An Appropriate Geometric Invariant for the C^2-Analysis of Subdivision Surfaces

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Abstract. We introduce the embedded Weingarten map as a geometric invariant of piecewise smooth surfaces. It is given by a (3×3)-matrix and provides complete curvature information in a continuous way. Thus, it is the appropriate tool for the C^2-analysis of subdivision surfaces near extraordinary points. We derive asymptotic expansions and show that the convergence of the sequence of embedded Weingarten maps to a constant limit is necessary and sufficient for curvature continuity.

1 Introduction

Locally, a subdivision surface $\mathbf{x}$ can be regarded as the union of the *central point* $\mathbf{x}^c$ and a sequence of smaller and smaller *spline rings* $\mathbf{x}^m, m \in \mathbb{N}$, which are converging to $\mathbf{x}^c$,

$$\mathbf{x}^c = \lim_{m\to\infty} \mathbf{x}^m.$$

While, at least for linear stationary schemes, the parametrizations of the $\mathbf{x}^m$ are explicitly known, the central point $\mathbf{x}^c$ itself is an isolated object so that standard formulas for computing normal vector and curvature properties at this point cannot be applied immediately. There are two ways to analyze $\mathbf{x}$ in a vicinity of $\mathbf{x}^c$:

First, one can try to express $\mathbf{x}$ locally as the graph of a real-valued height function h over the tangent plane at $\mathbf{x}^c$. Then h contains all curvature information of $\mathbf{x}$ at $\mathbf{x}^c$. In particular, by definition, $\mathbf{x}$ is *curvature continuous* at $\mathbf{x}^c$ if and only if h is C^2. Beyond its fundamental character as a definition, this approach can also be used for analytical purposes. For instance, in [1], convergence of the *anchored osculating paraboloid*, which is just the quadratic Taylor jet of h, is used to prove curvature continuity of surfaces generated by guided subdivision. However, explicit formulas for h and its derivatives are not easy to derive since a nonlinear pair of equations has to be solved in order to determine the value of h at a given point in the tangent plane.

Second, one can analyze convergence properties of geometric invariants to draw conclusions on $\mathbf{x}$ at $\mathbf{x}^c$. For instance, $\mathbf{x}$ is normal continuous at $\mathbf{x}^c$ if the sequence of normal vectors $\mathbf{n}^m$ of the spline rings is converging to a unique limit, called the *central normal*,

$$\mathbf{n}^c := \lim_{m\to\infty} \mathbf{n}^m.$$

Equally, it is suggested in [4] to establish curvature continuity based on the convergence of the principal curvatures $\kappa^m_{1,2}$ and the principal directions $\mathbf{r}^m_{1,2}$, which are the standard second order geometric invariants of smooth surfaces,

$$\kappa^{\mathrm{c}}_{1,2} := \lim_{m\to\infty} \kappa^m_{1,2}, \quad \mathbf{r}^{\mathrm{c}}_{1,2} := \lim_{m\to\infty} \mathbf{r}^m_{1,2}.$$

Favorably, these quantities are easy to derive from the underlying parametrizations of the spline rings $\mathbf{x}^m$. However, there is a nasty little problem with this approach: If $\mathbf{x}^{\mathrm{c}}$ happens to be an *umbillic point* of the surface, i.e., $\kappa^{\mathrm{c}}_1 = \kappa^{\mathrm{c}}_2$, then the corresponding principal directions are *not* convergent at $\mathbf{x}^{\mathrm{c}}$, even if the surface is curvature continuous. The reason for that phenomenon is the fact that at umbillic points any direction in the tangent plane is a principal direction. Consequently, we have to state that curvature analysis based on principal directions is a flawed approach which needs mending.

The principal directions are the eigenvectors of the *Weingarten map*, also called the *shape operator*, to the principal curvatures. Hence, one could try to consider convergence properties of the sequence W^m of Weingarten maps instead of principal curvatures and directions. In principle, this is possible, but point-wise, the Weingarten map lives in the tangent space of the surface, and a coordinate system in this tangent space is required to express W^m explicitly. A standard choice for such a coordinate system is given by the partial derivatives of the surface parametrization. This parametrization, however, is typically not C^1 for a subdivision surface, even it is geometrically smooth. Hence, with respect to these coordinates, the Weingarten map does not vary continuously on the surface. A continuous choice of coordinates might locally be difficult, and, for topological reasons, globally be impossible.

In this paper, we propose to extend the Weingarten map to the embedding space $\mathbb{R}^3$. Besides the principal directions, now regarded as vectors in $\mathbb{R}^3$, also the surface normal becomes an eigenvector corresponding to the trivial eigenvalue 0. The advantages of the *embedded Weingarten map*, which make it appropriate for the study of subdivision surfaces, are that

- it is a *geometric invariant* of the surface, i.e., it does not depend on the chosen parametrization but only on the shape of the surface;
- it *varies continuously* on the surface (even at umbillic points) if and only if the surface is C^2;
- it is *easy to compute* using the second fundamental form and the pseudo-inverse of the derivative of the parametrization.

In Chapter 2, we develop the concept of the embedded Weingarten map in some detail. Then, in Chapter 3, we apply it to subdivision surfaces and derive asymptotic expansions for the sequence E^m of embedded Weingarten maps of the spline rings. Finally, in Chapter 4, the convergence requirements on the sequence E^m are used to recall the well known conditions on curvature continuity.

2 The Extended Weingarten Map

Throughout the paper, points and vectors in $\mathbb{R}^3$ are denoted by boldface latin letters and understood as *rows* with three components, e.g., $\mathbf{x} = [x, y, z]$. Points in $\mathbb{R}^2$, such as parameters of surfaces, are denoted by boldface greek letters, e.g., $\boldsymbol{\sigma} = (s, t)$. For matrices A, B, a dot indicates multiplication by the transpose,

$$A \cdot B := AB^{\mathrm{t}}.$$

Equally, if one or both factors are points in $\mathbb{R}^3$, we define $A \cdot \mathbf{x} := A\mathbf{x}^{\mathrm{t}}$, and $\mathbf{x} \cdot \mathbf{y} := \mathbf{x}\mathbf{y}^{\mathrm{t}}$. The Euclidean norm is denoted $|\mathbf{x}| := \sqrt{\mathbf{x} \cdot \mathbf{x}}$. We consider a parametrized surface

$$\mathbf{x} : \Omega \ni (s, t) \to \mathbf{x}(s, t) \in \mathbb{R}^3$$

with domain $\Omega \subset \mathbb{R}^2$ which is twice differentiable. According to the above convention, we also write $\mathbf{x}(\boldsymbol{\sigma})$ instead of $\mathbf{x}(s, t)$, but mostly, the argument is simply omitted. The trace of $\mathbf{x}$ is denoted $\mathbf{X} := \mathbf{x}(\Omega)$. The first derivative of $\mathbf{x}$ is given by the (2×3)-matrix

$$D\mathbf{x} := \begin{bmatrix} \mathbf{x}_s \\ \mathbf{x}_t \end{bmatrix},$$

and we assume that the cross product of its rows does not vanish,

$${}^{\times}\!D\mathbf{x} := \mathbf{x}_s \times \mathbf{x}_t \neq \mathbf{0}.$$

In other words, $\mathbf{x}$ is a *regular C^2-surface*. The *Gauss map* of $\mathbf{x}$ is given by the normalized *normal vector*,

$$\mathbf{n} : \Omega \ni \boldsymbol{\sigma} \to \frac{{}^{\times}\!D\mathbf{x}(\boldsymbol{\sigma})}{|{}^{\times}\!D\mathbf{x}(\boldsymbol{\sigma})|} \in S^2.$$

Differentiating the identity $\mathbf{n} \cdot \mathbf{n} \equiv 1$, we obtain $D\mathbf{n} \cdot \mathbf{n} = 0$, what means that the row vectors of $D\mathbf{n}$ lie in the tangent plane of $\mathbf{x}$ at the corresponding point. Hence, there exists a (2×2)-matrix W, called the *Weingarten map*, with

$$D\mathbf{n} = -W D\mathbf{x}.$$

Multiplication with the transpose of $D\mathbf{x}$ yields

$$I\!I = W I,$$

where

$$I := D\mathbf{x} \cdot D\mathbf{x} = \begin{bmatrix} \mathbf{x}_s \cdot \mathbf{x}_s & \mathbf{x}_t \cdot \mathbf{x}_s \\ \mathbf{x}_s \cdot \mathbf{x}_t & \mathbf{x}_t \cdot \mathbf{x}_t \end{bmatrix}$$

and also

$$I\!I := -D\mathbf{n} \cdot D\mathbf{x} = -\begin{bmatrix} \mathbf{x}_s \cdot \mathbf{n}_s & \mathbf{x}_t \cdot \mathbf{n}_s \\ \mathbf{x}_s \cdot \mathbf{n}_t & \mathbf{x}_t \cdot \mathbf{n}_t \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{ss} \cdot \mathbf{n} & \mathbf{x}_{st} \cdot \mathbf{n} \\ \mathbf{x}_{st} \cdot \mathbf{n} & \mathbf{x}_{tt} \cdot \mathbf{n} \end{bmatrix}$$

are symmetric (2×2)-matrices, called the *first* and *second fundamental form of* $\mathbf{x}$, respectively. By $\det I = |{}^{\times}\!D\mathbf{x}|^2$ and regularity of the parametrization, I is invertible, thus

$$W = \mathit{II}\, I^{-1}.$$

Given a smooth curve $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma} + t\boldsymbol{\sigma}' + o(t)$ in the domain of $\mathbf{x}$, we define the related curves $\mathbf{c}_\mathbf{x}(t) := \mathbf{x}(\boldsymbol{\gamma}(t))$ on the surface, and $\mathbf{c}_\mathbf{n}(t) := \mathbf{n}(\boldsymbol{\gamma}(t))$ on the unit sphere. Dropping, as usual, the parameter $\boldsymbol{\sigma} = \boldsymbol{\gamma}(0)$, we obtain by the chain rule

$$\mathbf{c}'_\mathbf{x}(0) = \boldsymbol{\sigma}' D\mathbf{x}, \quad \mathbf{c}'_\mathbf{n}(0) = \boldsymbol{\sigma}' W D\mathbf{x}.$$

Now, we are looking for curves $\boldsymbol{\gamma}$ with the property that

- $\mathbf{r} := \mathbf{c}'_\mathbf{x}(0)$ has unit length, and
- $\mathbf{c}'_\mathbf{n}(0) = \kappa\mathbf{r}$ for some $\kappa \in \mathbb{R}$, i.e., $\mathbf{c}'_\mathbf{x}(0)$ and $\mathbf{c}'_\mathbf{n}(0)$ are parallel.

Then the vector $\mathbf{r} \in \mathbb{R}^3$ is called a *principal direction*, and κ is the corresponding *principal curvature* of $\mathbf{x}$ at the point $\mathbf{x}(\boldsymbol{\sigma})$. The condition $\mathbf{c}'_\mathbf{n}(0) = \kappa\mathbf{r}$ is equivalent to

$$\boldsymbol{\sigma}' W = \kappa\boldsymbol{\sigma}', \quad \mathbf{r} = \boldsymbol{\sigma}' D\mathbf{x}.$$

In other words, $\boldsymbol{\sigma}'$ is a left eigenvector of W to the eigenvalue κ. One can (and we will) show that W has always two real eigenvalues κ_1, κ_2, and that the corresponding pair $\mathbf{r}_1, \mathbf{r}_2$ of principal directions can be chosen orthonormal. It is well known that the principal curvatures and directions are *geometric invariants* in the sense that they do not depend on the parametrization, but only on the shape and the orientation of the surface. We state without proof

Theorem 1. *Let* $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *be two regular* C^2*-surfaces without self-intersections and equal trace,* $\mathbf{X} = \tilde{\mathbf{X}}$*. Then, if* $\mathbf{x}(\boldsymbol{\sigma}) = \tilde{\mathbf{x}}(\tilde{\boldsymbol{\sigma}})$,

$$\mathbf{n}(\boldsymbol{\sigma}) = s\tilde{\mathbf{n}}(\tilde{\boldsymbol{\sigma}})$$

either for $s = 1$ *or* $s = -1$*. Further, if* $\mathbf{r}$ *is a principal direction of* $\mathbf{x}$ *at* $\mathbf{x}(\boldsymbol{\sigma})$ *to the principal curvature* κ*, then it also a principal direction of* $\tilde{\mathbf{x}}$ *at* $\tilde{\mathbf{x}}(\tilde{\boldsymbol{\sigma}})$ *to the principal curvature* $s\kappa$*.*

The theory developed so far is well established, but not fully satisfactory for the analysis of subdivision surfaces. Clearly, if we assume for instance $\kappa_1 \le \kappa_2$, the principal curvatures depend continuously on the parameter $\boldsymbol{\sigma} \in \Omega$. But by contrast, the principal directions reveal discontinuities at *umbillic points* which are characterized by $\kappa_1 = \kappa_2$. Here, any direction in the tangent plane is a principal direction, and accordingly, $\mathbf{r}_1, \mathbf{r}_2$ do not converge when approaching such a point. Below, this phenomenon is illustrated at hand of a paraboloid of revolution, where the principal directions diverge near the vertex. In standard text books on differential geometry, this problem is not fully resolved, but merely addressed as a degenerate situation. In our context, however, continuity is an indispensable analytic tool since we want to study convergence properties of spline rings without having immediate access to a single surface parametrization near the extraordinary point. We suggest the following approach:

Definition 2. *For a regular C^2-surface* $\mathbf{x}$*, let*

$$D\mathbf{x}^+ := (I^{-1} D\mathbf{x})^{\mathrm{t}}$$

denote the pseudo-inverse of $D\mathbf{x}$*, which is a* (3×2)*-matrix. Then we define the* embedded Weingarten map *of* $\mathbf{x}$ *as the symmetric* (3×3)*-matrix*

$$E := D\mathbf{x}^+ \, II \cdot D\mathbf{x}^+.$$

We claim that this object is a geometric invariant which contains the complete curvature information in a continuous way. Because $E \cdot \mathbf{n} = \mathbf{0}$, the normal vector is always an eigenvector of E to the eigenvalue 0. The two other eigenvectors can be chosen orthonormal (mutually and with respect to $\mathbf{n}$), and collected in a (2×3)-matrix $\mathbf{R}$. The diagonal matrix of the corresponding pair of eigenvalues is denoted K. We obtain the factorization

$$E = [\mathbf{R}^{\mathrm{t}} \, \mathbf{n}^{\mathrm{t}}] \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{n} \end{bmatrix} = \mathbf{R}^{\mathrm{t}} K \mathbf{R},$$

and therefore

$$\mathbf{R} E = K \mathbf{R}.$$

By orthogonality of the eigenvectors, we have $\mathbf{R} \cdot \mathbf{n} = \mathbf{0}$. Hence, there exists a (2×2)-matrix Σ with $\mathbf{R} = \Sigma D\mathbf{x}$. Together with the definitions $E = D\mathbf{x}^{\mathrm{t}} I^{-1} II I^{-1} D\mathbf{x}$ and $W = II \, I^{-1}$, we conclude from the last display $\Sigma W D\mathbf{x} = W \Sigma D\mathbf{x}$, and eventually

$$\Sigma W = K \Sigma.$$

That is, the diagonal entries of $K = \operatorname{diag}(\kappa_1, \kappa_2)$ are the eigenvalues of W, and the rows of $\Sigma = [\boldsymbol{\sigma}'_1; \boldsymbol{\sigma}'_2]$ are the corresponding left eigenvectors which, via $[\mathbf{r}_1; \mathbf{r}_2] := \Sigma D\mathbf{x} = \mathbf{R}$, yield the principal directions.

So far, we have found the following: The embedded Weingarten map E is a symmetric (3×3)-matrix with a trivial eigenvalue 0 corresponding to the surface normal $\mathbf{n}$. The other two eigenvalues are the principal curvatures, and the corresponding eigenvectors are the principal directions of the surface. But unlike these directions, the matrix E depends continuously on the parameter $\boldsymbol{\sigma}$. The following theorem establishes E as a geometric invariant:

Theorem 3. *Let* $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *be two regular C^2-surfaces without self-intersections and equal trace,* $\mathbf{X} = \tilde{\mathbf{X}}$*, and let* $\mathbf{n}(\boldsymbol{\sigma}) = s\tilde{\mathbf{n}}(\tilde{\boldsymbol{\sigma}})$ *for* $\mathbf{x}(\boldsymbol{\sigma}) = \tilde{\mathbf{x}}(\tilde{\boldsymbol{\sigma}})$ *as in Theorem 1. Then the corresponding embedded Weingarten maps are equal up to sign,*

$$E(\boldsymbol{\sigma}) = s\tilde{E}(\tilde{\boldsymbol{\sigma}}).$$

Proof. Let $s = 1$. If $\mathbf{r}$ is a principal direction of $\mathbf{x}$ at $\mathbf{x}(\boldsymbol{\sigma})$ to the principal curvature κ, then, by Theorem 1, it is also a principal direction of $\tilde{\mathbf{x}}$ at $\tilde{\mathbf{x}}(\tilde{\boldsymbol{\sigma}})$ to κ. Further, $E(\boldsymbol{\sigma}) \cdot \mathbf{n}(\boldsymbol{\sigma}) = \tilde{E}(\tilde{\boldsymbol{\sigma}}) \cdot \tilde{\mathbf{n}}(\tilde{\boldsymbol{\sigma}}) = \mathbf{0}$. Hence, the eigenspaces and eigenvalues of $E(\boldsymbol{\sigma})$ and $\tilde{E}(\tilde{\boldsymbol{\sigma}})$ coincide so that $E(\boldsymbol{\sigma}) = \tilde{E}(\tilde{\boldsymbol{\sigma}})$. If $s = -1$, then again, the corresponding eigenspaces coincide. However, the eigenvalues and hence the matrices have opposite sign. □

The mean and the product of the principal curvatures are known as *mean curvature* and *Gaussian curvature* of $\mathbf{x}$, respectively,

$$K_m := \frac{\kappa_1 + \kappa_2}{2}, \quad K_g := \kappa_1 \kappa_2.$$

They can be computed easily form W,

$$K_m = \frac{1}{2} \operatorname{trace} W, \quad K_g = \det W = \det II / \det I,$$

and equally from E,

$$K_m = \frac{1}{2} \operatorname{trace} E, \quad K_g = \frac{1}{2} \operatorname{trace}^2 E - \frac{1}{2} \|E\|_{\mathrm{F}}^2,$$

where $\|E\|_{\mathrm{F}}^2 := \sum_{i,j} E_{i,j}^2$ is the squared Frobenius norm of E. A point $\mathbf{x}(\boldsymbol{\sigma})$ is called *elliptic* if $K_g(\boldsymbol{\sigma}) > 0$, *hyperbolic* if $K_g(\boldsymbol{\sigma}) < 0$, and *parabolic* if $K_g(\boldsymbol{\sigma}) = 0$. Now, we consider a continuous surface $\mathbf{x} : \Omega \to \mathrm{I\!R}^3$ which is regular and C^k everywhere with the exception of a single point, say $\boldsymbol{\sigma}_0 = \mathbf{0}$. Such an $\mathbf{x}$ is called a C_0^k-surface. More generally, we say that $\mathbf{x}$ is a C_r^k surface if there exists a regular C^r-surface $\tilde{\mathbf{x}}$ with equal trace, $\mathbf{X} = \tilde{\mathbf{X}}$. It is well known that $\mathbf{x}$ is a C_1^k-surface if the Gauss map converges,

$$\mathbf{n}^{\mathrm{c}} := \lim_{\boldsymbol{\sigma} \to \mathbf{0}} \mathbf{n}(\boldsymbol{\sigma}),$$

and if the projection of $\mathbf{x}$ to the tangent plane at $\mathbf{x}(\mathbf{0})$ is locally injective. The next theorem provides a similar result for curvature continuity and will be fundamental for the investigation of subdivision surfaces:

Theorem 4. *Let $\mathbf{x} : \Omega \to \mathrm{I\!R}^3$ be a C_1^k-surface, $k \geq 2$. It is a C_2^k-surface if and only if the embedded Weingarten map converges, i.e., if the limit*

$$E^{\mathrm{c}} = \lim_{\boldsymbol{\sigma} \to \mathbf{0}} E(\boldsymbol{\sigma})$$

exists.

Proof. Without loss of generality, let us assume that $\mathbf{n}^{\mathrm{c}} = \mathbf{e}_3 := [0, 0, 1]$ is the third unit vector, and that $\mathbf{x}(\mathbf{0}) = \mathbf{0}$. Since the projection of $\mathbf{x}$ to the tangent plane at $\mathbf{x}(\mathbf{0})$ is locally injective, is can be represented as graph over the tangent plane. That is, there exists a scalar function h such that, locally,

$$\tilde{\mathbf{x}}(u, v) := [u, v, h(u, v] = \mathbf{x}(\boldsymbol{\sigma}).$$

The function h is C^1, and, by the inverse function theorem, it is twice differentiable for $(u, v) \neq (0, 0)$. By Theorem 4, the embedded Weingarten maps of $\mathbf{x}$ and $\tilde{\mathbf{x}}$ are equal up to sign so that

$$\lim_{(u,v) \to (0,0)} \tilde{E}(u, v) = s \lim_{\boldsymbol{\sigma} \to \boldsymbol{\sigma}_0} E(\boldsymbol{\sigma}) = sE^{\mathrm{c}}, \quad s \in \{-1, 1\}.$$

Now, the mean value theorem shows that h is twice differentiable also at the origin, and that sE^{c} is the embedded Weingarten map at this point. $\square$

Let us illustrate the concepts developed so far at hand of a simple example. We consider the C_0^∞-surface

$$\mathbf{x}(s,t) := [2\sqrt{s},\ 2\sqrt{t},\ s+t], \quad (s,t) \in [0,1]^2.$$

The parametrization is not differentiable at the origin so that we do not know beforehand if the surface is normal or curvature continuous. Of course, the reparametrization $[x, y, (x^2+y^2)/4]$ shows that it is part of a paraboloid of revolution, what settles the problem immediately, but in more complicated situations, such knowledge is not readily available. With $w := (1+s+t)^{-1/2}$, we obtain for $(s,t) \neq (0,0)$

$$D\mathbf{x} = \begin{bmatrix} 1/\sqrt{s} & 0 & 1 \\ 0 & 1/\sqrt{t} & 1 \end{bmatrix}, \quad \mathbf{n} = w\,[-\sqrt{s},\ -\sqrt{t},\ 1].$$

The normal vector converges according to

$$\lim_{(s,t)\to(0,0)} \mathbf{n}(s,t) = [0,\ 0,\ 1],$$

and obviously, the projection of $\mathbf{x}$ to the xy-plane is injective so that $\mathbf{x}$ is a C_1^∞-surface. Further,

$$I = \begin{bmatrix} 1+1/s & 1 \\ 1 & 1+1/t \end{bmatrix}, \quad II = \frac{w}{2}\begin{bmatrix} 1/s & 0 \\ 0 & 1/t \end{bmatrix}, \quad W = \frac{w^3}{2}\begin{bmatrix} 1+t & -t \\ -s & 1+s \end{bmatrix}.$$

Hence, the principal curvatures are $\kappa_1 = w/2$ and $\kappa_2 = w^3/2$. For $(s,t) \neq (0,0)$, the corresponding left eigenvectors are unique up to orientation, and we obtain

$$\mathbf{r}_1 = \frac{1}{\sqrt{s+t}}\,[\sqrt{t},\ -\sqrt{s},\ 0], \quad \mathbf{r}_2 = \frac{w}{\sqrt{s+t}}\,[\sqrt{s},\ \sqrt{t},\ s+t].$$

Obviously, these vectors do *not* converge as $(s,t) \to (0,0)$. Rather, the origin is an umbillic point, and any direction in the xy-plane is a principal direction. Now we compute the embedded Weingarten map. With $v := 2+s+t$, we find for $(s,t) \neq (0,0)$

$$D\mathbf{x}^+ = w^2 \begin{bmatrix} \sqrt{s}(1+t) & -\sqrt{s}t \\ -\sqrt{t}s & \sqrt{t}(1+s) \\ s & t \end{bmatrix}, \quad E = \frac{w^5}{2}\begin{bmatrix} 1+vt & -v\sqrt{st} & \sqrt{s} \\ -v\sqrt{st} & 1+vs & \sqrt{t} \\ \sqrt{s} & \sqrt{t} & s+t \end{bmatrix}.$$

Its eigenvalues are $\kappa_1, \kappa_2, 0$, and the corresponding eigenvectors are $\mathbf{r}_1, \mathbf{r}_2, \mathbf{n}$. At the origin, we obtain the limit

$$\lim_{(s,t)\to(0,0)} E(s,t) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Unlike its first two eigenvectors, this matrix is uniquely determined. Because E depends continuously on (s,t) on the whole domain, we can conclude that $\mathbf{x}$ is a C_2^∞-surface, or in other words, it is curvature continuous.

One could argue that also the standard Weingarten map W converges as $(s,t) \to (0,0)$, and that curvature continuity follows equally from that. This is correct, but the situation changes if we attach to $\mathbf{x}$ a second piece of surface

$$\tilde{\mathbf{x}}(s,t) := [-2\sqrt{t},\ 2\sqrt{s},\ s+t], \quad (s,t) \in [0,1]^2$$

with

$$\tilde{\mathbf{n}} = w\,[-\sqrt{t},\, \sqrt{s},\, s{+}t],\ \tilde{W} = \frac{w^3}{2}\begin{bmatrix}1+t & -t\\ -s & 1+s\end{bmatrix},\ \tilde{E} = \frac{w^5}{2}\begin{bmatrix}1+vs & v\sqrt{st} & -\sqrt{t}\\ v\sqrt{st} & 1+vt & \sqrt{s}\\ -\sqrt{t} & \sqrt{s} & s+t\end{bmatrix}.$$

Then $\mathbf{x}$ and $\tilde{\mathbf{x}}$ join normal continuous according to

$$\begin{aligned}\mathbf{x}(0,u) &= \tilde{\mathbf{x}}(u,0) = [0,\ 2\sqrt{u},\ 0], \quad u \in [0,1]\\ \mathbf{n}(0,u) &= \tilde{\mathbf{n}}(u,0) = (1+u)^{-1/2}\,[0,\ -\sqrt{u},\ 1].\end{aligned}$$

The corresponding standard Weingarten maps differ at the common boundary,

$$W(0,u) = \frac{(1+u)^{-3/2}}{2}\begin{bmatrix}1+u & -u\\ 0 & 1\end{bmatrix}, \quad \tilde{W}(u,0) = \frac{(1+u)^{-3/2}}{2}\begin{bmatrix}1 & 0\\ -u & 1+u\end{bmatrix},$$

while the embedded Weingarten maps coincide,

$$E(0,u) = \tilde{E}(u,0) = \frac{(1+u)^{-5/2}}{2}\begin{bmatrix}(1+u)^2 & 0 & 0\\ 0 & 1 & \sqrt{u}\\ 0 & \sqrt{u} & u\end{bmatrix}.$$

From the latter observation, we conclude that the composed surface $\mathbf{X} \cup \tilde{\mathbf{X}}$ is curvature continuous.

The following sketchy considerations concerning the embedded Weingarten map are not directly related to the forthcoming analysis, but perhaps interesting in their own right:

- For a non-parabolic point, i.e, $K_g(\boldsymbol{\sigma}) \neq 0$, we define the pseudo-inverse of E by

  $$E^+ := \mathbf{R}^{\mathrm{t}} K^{-1} \mathbf{R}.$$

  Then the embedded Weingarten map is related to the first and second fundamental form by

  $$I = D\mathbf{n}\,(E^+)^2 \cdot D\mathbf{n}, \quad II = D\mathbf{n}\,E^+ \cdot D\mathbf{n}.$$

  Further, $D\mathbf{n}$ and $D\mathbf{x}$ are related by

  $$D\mathbf{n} = -D\mathbf{x}\,E, \quad D\mathbf{x} = -D\mathbf{n}\,E^+.$$

  Differentiating the last equation with respect to s and t, we find that the *integrability condition* $\mathbf{x}_{st} = \mathbf{x}_{ts}$ is equivalent to

  $$\mathbf{n}_s E^+_t = \mathbf{n}_t E^+_s,$$

  provided that $\mathbf{n}_{st} = \mathbf{n}_{ts}$. Compared with that, the integrability condition for the first and second fundamental form, also known as the Mainardi-Codazzi and Gauss equations, are slightly more complicated.

– If, as before, near a point $\mathbf{x}(\boldsymbol{\sigma})$ the surface $\mathbf{x}$ is locally described as the graph of a local height function h, and if coordinates are chosen such that $\mathbf{x}(\boldsymbol{\sigma}) = \mathbf{0}$ is the origin and the surface normal is the third unit vector, then the quadratic Taylor jet

$$h_2(u,v) = au^2 + 2buv + cv^2 \tag{1}$$

of h is called the *osculating paraboloid* of $\mathbf{x}$ at $\mathbf{x}(\boldsymbol{\sigma})$. This surface can be regarded as a degenerate quadric. Its vertex is the origin, the two main axes in the xy-plane are the principal directions of $\mathbf{x}$, and the corresponding eigenvalues are the principal curvatures. Further, its symmetry axis, which is the main axes corresponding to the eigenvalue 0, is the normal vector $\mathbf{n}(\boldsymbol{\sigma})$. These informations are sufficient to verify that, in implicit form, the osculating paraboloid is the set of all $\mathbf{y} \in \mathbb{R}^3$ satisfying

$$\big((\mathbf{y}-\mathbf{x})E - 2\mathbf{n}\big)\cdot(\mathbf{y}-\mathbf{x}) = 0, \tag{2}$$

where of course, $\mathbf{x}, \mathbf{n}$ and E are evaluated at $\boldsymbol{\sigma}$. The advantage of (2) over (1) is that it does not refer to special coordinates.

– Let $\mathbf{X}$ be a surface given in implicit form,

$$\mathbf{X} := \{\mathbf{y} \in \mathbb{R}^3 : f(\mathbf{y}) = 0\}.$$

A regular point $\mathbf{x} \in \mathbf{X}$ of the surface is characterized by

$$f(\mathbf{x}) = 0, \quad \mathbf{g} := \nabla f(\mathbf{x}) \neq \mathbf{0}.$$

Then $\mathbf{n} := -\mathbf{g}/|\mathbf{g}|$ is the normal vector (or more precisely one of the two normal vectors) of $\mathbf{X}$ at $\mathbf{x}$. The quadratic Taylor expansion of $2f$ at $\mathbf{x}$ reads

$$f_2(\mathbf{y}) = \big((\mathbf{y}-\mathbf{x})H + 2\mathbf{g}\big)\cdot(\mathbf{y}-\mathbf{x}),$$

where H is the Hessian H of f at $\mathbf{x}$. Hence, the quadric $\mathbf{Q} := \{\mathbf{y} : f_2(\mathbf{y}) = 0\}$ is osculating $\mathbf{X}$ at $\mathbf{x}$. Now let us consider the modified function

$$\tilde{f}_2(\mathbf{y}) = \big((\mathbf{y}-\mathbf{x})\tilde{H} + 2\mathbf{g}\big)\cdot(\mathbf{y}-\mathbf{x}),$$

where

$$\tilde{H} := (\mathrm{Id} - N)H(\mathrm{Id} - N), \quad N := \mathbf{n}^{\mathrm{t}}\mathbf{n},$$

yielding the quadric $\tilde{\mathbf{Q}} := \{\mathbf{y} : \tilde{f}_2(\mathbf{y}) = 0\}$. First, we observe that also $\tilde{\mathbf{Q}}$ is osculating $\mathbf{X}$ at $\mathbf{x}$ because $(H - \tilde{H})\cdot\mathbf{t} = 0$ for all vectors $\mathbf{t}$ with $\mathbf{t}\cdot\mathbf{n} = 0$. Second, $\tilde{\mathbf{Q}}$ is a paraboloid with symmetry axes $\mathbf{n}$ because $\tilde{H}\cdot\mathbf{n} = 0$. Hence, $\tilde{\mathbf{Q}}$ is the osculating paraboloid of $\mathbf{X}$ at $\mathbf{x}$, and comparison with (2), using $\mathbf{g} = -|\mathbf{g}|\mathbf{n}$, shows that

$$E := \frac{\tilde{H}}{|\mathbf{g}|}$$

is the embedded Weingarten map of $\mathbf{X}$ at $\mathbf{x}$. The formulas derived here can also be found in the unpublished manuscript [5].

3 Asymptotic Expansions

In this section, we derive asymptotic expansions for the fundamental forms and the embedded Weingarten map of the sequence of spline rings. The details of the setup can be found in [9] or [4].

We consider a subdivision surface $\mathbf{X}$ composed of spline rings

$$\mathbf{x}^m = B\mathbf{P}^m, \quad \mathbf{P}^m = A^m\mathbf{P},$$

where B is a row vector of basis functions forming a partition of unity, $\mathbf{P}$ is a column vector of initial control points, and A is a square *subdivision matrix* with rows summing to one. We assume a standard scheme, i.e., the eigenvalues of A are

$$\lambda_0 = 1, \quad \lambda := \lambda_1 = \lambda_2, \quad \mu := \lambda_3 = \cdots = \lambda_r,$$

where

$$1 > \lambda > \mu > \lambda_i, \quad i > r.$$

Denoting the right eigenvector to λ_i by v_i, we define the *eigenfunctions*

$$f_i := Bv_i$$

and the *eigencoefficients* $\mathbf{q}_i$ by the expansion

$$\mathbf{P} = \sum_i v_i\mathbf{q}_i.$$

Now, we obtain

$$\mathbf{x}^m = \sum_i \lambda_i^m f_i\mathbf{q}_i.$$

Since $f_0 = 1$, we obtain

$$\mathbf{x}^m = \mathbf{q}_0 + \lambda^m[f_1, f_2]\mathbf{Q} + o(\lambda^m), \quad \mathbf{Q} = \begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\end{bmatrix}.$$

The planar spline ring

$$\boldsymbol{\Psi} := [f_1, f_2]$$

is called the *characteristic map* of the scheme [6], and it is well known that $\mathbf{X}$ is a C_1^k-surface for almost all initial data if $\boldsymbol{\Psi}$ is regular and injective. In particular, the *central normal vector* is

$$\mathbf{n}^{\mathrm{c}} := \lim_{m\to\infty} \frac{{}^{\times}\!D\mathbf{x}^m}{|{}^{\times}\!D\mathbf{x}^m|} = \frac{\mathbf{q}_1 \times \mathbf{q}_2}{|\mathbf{q}_1 \times \mathbf{q}_2|},$$

provided that $\mathbf{q}_1, \mathbf{q}_2$ are linearly independent. In the following, we will assume *generic initial data*, i.e.,

$$\det[\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k] \neq 0, \quad i \neq j \neq k,$$

and note that the set of non-generic data has measure zero so that it can be excluded from our analysis without an essential loss of generality. In particular, $\mathbf{q}_1, \mathbf{q}_2$ are linearly independent for generic initial data so that the central normal is well defined.

To efficiently deal with asymptotic expansions, we introduce an equivalence relation for sequences of functions with coinciding leading terms. We write

$$a^m \overset{c^m}{=} b^m \quad \text{iff} \quad a^m - b^m = o(c^m),$$

where $o(c^m)/c^m$ converges uniformly to zero as $m \to \infty$. For example, $a^m \overset{1}{=} a$ means that a^m converges to a. For vector-valued expressions, the equivalence relation is understood component-wise. For simplicity, $\overset{c^m}{=}$ is mostly replaced by the symbol $\doteq$ with the understanding that the dot refers to the lowest order term specified explicitly on the right hand side of a relation. Hence, the expansion of the sequence of spline rings above now simply reads $\mathbf{x}^m \doteq \mathbf{q}_0 + \lambda^m \boldsymbol{\Psi}\mathbf{Q}$, meaning that the omitted remainder term decays faster than λ^m.

For the following analysis of curvature, we assume for the sake of simplicity (but without loss of generality) that Cartesian coordinates are chosen such that $\mathbf{q}_0 = \mathbf{0}$ is the origin and $\mathbf{n}^{\mathrm{c}} = \mathbf{e}_3$ is the third unit vector. Hence, the eigencoefficients $\mathbf{q}_1, \mathbf{q}_2$ lie in the xy-plane, and there exists a (2×2)-matrix L with

$$\mathbf{Q} = L\mathbf{T}, \quad \mathbf{T} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

For generic initial data, $\det L \neq 0$. With $\tilde{\boldsymbol{\Psi}} := \boldsymbol{\Psi} L$ and

$$\varphi := \sum_{\ell=3}^{r} f_\ell \, \mathbf{q}_\ell \cdot \mathbf{n}^{\mathrm{c}},$$

the asymptotic expansion of $\mathbf{x}^m$ involving also the behavior of the normal component reads

$$\mathbf{x}^m \doteq [\lambda^m \tilde{\boldsymbol{\Psi}}, \mu^m \varphi] = \tilde{\mathbf{x}} \operatorname{diag}([\lambda^m, \lambda^m, \mu^m]).$$

The spline ring $\tilde{\mathbf{x}}$ is called the *central surface* of $\mathbf{X}$, and many shape properties of $\mathbf{X}$ are related to those of $\tilde{\mathbf{x}}$, see [4, 3].

From [4], we recall without proof the following expansions:

Theorem 5. *The asymptotic expansions of the partial derivatives and the fundamental forms of the spline rings* $\mathbf{x}^m$ *are*

$$\begin{aligned} D\mathbf{x}^m &\doteq \lambda^m D\tilde{\boldsymbol{\Psi}}\,\mathbf{T} = \lambda^m D\boldsymbol{\Psi} L\mathbf{T} \\ I^m &\doteq \lambda^{2m} I, \quad I^{-1} \doteq \lambda^{-2m} I^{-1}, \quad I := D\tilde{\boldsymbol{\Psi}} \cdot D\tilde{\boldsymbol{\Psi}} \\ I\!I^m &\doteq \mu^m I\!I, \quad I\!I := \sqrt{\frac{\det \tilde{I}}{\det I}}\, \tilde{I\!I}, \end{aligned}$$

where $\tilde{I}$ *and* $\tilde{I\!I}$ *are the fundamental forms of the central surface* $\tilde{\mathbf{x}}$.

In particular, one easily concludes that the sign of the Gaussian curvature of the spline rings and the central surface are asymptotically equal at non-parabolic points,

$$\operatorname{sign} K_g^m = \operatorname{sign}\det II^m \doteq \operatorname{sign}\det \tilde{II} \quad \text{if} \quad \det \tilde{II} \neq 0.$$

Now we are prepared to derive the asymptotic expansion of the embedded Weingarten map of the spline rings.

Theorem 6. *The embedded Weingarten maps E^m of the spline rings $\tilde{\mathbf{x}}$ satisfy*

$$E^m \doteq (\mu/\lambda^2)^m \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad E := D\tilde{\boldsymbol{\Psi}}^{-1} II \cdot D\tilde{\boldsymbol{\Psi}}^{-1}.$$

Proof. By Theorem 5, we obtain for the pseudo-inverse of $D\mathbf{x}^m$

$$(D\mathbf{x}^m)^+ = (D\mathbf{x}^m)^{\mathrm{t}}(I^m)^{-1} \doteq \lambda^{-m}\, \mathbf{T}^{\mathrm{t}} D\tilde{\boldsymbol{\Psi}}^{\mathrm{t}} I^{-1} = \lambda^{-m}\, \mathbf{T}^{\mathrm{t}} D\tilde{\boldsymbol{\Psi}}^{-1}.$$

Hence,

$$\begin{aligned} E^m &= (D\mathbf{x}^m)^+\, II^m \cdot (D\mathbf{x}^m)^+ \doteq (\mu/\lambda^2)^m\, \mathbf{T}^{\mathrm{t}} \big(D\tilde{\boldsymbol{\Psi}}^{-1} II \cdot D\tilde{\boldsymbol{\Psi}}^{-1}\big)\mathbf{T} \\ &= (\mu/\lambda^2)^m\, \mathbf{T}^{\mathrm{t}} E \mathbf{T}, \end{aligned}$$

as stated.

4 Conditions for Curvature Continuity

By Theorem 4, a subdivision surface is C_2^k if and only if $k \geq 2$, and if

$$E^{\mathrm{c}} := \lim_{m\to\infty} E^m$$

exists and is constant. Obviously, a limit can exist only in the following cases:

1. $\mu/\lambda^2 > 1$ and $E = 0$. Although it is often taken for granted that the first condition alone implies divergence of the principal curvatures, it has to be excluded that the second condition cannot be satisfied for generic initial data. This part of the argument, which is not completely trivial, is rarely carried out in detail. The condition $E = 0$ implies $II = \tilde{II} = 0$ since $D\tilde{\boldsymbol{\Psi}} = D\boldsymbol{\Psi}\, L$ has full rank. Let us consider the central surface $\tilde{\mathbf{x}} = [\tilde{\boldsymbol{\Psi}}, \varphi]$. Its second fundamental form vanishes if and only if its trace is part of a plane. Hence, there exists a vector $\tilde{\mathbf{n}}$ with unit length and a number $\tilde{c}$ such that

   $$[1,\, \tilde{\mathbf{x}}] \cdot [\tilde{c},\, \tilde{\mathbf{n}}] = B[e,\, \tilde{v}_1,\, \tilde{v}_2,\, w] \cdot [\tilde{c},\, \tilde{\mathbf{n}}] = 0,$$

   where $e = v_0 = [1; \ldots; 1]$ are the control points of the 1-function, $[\tilde{v}_1, \tilde{v}_2] = [v_1, v_2]L$ are the control points of $\tilde{\boldsymbol{\Psi}}$, and

   $$w := \sum_{\ell=3}^{r} v_\ell\, \mathbf{q}_\ell \cdot \mathbf{n}^{\mathrm{c}}$$

are the control points of φ. Since the functions in B are assumed to form a basis[1], we have

$$[e,\, \tilde v_1,\, \tilde v_2,\, w] \cdot [\tilde c,\, \tilde{\mathbf{n}}] = 0.$$

For generic initial data, $\mathbf{q}_\ell \cdot \mathbf{n}^c \neq 0$ for all $\ell \in \{3, \ldots, r\}$. Hence, linear independence of the eigenvectors $v_0, \ldots, v_r$ implies that the third component $\tilde{\mathbf{n}}_3$ of $\tilde{\mathbf{n}}$ must vanish. This, however, contradicts

$$|\tilde{\mathbf{n}}_3| = \frac{|\det D\tilde{\boldsymbol{\Psi}}|}{|\tilde{\mathbf{x}}_s \times \tilde{\mathbf{x}}_t|} = \frac{|\det D\boldsymbol{\Psi} \cdot \det L|}{|\tilde{\mathbf{x}}_s \times \tilde{\mathbf{x}}_t|} \neq 0.$$

This verifies in a rigorous way that the case $\mu/\lambda^2 > 1$ cannot yield curvature continuity for generic initial data.

2. $\mu/\lambda^2 < 1$. This case yields curvature continuity with $E^c = 0$. However, the enforced flat spot at the central point is not acceptable in most applications.
3. $\mu/\lambda^2 = 1$. This is the well known case of bounded curvature, and one can show that E is constant if and only if

$$f_\ell \in \{f_1^2, f_1 f_2, f_2^2\}, \quad \ell = 3, \ldots, r,$$

i.e., if the subsub-dominant eigenfunctions are quadratic polynomials in the subdominant ones.

Recently, a new class of C_2^2-schemes was devised by Karciauskas and Peters [1, 2] under the label of *guided subdivision*. Generalizing the idea of TURBS [8] in a striking way, these subdivision schemes combine for the first time analytic smoothness with geometric fairness and ease of use. Still, they are more complicated than standard schemes, like the algorithms of Catmull-Clark or Loop, and in particular, according to the results in [7], the polynomial bi-degree of the patches is at least $(6, 6)$ if the setup described here is used. Hence, curvature discontinuous schemes are not yet obsolete, and their shape properties should be further investigated and optimized. As a result of this paper, the embedded Weingarten limit map, given by the symmetric (2×2) matrix E, is the appropriate tool to do that.

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[1] If the functions in B are not linearly independend, the argument given here is not valid unless the matrix A is modified in such a way that *ineffective eigenvectors* are removed [9].

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