Moreover, let $W^{\mathrm{ext}}(m)$ be the linear work of applied external forces with f_b being the resultant body forces and f_s the resultant surface traction and let $g_{\mathrm{d}} : \omega \mapsto \mathbb{R}^3$ denote the prescribed Dirichlet boundary conditions for the membrane,

$$\begin{aligned} W^{\mathrm{ext}}(m) &= \int_\omega \langle f_b, m\rangle\, \mathrm{d}\omega - \int_{\gamma_s} \langle f_s, m\rangle\, \mathrm{ds}\,, \\ m_{|\gamma_0}(t,x,y) &= g_{\mathrm{d}}(t,x,y) \qquad x, y \in \gamma_0 \subset \partial\omega\,. \end{aligned} \tag{2.5}$$

The field of local viscoelastic rotation follows an **evolution equation**

$$\frac{\mathrm{d}}{\mathrm{dt}} \overline{R}(t) = \nu^+ \cdot \mathrm{skew}\,(B) \cdot \overline{R}(t) \quad \text{with} \quad \nu^+ := \frac{1}{\eta} \nu^+(F, \overline{R}), \quad \text{and } B = F\overline{R}^T. \tag{2.6}$$

Here $\nu^+ \in \mathbb{R}^+$ represents a scalar valued function introducing an *artificial viscosity* and η plays the role of an *artificial relaxation time* (with units [sec]). The evolution equation (2.6) and parameter ν^+ are introduced into the model to preserve ellipticity of the force balance. Physically, one may imagine the viscoelastic rotation $\overline{R}$ as *shadowing* the exact continuum rotation in a viscous sense.

The **stresses** generated **in the membrane** are known to be the derivative of the strain energy density w.r.t the corresponding deformation tensor. Consequently, we obtain here for the first Piola-Kirchhoff stress tensor

$$S(F,\overline{R}) = D_F W(F,\overline{R}) = \mu\,\overline{R}\left(F^T\overline{R} + \overline{R}^T F - 2\cdot 1\!\!1\right) + \lambda\,\langle F^T\overline{R} - 1\!\!1, 1\!\!1\rangle \cdot \overline{R}\,, \tag{2.7}$$

where F is the reconstructed deformation gradient (2.2). Then the Cauchy stresses as well as other stress tensors follow by the well known transformation rules, [Hol00]. Note that the Cauchy stresses are in principal non-symmetric (due to the independent field of rotations $\overline{R}$) but turn out to be close to symmetry by our choice of a small artificial viscosity.

Before we proceed with a temporal and spatial discretisation of the presented model we add here some remarks to associate our model with other approaches of dimensional reduction.

Remark 2.1: Note that for the reduced term $\widehat{F} = (\nabla m|\overline{R}_3)$ instead of (2.2) the elastic energy can be written in fact as

$$\begin{aligned} W(F,\overline{R}) &= \mu\,\|\,\mathrm{sym}\left(F^T\overline{R} - 1\!\!1\right)\|^2 + \frac{\lambda}{2}\,\mathrm{tr}\left[\mathrm{sym}\left(F^T\overline{R} - 1\!\!1\right)\right]^2 \\ &= \mu\,\|\,\mathrm{sym}\left(\widehat{F}^T\overline{R} - 1\!\!1\right)\|^2 + \frac{\mu\lambda}{(2\mu+\lambda)}\,\mathrm{tr}\left[\mathrm{sym}\left(\widehat{F}^T\overline{R} - 1\!\!1\right)\right]^2, \end{aligned} \tag{2.8}$$

showing the characteristic apparent change of the Lamé moduli μ, λ for the two-dimensional structure due to the plain stress state. Here $\mu\lambda/(2\mu+\lambda)$ is half of the harmonic mean of μ and $\lambda/2$. Observe that it is not expedient to use $\widehat{F}$ in the condensed form (2.8) since in the coupled evolution equation (2.6) it is F of equation (2.2) which appears.

Remark 2.2: The three-dimensional deformation (2.1) can be reconstructed through

$$\phi_s(x,y,z) := m(x,y) + z\varrho_m(x,y)\,\overline{R}(x,y).e_3 \quad \text{and} \quad \nabla\varphi_s(x,y,0) = (\nabla m|\varrho_m\,\overline{R}_3)\,. \tag{2.9}$$

Inserting the ansatz for the reconstructed deformation (2.9) into the underlying continuum model and enforcing traction-free boundary conditions at the upper and lower face of the membrane in an averaged sense determines the analytical expression for ϱ_m used in $(2.2)_2$. We note that it is not possible to prescribe boundary conditions for the rotations $\overline{R}$ in this viscoelastic membrane formulation.

Remark 2.3: Due to the underlying isotropy the model (2.3) approaches in the vanishing viscosity limit $\nu^+ \to \infty$ (or zero relaxation time limit $\eta \to 0$) formally the intrinsic two-dimensional membrane-shell problem

$$\begin{aligned} &\int_\omega h\,W_\infty(U((\nabla m|\vec{n}_m))\,\mathrm{d}\omega - W^{\mathrm{ext}}(m) \mapsto \ \text{stat}\,.\ \text{w.r.t.}\ m\,, \\ &W_\infty(U) = \mu\,\|U - 1\!\!1\|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\,[U - 1\!\!1]^2\,, \end{aligned} \tag{2.10}$$

with $\vec{n}_m$ being the unit normal on the parameterized membrane surface. Thus, U is the classical symmetric elastic stretch and $U - \mathbb{1}$ is the elastic Biot strain tensor,

$$U = U((\nabla m|\vec{n}_m)) = \begin{pmatrix} \sqrt{\nabla m^T \nabla m} & 0 \\ 0 & 1 \end{pmatrix} . \tag{2.11}$$

Problem (2.10) is a geometrically exact equilibrium membrane model for small elastic strains and finite deformations in the *classical* sense, i.e., with no extra internal process. The transition from (2.3) to (2.10) in the formal relaxation limit $\eta \to 0$, however, is not entirely trivial since it is not just the replacement of the independent rotation $\overline{R}$ by the continuum rotation $R = (\nabla m|\vec{n}_m)U^{-1}$. Moreover, note the subtle change from global minimization to a stationarity requirement only. As well, it must be noted that the elastic equilibrium energy $W_\infty(U)$ is non-quasiconvex and non-elliptic w.r.t. $\nabla m \in \mathbb{M}^{3\times 2}$ but it is convex in the classical symmetric stretch U. Currently there are no mathematical theorems available establishing the existence of minimizers or equilibria solutions based directly on W_∞. In this sense, the viscoelastic formulation (2.3) provides a physical *regularization* of the occurring *loss of ellipticity* in the more classical formulation (2.10).

3 Discretization of the model

3.1 Temporal discretization

Let us now consider a fully implizit time discretized version of model (2.3). In principle, the simplest method for one time step is the following staggered scheme: let $(m^{n-1}, \overline{R}^{n-1})$ be the given solution for the deformation of the midsurface and the rotations at time t_{n-1}. Now, compute the new solution $(m^n, \overline{R}^n) \in \mathbb{V}$ at time t_n such that [1]

$$\int_\omega h\, W(F^n, \overline{R}^n)\, \mathrm{d}\omega - W^{\mathrm{ext,n}}(m^n) \mapsto \min . , \tag{3.1}$$

w.r.t. m^n at fixed $\overline{R}^n$ and with strain energy density function (2.4). The current deformation gradient $F^n = F(t_n)$ is

$$F^n = (\nabla m^n | \varrho_m^n\, \overline{R}_3^n) \quad \text{with } \varrho_m^n = 1 - \frac{\lambda}{2\mu + \lambda} \left[\langle (\nabla m^n|0), \overline{R}^n \rangle - 2 \right] ,$$

and the current boundary conditions are

$$m^n_{|\gamma_0}(t_n, x, y) = g_{\mathrm{d}}(t_n, x, y) , \qquad x, y \in \gamma_0 \subset \partial\omega . \tag{3.2}$$

The evolution equation for the rotations is now mapped by a **local exponential update**. This implies that $\overline{R}^n = \overline{R}^n(\nabla m^n)$ solves the following highly nonlinear problem

$$\overline{R}^n = \exp\left(\vartriangle\! t\, \nu_n^+ \operatorname{skew}\left(F^n \overline{R}^{n,T} \right) \right) \cdot \overline{R}^{n-1} \quad \text{with } \nu_n^+ = \frac{1}{\eta} \left(1 + \| \operatorname{skew} F^n \overline{R}^{n,T} \| \right)^2 . \tag{3.3}$$

[1] We abbreviate here $\mathbb{V} = H^{1,2}_\circ(\omega, \mathbb{R}^3; \gamma_0) \times C^0(\omega, \mathrm{SO}(3))$, where the space $H^{1,2}_\circ(\omega, \mathbb{R}^3; \gamma_0)$ is the set of all functions $m : \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ which have square integrable weak derivatives and vanish on $\gamma_0 \subset \partial\omega$ in the sense of traces and $C^0(\overline{\omega}, \mathrm{SO}(3))$ is the set of all proper rotations $R : \omega \mapsto \mathrm{SO}$ that are continuous up to the boundary of ω. We employ the standard notation of Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H^{1,2}_\circ(\Omega)$, indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions.

By the properties of logarithmic and exponential mapping it can easily be shown that (3.3) converges to (2.6) for the limit $\vartriangle t \to 0$. We will come back to this in Section 4.

Furthermore, for each load- and time step $\vartriangle t = [t_{n-1}, t_n]$ we compute the new solution $m^n, \overline{R}^n$ by an iteration $m^{n,j}, \overline{R}^{n,j}$ with the understanding that

$$\lim_{j\to\infty} m^{n,j} = m^n\,, \qquad \lim_{j\to\infty} \overline{R}^{n,j} = \overline{R}^n\,. \tag{3.4}$$

Remark: The elastic trial solution. The iteration (we employ here a classical Newton scheme) is very sensitive as far as proper startvalues for the current time step are concerned. This is all the more the case since our nonlinear problem admits multiple stationary solutions, e.g. uniform compression versus bulging out. In order to capture the "interesting" minimizing solution we proceed as follows: first we compute locally the orthogonal part of the reconstructed deformation gradient in the previous time step,

$$\overline{R}^n_{-1} := \operatorname{polar}((\nabla m^{n-1}|\varrho_m^{n-1}\overline{R}_3^{n-1}))\,. \tag{3.5}$$

Then we solve the following (modified) minimization problem for $m^{n,0}$

$$\int_\omega h\,W(F^{n,0}, \overline{R}^n_{-1})\,\mathrm{d}\omega - W^{\mathrm{ext,n}}(m^{n,0}) \mapsto \min\,. \quad \text{w.r.t. } m^{n,0} \text{ at fixed } \overline{R}^n_{-1}\,. \tag{3.6}$$

We set now

$$\overline{R}^{n,0} = \operatorname{polar}((\nabla m^{n,0}|\varrho_m(\nabla m^{n,0}, \overline{R}^n_{-1})\overline{R}^n_{-1}.e_3)\,, \tag{3.7}$$

and take the pair $(m^{n,0}, \overline{R}^{n,0})$ as initial elastic trial solution for a subsequent global Newton iteration.

3.2 Spatial discretization

The finite element discretization of problem (2.3) considers discrete subspaces of the continuous solution spaces for the membrane's deformation. Thus the **discrete problem** reads: find the deformation of the midsurface of the membrane and the independent local viscoelastic rotation $(m_\mathrm{h}, \overline{R}_\mathrm{h}) : [0, T] \times \mathbb{V}_\mathrm{h}$ such that

$$\int_\omega h\,W(F(m_\mathrm{h}), \overline{R}_\mathrm{h})\,\mathrm{d}\omega - W^{\mathrm{ext}}(m_\mathrm{h}, \overline{R}_{\mathrm{h}3}) \mapsto \min.\,, \tag{3.8}$$

w.r.t. m_h at fixed rotation $\overline{R}_\mathrm{h}$.

The discrete space $\mathbb{V}_\mathrm{h}$ is the space of $\mathcal{T}$-piecewise polynomials based on a regular triangulation $\mathcal{T}$ of ω in (closed) triangles or parallelograms. We assume that the triangulation matches the domain exactly, i.e., $\cup\mathcal{T} = \omega$ and two distinct elements T_1 and T_2 in $\mathcal{T}$ are either disjoint, or $T_1 \cap T_2$ is a complete edge or a common vertex of both (there are no hanging nodes). Let $\mathcal{P}_k(\mathcal{T})$ be the linear space of $\mathcal{T}$-piecewise polynomials of degree $\leq k$, and, let $\mathcal{P}^\circ_k(\mathcal{T})$ denote the continuous discrete functions in $\mathcal{P}_k(\mathcal{T})$ with homogeneous boundary values, i.e.,

$$\mathcal{P}_k(\mathcal{T}) := \left\{u_\mathrm{h} \in L^2(\omega) : \forall T \in \mathcal{T},\ u_\mathrm{h}|_T \in \mathcal{P}_k(T)\right\} \quad \text{and} \quad \mathcal{P}^\circ_k(\mathcal{T}) := \mathcal{P}_k(\mathcal{T}) \cap H^1_0(\omega). \tag{3.9}$$

If $T \in \mathcal{T}$ is a triangle, $\mathcal{P}_k(T)$ denotes the space of polynomials of total degree $\leq k$; while $\mathcal{P}_k(T)$ denotes the space of polynomials of partial degree $\leq k$ if T is a parallelogram. Consequently, the discrete subspace of the deformation of the midsurface identifies with $\mathcal{P}^{\circ}_{k+1}(\mathcal{T})^3$ for any non-negative integer k. We apply here $k = 0$ but an extension of the subspace, i.e., an (adaptive) mesh refinement, may raise computational efficiency, cf. [CW03, Wei01]. In every time step $\vartriangle t$ the spatial discretization reads

$$\mathbb{V}_{\mathrm{h}} = \mathcal{P}^{\circ}_1(\mathcal{T})^3 \times \mathcal{P}_0(\mathcal{T})^{3x3} . \tag{3.10}$$

In the following we omit the subindex h for readability.

3.3 Weak form of the thin membrane model

To formulate the finite element equations the weak form of problem (2.3), i.e., the principle of virtual work, is derived here. Let $\delta m :\in \mathbb{M}^{3\times 2}$ denote the virtual displacement of the midsurface with $\delta m \in H^{1,2}_{\circ}(\omega, \mathbb{R}^3; \gamma_0)$. Then we obtain for fixed rotations $\overline{R} \equiv \overline{R}^n$

$$\int_\omega h \left\langle D_{\nabla m}\left[W(F^n, \overline{R})\right], \nabla(\delta m)\right\rangle \mathrm{d}\omega - W^{\mathrm{ext,n}}(\delta m) = 0, \tag{3.11}$$

where we used the linearity of W^{ext}. Performing the differentiation w.r.t. ∇m this is equivalent to

$$\int_\omega h \,\langle D_F W(F^n, \overline{R}), (\nabla(\delta m)| - \frac{\lambda}{2\mu+\lambda}\langle(\nabla(\delta m)|0), \overline{R}\rangle\, \overline{R}_3)\rangle \,\mathrm{d}\omega - W^{\mathrm{ext,n}}(\delta m) = 0. \tag{3.12}$$

For the strain energy density function (2.4) it holds that for an arbitrary three-dimensional increment, $H \in \mathbb{M}^{3\times 3}$, the differential is given by

$$\langle D_F W(F, \overline{R}), H\rangle = \mu\,\langle F^T\overline{R} + \overline{R}^T F - 2\cdot 1\!\!1, \overline{R}^T H\rangle + \lambda\,\langle F^T\overline{R} - 1\!\!1, 1\!\!1\rangle \cdot \langle \overline{R}^T H, 1\!\!1\rangle\,. \tag{3.13}$$

Therefore, taking as increment

$$H \equiv \delta F^n = (\nabla(\delta m)|\delta\varrho^n_m\, \overline{R}^n_3) \qquad \text{with} \quad \delta\varrho^n_m = -\frac{\lambda}{2\mu+\lambda}\langle(\nabla(\delta m)|0), \overline{R}^n\rangle\,, \tag{3.14}$$

we infer that

$$\delta F^n = (\nabla(\delta m)| - \frac{\lambda}{2\mu+\lambda}\langle(\nabla(\delta m)|0), \overline{R}^n\rangle\, \overline{R}^n_3)\,, \tag{3.15}$$

and we obtain for (3.12) the expression

$$\begin{aligned}\int_\omega h\, \Big[&\mu\,\langle F^{T,n}\overline{R} + \overline{R}^T F^n - 2\cdot 1\!\!1, \overline{R}^T(\nabla(\delta m)| - \frac{\lambda}{2\mu+\lambda}\langle(\nabla(\delta m)|0), \overline{R}^n\rangle\, \overline{R}^n_3)\rangle \\ &+\lambda\,\langle F^{T,n}\overline{R} - 1\!\!1, 1\!\!1\rangle \cdot \langle \overline{R}^T(\nabla(\delta m)| - \frac{\lambda}{2\mu+\lambda}\langle(\nabla(\delta m)|0), \overline{R}^n\rangle\, \overline{R}^n_3), 1\!\!1\rangle\Big]\, \mathrm{d}\omega \\ &- W^{\mathrm{ext,n}}(\delta m) = 0\,.\end{aligned} \tag{3.16}$$

To obtain the weak form *consistent* with the local exponential update the rotations $\overline{R}$ need to be replaced with $\overline{R}^n(\nabla m^n)$ according to equation (3.3). Finally, the consistent weak form of problem (2.3) reads with (3.15)

$$\begin{aligned}\int_\omega h \Big[&\mu\,\langle F^{T,n}\overline{R}^n(\nabla m^n) + \overline{R}^{T,n}(\nabla m^n)F^n - 2\cdot 1\!\!1, \overline{R}^{T,n}(\nabla m^n)\cdot \delta F^n\rangle \\ &+\lambda\,\langle F^{T,n}\overline{R}^n(\nabla m^n) - 1\!\!1, 1\!\!1\rangle \cdot \langle \overline{R}^{T,n}(\nabla m^n)\cdot \delta F^n, 1\!\!1\rangle\Big]\, \mathrm{d}\omega - W^{\mathrm{ext,n}}(\delta m) = 0,\end{aligned} \tag{3.17}$$

where it should be emphazised again that F^n is itself a nonlinear function of ∇m^n,

$$F^n = (\nabla m^n | \varrho_m(\nabla m^n, \overline{R}^n(\nabla m^n))\, \overline{R}^n(\nabla m^n).e_3).$$

Let us remark that by our approach (3.8)-(3.10) the rotations are discontinuous along the element edges. Therefore, the rotations jump over elemental interfaces and we expect and observe kinks of the shell's midsurface. The exact positions of these kinks in the numerical solution depend, to a certain extend, on the triangulation.

4 The implicit local exponential update for the rotations in three dimensions

This section is devoted to a detailed discussion of the key feature of our model, the local exponential update of the rotations $\overline{R}(t)$. To this end let us first consider the following local, nonlinear subproblem: given an arbitrary, fully three-dimensional deformation history $F \in C^1(\mathbb{R}^+, \mathrm{GL}^+(3))$, determine a time-dependent rotation $R \in C^1(\mathbb{R}^+, \mathrm{SO}(3))$, such that

$$\frac{\mathrm{d}}{\mathrm{dt}} R(t) = \nu^+ \cdot \mathrm{skew}\left(F(t)R^T\right) \cdot R(t)\,, \quad R(0) = R_0 \in \mathrm{SO}(3)\,, \tag{4.1}$$

with

$$\nu^+ = \frac{1}{\eta}\left(1 + \|\,\mathrm{skew}(F(t)R^T(t))\|\right)^2 .$$

This problem is at first sight independent of (3.1). In [Nef03] it has been shown that for **fixed in time** F the solution $R(t)$ converges asymptotically to the continuum rotation $R = \mathrm{polar}(F)$. We first validate this general analytical result for our numerical algorithm.

In Section 3.1 we proposed to integrate the local evolution equation (4.1) numerically with an **implicit exponential update**, i.e. we solve the time-discretized version (3.3). Set now for readability

$$\overline{R} \equiv \overline{R}^{n-1}\,, \quad X \equiv \overline{R}^n\,, \quad F \equiv F^n\,, \quad B \equiv F\overline{R}^T = F^n\overline{R}^{n-1,T}\,, \tag{4.2}$$

and let us define a function $\gamma(s)$ with $s = \|\,\mathrm{skew}(F\overline{R}^T)\|$ and

$$\gamma(s) = \frac{1}{\eta}\,(1+s)^2\,, \quad \gamma'(s) = \frac{2(1+s)}{\eta}\,. \tag{4.3}$$

Then the proposed implicit local update for the rotations consists of solving

$$X = \exp\left(\vartriangle\! t\, \gamma(\|\,\mathrm{skew}\left(FX^T\right)\|)\, \mathrm{skew}(FX^T)\right) \cdot \overline{R}\,, \tag{4.4}$$

for $X \in \mathrm{SO}(3)$. If we set $X = Q \cdot \overline{R}$, we need to solve

$$Q = \exp\left(\vartriangle\! t\, \gamma(\|\,\mathrm{skew}\left(F\overline{R}^T Q^T\right)\|)\, \mathrm{skew}(F\overline{R}^T Q^T)\right)\,, \tag{4.5}$$

where $Q = \overline{R}^n\overline{R}^{n-1,T}$ is the time-incremental change of the rotations in one step. Set now $Q = \exp(A)$ for some $A \in \mathfrak{so}(3)$ by the surjectivity of the exponential function. This turns (4.5) into

$$\exp(A) = \exp\left(\vartriangle\! t\, \gamma(\|\,\mathrm{skew}\left(F\overline{R}^T\exp(A)^T\right)\|)\, \mathrm{skew}(F\overline{R}^T\exp(A)^T)\right)\,, \tag{4.6}$$

for the new unknown $A \in \mathfrak{so}(3)$. Since $\exp : \mathfrak{so}(3) \mapsto \mathrm{SO}(3)$ is bijective on a large ball around zero we have equivalently and still exact

$$A = \vartriangle t\, \gamma(\| \operatorname{skew}\left(B \exp(A)^T\right) \|) \operatorname{skew}(B \exp(A)^T) \,. \tag{4.7}$$

The solution of (4.7) at given B will be denoted by A^{ex}. The nonlinear equation (4.7) will here be solved with a **local Newton iteration**. If A^{ex} is determined, then $\overline{R}^n = X = \exp(A^{\mathrm{ex}})\, \overline{R}^{n-1}$.

Since we expect the incremental change in one time step to be small anyhow (equivalent to small A) we may approximate $\exp(A)^T \approx 1\!\!1 - A$ and introduce this approximation into (4.7). This leads us to consider

$$\begin{aligned} & A = \vartriangle t\, \gamma(\| \operatorname{skew}(B(1\!\!1 - A)) \|) \operatorname{skew}(B(1\!\!1 - A)) \\ \Rightarrow \quad & A = \vartriangle t\, \gamma \operatorname{skew}(B) - \vartriangle t\, \gamma \operatorname{skew}(B\,A) \,. \end{aligned} \tag{4.8}$$

Assuming for the moment that γ is already given this equation has a unique solution A whenever $A \mapsto A + \vartriangle t\, \gamma \operatorname{skew}(B\,A)$ is strictly monotone, i.e., $\forall\, A \in \mathfrak{so}(3)$:

$$\langle A + \vartriangle t\, \gamma \operatorname{skew}(B\,A), A\rangle > 0 \,. \tag{4.9}$$

Since

$$\begin{aligned} \langle A + \vartriangle t\, \gamma \operatorname{skew}(B\,A), A\rangle &= \|A\|^2 + \vartriangle t\, \gamma\, \langle B\,A, A\rangle \\ &\geq \|A\|^2 - \vartriangle t\, \gamma\, \|A\|^2\, \|B\| = \|A\|^2\, [1 - \vartriangle t\, \gamma\, \|B\|] \,, \end{aligned} \tag{4.10}$$

we obtain a useful bound on the size of **time step** $\vartriangle t$

$$\vartriangle t\, \gamma < \|B\|^{-1} \,, \tag{4.11}$$

which ensures condition (4.9) and implies algorithmically that (4.7) has a unique solution. This bound will be used in the implementation throughout. Concluding let us emphasize that the actual computation of A is performed with equation (4.7), the linearized version $(4.8)_1$ is only applied to derive time step bound (4.11).

4.1 The startvalue for the local Newton-iteration

At given $F, \overline{R}$ we want to solve equation (4.4). In view of the local relaxation limit $R = \mathrm{polar}(F)$, a good startvalue X_0 for X should be given by $X^0 = \mathrm{polar}(F)$. Since the final Newton method will be based on equation (4.7) and since $X = Q\,\overline{R} = \exp(A)\overline{R}$ we consider

$$\begin{aligned} & \mathrm{polar}(F) = X^0 = Q^0\, \overline{R} = \exp(A^0)\, \overline{R} \Rightarrow \\ & \qquad A^0 = \log\left(\mathrm{polar}(F)\, \overline{R}^T\right) = \log\left(\mathrm{polar}(F^n)\, \overline{R}^{n-1,T}\right) \,. \end{aligned} \tag{4.12}$$

4.2 The local Newton-iteration for the rotational update

In this paragraph we derive the incremental linearisation of the local exponential update in detail. According to (4.7) we have to solve the nonlinear matrix equation

$$A = \vartriangle t\, \nu^+(B \exp(A)^T) \cdot \operatorname{skew}(B \exp(A)^T) \tag{4.13}$$

with $\nu^+(B\exp(A)^T) = \gamma(\|\operatorname{skew}(B\exp(A)^T)\|)$ and function (4.3). A local Newton step corresponding to the nonlinear equation (4.7) is obtained by inserting $A^k + \Delta A$ into (4.7) in lieu of A and expanding it with respect to the increment ΔA up to first order. Set for the moment $A = A^k + \Delta A$. Then

$$\begin{aligned} A^k + \Delta A &= \vartriangle t\, \nu(B\exp(A^k + \Delta A)^T)) \cdot \operatorname{skew}(B\exp(A^k + \Delta A)^T)) \\ &= \vartriangle t\, \big(\nu(B\exp(A^k)^T) + D_A[\nu(B\exp(A^k)^T)].\Delta A + \ldots\big) \cdot \\ &\qquad\qquad \big(\operatorname{skew}(B\exp(A^k)^T) + \operatorname{skew}(B[D\exp(A^k).\Delta A]^T)\big)\,. \end{aligned} \tag{4.14}$$

Let us first collect some facts from Lie-Group theory. We define the matrix exponential function

$$\exp : \mathfrak{so}(3) \mapsto \mathrm{SO}(3)\,, \qquad \exp(X) = \sum_{i=1}^{\infty} \frac{1}{i!} X^i\,. \tag{4.15}$$

This function is bijective on a large ball around $0 \in \mathfrak{so}(3)$, more precisely, for $\|X\|^2 = -\mathrm{tr}\,[X^2] < 2\pi^2$, [HN91]. Let us also introduce the **adjoint operator**

$$\mathrm{ad} : \mathfrak{so}(3) \mapsto \mathrm{Lin}(\mathfrak{so}(3), \mathfrak{so}(3))\,, \qquad \mathrm{ad}(X).Y = [X, Y] = XY - YX\,. \tag{4.16}$$

The analytical form of the differential of the matrix exponential function can be written as

$$D\exp(X).H = \exp(X) \cdot \left[\sum_{i=1}^{\infty} \frac{1}{i!}(-\,\mathrm{ad}(X))^{i-1}\right].H\,, \tag{4.17}$$

with the series expanded to

$$\begin{aligned} \left[\sum_{i=1}^{\infty} \frac{1}{i!}(-\,\mathrm{ad}(X))^{i-1}\right].H &= \left[1\!\!1 - \frac{1}{2}\,\mathrm{ad}(X) + \frac{1}{3!}\,\mathrm{ad}(X)^2 + \ldots\right].H \\ &= H - \frac{1}{2}\,\mathrm{ad}(X).H + \frac{1}{3!}\,\mathrm{ad}(X).(\mathrm{ad}(X).H) + \ldots\,. \end{aligned} \tag{4.18}$$

For commutating pairs $(X, H) \in \mathfrak{so}(3) \times \mathfrak{so}(3)$ we have $\mathrm{ad}(X).H = 0$ and we recover the classical derivative formula $D\exp(X).H = \exp(X) \cdot H$. For brevity let us define an operator $P_l(\mathrm{ad}(X)).H \in \mathfrak{so}(3)$ with

$$P_l(\mathrm{ad}(X)).H = \left[\sum_{i=1}^{l} \frac{1}{i!}(-\,\mathrm{ad}(X))^{i-1}\right].H\,. \tag{4.19}$$

The index refers to the order of approximation $l \in \mathbb{N}$ in (4.18). An approximation of the differential is obtained by using

$$D_{.,l}\exp(X).H = \exp(X) \cdot P_l(\mathrm{ad}(X)).H\,. \tag{4.20}$$

Since $\exp(A)^T \cdot \exp(A) = 1\!\!1$ for all $A \in \mathfrak{so}(3)$[2] it holds that $\exp(A)^T \cdot D\exp(A).H \in \mathfrak{so}(3)$ for arbitrary $H \in \mathfrak{so}(3)$. This feature is shared by the approximate formula, since

$$\exp(A)^T \cdot D_{.,l}\exp(A).H = \exp(A)^T \cdot \exp(A) \cdot P_l(\mathrm{ad}(A)).H = P_l(\mathrm{ad}(A)).H\,. \tag{4.21}$$

[2] $\exp(A)^T = \exp(-A)$ and $\exp(-A)\exp(A) = \exp(-A + A) = \exp(0) = 1\!\!1$ for all $A \in \mathfrak{so}(3)$.

With formulas (4.15 -4.20) we evaluate equation (4.14) to

$$
\begin{aligned}
&A^k + \Delta A \\
&\approx {\scriptstyle\Delta} t\, \nu(B\exp(A^k)^T)\operatorname{skew}(B\exp(A^k)^T) + {\scriptstyle\Delta} t\, \nu(B\exp(A^k)^T)\operatorname{skew}(B[D\exp(A^k).\Delta A]^T) \\
&\qquad + {\scriptstyle\Delta} t\, \big(D_A[\nu(B\exp(A^k)^T)].\Delta A\big)\cdot \operatorname{skew}(B\exp(A^k)^T) \qquad (4.22)\\
&= {\scriptstyle\Delta} t\, \nu(B\exp(A^k)^T)\operatorname{skew}(B\exp(A^k)^T) + {\scriptstyle\Delta} t\, \nu(B\exp(A^k)^T)\operatorname{skew}(B[\exp(A^k)P_l(\operatorname{ad}(A^k)).\Delta A]^T) \\
&\qquad + {\scriptstyle\Delta} t\, \big(D_A[\nu(B\exp(A^k)^T)].\Delta A\big)\cdot \operatorname{skew}(B\exp(A^k)^T) \\
&= {\scriptstyle\Delta} t\, \nu(B\exp(A^k)^T)\operatorname{skew}(B\exp(A^k)^T) + {\scriptstyle\Delta} t\, \nu(B\exp(A^k)^T)\operatorname{skew}(B[P_l(\operatorname{ad}(A^k)).\Delta A]^T\exp(A^k)^T) \\
&\qquad + {\scriptstyle\Delta} t\, \big(D_A[\nu(B\exp(A^k)^T)].\Delta A\big)\cdot \operatorname{skew}(B\exp(A^k)^T)\,.
\end{aligned}
$$

The linear approximation for the nonlinear viscosity term $\gamma(\|\operatorname{skew}(B\exp(A^k)^T)\|)$ is given by

$$
\begin{aligned}
&D_A[\gamma(\|\operatorname{skew}(B\exp(A^k)^T)\|)].\Delta A \\
&= \gamma'(\|\operatorname{skew}(B\exp(A^k)^T)\|)\,\langle \frac{\operatorname{skew}(B\exp(A^k)^T)}{\|\operatorname{skew}(B\exp(A^k)^T)\|}, \operatorname{skew}(B\cdot[D\exp(A^k).\Delta A]^T)\rangle \\
&= \gamma'(\|\operatorname{skew}(B\exp(A^k)^T)\|)\,\langle \frac{B^T\operatorname{skew}(B\exp(A^k)^T)}{\|\operatorname{skew}(B\exp(A^k)^T)\|}, [D\exp(A^k).\Delta A]^T\rangle \\
&\approx \gamma'(\|\operatorname{skew}(B\exp(A^k)^T)\|)\,\langle \frac{B^T\operatorname{skew}(B\exp(A^k)^T)}{\|\operatorname{skew}(B\exp(A^k)^T)\|}, [D_{\sharp,l}\exp(A^k).\Delta A]^T\rangle \\
&= \gamma'(\|\operatorname{skew}(B\exp(A^k)^T)\|)\,\langle \frac{B^T\operatorname{skew}(B\exp(A^k)^T)}{\|\operatorname{skew}(B\exp(A^k)^T)\|}, [\exp(A^k)P_l(\operatorname{ad}(A^k)).\Delta A]^T\rangle \\
&= \gamma'(\|\operatorname{skew}(B\exp(A^k)^T)\|)\,\langle \frac{B^T\operatorname{skew}(B\exp(A^k)^T)}{\|\operatorname{skew}(B\exp(A^k)^T)\|}, [P_l(\operatorname{ad}(A^k)).\Delta A]^T\exp(A^k)^T\rangle \\
&= \gamma'(\|\operatorname{skew}(B\exp(A^k)^T)\|)\,\langle \frac{B^T\operatorname{skew}(B\exp(A^k)^T)\exp(A^k)}{\|\operatorname{skew}(B\exp(A^k)^T)\|}, [P_l(\operatorname{ad}(A^k)).\Delta A]^T\rangle \qquad (4.23)\\
&= -\gamma'(\|\operatorname{skew}(B\exp(A^k)^T)\|)\,\langle \frac{B^T\operatorname{skew}(B\exp(A^k)^T)\exp(A^k)}{\|\operatorname{skew}(B\exp(A^k)^T)\|}, P_l(\operatorname{ad}(A^k)).\Delta A]\rangle\,.
\end{aligned}
$$

Combining this with the former computation, we obtain

$$
\begin{aligned}
&A^k + \Delta A \\
&= {\scriptstyle\Delta} t\, \gamma(\|\operatorname{skew}(B\exp(A^k)^T)\|)\operatorname{skew}(B\exp(A^k)^T) \\
&- {\scriptstyle\Delta} t\, \gamma(\|\operatorname{skew}(B\exp(A^k)^T)\|)\operatorname{skew}(B[P_l(\operatorname{ad}(A^k)).\Delta A]\exp(A^k)^T) \qquad (4.24)\\
&- {\scriptstyle\Delta} t\, \gamma'(\|\operatorname{skew}(B\exp(A^k)^T)\|)\,\langle \frac{B^T\operatorname{skew}(B\exp(A^k)^T)\exp(A^k)}{\|\operatorname{skew}(B\exp(A^k)^T)\|}, P_l(\operatorname{ad}(A^k)).\Delta A\rangle\cdot\operatorname{skew}(B\exp(A^k)^T)\,.
\end{aligned}
$$

Abbreviate now

$$
S^{n,k} = \operatorname{skew}(B^n\exp(A^k)^T), \quad N^{\mathrm{ex}} = \|S^{n,k}\|\,, \tag{4.25}
$$

and set for the **local residuum** (the defect of (4.7))

$$
\operatorname{Res}(B, A^k) = A^k - {\scriptstyle\Delta} t\, \gamma(\|\operatorname{skew}(B\exp(A^k)^T)\|)\,\operatorname{skew}(B\exp(A^k)^T)\,. \tag{4.26}
$$

The local Newton method consists in iterating $A^{k+1} := A^k + \Delta A$ until $\|\operatorname{Res}^k\| \le$ tolerance. [3] Using these abbreviations yields the **linear matrix equation** for $\Delta A \in \mathfrak{so}(3)$

$$
\begin{aligned}
\operatorname{Res}(B, A^k) + \Delta A = &- {\scriptstyle\Delta} t\, \gamma(N^{\mathrm{ex}})\,\operatorname{skew}(B[P_l(\operatorname{ad}(A^k)).\Delta A]\exp(A^k)^T) \\
&- {\scriptstyle\Delta} t\, \frac{\gamma'(N^{\mathrm{ex}})}{N^{\mathrm{ex}}}\,\langle B^T S\exp(A^k), P_l(\operatorname{ad}(A^k)).\Delta A\rangle\cdot S\,, \qquad (4.27)
\end{aligned}
$$

[3] One may also use as criterion $\Delta A \le$ tolerance.

or

$$\Delta A + {\vartriangle}t\, \gamma(N^{\mathrm{ex}})\ \mathrm{skew}(B[P_l(\mathrm{ad}(A^k)).\Delta A]\exp(A^k)^T) \\ + {\vartriangle}t\, \frac{\gamma'(N^{\mathrm{ex}})}{N^{\mathrm{ex}}}\ \langle B^T S\exp(A^k), P_l(\mathrm{ad}(A^k)).\Delta A\rangle \cdot S = -\,\mathrm{Res}(B, A^k)\,. \quad (4.28)$$

Now we reduce this equation to a linear vector equation for $\Delta a = \mathrm{axl}(\Delta A)$ applying on both sides of the equation the operator

$$\mathrm{axl} : \mathfrak{so}(3) \mapsto \mathbb{R}^3, \quad \mathrm{axl}\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} = \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad (4.29)$$

with inverse $\mathrm{anti} : \mathbb{R}^3 \mapsto \mathfrak{so}(3)$. This yields

$$\mathrm{axl}(\Delta A) + {\vartriangle}t\, \gamma(N^{\mathrm{ex}})\ \mathrm{axl}(\mathrm{skew}(B[P_l(\mathrm{ad}(A^k)).\Delta A]\exp(A^k)^T)) \\ + {\vartriangle}t\, \frac{\gamma'(N^{\mathrm{ex}})}{N^{\mathrm{ex}}}\ \langle B^T S\exp(A^k), P_l(\mathrm{ad}(A^k)).\Delta A\rangle \cdot \mathrm{axl}(S) = -\,\mathrm{axl}(\mathrm{Res}(B, A^k))\,, \quad (4.30)$$

or

$$\Delta a + {\vartriangle}t\, \gamma(N^{\mathrm{ex}})\ \mathrm{axl}(\mathrm{skew}(B[P_l(\mathrm{ad}(A^k)).\,\mathrm{anti}(\Delta a)]\exp(A^k)^T)) \\ + {\vartriangle}t\, \frac{\gamma'(N^{\mathrm{ex}})}{N^{\mathrm{ex}}}\ \langle B^T S\exp(A^k), P_l(\mathrm{ad}(A^k)).\,\mathrm{anti}(\Delta a)\rangle \cdot \mathrm{axl}(S) = -\,\mathrm{axl}(\mathrm{Res}(B, A^k))\,. \quad (4.31)$$

The linear operators on the left hand side, which act on Δa will be expressed with matrix-operations applied to $\Delta a \in \mathbb{R}^3$. The corresponding matrix representation is easily obtained by putting into the i.th column the image of the i.th base vector $e_i \in \mathbb{R}^3$ under the action of the linear operator. Thus we obtain

$$\begin{pmatrix} \Delta a_1 \\ \Delta a_2 \\ \Delta a_3 \end{pmatrix} + {\vartriangle}t\, \gamma(N^{\mathrm{ex}}) \begin{pmatrix} \mathrm{axl}(\mathrm{skew}(B[P_l(\mathrm{ad}(A^k)).\,\mathrm{anti}(e_1)]\exp(A^k)^T)) \\ \mathrm{axl}(\mathrm{skew}(B[P_l(\mathrm{ad}(A^k)).\,\mathrm{anti}(e_2)]\exp(A^k)^T)) \\ \mathrm{axl}(\mathrm{skew}(B[P_l(\mathrm{ad}(A^k)).\,\mathrm{anti}(e_3)]\exp(A^k)^T)) \end{pmatrix}^T \begin{pmatrix} \Delta a_1 \\ \Delta a_2 \\ \Delta a_3 \end{pmatrix} \\ + {\vartriangle}t\, \frac{\gamma'(N^{\mathrm{ex}})}{N^{\mathrm{ex}}} \begin{pmatrix} \langle B^T S\exp(A^k), P_l(\mathrm{ad}(A^k)).\,\mathrm{anti}(e_1)\rangle \cdot \mathrm{axl}(S) \\ \langle B^T S\exp(A^k), P_l(\mathrm{ad}(A^k)).\,\mathrm{anti}(e_2)\rangle \cdot \mathrm{axl}(S) \\ \langle B^T S\exp(A^k), P_l(\mathrm{ad}(A^k)).\,\mathrm{anti}(e_3)\rangle \cdot \mathrm{axl}(S) \end{pmatrix}^T \begin{pmatrix} \Delta a_1 \\ \Delta a_2 \\ \Delta a_3 \end{pmatrix} \quad (4.32) \\ = -\,\mathrm{axl}(\mathrm{Res}(B, A^k))\,.$$

Then we compute the increment $\Delta A = \mathrm{anti}(\Delta a)$ and the final local rotation X follows by $X = \exp(A^{\mathrm{ex}}) \cdot \overline{R}$ with the converged local solution

$$A^{k+1} := A^k + \Delta A \qquad A^{\mathrm{ex}} = \lim_{k\to\infty} A^k\,. \quad (4.33)$$

In view of $S \in \mathfrak{so}(3)$ we have $\|S\| = \sqrt{2}\sqrt{S_{12}^2 + S_{13}^2 + S_{23}^2}$.

4.3 Convergence of the local Newton-iteration

In the preceding section we proposed a numerically robust implicit exponential update as a consistent discretization for the evolution equation for the rotations which preserves the Lie-group structure of $\mathrm{SO}(3, \overline{R})$ on the discrete level. This scheme takes advantage of the underlying property that the quadratic minimization problem is uniquely solvable at given $\overline{R}^n$. In order to validate this update numerically we now study carefully the convergence properties of the evolution equation (4.1). Exemplarily we present here results choosing a deformation gradient $\hat{F}$ such that we know the exact rotation $\hat{R}$ immediately.

$$
\begin{aligned}
\hat{F}(\varphi) &= \hat{R}(\varphi)\cdot\hat{U}(\varphi),\\
\hat{R}(\varphi) &= \begin{pmatrix} 1 & 0 & 0\\ 0 & \cos(\varphi) & \sin(\varphi)\\ 0 & -\sin(\varphi) & \cos(\varphi)\end{pmatrix}\\
\hat{U}(\varphi) &= \begin{pmatrix} \left(1+\frac{1}{2}\sin(\varphi)\right)^2 & 0 & \sin^2(\varphi)\\ 0 & \frac{1}{1+\frac{1}{2}\sin(\varphi)} & 0\\ \sin^2(\varphi) & 0 & \frac{1}{1+\frac{1}{2}\sin(\varphi)}\end{pmatrix}
\end{aligned}
\tag{4.34}
$$

The computed rotation R^n is now compared to the exact rotation for different values of viscosity η. To this end we monitor the error

$$
e = \|R^{nT}\hat{R} - 1\!\!1\|\,. \tag{4.35}
$$

In Figure 2(a) error (4.35) is plotted versus the total time for $\varphi = \pi/3$ in equation (4.34). Here and below the start value for the rotational update is $R^0 = 1\!\!1$. The error decreases rapidly for all values of viscosity η, however, the larger the value of η the slower is the rate of convergence. This result nicely reflects the special feature of our proposed membrane model (2.3).

Unfortunately, the numerical computation of exponential and logarithmic matrix mappings (which require eigenvalue analyzes and transformations) has a relatively large inherent uncertainty, i.e., for any $A \in \mathrm{GL}(3)$ we compute (instead of zero)

$$
||\log(\exp(A)) - A|| \approx 10^{-15}\ldots 10^{-13}\,. \tag{4.36}
$$

For that reason we stop all convergence studies if error (4.35) reaches values in that range. Let us emphasize that we employ in the implementation of our model Matlab's standard routines for exponential and logarithmic mappings but the magnitude of the mapping error is the *same* in the *LAPack*-based Fortran implementations of [ORR01].

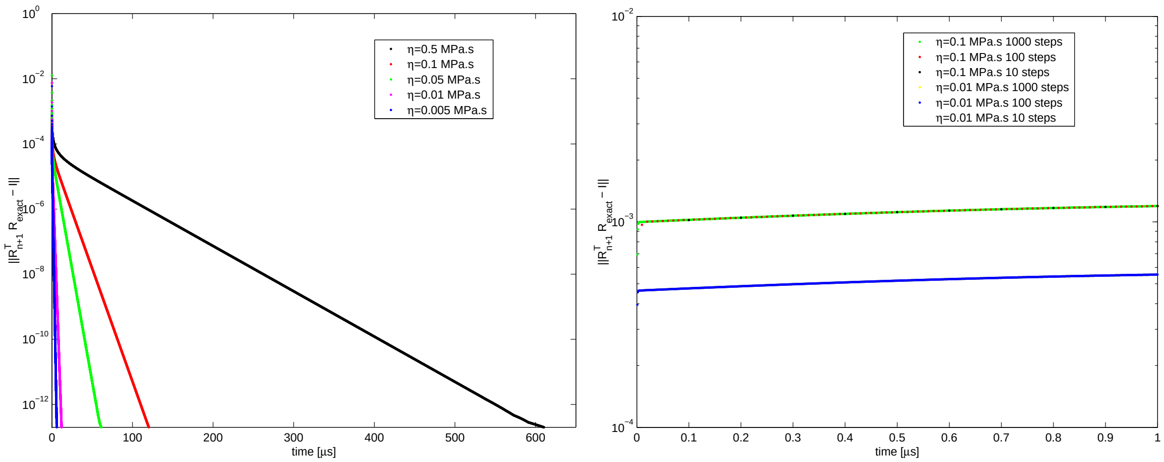
Figure 2(b) illustrates the (in)dependence of the rotational update on time increment $\vartriangle t$. Here, the deformation gradient (4.34) is chosen time dependent by $\varphi(t) = \pi\cdot t/(3t_{\text{total}})$, $t_{\text{total}} = 1\mu$s and the time steps are varied, $\vartriangle t \in [0.0001, 0.1]\mu$s. All time incrementations fulfil condition (4.11) or, equivalently,

$$
\vartriangle t \le \frac{\eta}{\|F\overline{R}^T\|\left(1+\|\operatorname{skew}(F\overline{R}^T)\|\right)^2}\,. \tag{4.37}
$$

The two curves in Figure 2(b) correspond to two different values of viscosity η with the slightly bigger error for higher viscosity. For all time increments the computed curves lay on top of each other, the actual size of step $\vartriangle t$ has no influence on the accuracy of solution. The small increase of the rotational error corresponds to bigger rotations.

4.4 Local response tangent for A^{ex} in three dimensions

We conclude this section with the differential of the local response of the material with respect to the rotations. Let $A^{\text{ex}}(F^n\overline{R}^{n-1,T})$ solve (4.7). Then the exponential update is characterized



(a) variation of viscosity η (b) variation of time step $\vartriangle t$

Figure 2: Error of the rotational update $\|R^{nT}\hat{R} - I\|$ versus time for different values of viscosity η and a constant time step $\vartriangle t$ (left) and for different sizes of time step $\vartriangle t$ and two different values of viscosity η (right).

by

$$\overline{R}^n(F^n) = \exp\left(A^{\mathrm{ex}}(F^n \overline{R}^{n-1,T})\right) \cdot \overline{R}^{n-1}, \tag{4.38}$$

The exact differential of this local response, which is needed in a subsequent global Newton step, is given by

$$D_F \overline{R}(F).H = \left[D\exp\left(A^{\mathrm{ex}}(F\overline{R}^{n-1,T})\right).\left[DA^{\mathrm{ex}}(F\overline{R}^{n-1,T}).(H\overline{R}^{n-1,T})\right]\right] \cdot \overline{R}^{n-1}, \tag{4.39}$$

cf. (4.2) for the notation. Since A^{ex} satisfies the equation (4.7)

$$A(B) = \vartriangle t\, \gamma(\|\operatorname{skew}(B\exp(A(B))^T)\|) \cdot \operatorname{skew}(B\exp(A(B))^T), \quad B := F^n \overline{R}^{n-1,T}, \tag{4.40}$$

we get, by differentiating w.r.t. $B \in \mathbb{M}^{3\times 3}$ and for any increment $H \in \mathbb{M}^{3\times 3}$

$$\begin{aligned}
DA(B).H &= \vartriangle t\, D_B\left[\nu(B\exp(A(B)^T) \cdot \operatorname{skew}(B\exp(A(B))^T)\right].H \\
&= \vartriangle t\, D\nu(B\exp(A(B))^T).\left[H\exp(A(B))^T + B\left[D\exp(A(B)).[DA(B).H]\right]^T\right] \\
&\qquad \cdot \operatorname{skew}(B\exp(A(B))^T) + \vartriangle t\, \nu(B\exp(A(B))^T) \\
&\qquad \cdot \operatorname{skew}(\left[H\exp(A(B))^T + B\left[D\exp(A(B)).[DA(B).H]\right]^T\right]) \\
&= \vartriangle t\, D\nu(B\exp(A(B))^T).\left[H\exp(A(B))^T + B\left[\exp(A(B))P_l(\operatorname{ad}(A(B)).[DA(B).H]\right]^T\right] \\
&\qquad \cdot \operatorname{skew}(B\exp(A(B))^T) + \vartriangle t\, \nu(B\exp(A(B))^T) \\
&\qquad \cdot \operatorname{skew}(\left[H\exp(A(B))^T + B\left[\exp(A(B))P_l(\operatorname{ad}(A(B)).[DA(B).H]\right]^T\right]) \qquad (4.41) \\
&= \vartriangle t\, D\nu(B\exp(A(B))^T).\left[H\exp(A(B))^T + B\left[P_l(\operatorname{ad}(A(B)).[DA(B).H]\right]^T\exp(A(B))^T\right] \\
&\qquad \cdot \operatorname{skew}(B\exp(A(B))^T) + \vartriangle t\, \nu(B\exp(A(B))^T) \\
&\qquad \cdot \operatorname{skew}(\left[H\exp(A(B))^T + B\left[P_l(\operatorname{ad}(A(B)).[DA(B).H]\right]^T\exp(A(B))^T\right]).
\end{aligned}$$

Set $\mathfrak{H} = DA^{\mathrm{ex}}(B).H$ where $\mathfrak{H} \in \mathfrak{so}(3)$ and abbreviate

$$S^{\mathrm{ex}} = \operatorname{skew}(B\exp(A(B))^T), \quad N^{\mathrm{ex}} = \|S^{\mathrm{ex}}\|. \tag{4.42}$$

This yields

$$\begin{aligned}
\mathfrak{H} &= \vartriangle t\,\frac{\gamma'(N^{\mathrm{ex}})}{N}\langle S^{\mathrm{ex}}, \big[H\exp(A(B))^T + B\,[P_l(\mathrm{ad}(A(B))).\mathfrak{H}]^T\exp(A(B))^T\big]\rangle\cdot S^{\mathrm{ex}} + \\
&\quad \vartriangle t\,\gamma(N^{\mathrm{ex}})\cdot\mathrm{skew}(\big[H\exp(A(B))^T + B\,[P_l(\mathrm{ad}(A(B))).\mathfrak{H}]^T\exp(A(B))^T\big]) \\
\mathfrak{H} &= \vartriangle t\,\frac{\gamma'(N^{\mathrm{ex}})}{N}\langle S^{\mathrm{ex}}, \big[H\exp(A(B))^T - B\,[P_l(\mathrm{ad}(A(B))).\mathfrak{H}]\exp(A(B))^T\big]\rangle\cdot S^{\mathrm{ex}} + \\
&\quad \vartriangle t\,\gamma(N^{\mathrm{ex}})\cdot\mathrm{skew}(\big[H\exp(A(B))^T - B\,[P_l(\mathrm{ad}(A(B))).\mathfrak{H}]\exp(A(B))^T\big])\,. \qquad (4.43)
\end{aligned}$$

Define as well

$$\mathrm{Res}^{\mathrm{ex}}(H, A(B)) = \vartriangle t\,\frac{\gamma'(N^{\mathrm{ex}})}{N}\langle S^{\mathrm{ex}}, \big[H\exp(A(B))^T\big]\rangle\cdot S^{\mathrm{ex}} + \vartriangle t\,\gamma(N^{\mathrm{ex}})\cdot\mathrm{skew}(\big[H\exp(A(B))^T\big])\,. \qquad (4.44)$$

Thus (with $A(B) = A^{\mathrm{ex}}$)

$$\begin{aligned}
&\mathfrak{H} + \vartriangle t\,\frac{\gamma'(N^{\mathrm{ex}})}{N}\langle S^{\mathrm{ex}}, B\,[P_l(\mathrm{ad}(A(B))).\mathfrak{H}]\exp(A^{\mathrm{ex}})^T\rangle\cdot S^{\mathrm{ex}} \\
&\qquad + \vartriangle t\,\gamma(N^{\mathrm{ex}})\cdot\mathrm{skew}(B\,[P_l(\mathrm{ad}(A^{\mathrm{ex}})).\mathfrak{H}]\exp(A^{\mathrm{ex}})^T) \\
&= \mathrm{Res}^{\mathrm{ex}}(H, A^{\mathrm{ex}})\,. \qquad (4.45)
\end{aligned}$$

As before, we transform this linear matrix equation into a corresponding vector format for $x = \mathrm{axl}(\mathfrak{H}) \in \mathbb{R}^3$:

$$\begin{aligned}
&x + \vartriangle t\,\frac{\gamma'(N^{\mathrm{ex}})}{N}\langle S^{\mathrm{ex}}, B\,[P_l(\mathrm{ad}(A^{\mathrm{ex}})).\,\mathrm{anti}(x)]\exp(A^{\mathrm{ex}})^T\rangle\cdot\mathrm{axl}(S^{\mathrm{ex}}) \\
&\quad + \vartriangle t\,\gamma(N^{\mathrm{ex}})\cdot\mathrm{axl}(\mathrm{skew}(B\,[P_l(\mathrm{ad}(A^{\mathrm{ex}})).\,\mathrm{anti}(x)]\exp(A^{\mathrm{ex}})^T)) = \mathrm{axl}(\mathrm{Res}^{\mathrm{ex}}(H, A^{\mathrm{ex}}))\,. \qquad (4.46)
\end{aligned}$$

Thus

$$\begin{aligned}
&\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \vartriangle t\,\gamma(N^{\mathrm{ex}})\begin{pmatrix} \mathrm{axl}(\mathrm{skew}(B[P_l(\mathrm{ad}(A^{\mathrm{ex}})).\,\mathrm{anti}(e_1)]\exp(A^{\mathrm{ex}})^T)) \\ \mathrm{axl}(\mathrm{skew}(B[P_l(\mathrm{ad}(A^{\mathrm{ex}})).\,\mathrm{anti}(e_2)]\exp(A^{\mathrm{ex}})^T)) \\ \mathrm{axl}(\mathrm{skew}(B[P_l(\mathrm{ad}(A^{\mathrm{ex}})).\,\mathrm{anti}(e_3)]\exp(A^{\mathrm{ex}})^T)) \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&\quad + \vartriangle t\,\frac{\gamma'(N^{\mathrm{ex}})}{N^{\mathrm{ex}}}\begin{pmatrix} \langle B^T S\exp(A^{\mathrm{ex}}), P_l(\mathrm{ad}(A^{\mathrm{ex}})).\,\mathrm{anti}(e_1)\rangle\cdot\mathrm{axl}(S) \\ \langle B^T S\exp(A^{\mathrm{ex}}), P_l(\mathrm{ad}(A^{\mathrm{ex}})).\,\mathrm{anti}(e_2)\rangle\cdot\mathrm{axl}(S) \\ \langle B^T S\exp(A^{\mathrm{ex}}), P_l(\mathrm{ad}(A^{\mathrm{ex}})).\,\mathrm{anti}(e_3)\rangle\cdot\mathrm{axl}(S) \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (4.47) \\
&\quad = \mathrm{axl}(\mathrm{Res}^{\mathrm{ex}}(H, A^{\mathrm{ex}}))\,,
\end{aligned}$$

similar to (4.32) apart from the modified right hand side. Whence the searched differential is obtained as $DA^{\mathrm{ex}}(B).H = \mathrm{anti}(x(H))$.

This result is the basis for a fully algorithmically treatment of the consistent linearization of the consistent weak form (3.17). However, the resulting algorithmic tangent is a cumbersome expression and is, therefore, not listed here. Moreover, our numerical experience showed that applying the consistent linearization has no pay off compared to a numerical tangent.

5 Numerical examples

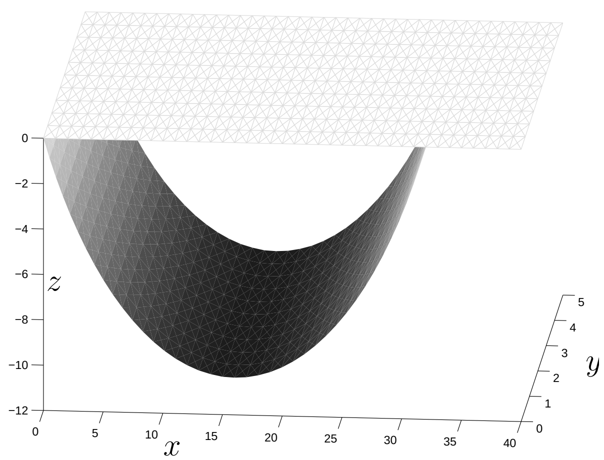
In the remaining of this text we shall illustrate the special features of our model by some illustrative examples. The material data of all models are summarized in Table 1. Furthermore

	μ	λ	ρ	h
hard synthetic sheet (Sect.5.1)	26316 MPa	51084 MPa	2.7 kg/m^3	0.1 mm
soft elastic foil (Sect.5.2)	1358 MPa	2036 MPa	-	1 mm
elastic Kapton foil (Sect.5.3)	1358 MPa	2036 MPa	1.5 kg/m^3	25 μm

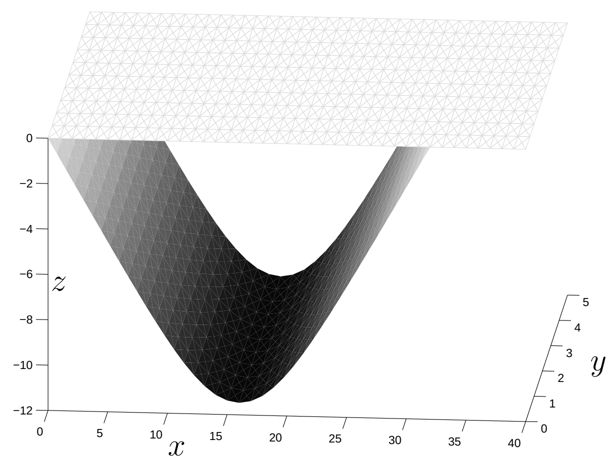
Table 1: Material data and membrane thickness of the examples.

let us mention that the viscosity parameter η controls the convergence of the rotational update (3.3) as shown in Section 4. It needs to fulfil the stability condition (4.11) but is arbitrarily otherwise and has no actual influence on the material's response. Therefore, we do not list η as a material parameter but set it here and below to be $\eta = 0.01$ MPa·s.

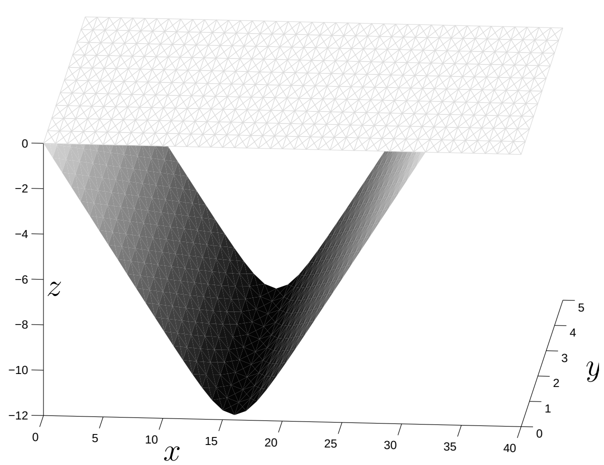
5.1 Rectangular sheet



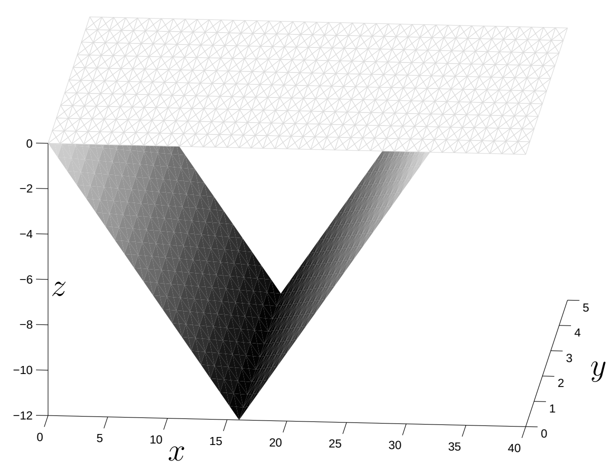
(a) no relaxation



(b) relaxation time 500s



(c) relaxation time 1000s



(d) relaxation time 5000s

Figure 3: Initial and deformed state of a hard sheet loaded by dead load and subjected to in-plane displacement of one side after different time periods of relaxation.

At first we compute the bending of a dead-loaded sheet made of a relatively hard synthetic polymer. The midsurface $m(x, y)$ of the 40×5 mm sheet is discretized with a regular triangulation of 1600 elements, $u_h \in \mathcal{P}_1(\mathcal{T})$. The membrane is completely fixed at the left boundary (where $x = 0$, see Figure 3a), constrained in y-direction along its sides parallel to the x-axes and constrained in vertical z-direction at the right boundary. Clearly, by nature of the model only the displacement can be constrained along the boundaries, cf. Remark 2.3.

A volumetric load (corresponding to a dead load) is applied on the sheet within $t_{\text{load}} = 100$ s. Then, within 1000 s the right boundary is moved to the left, i.e., a displacement $u_x(t) = -\bar{u} \cdot (t - t_{\text{load}})$ is prescribed with maximal displacement $\bar{u} = 8$ mm. After that period of time load and displacement are kept constant and the material starts to relax. In Figure 3(b-d) the initial and the deformed state of the sheet are displayed for different times of relaxation. At the beginning we clearly observe a bent sheet. Due to energy minimization the rotations relax and finally run into one (central) kink.

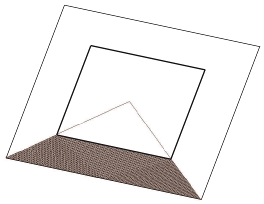
The specific feature of the model to result in kinks of the membrane's mid-surface is a direct consequence of the completely neglected bending stiffness in the presented model. Discontinuities of rotations R along the finite element edges are a valid solution of the discrete problem (3.8). As outlined previously, this is not a shell-like theory of thin continua but the model is capable of representing the essential features of gossamer structures undergoing large deformations! Furthermore, results computed in that way are to a certain extend mesh dependent since the model is able to kink only along the finite element boundaries. The regular triangulation in Figure 3 obviously allows for a central kink but variations of the mesh gave, provided that they were sufficiently fine, basically the same result.

5.2 Wrinkling of a thin foil

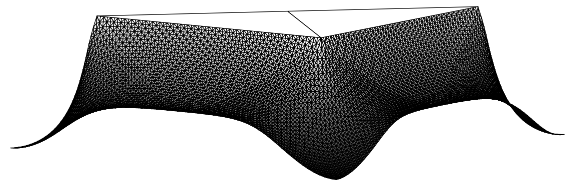
Let us now apply our model to the problem of an elastic foil under pressure load. The square foil has a side length of 2 m and a thickness of 1 mm; we think of a gossamer material like Kapton foil. The foil lays on a 1.2×1.2 m square obstacle (think of a cloths on a table) and only the unsupported part of it can deform. The foil is loaded from above with a pressure of $p_0 = 0.75$ MPa. The initial situation displays Figure 4a. For reason of symmetry only one quarter of the unsupported part is meshed with 6144 triangular elements. The displacements are constraint along the obstacles's edge and, moreover, symmetry conditions apply.

The computation started with a plane initial placement, the pressure is applied within 100 time steps of 10s. Two intermediate states as well as the final deformation are displayed in Figure 4b-d. The computed results are reflected on a symmetry axis and pictured for all models from the same point of view. Starting with the initial situation we observe a downwards folding of the model. Because the membrane cannot handle a significant compressive stress it starts to folds at the corner sides of the foil under further rising of the pressure. Relaxing this state (and keeping the pressure constant) results in kinks along these sides.

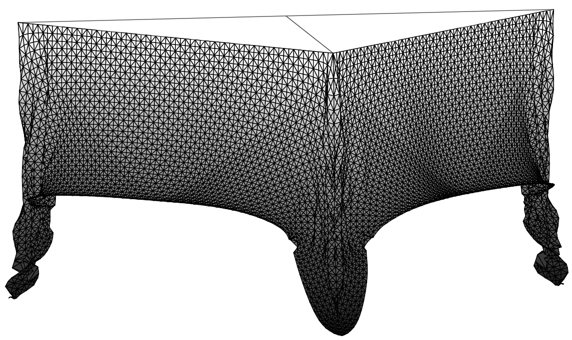
Thinking of a typical wrapping foil, the observed pattern of wrinkles and folds seem realistic and the example nicely illustrates that a common limitation of many finite element analysis programs, namely, the inability to handle tensionless states of a membrane-like structure is not encountered here.



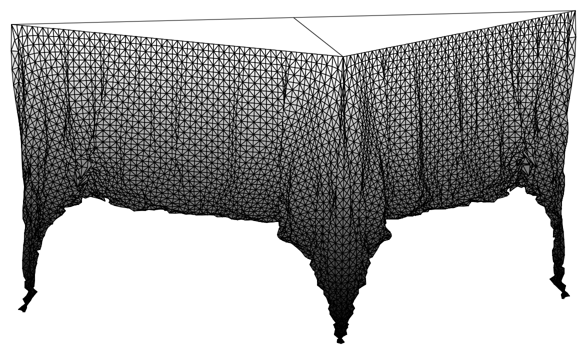
(a) computed model



(b) $p(t) = 0.5p_0$



(c) $p(t) = p_0$, no relaxation



(d) $p(t) = p_0$, relaxed state

Figure 4: Wrinkling of a foil blown over a square obstacle.

5.3 Twisting a band of elastic foil

Finally we analyze the membrane's stress state turning a band of Kapton foil upside down. To this end a 100×400 mm strip of thin foil (with material data from [YP03], see Table 1) is discretized. Within a time period of one hour the band is stretched of 25% in axial direction and both ends are turned against each other by 180 degrees. The deformed membrane is shown in Figure 5, the black lines mark the controlled boundaries. (The finite element triangulation has 16384 elements and is to dense to be displayed.) Along the marked edges the band is hold tight and the displacements are prescribed resulting in a twisted stretch of the band. The deformation is computed in 30 time steps, the final (relaxed) state is reached within 10 additional time steps. The averaged band thickness reduces from initially 0.25 mm to 0.0241 mm in the final state.

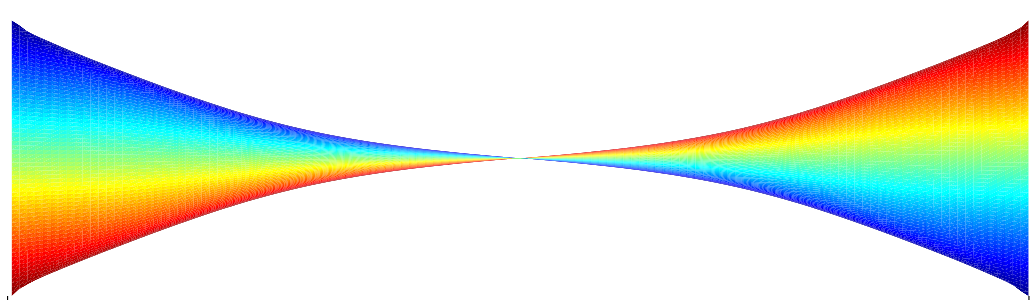
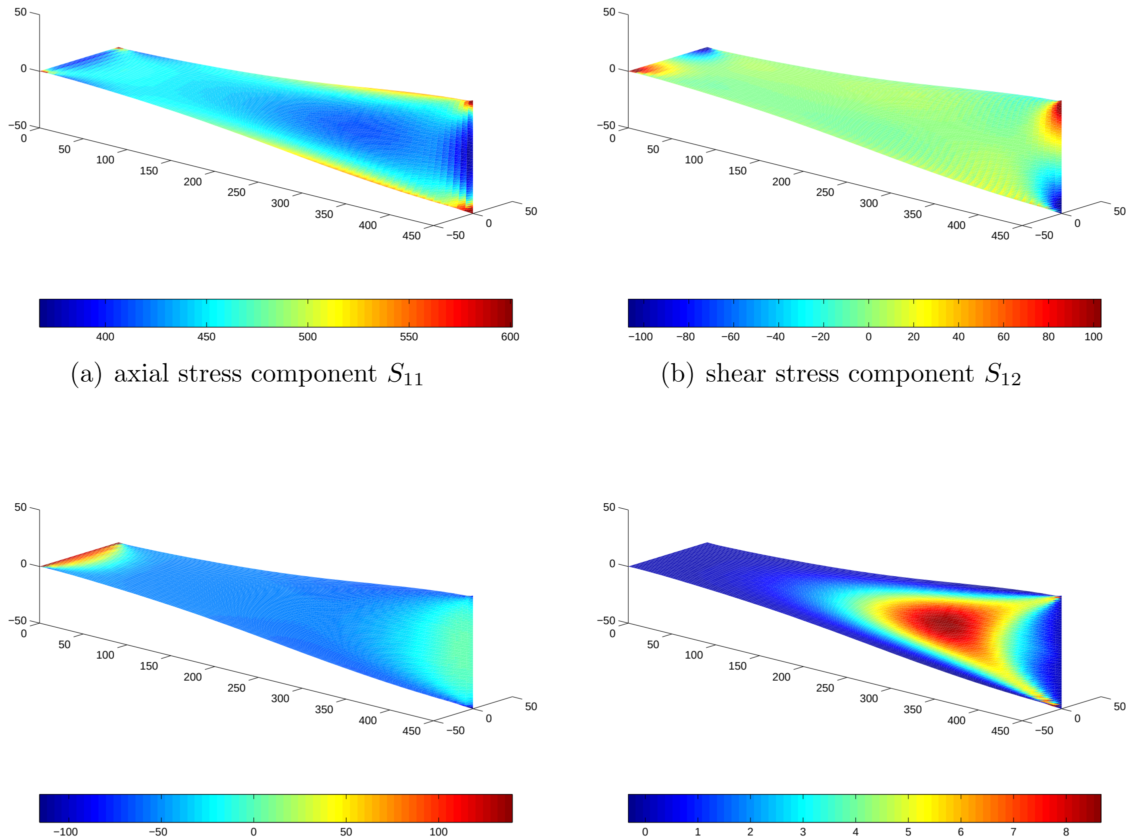


Figure 5: Twisted and stretched band of Kapton foil.



(a) axial stress component S_{11}

(b) shear stress component S_{12}

(c) nominal stress component S_{22}

(d) nominal stress component S_{33}

Figure 6: Axial stress, in-plane shear stress and in-plane and out-of-plane nominal stress components in the twisted foil strip.

As Figure 5 shows, the presented membrane model is capable of *exactly* capturing such large deformation states. The only drawback to mention here is the relatively slow convergence of the equilibrium iteration; depending on the criteria to stop the iteration several numbers of Newton steps may be necessary. One reason for that may be the previously mentioned error of logarithmic and exponential mapping routines but more significant seems the fact that the linearization of the discretized weak form with respect to the deformation (which is performed here numerically) gives not the precise consistent tangent. Since by linearization w.r.t. the full gradient F^n the evolution of rotations, precisely, the increment of $\varrho_m R_3$ is not fully taken into account, the resulting tangent may deviate from an exact (but unknown) linearization. This, in turn, slows down convergence to the equilibrium solution in particular for large rotations.

Figure 6 illustrate the resulting stress state in the twisted band. Here, for purpose of illustration, only one half of the band is computed and twisted by 90 degrees, displayed are the components of the first Piola-Kirchhoff stress tensor (2.7). Evidently, the dominating stress component is the nominal stress in axial direction S_{11} with average values of 450 MPa and maxima along the boundaries. Shear stress S_{12} and the nominal out of plane stresses S_{33} are close to zero, only as a result of the clamped edges we observe such stress components.

6 Summary

In this paper we proposed a geometrically exact model for gossamer structures undergoing large deformations. Contrary to many other formulations the underlying theory of a thin membrane with viscoelastic transverse shear resistance is well-posed even in a tensionless membrane state. A key feature of the model is an evolution equation for an independent field of rotations. These rotations adjust viscoelastically to the actual continuum rotations. A time-discretization of the model is performed whereby the evolution equation for the rotations is locally integrated with an exponential update algorithm. By means of numerical studies the convergence of the proposed scheme is validated.

Numerical examples illustrate that in our model a common limitation of standard solution procedures, namely the inability to handle a locally slack membrane state is not encountered. The membrane model incorporates full geometric nonlinear capabilities, only for simplicity we presume it to be initially plane. Note that the model is *not* restricted to such a plane initial state, an extension to curved structures is straightforward and, together with an extension to irreversible material behavior, subject of ongoing work. Typical applications of the model we have in mind are very thin sheets with negligible bending stiffness, e.g., synthetic foil, thin fabric or tissue.

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