Local space-time regularity criteria for weak solutions of the Navier-Stokes equations beyond Serrin's condition

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Abstract

Consider a weak solution u of the Navier-Stokes equations for a general domain $\Omega \subseteq \mathbb{R}^3$ on the time interval $[0, \infty)$ and a parabolic cylinder $Q_r = Q_r(t_0, x_0) \subseteq (0, \infty) \times \Omega$ with $r > 0$, $t_0 \in (0, \infty)$, $x_0 \in \Omega$. Then we show that there exists an absolute constant $\varepsilon_* > 0$ such that the local condition $\|u\|_{L^q(Q_r)} \le \varepsilon_* \, r^{\frac{2}{q}+\frac{3}{q}-1}$, $\frac{2}{q} + \frac{3}{q} \le 1 + \frac{1}{4}$, implies the regularity of u in the smaller cylinder $Q_{r/2}$. The special case $\frac{2}{q} + \frac{3}{q} = 1$ yields the well-known local Serrin condition $\|u\|_{L^q(Q_r)} \le \varepsilon_*$. Thus our criterion extends Serrin's condition admitting smaller exponents q and replacing the barrier 1 by $1 + \frac{1}{4}$.

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1 Main Result

In our main result, see Theorem 1.1 below, we consider a completely general domain $\Omega \subseteq \mathbb{R}^3$, i.e. a connected open subset of $\mathbb{R}^3$ with boundary $\partial\Omega$, and the Navier-Stokes system on $[0, \infty) \times \Omega$ in the usual form

$$\begin{aligned} &u_t - \Delta u + u \cdot \nabla u + \nabla p = f, \quad \operatorname{div} u = 0, \\ &u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0 \end{aligned} \tag{1.1}$$

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with external force f and initial value u_0. We are interested in regularity properties of a weak solution u in parabolic cylinders $Q_r \subseteq (0,\infty)\times\Omega$ defined by

$$Q_r = Q_r(t_0,x_0) := \{(t,x); t_0 - r^2 < t < t_0, |x-x_0| < r\} = (t_0-r^2,t_0)\times B_r(x_0), \tag{1.2}$$

where $B_r(x_0)\subseteq\Omega$ means the open ball with radius $r>0$ and center $x_0\in\Omega$, and $t_0\in(0,\infty)$. See Section 2 concerning further definitions.

Theorem 1.1 *Let $\Omega\subseteq\mathbb{R}^3$ be a general domain, let u be a weak solution of the Navier-Stokes system* (1.1) *with data $f=\operatorname{div}F$, $F\in L^2\big(0,\infty;L^2(\Omega)\big)$, $u_0\in L^2_\sigma(\Omega)$, and let $Q_r=Q_r(t_0,x_0)\subseteq(0,\infty)\times\Omega$ be a parabolic cylinder with $t_0\in(0,\infty)$, $x_0\in\Omega$, $r>0$.*

Then there is an absolute constant $\varepsilon_*>0$ with the following property: If

$$\|u\|_{L^q(Q_r)} \le \varepsilon_*\, r^{\frac{2}{q}+\frac{3}{q}-1}, \quad 1<q<\infty,\ \frac{2}{q}+\frac{3}{q}\le 1+\frac{1}{4}, \tag{1.3}$$

and

$$\|F\|_{L^4(t_0-r^2,t_0;L^2(B_r(x_0)))} \le \varepsilon_*, \tag{1.4}$$

then u is regular in $Q_{r/2}=Q_{r/2}(t_0,x_0)$ in the sense that Serrin's condition

$$u\in L^4\Big(t_0-\big(\frac{r}{2}\big)^2,t_0;L^6\big(B_{r/2}(x_0)\big)\Big),\quad \frac{2}{4}+\frac{3}{6}=1, \tag{1.5}$$

is satisfied in $Q_{r/2}$.

In (1.3) we wrote $\frac{2}{q}+\frac{3}{q}$ instead of $\frac{5}{q}$ in order to point out the analogy with the classical Serrin number $\frac{2}{s}+\frac{3}{q}$ where $s,q\in(1,\infty)$ denote possibly different exponents of integration with respect to time and space. For technical reasons we have to restrict ourselves to the case $s=q$ in this paper.

A result similar to Theorem 1.1 holds when we replace the cylinder $Q_r(t_0,x_0)$ by the slightly modified parabolic cylinder of the form

$$Q_r^* = Q_r^*(t_0,x_0) = \Big(t_0-\frac{7}{8}r^2, t_0+\frac{1}{8}r^2\Big)\times B_r(x_0) \tag{1.6}$$

with $r>0$, $t_0\in(0,\infty)$, $t_0-\frac{7}{8}r^2>0$, $x_0\in\Omega$. See [2] concerning these cylinders.

If for given $(t_0,x_0)\in(0,\infty)\times\Omega$ there is at least one $Q_r^*(t_0,x_0)\subseteq(0,\infty)\times\Omega$, $r>0$, such that u is regular in $Q_r^*(t_0,x_0)$, then (t_0,x_0) is a regular point of u; see Remark 2.2 below. The next corollary yields a criterion for regular points.

Corollary 1.2 *Let $\Omega\subseteq\mathbb{R}^3$ be a general domain, let u be a weak solution of the Navier-Stokes system* (1.1) *with data $f=\operatorname{div}F$, $F\in L^4\big(0,\infty;L^2(\Omega)\big)\cap L^2\big(0,\infty;L^2(\Omega)\big)$, $u_0\in L^2_\sigma(\Omega)$, let $t_0\in(0,\infty)$, $x_0\in\Omega$, and consider the cylinders $Q_r^*(t_0,x_0)$ contained in $(0,\infty)\times\Omega$ for $r>0$.*

Suppose

$$\liminf_{r\to 0} r^{1-5/q}\|u\|_{L^q(Q_r^*)} < \varepsilon_*, \quad 1<q<\infty,\ \frac{2}{q}+\frac{3}{q}\le 1+\frac{1}{4} \tag{1.7}$$

with $\varepsilon_* > 0$ as in Theorem 1.1. Then (t_0, x_0) is a regular point of u.

Remark 1.3 Note that $\frac{2}{q}+\frac{3}{q}\le 1+\frac{1}{4}$, $1<q<\infty$, is equivalent to $4\le q<\infty$, whereas the case $5\le q<\infty$ is well-known by Serrin's condition $\frac{2}{q}+\frac{3}{q}\le 1$. Thus within the region

$$4\le q<5 \tag{1.8}$$

we obtain a new local regularity condition beyond Serrin's condition $\frac{2}{q}+\frac{3}{q}\le 1$, since $1<\frac{2}{q}+\frac{3}{q}\le 1+\frac{1}{4}$ is equivalent to (1.8) with Serrin's barrier strictly larger than 1.

The weakest possible regularity condition in (1.3) is obtained for $q=4$ in the form

$$\|u\|_{L^4(Q_r)}\le \varepsilon_* r^{\frac{1}{4}}. \tag{1.9}$$

Further we note that the condition for the regularity of (t_0, x_0) in Corollary 1.2 does not depend on the local behavior of the external force f.

It is interesting to compare Theorem 1.1 with a local regularity result in [2], Proposition 1 and Corollary 1; for simplicity we will perform this comparison in a slightly different formulation and with $f=0$. The authors of [2] need a special type of weak solutions, the so-called suitable weak solutions, see (2.6), (2.7) below, and their local regularity condition contains the associated pressure p. The existence of such a weak solution is non-trivial and was shown in [2] for $\mathbb{R}^3$ and for smooth bounded Ω; see [5] for an existence proof for uniform C^2-domains. On the other hand, the existence of a weak solution u in Theorem 1.1 in the sense of Definition 2.1 below is well-known for general domains.

Lemma 1.4 ([2]) *Let $\Omega\subseteq\mathbb{R}^3$ be a general domain, let u be a suitable weak solution of the Navier-Stokes system* (1.1) *with data $f=0$, $u_0\in L^2_\sigma(\Omega)$, associated pressure term ∇p, and let $Q_r = Q_r(t_0,x_0)\subseteq(0,\infty)\times\Omega$ be a parabolic cylinder.*

Then there is an absolute constant $\varepsilon_*>0$ with the following property: If

$$\|u\|^3_{L^3(Q_r)}+\|up\|_{L^1(Q_r)}+\|p\|_{L^{5/4}(Q_r)}\le\varepsilon_* r^2, \tag{1.10}$$

then u is regular in $Q_{r/2}=Q_{r/2}(t_0,x_0)$ in the sense that $|u(t,x)|\le C_1 r^{-1}$ holds for almost all $(t,x)\in Q_{r/2}$ and some absolute constant $C_1>0$.

Note that the regularity condition of Theorem 1.1 for $f=0$, $q=4$ can be written with ε_* as in (1.3) in the form

$$\|u\|^4_{L^4(Q_r)}\le(\varepsilon_*)^4\, r \tag{1.11}$$

which is completely independent of (1.10); the same holds for the corresponding proofs.

2 Notations and Preliminaries

In the first part of this section, where we look at usual weak solutions, $\Omega \subseteq \mathbb{R}^3$ means a general domain as in Theorem 1.1. In the second part we consider another type of weak solutions, investigated recently in [1], [4], the so-called very weak solutions with inhomogeneous boundary values, where $\Omega \subseteq \mathbb{R}^3$ is a smooth bounded domain in the sense that the boundary $\partial\Omega$ is of class $C^{2,1}$.

Definition 2.1 Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, and let

$$f = \operatorname{div} F, \quad F = \big(F_{ij}\big)_{i,j=1,\dots,3} \in L^2\big(0,\infty; L^2(\Omega)\big), \quad u_0 \in L^2_\sigma(\Omega). \tag{2.1}$$

Then a function

$$u \in L^\infty\big(0,\infty; L^2_\sigma(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0,\infty); W^{1,2}_0(\Omega)\big),\ \nabla u \in L^2\big(0,\infty; L^2(\Omega)\big), \tag{2.2}$$

is called a weak solution of the Navier-Stokes system (1.1) if

$$u : [0,\infty) \to L^2_\sigma(\Omega) \ \text{ is weakly continuous,} \tag{2.3}$$

and the condition

$$-\langle u, w_t\rangle_{\Omega,\infty} + \langle \nabla u, \nabla w\rangle_{\Omega,\infty} - \langle uu, \nabla w\rangle_{\Omega,\infty} = \langle u_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,\infty} \tag{2.4}$$

is satisfied for all $w \in C^2_0\big([0,\infty); C^2_{0,\sigma}(\Omega)\big)$.

Here we use the following standard notations: $\nabla = (\partial_1, \partial_2, \partial_3)$ where $\partial_j = \partial/\partial x_j$, $j = 1,2,3$, $\operatorname{div} u = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3$ for a vector field $u = (u_1, u_2, u_3)$ and $\operatorname{div} F = (\partial_1 F_{1j} + \partial_2 F_{2j} + \partial_3 F_{3j})_{j=1,2,3}$ for a matrix field $F = (F_{ij})_{i,j=1,2,3}$. Moreover, $uu = (u_i u_j)_{i,j=1,2,3}$ and $u \cdot \nabla u = (u \cdot \nabla)u = \operatorname{div}(uu)$ provided that $\operatorname{div} u = 0$.

By $C^j(\Omega)$, $C^j(\overline{\Omega})$, $C^j_0(\Omega)$ and $C^j_0(\overline{\Omega})$, $j \in \mathbb{N}$ or $j = \infty$, we denote the usual spaces of smooth functions. In particular, a function $u \in C^2_0(\overline{\Omega})$ vanishes on $\partial\Omega$, but ∇u may be different from zero on $\partial\Omega$. Let

$$C^j_{0,\sigma}(\Omega) = \{w \in C^j_0(\Omega);\ \operatorname{div} w = 0\},$$

$C^j_{0,\sigma}(\overline{\Omega}) = \{w \in C^j_0(\overline{\Omega}); \operatorname{div} w = 0\}$ and $L^q_\sigma = L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$. Furthermore, $C^j_0\big([0,T); X\big)$, $j \in \mathbb{N}$, is the space of continuous functions $w : [0,T) \to X$ with compact support $\operatorname{supp} w \subset [0,T)$ such that $w_t = dw/dt, \dots, d^j w/dt^j$ are continuous. The usual Sobolev spaces are denoted by $W^{j,q} = W^{j,q}(\Omega)$, $j \in \mathbb{N}$, and in particular $W^{1,2}_0 = W^{1,2}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{W^{1,2}}}$.

For a Banach space X with norm $\|\cdot\|_X$ we need for $1 \le s \le \infty$ the usual Bochner spaces $L^s(T_0, T_1; X)$, $0 \le T_0 < T_1 \le \infty$, with norm

$$\|u\|_{L^s(T_0,T_1;X)} = \begin{cases} \Big(\displaystyle\int_{T_0}^{T_1} \|u\|_X^s \, dt\Big)^{1/s}, & 1 \le s < \infty \\ \operatorname*{ess\,sup}_{T_0 \le t \le T_1} \|u(t)\|_X, & s = \infty\,. \end{cases}$$

If $X = L^q = L^q(\Omega)$, $1 < q < \infty$, is the usual Lebesgue space we set $\|\cdot\|_X = \|\cdot\|_{L^q} = \|\cdot\|_q = \|\cdot\|_{q,\Omega}$, and $\|u\|_{L^s(T_0,T_1;L^q(\Omega))} = \|u\|_{q,s}$ if T_0, T_1 are known from the context. For $1 < q,\ s < \infty$ and $0 < T \leq \infty$ let $\langle u, v\rangle_\Omega = \int_\Omega u \cdot v\, dx$ denote the usual duality pairing of functions or vector fields $u \in L^q$, $v \in L^{q'}$, where $q' = \frac{q}{q-1}$. Further, for $u \in L^s(0,T;L^q)$, $v \in L^{s'}(0,T;L^{q'})$, $s' = \frac{s}{s-1}$,

$$\langle u, v\rangle_{\Omega,T} = \langle u, v\rangle_T = \int_0^T \langle u, v\rangle_\Omega\, d\tau$$

means the duality pairing in $[0,T) \times \Omega$. In (2.4) we used $q = q' = 2$, $s = s' = 2$ and $T = \infty$. Moreover, $L^2_{\text{loc}}\big([0,T);\, W_0^{1,2}(\Omega)\big)$, $0 < T \leq \infty$, is the space of all $W_0^{1,2}(\Omega)$-valued functions $w : t \mapsto w(t)$ such that $w \in L^2\big(0,T';\, W_0^{1,2}(\Omega)\big)$ for all $T' \in (0,T)$.

Note that the existence of a weak solution u as in Definition 2.1 is well-known for general domains, see, e.g., [10], V.3. Using (2.3) we see that

$$u(0) = u\big|_{t=0} = u_0$$

in (1.1) is well-defined. Because of (2.2) the condition $u\big|_{\partial\Omega} = 0$ in (1.1) is well-defined in the sense that the trace $u(t)\big|_{\partial\Omega} = 0$ for almost all $t \in [0,\infty)$. Further we get from (2.2) that the condition $\operatorname{div} u = 0$ is well-defined in the sense of distributions. Finally, we find a unique distribution of the form ∇p, the pressure term associated with u, such that

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = f$$

holds in $(0,\infty) \times \Omega$ in the sense of distributions. Thus the system (1.1) is well-defined in a certain weak sense for each weak solution u.

Usually the notion of a weak solution u includes the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2\, d\tau \leq \frac{1}{2}\, \|u_0\|_2^2 - \int_0^t \langle F, \nabla u\rangle_\Omega\, d\tau, \quad 0 \leq t < \infty. \tag{2.5}$$

However, we do not need (2.5) in our method.

The more special notion of a suitable weak solution u plays an important role in the local regularity theory of weak solutions, see [2]. In this case the existence of u is non-trivial and has been shown for $\Omega = \mathbb{R}^3$, for bounded domains $\Omega \subseteq \mathbb{R}^3$ with smooth boundary $\partial\Omega$, see [2], p. 822, and for exterior domains $\Omega \subseteq \mathbb{R}^3$, see [7], [11]; recently, the existence has been proved for general domains $\Omega \subseteq \mathbb{R}^3$ with uniform C^2-boundary $\partial\Omega$, see [5].

Let Ω, $f = \operatorname{div} F$, and u_0 be as in Definition 2.1, and assume additionally that $f \in L^2\big(0,\infty; L^2(\Omega)\big)$. Then a weak solution u satisfying (2.2) – (2.4) is called a *suitable weak solution* of the system (1.1) with data f, u_0, if the associated pressure term satisfies

$$\nabla p \in L^q_{\text{loc}}\big((0,\infty); L^q_{\text{loc}}(\overline{\Omega})\big) \ \text{ with } q = \frac{5}{4}, \tag{2.6}$$

and if the *local energy inequality*

$$\frac{1}{2}\,\|\varphi u(t)\|_2^2 + \int_{t_0}^{t} \|\varphi \nabla u\|_2^2\,d\tau \le \frac{1}{2}\,\|\varphi u(t_0)\|_2^2 + \int_{t_0}^{t} \langle \varphi f, \varphi u\rangle_\Omega\,d\tau \qquad (2.7)$$
$$-\frac{1}{2}\int_{t_0}^{t} \langle \nabla |u|^2, \nabla \varphi^2\rangle_\Omega\,d\tau + \int_{t_0}^{t} \langle \frac{1}{2}\,|u|^2 + p,\, u\cdot\nabla\varphi^2\rangle_\Omega\,d\tau$$

is satisfied for almost all $t_0 \in [0,\infty)$, all $t \in [t_0,\infty)$, and all $\varphi \in C_0^\infty(\mathbb{R}^3)$.

Using a standard mollification, see e.g., [10], II, 1.7, we obtain from (2.7) in particular the inequality

$$\int_{(0,T)\times\Omega} |\nabla u|^2\phi\,dt\,dx \le \frac{1}{2}\int_{(0,T)\times\Omega} \big(|u|^2(\phi_t+\Delta\phi)+(|u|^2+2p)u\cdot\nabla\phi+2(u\cdot f)\phi\big)dt\,dx \qquad (2.8)$$

for all $\phi \in C_0^\infty\big((0,T)\times\Omega\big)$ with $\phi \ge 0$. This special formulation has been used in [2], (2.5), in the definition of suitable weak solutions.

Remark 2.2 (i) If a weak solution u satisfies a local Serrin condition as in (1.5) then we know higher regularity properties in space direction in each subdomain $D \subset Q_{r/2}$ with $\overline{D} \subseteq Q_{r/2}$, if f and Ω as in Theorem 1.1 are smooth in the sense that $f \in C_0^\infty\big((0,\infty)\times\Omega\big)$ and that Ω has a uniform C^2-boundary $\partial\Omega$; see [2], p. 780, and [13], p. 453, concerning such properties. Indeed, first we obtain integrability properties of ∇p and u_t in some L^q-spaces, see [5]; then we use a standard localization procedure with a cut-off function to prove that each space derivative of u is essentially bounded in D. This justifies to say that u is regular in $Q_{r/2}$ if (1.5) is satisfied.

(ii) If instead of (1.5) the condition

$$u \in L^4\Big(t_0 - \frac{7}{8}\big(\frac{r}{2}\big)^2,\; t_0 + \frac{1}{8}\big(\frac{r}{2}\big)^2;\; L^6\big(B_{r/2}(x_0)\big)\Big) \qquad (2.9)$$

is satisfied, where now $Q_{r/2}$ is replaced by $Q^*_{r/2}$, then the regularity properties above hold with $(t_0,x_0) \in D \subset Q^*_{r/2}$. Therefore, (t_0,x_0) is called a *regular point* in this case, cf. Corollary 1.2 .

To prove Theorem 1.1 we use the theory of very weak solutions for smooth bounded domains, which has been introduced in [1] and generalized in [4], see Definition 2.3 below. For this purpose, we assume in the next part of this section that $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with boundary of class $C^{2,1}$.

Let $P_q : L^q \to L^q_\sigma$, $1 < q < \infty$, be the Helmholtz projection, and let $A_q : \mathcal{D}(A_q) \to L^q_\sigma(\Omega)$ with domain $\mathcal{D}(A_q) = W^{2,q}(\Omega)\cap W^{1,q}_0(\Omega)\cap L^q_\sigma(\Omega)$ be the Stokes operator. It is well-known that $-A_q$ generates a bounded analytic semigroup e^{-tA_q}, $t \ge 0$, on L^q_σ, and that the fractional powers A_q^α, $-1 \le \alpha \le 1$, of A_q are well-defined, see, e.g., [3], [4], [5], [6], [12]. In particular, we need the following embedding properties, see [4]:

$$\|u\|_q \le c\|A_\gamma^\alpha u\|_\gamma \quad \text{for } u \in \mathcal{D}(A_\gamma^\alpha),\ 1<\gamma\le q,\ 2\alpha + \frac{3}{q} = \frac{3}{\gamma}, \tag{2.10}$$

$$\|A_q^\alpha e^{-tA_q}u\|_q \le ce^{-\delta t}t^{-\alpha}\|u\|_q \quad \text{for } u \in L^q_\sigma(\Omega),\ t>0,\ 0\le\alpha\le 1 \tag{2.11}$$

for some $\delta = \delta(q,\Omega) > 0$ and constants $c = c(\alpha,q,\Omega) > 0$ and $c = c(\alpha,\delta,q,\Omega) > 0$, respectively.

Let $0 < T \le \infty$, $1 < q, s < \infty$. Then the maximal regularity estimate

$$\|(v_t, A_q v)\|_{L^s(0,T;L^q)} \le c\big(\|A_q e^{-\cdot A_q}v_0\|_{L^s(0,T;L^q)} + \|f\|_{L^s(0,T;L^q)}\big), \tag{2.12}$$

$c = c(q,\Omega) > 0$, holds for the unique solution v of the evolution system

$$v_t + A_q v = f, \quad v(0) = v_0$$

with data $f \in L^s(0,T;L^q_\sigma)$ and $v_0 \in L^q_\sigma$ such that $A_q e^{-\cdot A_q}v_0 \in L^s(0,T;L^q_\sigma)$.

To deal with traces on the boundary $\partial\Omega$ let $N = N(x)$ denote the exterior normal unit vector at $x \in \partial\Omega$. Let $L^q(\partial\Omega)$ be the usual L^q-space on $\partial\Omega$ with norm $\|\cdot\|_{L^q(\partial\Omega)} = \|\cdot\|_{q,\partial\Omega}$. Then

$$\langle g,h\rangle_{\partial\Omega} = \int_{\partial\Omega} g\cdot h\,dS, \quad g\in L^q(\partial\Omega),\ h\in L^{q'}(\partial\Omega),$$

means the duality pairing on $\partial\Omega$ where dS is the surface element. Analogously, we define the duality pairing $\langle g,h\rangle_{\partial\Omega,T} = \int_0^T \langle g,h\rangle_{\partial\Omega}\,d\tau$ for $g \in L^s\big(0,T;L^q(\partial\Omega)\big)$, $h \in L^{s'}\big(0,T;L^{q'}(\partial\Omega)\big)$, $s' = \frac{s}{s-1}$. Further we need the Sobolev spaces $W^{\alpha,q}(\partial\Omega)$, $-2 \le \alpha \le 2$, of fractional order α with norm $\|\cdot\|_{W^{\alpha,q}(\partial\Omega)} = \|\cdot\|_{\alpha;q,\partial\Omega}$. Here, the space of negative order is defined as the dual space

$$W^{-\alpha,q}(\partial\Omega) = \big(W^{\alpha,q'}(\partial\Omega)\big)', \quad 0<\alpha\le 2, \tag{2.13}$$

of the space $W^{\alpha,q'}(\partial\Omega)$ of positive order. The corresponding duality pairing is again denoted by $\langle\cdot,\cdot\rangle_{\partial\Omega}$.

Next we mention the embedding estimate

$$\|g\|_{q,\partial\Omega} \le c\|g\|_{\alpha;\gamma,\partial\Omega}, \quad 1<\gamma\le q,\ \alpha + \frac{2}{q} = \frac{2}{\gamma},\ 0\le\alpha\le 2, \tag{2.14}$$

for all $g \in W^{\alpha,\gamma}(\partial\Omega)$ where $c = c(\alpha,q,\partial\Omega) > 0$. By a standard duality argument we get form (2.14) the embedding estimate

$$\|g\|_{-\alpha;q,\partial\Omega} \le c\|g\|_{\gamma,\partial\Omega}, \quad 1<\gamma\le q,\ \alpha + \frac{2}{q} = \frac{2}{\gamma},\ 0\le\alpha\le 2, \tag{2.15}$$

for all $g \in L^\gamma(\partial\Omega)$ where $c = c(\alpha,q,\partial\Omega) > 0$.

The following definition of very weak solutions, see Definition 2.3 below, is for simplicity a special version of a more general notion introduced in [4]. Here we are mainly interested in boundary values as weak as possible. Note that a very weak solution v need not have any differentiability property in space, besides of $\operatorname{div} v = 0$. However, v satisfies a Serrin condition and is therefore uniquely determined and regular if the data and $\partial\Omega$ are smooth. On the other hand, the usual weak solution u of Definition 2.1 has a finite gradient in L^2, but we do not know uniqueness and global regularity properties.

In the following the set

$$\mathcal{J}^{q,s} = \mathcal{J}^{q,s}(\Omega) := \Big\{ v_0 \in L^2(\Omega);\ \|A_2^{-1}P_2 v_0\|_q + \Big(\int_0^\infty \|e^{-tA_2}P_2 v_0\|_q^s \, dt \Big)^{\frac{1}{s}} < \infty \Big\}$$

where $\|v_0\|_{\mathcal{J}^{q,s}} := \|A_2^{-1}P_2 v_0\|_{q,\Omega} + \big(\int_0^\infty \|e^{-tA_2}P_2 v_0\|_q^s \, dt \big)^{\frac{1}{s}}$, $1 < q, s < \infty$, plays the role as the space of initial values; for simplicity this space is not defined in the most general form as in [4], (2.18). Note that $\|v_0\|_{\mathcal{J}^{q,s}} = 0$ only means that $P_2 u_0 = 0$, see [4]. Therefore, $\|\cdot\|_{\mathcal{J}^{q,s}}$ is the norm of the quotient space of $\mathcal{J}^{q,s}$ modulo v_0 with $P_2 v_0 = 0$, i.e., modulo such gradients.

Definition 2.3 ([4]) *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary of class $C^{2,1}$, and let $0 < T \le \infty$, $3 < q < \infty$, $2 < s < \infty$, $1 < \gamma < q$, such that $\frac{1}{3} + \frac{1}{q} = \frac{1}{\gamma}$, $\frac{2}{s} + \frac{3}{q} = 1$.*

Then a function $v \in L^s\big(0,T; L^q(\Omega)\big)$ is called a very weak solution *of the Navier-Stokes system*

$$\begin{aligned} &v_t - \Delta v + v \cdot \nabla v + \nabla h = f,\ \operatorname{div} v = 0, \\ &v|_{\partial\Omega} = g,\ v|_{t=0} = v_0 \end{aligned} \tag{2.16}$$

on $[0,T) \times \Omega$ with data f, g, v_0 satisfying

$$\begin{aligned} &f = \operatorname{div} F,\ F = (F_{ij})_{i,j=1,2,3} \in L^s\big(0,T; L^\gamma(\Omega)\big), \\ &g \in L^s\big(0,T; W^{-\frac{1}{q},q}(\partial\Omega)\big), \\ &\int_{\partial\Omega} N \cdot g \, dS = \langle N, g \rangle_{\partial\Omega} = 0, \\ &v_0 \in \mathcal{J}^{q,s}(\Omega), \end{aligned} \tag{2.17}$$

if the relation

$$\begin{aligned} -\langle v, w_t \rangle_{\Omega,T} - \langle v, \Delta w \rangle_{\Omega,T} + \langle g, N \cdot \nabla w \rangle_{\partial\Omega,T} - \langle vv, \nabla w \rangle_{\Omega,T} \\ = \langle v_0, w(0) \rangle_\Omega - \langle F, \nabla w \rangle_{\Omega,T} \end{aligned} \tag{2.18}$$

is satisfied for all $w \in C_0^2\big([0,T); C_{0,\sigma}^2(\overline{\Omega})\big)$, and if the conditions

$$\operatorname{div} v = 0, \quad N \cdot v|_{\partial\Omega} = N \cdot g \tag{2.19}$$

are satisfied in $(0,T) \times \Omega$ and $(0,T) \times \partial\Omega$, respectively.

Note that the system (2.16) is well-defined in a weak sense. Using (2.18) with $w \in C_0^2\big((0,T);C^2_{0,\sigma}(\Omega)\big)$ we conclude that there is a unique (associated) pressure term ∇h such that $(2.16)_1$ holds in the sense of distributions in $(0,T)\times\Omega$. Further we conclude from (2.18), (2.19) that the boundary condition $v|_{\partial\Omega} = g$ is well-defined, and that the initial condition $v|_{t=0} = v_0$ is well-defined up to a gradient, see [4]. Moreover, v is uniquely determined and arbitrarily smooth in $(0,T)\times\overline{\Omega}$ if $\partial\Omega$ and the data are sufficiently smooth.

The following lemma yields the existence of v under a smallness condition on the data, see [4], Theorem 1.

Lemma 2.4 ([4]) *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain of class $C^{2,1}$, let $0 < T \le \infty$, and let f, g, v_0 be as in* (2.17) *with $3 < q < \infty$, $2 < s < \infty$, $1 < \gamma < q$, $\frac{1}{3}+\frac{1}{q}=\frac{1}{\gamma}$, $\frac{2}{s}+\frac{3}{q}=1$. Then there is a constant $\varepsilon = \varepsilon(\Omega,q) > 0$ with the following property: If*

$$\Big(\int_0^T \|e^{-tA_2}P_2v_0\|^s_{q,\Omega}\,dt\Big)^{1/s} + \Big(\int_0^T \|F\|^s_{\gamma,\Omega}\,dt\Big)^{1/s} + \Big(\int_0^T \|g\|^s_{-\frac{1}{q};q,\partial\Omega}\,dt\Big)^{1/s} \le \varepsilon, \tag{2.20}$$

then there exists a unique very weak solution $v \in L^s\big(0,T;L^q(\Omega)\big)$ of the system (2.16)*.*

Our method to prove Theorem 1.1 rests on the local identification of the given weak solution u with a certain very weak solution v, see Section 3.

Omitting the nonlinear term $v\cdot\nabla v$ in (2.16), we obtain the linear nonstationary Stokes system. The corresponding notion of a very weak solution is obtained by omitting the term $\langle vv, \nabla w\rangle_{\Omega,T}$ in (2.18), and the existence of a unique solution is obtained in this case without any smallness condition, see [4], Theorem 4.

Lemma 2.5 ([4]) *Let Ω, T be as in Lemma 2.4, assume $1 < \gamma < q < \infty$, $\frac{1}{3}+\frac{1}{q}=\frac{1}{\gamma}$ and $1 < s < \infty$, and let f, g, v_0 be as in this lemma, but omit the condition $\frac{2}{s}+\frac{3}{q}=1$. Then the linearized system* (2.16) *has a unique very weak solution $E \in L^s\big(0,T;L^q(\Omega)\big)$, i.e., by definition,*

$$-\langle E, w_t\rangle_{\Omega,T} - \langle E, \Delta w\rangle_{\Omega,T} + \langle g, N\cdot\nabla w\rangle_{\partial\Omega,T} = \langle v_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T} \tag{2.21}$$

for all $w \in C_0^2\big([0,T);C^2_{0,\sigma}(\overline{\Omega})\big)$, and the conditions

$$\operatorname{div} E = 0, \quad N\cdot E|_{\partial\Omega} = N\cdot g, \tag{2.22}$$

hold. Moreover, E satisfies the estimate

$$\begin{aligned} &\|A_q^{-1}P_qE_t\|_{L^s(0,T;L^q(\Omega))} + \|E\|_{L^s(0,T;L^q(\Omega))} \\ &\le C\big(\|v_0\|_{\mathcal{J}^{q,s}} + \|F\|_{L^s(0,T;L^\gamma(\Omega))} + \|g\|_{L^s(0,T;W^{-\frac{1}{q},q}(\partial\Omega))}\big) \end{aligned}$$

with some constant $C = C(\Omega,q,\gamma,s) > 0$ independent of T.

Further we note, see [4], (4.19), that v from Lemma 2.4 and E from Lemma 2.5 satisfy the (well-defined) semigroup relation

$$v(t) - E(t) = -\int_0^t A_q e^{-(t-\tau)A_q} A_q^{-1} P_q \operatorname{div}(vv)\, d\tau,\ 0 \le t < T. \tag{2.23}$$

In the following we will also apply Lemma 2.4, Lemma 2.5, and formula (2.23) with $[0,T)$ replaced by any other interval $[T_0, T_1)$, $0 \le T_0 < T_1 \le \infty$. Then $[0,T)\times\Omega$ is replaced by $[T_0,T_1)\times\Omega$, the initial condition $v|_{t=0} = v_0$ is replaced by $v|_{t=T_0} = v_0$, and, instead of (2.23), we get the relation

$$v(t) - E(t) = -\int_{T_0}^t A_q e^{-(t-\tau)A_q} A_q^{-1} P_q \operatorname{div}(vv)\, d\tau, \quad T_0 \le t < T_1. \tag{2.24}$$

Next assume that $vv \in L^2\big(T_0, T_1; L^2(\Omega)\big)$ is satisfied in (2.24). Then (2.24) is the well-known representation formula, see, e.g., [10], IV, (2.4.4), yielding the usual weak solution $v - E$ of the linear Stokes system

$$\begin{array}{rcl} (v-E)_t - \Delta(v-E) + \nabla h &=& -\operatorname{div}(vv), \quad \operatorname{div}(v-E) = 0, \\ v - E|_{\partial\Omega} = 0,\ v - E|_{t=T_0} &=& 0 \end{array} \tag{2.25}$$

in $[T_0,T_1)\times\Omega$, and satisfying the usual energy relation

$$\frac{1}{2}\|v(t)-E(t)\|_2^2 + \int_{T_0}^t \|\nabla(v-E)\|_2^2\, d\tau = \int_{T_0}^t \langle vv, \nabla(v-E)\rangle_\Omega\, d\tau, \tag{2.26}$$

$T_0 \le t \le T_1$. An easy consequence is the energy estimate

$$\|v-E\|^2_{L^\infty(T_0,T_1;L^2(\Omega))} + \|\nabla(v-E)\|^2_{L^2(T_0,T_1;L^2(\Omega))} \le \|vv\|^2_{L^2(T_0,T_1;L^2(\Omega))} < \infty, \tag{2.27}$$

and it follows that

$$v - E \in L^\infty\big(T_0,T_1;L^2(\Omega)\big), \quad \nabla(v-E) \in L^2\big(T_0,T_1;L^2(\Omega)\big), \tag{2.28}$$

see [10], Theorem IV, 2.3.1.

3 Proof of Theorem 1.1

In the following let $Q_r(t_0,x_0) \subseteq (0,\infty)\times\Omega$, $r > 0$, $t_0 \in (0,\infty)$, $x_0 \in \Omega$, be a parabolic cylinder and let u be a weak solution of (1.1) with data $f = \operatorname{div} F$, u_0 as in Theorem 1.1. Our aim is to prove the following result:

There exists an absolute constant $\varepsilon_* > 0$ such that if (3.1)

$$\|u\|_{L^q(Q_r)} \le \varepsilon_*\, r^{\frac{2}{q}+\frac{3}{q}-1},\ \frac{2}{q}+\frac{3}{q} \le 1 + \frac{1}{4},\ \|F\|_{L^4(t_0-r^2,t_0;L^2(B_r(x_0)))} \le \varepsilon_*\,,$$

then $u \in L^4\Big(t_0 - (\frac{r}{2})^2, t_0; L^6\big(B_{r/2}(x_0)\big)\Big)$.

To this end we need several steps.

a) *Reduction to the special case* $q = 4$, $r = 1$, $x_0 = 0$

Hölder's inequality leads to the estimate

$$\|u\|_{L^4(Q_r)} \le \Big(\frac{4}{3}\pi r^5\Big)^{\frac14 - \frac1q} \|u\|_{L^q(Q_r)} \le 2\pi r^{\frac54 - \frac5q} \|u\|_{L^q(Q_r)}.$$

Therefore, if we know the result (3.1) with some ε_* for $q = 4$, then this result holds for the general case with ε_* replaced by $\frac{\varepsilon_*}{2\pi}$. Thus we may set $q = 4$ in the following.

Next we use a well-known scaling procedure, introduced in principle in [2], in order to reduce the problem (3.1) with $q = 4$ to the case $r = 1$. For $\lambda > 0$ let $\tilde\Omega = \{y \in \mathbb{R}^3; \lambda y + x_0 \in \Omega\}$, and let $\tilde u, \tilde p, \tilde f, \tilde F, \tilde u_0$ be defined in the variables $\tau = \lambda^{-2} t \in [0,\infty)$, $y = \lambda^{-1}(x - x_0) \in \tilde\Omega$ where $t \in (0,\infty)$, $x \in \Omega$, by setting

$$\begin{aligned} &\tilde u(\tau, y) = \lambda u(t,x), \quad \tilde p(\tau,y) = \lambda^2 p(t,x), \quad \tilde f(\tau,y) = \lambda^3 f(t,x), \\ &\tilde F(\tau,y) = \lambda^2 F(t,x), \quad \tilde u_0(y) = \lambda u_0(x). \end{aligned} \tag{3.2}$$

Then an elementary calculation shows that $\tilde u$ is a weak solution in $[0,\infty) \times \tilde\Omega$ of the system

$$\begin{aligned} \tilde u_\tau - \Delta \tilde u + \tilde u \cdot \nabla \tilde u + \nabla \tilde p &= \tilde f, & \operatorname{div} \tilde u &= 0, \\ \tilde u\big|_{\partial\tilde\Omega} &= 0, & \tilde u\big|_{\tau=0} &= \tilde u_0 \end{aligned} \tag{3.3}$$

with data $\tilde f = \operatorname{div} \tilde F$, $\tilde u_0$, if and only if u is a weak solution in $[0,\infty) \times \Omega$ of (1.1) with data $f = \operatorname{div} F, u_0$. It holds $(t,x) \in Q_r(t_0,x_0)$ if and only if $(\tau,y) \in Q_{r/\lambda}(\tau_0,y_0)$ where $\tau_0 = \lambda^{-2} t_0$, $y_0 = 0$. Moreover,

$$\begin{aligned} \|\tilde u\|_{L^4(Q_{r/\lambda})} &= \Big(\int_{\tau_0 - (\frac{r}{\lambda})^2}^{\tau_0} \int_{|y| < \frac{r}{\lambda}} |\tilde u(\tau,y)|^4 \, dy \, d\tau \Big)^{\frac14} \\ &= \lambda^{-\frac14} \Big(\int_{t_0 - r^2}^{t_0} \int_{|x - x_0| < r} |u(t,x)|^4 \, dx \, dt \Big)^{\frac14} = \lambda^{-\frac14} \, \|u\|_{L^4(Q_r)} \end{aligned} \tag{3.4}$$

and, using the notation $B_r = B_r(0)$,

$$\begin{aligned} \|\tilde F\|_{L^4(\tau_0 - (\frac{r}{\lambda})^2, \tau_0; L^2(B_{r/\lambda}))} &= \|F\|_{L^4(t_0 - r^2, t_0; L^2(B_r(x_0)))}, \\ \|\tilde u\|_{L^4(\tau_0 - (\frac{r}{2\lambda})^2, \tau_0; L^6(B_{r/2\lambda}))} &= \|u\|_{L^4(t_0 - (\frac{r}{2})^2, t_0; L^6(B_{r/2}(x_0)))}. \end{aligned} \tag{3.5}$$

Therefore, setting $\lambda = r$, we see that $Q_{r/\lambda}(\tau_0, 0) = Q_1(\tau_0, 0)$, and that the condition (1.3) with $q = 4$ now has the form $\|\tilde u\|_{L^4(Q_1)} \le \varepsilon_*$. Hence it suffices to assume that $x_0 = 0 \in \Omega$, and to solve the problem (3.1) for given u, $f = \operatorname{div} F$ and u_0 on $Q_1 = Q_1(t_0, 0) \subseteq (0,\infty) \times \Omega$ in the following reduced form.

There exists an absolute constant $\varepsilon_* > 0$ such that if (3.6)

$$\|u\|_{L^4(Q_1(t_0,0))} \le \varepsilon_*, \quad \|F\|_{L^4(t_0 - 1, t_0; L^2(B_1))} \le \varepsilon_*,$$

then $u \in L^4\big(t_0 - \frac{1}{2}, t_0; L^6(B_{1/2})\big)$.

b) *Construction of a local very weak solution v*

In order to solve the problem (3.6) we first construct a very weak solution v on a cylinder $Q' = [t_0', t_0) \times B_{r'}$ with appropriate values $t_0' \in (t_0 - 1, t_0 - \frac{1}{2})$, $r' \in (\frac{1}{2}, 1)$, and appropriate boundary and initial conditions. Then in part c) below we will prove that $u = v$ on Q' leading to the desired regularity of u in $B_{1/2}$.

For this purpose we choose t_0' and r' in such a way that

$$u(t_0') \in L^4(B_{r'}), \quad u|_{(t_0-1,t_0)\times\partial B_{r'}} \in L^4\big(t_0 - 1, t_0; L^4(\partial B_{r'})\big) \tag{3.7}$$

are well-defined and satisfy the estimates

$$\|u(t_0')\|_{\mathcal{J}^{6,4}(B_{r'})} \le C_1 \|u(t_0')\|_{L^4(B_{r'})} \le C_2 \|u\|_{L^4(Q_1)} \tag{3.8}$$

and

$$\|u\|_{L^4(t_0',t_0;W^{-\frac{1}{6},6}(\partial B_{r'}))} \le C_3 \|u\|_{L^4(t_0',t_0;L^4(\partial B_{r'}))} \le C_4 \|u\|_{L^4(Q_1)}, \tag{3.9}$$

where $C_j = C_j(r') > 0$, $j = 1, \dots, 4$.

To find such values t_0', r' we argue as follows. Since we have to find a (sufficiently small) constant ε_* in (3.6) with the desired property, we assume that

$$\|u\|_{L^4(Q_1)} < \infty, \quad \|F\|_{L^4(t_0-1,t_0;L^2(B_1))} < \infty.$$

Then $u(t_0')|_{B_1} \in L^4(B_1)$ is well-defined for almost all $t_0' \in \big(t_0 - 1, t_0 - \frac{1}{2}\big)$. If there is no such value t_0' satisfying additionally

$$\|u(t_0')\|^4_{L^4(B_1)} \le 2\|u\|^4_{L^4(t_0-1,t_0-\frac{1}{2};L^4(B_1))} \tag{3.10}$$

we conclude that

$$\|u\|^4_{L^4(t_0-1,t_0-\frac{1}{2};L^4(B_1))} = \int_{t_0-1}^{t_0-\frac{1}{2}} \|u\|^4_{B_1}\, dt > \frac{1}{2} \cdot 2\|u\|^4_{L^4(t_0-1,t_0-\frac{1}{2};L^4(B_1))} \tag{3.11}$$

which is a contradiction. Using (2.10), (2.11) with $\alpha = \frac{1}{8}$, $q = 6$, $\delta = \delta(q, B_{r'}) > 0$, we thus obtain for some $t_0' \in (t_0 - 1, t_0 - \frac{1}{2})$ and - first of all - for each $r' \in (\frac{1}{2}, 1)$ the estimate

$$\begin{aligned}
&\|u(t_0')\|_{\mathcal{J}^{6,4}(B_{r'})} \qquad\qquad (3.12)\\
&= \|A_2^{-1}P_2 u(t_0')\|_{6,B_{r'}} + \left(\int_0^\infty \|A_2^{1/8} e^{-tA_2} A_2^{-1/8} P_2 u(t_0')\|^4_{6,B_{r'}}\, dt\right)^{1/4}\\
&\le C_1 \left(\|A_2^{-1/8} P_2 u(t_0')\|_{6,B_{r'}} + \left(\int_0^\infty e^{-4\delta t} t^{-4/8} \|A_2^{-1/8} P_2 u(t_0')\|^4_{6,B_{r'}}\, dt\right)^{1/4}\right)\\
&\le C_2 \|u(t_0')\|_{4,B_{r'}} \le C_2 \|u(t_0')\|_{4,B_1} \le C_2 2^{1/4} \|u\|_{L^4(Q_1)},
\end{aligned}$$

$C_j = C_j(r') > 0$, $j = 1,2$, which yields (3.8). Concerning r' we argue in the same way as for (3.10), and find at least one $r' \in (\frac{1}{2}, 1)$ such that

$$\|u\|^4_{L^4(t_0-1,t_0;L^4(\partial B_{r'}))} \le 2\|u\|^4_{L^4(t_0-1,t_0;L^4(B_1\setminus B_{1/2}))}. \tag{3.13}$$

Using (2.15) with $\alpha = \frac{1}{6}$, $q = 6$, $\gamma = 4$, we thus obtain the estimate

$$\begin{aligned} \|u\|_{L^4(t_0-1,t_0;W^{-\frac{1}{6},6}(\partial B_{r'}))} &\le C_3\|u\|_{L^4(t_0-1,t_0;L^4(\partial B_{r'}))} \\ &\le C_3 2^{1/4}\|u\|_{L^4(t_0-1,t_0;L^4(B_1\setminus B_{1/2}))} \le C_3 2^{1/4}\|u\|_{L^4(Q_1)}, \end{aligned} \tag{3.14}$$

$C_3 = C_3(r') > 0$, which yields (3.9).

Let $Q' = [t_0', t_0) \times B_{r'}$. Then we are able to apply Lemma 2.4 with $\Omega = B_{r'}$, $q = 6$, $s = 4$, $\gamma = 2$ and with $[0,T)$ replaced by $[t_0', t_0)$. Thus we obtain a constant $\varepsilon(r') > 0$ and a unique very weak solution $v \in L^4(t_0', t_0; L^6(B_{r'}))$ in Q' of the system

$$\begin{aligned} v_t - \Delta v + v\cdot\nabla v + \nabla h &= f, & \operatorname{div} v &= 0, \\ v\big|_{\partial B_{r'}} &= g, & v\big|_{t=t_0'} &= v_0 \end{aligned} \tag{3.15}$$

with data

$$f = \operatorname{div} F,\ \ F \in L^4(t_0', t_0; L^2(B_{r'})), \quad g = u\big|_{(t_0',t_0)\times\partial B_{r'}}, \quad v_0 = u(t_0')\big|_{B_{r'}}, \tag{3.16}$$

if

$$\|u\|_{L^4(Q_1)} + \|F\|_{L^4(t_0-1,t_0;L^2(B_1))} \le \varepsilon(r'). \tag{3.17}$$

Identifying $u = v$ on $Q' \supset Q_{1/2}$, see Part c) below, we are led to the desired property $u \in L^4(t_0 - \frac{1}{2}, t_0; L^6(B_{1/2}(x_0)))$. However, in order to prove (3.6) we need that the constant in (3.17) does *not* depend on r'. To obtain an absolute constant in (3.17) we modify the system (3.15), using again the scaling procedure, as follows.

With $\lambda = r'$ let $\tau = \lambda^{-2}t$, $\tau_0' = \lambda^{-2}t_0'$, $\tau_0 = \lambda^{-2}t_0$ for $t \in [t_0', t_0)$, $\tau \in [\tau_0', \tau_0)$, and let $y = \lambda^{-1}x \in B_1$, $x \in B_{r'}$, $y_0 = 0$. Then $\tilde v, \tilde F, \tilde g, \tilde v_0$ are defined by $\tilde F(\tau, y) = \lambda^2 F(t,x)$, $\tilde g(\tau, y) = \lambda g(t,x)$ and $\tilde v_0(y) = \lambda v_0(x)$. Obviously the scaling argument as in (3.3) shows that $\tilde v \in L^4\big(\tau_0', \tau_0; L^6(B_1)\big)$ is a very weak solution in $\tilde Q' = [\tau_0', \tau_0] \times B_1$ of the system

$$\begin{aligned} \tilde v_\tau - \Delta\tilde v + \tilde v\cdot\nabla\tilde v + \nabla\tilde h &= \tilde f, & \operatorname{div}\tilde v &= 0, \\ \tilde v\big|_{\partial B_1} &= \tilde g, & \tilde v\big|_{\tau=\tau_0'} &= \tilde v_0 \end{aligned} \tag{3.18}$$

with data

$$\tilde f = \operatorname{div}\tilde F, \quad \tilde g = \tilde u\big|_{(\tau_0',\tau_0)\times\partial B_1}, \quad \tilde v_0 = \tilde u(\tau_0')\big|_{B_1}, \tag{3.19}$$

if and only if $v \in L^4\big(t_0', t_0; L^6(B_{r'})\big)$ is a very weak solution in Q' of the system (3.15) with data (3.16). The same calculation as for (3.8), (3.9), see (3.12) - (3.14), now yields the estimates

$$\|\tilde{u}(\tau_0')\|_{\mathcal{J}^{6,4}(B_1)} \leq C_1 \|\tilde{u}(\tau_0')\|_{L^4(B_1)} \leq C_2 \|u\|_{L^4(Q_1)} \tag{3.20}$$

and

$$\|\tilde{u}\|_{L^4(\tau_0', \tau_0; W^{-\frac{1}{6},6}(\partial B_1))} \leq C_3 \|\tilde{u}\|_{L^4(\tau_0', \tau_0; L^4(\partial B_1))} \leq C_4 \|u\|_{L^4(Q_1)}\,; \tag{3.21}$$

here, replacing B_r by B_1, the constants $C_1, \ldots, C_4$ depend on B_1 and therefore are absolute constants. Hence the smallness condition (3.17) corresponding to the system (3.18) with data (3.19) is satisfied with some absolute constant $\varepsilon_* > 0$. Thus we can return to (3.15) and obtain by virtue of Lemma 2.4 the following result:

There exists an absolute constant $\varepsilon_* > 0$ such that if (3.22)

$$\|u\|_{L^4(Q_1)} + \|F\|_{L^4(t_0-1, t_0; L^2(B_1))} \leq \varepsilon_*,$$

then the system (3.15) with data (3.16) has a unique very weak solution $v \in L^4\big(t_0', t_0; L^6(B_{r'})\big)$.

c) *Identification $u = v$ on Q'*

It remains to prove this identification. Assuming $u = v$ on Q' we conclude, since $t_0' < t_0 - \frac{1}{2}$, $\frac{1}{2} < r'$, that (3.6) is true, and we complete the proof. To this end we need several arguments as follows.

First we consider the very weak solution $E \in L^4\big(t_0', t_0; L^6(B_{r'})\big)$ of the linearized system (3.15), omitting the term $v \cdot \nabla v$; see (2.21), (2.22) with $s = 4$, $q = 6$, $\Omega = B_{r'}$, $[0, T)$ replaced by $[t_0', t_0)$. Then formula (2.24) can be written in the form

$$v(t) - E(t) = -\int_{t_0'}^{t} e^{-(t-\tau)A_2}\, P_2 \mathrm{div}\,\big(v(\tau)v(\tau)\big)\, d\tau. \tag{3.23}$$

Using Hölder's inequality in $L^2(Q') = L^2\big(t_0', t_0; L^2(B_{r'})\big)$, we obtain that

$$\|vv\|_{L^2(Q')} \leq C\|v\|^2_{L^4(Q')} \leq C\|v\|^2_{L^4(Q_1)} < \infty$$

with an absolute constant $C > 0$, and thus that $v - E$ in (3.23) has the properties (2.25) – (2.28) with $[T_0, T_1) \times \Omega$ replaced by $[t_0', t_0) \times B_{r'}$.

Next we use that $uu \in L^2\big(t_0', t_0; L^2(B_{r'})\big)$, and we argue for $u - E$ in the same way as for $v - E$ in (2.25) – (2.28). Indeed, using for u, E the relations (2.4), (2.18), we conclude that (2.25) – (2.28) is true if v is replaced by u, and $[T_0, T_1) \times \Omega$ is replaced by $[t_0', t_0) \times B_{r'}$.

Further we conclude that $u-v$ is the weak solution of the Stokes system

$$(u-v)_t - \Delta(u-v) + \nabla h = -\mathrm{div}\,(uu-vv), \quad \mathrm{div}\,(u-v) = 0,$$
$$u-v\big|_{\partial B_{r'}} = 0, \quad u-v\big|_{t=t_0'} = 0$$

in $[t_0', t_0) \times B_{r'}$, and that the corresponding energy properties as in (2.26) – (2.28) hold for $u-v$.

Since $u(u-v) \in L^2(Q')$ and $\nabla(u-v) \in L^2(Q')$ we see by Hölder's inequality that $\int_{t_0'}^{t_0} \langle u(u-v), \nabla(u-v)\rangle_{B_{r'}}\, d\tau$ is well-defined, and that

$$\langle u(u-v), \nabla(u-v)\rangle_{B_{r'}} = \frac{1}{2}\langle u, \nabla|u-v|^2\rangle_{B_{r'}} = 0.$$

Therefore, using (2.26) for $u-v$ as explained above, we obtain that

$$\frac{1}{2}\|u(t)-v(t)\|_2^2 + \int_{t_0'}^{t_0} \|\nabla(u-v)\|_2^2\, d\tau = \int_{t_0'}^{t} \langle uu - vv, \nabla(u-v)\rangle_{B_{r'}}\, d\tau$$
$$= \int_{t_0'}^{t} \langle u(u-v) + (u-v)v, \nabla(u-v)\rangle_{B_{r'}}\, d\tau = \int_{t_0'}^{t} \langle (u-v)v, \nabla(u-v)\rangle_{B_{r'}}\, d\tau.$$

A consequence of this relation as in (2.27) is the energy estimate

$$\begin{aligned} |||u-v|||_{t_0',t_0}^2 &:= \|u-v\|^2_{L^\infty(t_0',t_0;L^2(B_{r'}))} + \|\nabla(u-v)\|^2_{L^2(t_0',t_0;L^2(B_{r'}))} \\ &= \|u-v\|^2_{2,\infty} + \|\nabla(u-v)\|^2_{2,2} \\ &\le C_1\|(u-v)v\|^2_{2,2}\,. \end{aligned} \tag{3.24}$$

Next we use the standard Sobolev estimate $\|u-v\|_3 \le C_2\|\nabla(u-v)\|_2^{\frac{1}{2}}\,\|u-v\|_2^{\frac{1}{2}}$, see e.g. [10], II, (1.3.2), and Hölder's inequality, and obtain that

$$\begin{aligned} \|(u-v)v\|^2_{2,2} &\le C_3\|v\|^2_{6,4}\,\|u-v\|^2_{3,4} \\ &\le C_4\|v\|^2_{6,4}\big(\|\nabla(u-v)\|^2_{2,2} + \|u-v\|^2_{2,\infty}\big) \\ &= C_4\|v\|^2_{6,4}\,|||u-v|||^2_{t_0',t_0}\,. \end{aligned}$$

In these estimates $C_1, \ldots, C_4 > 0$ are absolute constants. Thus (3.24) leads to the estimate

$$|||u-v|||_{t_0',t_0} \le C\|v\|_{L^4(t_0',t_0;L^6(B_{r'}))}\,|||u-v|||_{t_0',t_0} \tag{3.25}$$

with some absolute constant $C > 0$.

Our purpose is to apply to (3.25) the well-known absorption principle as follows. Consider a decomposition $t_0' = t_1 < t_2 < \ldots < t_{m-1} < t_m = t_0$, $m \in \mathbb{N}$, in such a way that

$$C\|v\|_{L^4(t_{j-1},t_j;L^6(B_{r'}))} \le \frac{1}{2} \tag{3.26}$$

for $j = 2, \ldots, m$. The estimate (3.25) also holds with t_0 replaced by t_1, and inserting (3.26) in (3.25) we get that

$$|||u - v|||_{t_0', t_1} \leq \frac{1}{2} |||u - v|||_{t_0', t_1}$$

which means that $u = v$ in $[t_0', t_1)$. Repeating this argument with $[t_0', t_1)$ replaced by $[t_1, t_2)$ yields $u = v$ in $[t_1, t_2)$, and so on. In a finite number of such steps we conclude that $u = v$ in Q'. This completes the proof of Theorem 1.1. ■

Proof of Corollary 1.2 The proof of Theorem 1.1 can be carried out in the same way with $Q_r(t_0, x_0)$ replaced by $Q_r^*(t_0, x_0)$, which means that (1.5) is replaced by $u \in L^4\big(t_0 - \frac{7}{8}\big(\frac{r}{2}\big)^2, t_0 + \frac{1}{8}\big(\frac{r}{2}\big)^2; L^6\big(B_{r/2}(x_0)\big)$, after the corresponding modifications in this theorem. This means in particular that (t_0, x_0) is a regular point of u. If the condition (1.7) is satisfied, we find a sequence $Q_{r_j}^*(t_0, x_0) \subseteq (0, \infty) \times \Omega$, $r_j > 0$, $j \in \mathbb{N}$, with $\lim_{j \to \infty} r_j = 0$, in such a way that

$$\lim_{j \to \infty} r_j^{q-5} \, \|u\|_{L^q(Q_{r_j}^*)}^q < (\varepsilon_*)^q.$$

Then there is at least one radius $r_0 > 0$ among the values $r_1, r_2, \ldots,$ such that

$$\|u\|_{L^q(Q_{r_0}^*)} \leq \varepsilon_* \, r_0^{\frac{2}{q} + \frac{3}{q} - 1},$$

cf. (1.3), holds in Theorem 1.1 suitably modified. Furthermore, using $F \in L^4\big(0, \infty; L^2(\Omega)\big)$, we can choose r_0 sufficiently small in such a way that (1.4) is satisfied after the corresponding modification. This shows that (t_0, x_0) is a regular point, and the proof is complete.

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