On the Dirichlet problem for the prescribed mean curvature equation over nonconvex domains

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Abstract

We study and solve the Dirichlet problem for graphs of prescribed mean curvature H in $\mathbb{R}^{n+1}$ over general domains Ω without requiring a mean convexity assumption. By using pieces of nodoids as barriers we first give sufficient conditions for the solvability in case of zero boundary values. Applying a result by Schulz and Williams we can then also solve the Dirichlet problem for boundary values satisfying a Lipschitz condition.

Introduction

In this paper we study and solve the Dirichlet problem for n-dimensional graphs of prescribed mean curvature in $\mathbb{R}^{n+1}$: Given a domain $\Omega \subset \mathbb{R}^n$ and Dirichlet boundary values $g \in C^0(\partial\Omega, \mathbb{R})$ we want to find a solution $f \in C^2(\Omega, \mathbb{R}) \cap C^0(\overline{\Omega}, \mathbb{R})$ of

$$\operatorname{div} \frac{\nabla f}{\sqrt{1+|\nabla f|^2}} = nH(x, f) \quad \text{in } \Omega \,, \; f = g \quad \text{on } \partial\Omega \,. \tag{1}$$

The given function $H : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is called the prescribed mean curvature. At each point $x \in \Omega$ the geometric mean curvature of the graph f, defined as the average of the principal curvatures, is equal to the value $H(x, f(x))$, thus a solution f is also called a graph of prescribed mean curvature H.

For the minimal surface case, i.e. $H \equiv 0$, it is known that the mean convexity of the domain Ω yields a necessary and sufficient condition for the Dirichlet problem to be solvable for all Dirichlet boundary values (see [7]). Here, mean convexity means that $\hat{H}(x) \geq 0$ for the mean curvature of $\partial\Omega$ w.r.t. the inner normal. For the prescribed mean curvature case, a stronger assumption is needed on the domain Ω in order to solve the boundary value problem for all Dirichlet boundary values g. A necessary condition on the domain Ω and the prescribed mean curvature H is

$$|H(x, z)| \leq \frac{n-1}{n} \hat{H}(x) \quad \text{for } (x, z) \in \partial\Omega \times \mathbb{R} \tag{2}$$

(see [4, Corollary 14.13]). Additionally requiring a smallness condition on H implying the existence of a C^0-estimate, Gilbarg and Trudinger [4, Theorem 16.9] could then solve the Dirichlet problem in case $H = H(x)$.

It is now a natural question to ask if we can relax the mean convexity assumption (2) if we only consider very special boundary values, for example zero boundary values. This is indeed possible, as our first existence result demonstates.

Theorem 1: *Assumptions:*

a) *Let the bounded $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^n$ satisfy a uniform exterior sphere condition of radius $r > 0$ and be included in the annulus $\{x \in \mathbb{R}^n \mid r < |x| < r + d\}$ for some constant $d > 0$.*

b) *Let the prescribed mean curvature $H = H(x, z) \in C^{1+\alpha}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfy $H_z \geq 0$ and the smallness assumption*

$$h := \sup_{\Omega \times \mathbb{R}} |H(x, z)| < \frac{2(2r)^{n-1}}{(2r + d)^n - (2r)^n} \,. \tag{3}$$

Then the Dirichlet problem (1) has a unique solution $f \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ for zero boundary values.

For dimension $n = 2$ and constant mean curvature, similar existence theorems, again for zero boundary values, can be found in [10], [11] or [12]. Note that Theorem 1 can be applied to the annulus $\Omega := \{x \in \mathbb{R}^n \,|\, r < |x| < r + d\}$ which does not satisfy the mean convexity assumption (2). More generally, given any bounded C^2-domain Ω we can find constants $r > 0$ and $d > 0$ such that assumption a) of Theorem 1 is satisfied for a suitable translation of Ω.

The uniqueness part of Theorem 1 follows directly from the assumption $H_z \geq 0$ together with the maximum principle. However, $H_z \geq 0$ is not only needed for the uniqueness but also for the existence of a solution. More precisely, it is needed to show a global gradient estimate for solutions of Dirichlet problem (1) (see Section 2).

Note that some kind of smallness assumption on h in Theorem 1 is needed since there exists the following necessary condition: If there exists a graph of constant mean curvature $h > 0$ over a domain Ω containing a disc of radius $\varrho > 0$, then we have necessarily $h \leq \frac{1}{\varrho}$. This follows from a comparision with spherical caps of constant mean curvature $\frac{1}{\varrho}$ together with the maximum principle. Consequently, the smallness condition on h in Theorem 1 cannot depend on the radius r of the exterior sphere condition alone.
Furthermore, the smallness condition on h also cannot solely depend on the diameter of the domain: Consider the annulus $\Omega = \{x \in \mathbb{R}^n \mid \varepsilon < |x| < 1\}$ for some $0 < \varepsilon < 1$ with $\operatorname{diam}(\Omega) = 2$. A calculation will show that a graph of constant mean curvature $h > 0$ having zero boundary values does not exist if one chooses $\varepsilon > 0$ sufficiently small.

Theorem 1 specifically applies to convex domains. Note that a convex domain satisfies a uniform exterior sphere condition of any radius $r > 0$. By letting $r \to +\infty$, we then obtain the following corollary, which for dimension $n = 2$ and constant mean curvature can also be found in [12, Corollary 3] or [10, Theorem 1.4].

Corollary 1: *Let a bounded convex $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^n$ be given such that $\overline{\Omega}$ is included within the strip $\{x \in \mathbb{R}^n \mid 0 < x_1 < d\}$ of width $d > 0$. Let the prescribed mean curvature $H \in C^{1+\alpha}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfy $H_z \geq 0$ as well as*

$$h := \sup_{\Omega \times \mathbb{R}} |H(x, z)| < \frac{2}{nd} \,.$$

Then the Dirichlet problem (1) has a unique solution $f \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ for zero boundary values.

Note that in Corollary 1 the diameter of the domain Ω can be arbitrarily large, while in Theorem 1 the diameter is bounded by $2(r + d)$.

In case of arbitrary boundary values g, Williams [16] could show that the Dirichlet problem (1) for $H \equiv 0$ is still solvable over domains not being mean convex domains, if one requires certain smallness assumptions on g. More precisely he showed: For any Lipschitz constant $0 \leq L < \frac{1}{\sqrt{n-1}}$ there exists some $\varepsilon = \varepsilon(L, \Omega) > 0$ such that the Dirichlet problem (1) is solvable for the minimal surface equation if the boundary values g satisfy

$$|g(x) - g(y)| \leq L|x - y| \quad \text{for } x, y \in \partial\Omega \quad \text{and} \quad |g(x)| \leq \varepsilon \quad \text{for } x \in \partial\Omega \,. \tag{4}$$

For the proof Williams first considers weak solutions of the minimal surface equation. Constructing suitable barriers he then shows that these weak solutions are continuous up to the boundary and that the Dirichlet boundary values are attained.

Schulz and Williams [15] generalised the result of Williams [16] from the minimal surface case to the prescribed mean curvature case $H = H(x, z)$. However, two more assumptions are needed there: As in Theorem 1, the prescribed mean curvature function H must satisfy the monotonocity assumption $H_z \geq 0$. This assumption is needed for the existence of weak solutions (see [9]). Moreover, they require the existence of an initial solution $f_0 \in C^2(\Omega, \mathbb{R}) \cap C^1(\overline{\Omega}, \mathbb{R})$ for Dirichlet boundary values g_0, which must be Lipschitz continuous with a Lipschitz constant smaller than $\frac{1}{\sqrt{n-1}}$.

Using our solution of Theorem 1 and Corollary 1 as an initial solution with zero boundary values, we can apply the result of Schulz and Williams to solve the Dirichlet problem for Lipschitz continuous boundary values as well:

Theorem 2: *Let the assumptions of Theorem 1 or Corollary 1 be satisfied. Then for any Lipschitz constant* $0 \leq L < \frac{1}{\sqrt{n-1}}$ *there exists some* $\varepsilon = \varepsilon(\Omega, H, L) > 0$ *such that the Dirichlet problem (1) has a solution* $f \in C^{2+\alpha}(\Omega, \mathbb{R}) \cap C^{0,1}(\overline{\Omega}, \mathbb{R})$ *for all Lipschitz continuous boundary values* $g : \partial\Omega \to \mathbb{R}$ *satisfying assumption (4).*

As demonstrated in [15], the smallness assumption on the Lipschitz constant L is sharp. One can construct domains Ω for which the result of Theorem 2 is false for any Lipschitz constant $L > \frac{1}{\sqrt{n-1}}$. For the minimal surface case, this actually holds for all domains Ω which are not mean convex (see [16, Theorem 4]).

This paper is organized as follows: In Section 1 we first we show that solutions satisfy a height as well as a boundary gradient estimate. As barriers we use a piece of a rotationally symmetric surface of constant mean curvature h, a so-called Delaunay nodoid. This surface is constructed in Proposition 1 by solving an ordinary differential equation. There we need a smallness assumption on h corresponding to assumption (3) of Theorem 1. In Section 2 we first give a global gradient estimate in terms of the boundary gradient (see Corollary 2). The monotonocity assumption $H_z \geq 0$ plays an important role there. We then give the proof of Theorem 1 and Corollary 1 using the Leray-Schauder method from [4].

1. Estimates of the height and the boundary gradient

To obtain a priori C^0 estimates as well as boundary gradient estimates for solutions of problem (1), it is essential to have certain super and subsolutions at hand serving us upper and lower barriers. In this paper we will use a rotationally symmetric surface of constant mean curvature h, a so-called Denaunay surface as barrier. For $h = 0$ we have the family of catenoids and for $h \neq 0$ a family consisting of two types of surfaces: the embedded unduloids and the immersed nodoids (see [6]; [8] for $n = 2$). We will now construct a piece of the n-dimensional catenoid (if $h = 0$) and n-dimensional nodoid (if $h \neq 0$) which is given as a graph defined over the annulus

$$\{x \in \mathbb{R}^n \mid r \leq |x| \leq R\} .$$

It can be represented almost explicitely by solving a second order ordinary differential equation.

Proposition 1: *Let the numbers $r > 0$, $h \geq 0$ and $R > r$ be given satisfying*

$$h < \frac{2(2r)^{n-1}}{(R+r)^n - (2r)^n} . \tag{5}$$

Then there exists a function $p \in C^2([r, R], [0, +\infty))$ with $p(r) = 0$ and $p(t) > 0$ for $t \in (r, R]$ such that the rotationally symmetric graph $f(x) := p(|x|)$ defined on the annulus $r \leq |x| \leq R$ has constant mean curvature $-h$. Furthermore, there exists some $t_0 \in (r, R]$ such that $p(t)$ is increasing for $t \in [r, t_0]$ and decreasing for $t \in [t_0, R]$.

Proof:

1.) Inserting $p(|x|) = f(x)$ into the mean curvature equation

$$\operatorname{div} \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} = -nh$$

we obtain for p the second order differential equation

$$\frac{p''}{(1 + p'^2)^{\frac{3}{2}}} + \frac{(n-1)p'}{t(1 + p'^2)^{\frac{1}{2}}} = -nh .$$

Multiplying this equation by t^{n-1} and integrating this yields the first order differential equation

$$\frac{t^{n-1}p'}{\sqrt{1 + p'^2}} = c - ht^n \tag{6}$$

where $c \in \mathbb{R}$ is some integration constant serving as a parameter. We focus here on the case $c > 0$, corresponding to the choice of a nodoid. The case $c = 0$ yields a sphere and $c < 0$ an unduloid. Solving equation (6) for p' we obtain

$$p'(t) = \frac{c - ht^n}{\sqrt{t^{2n-2} - (c - ht^n)^2}} . \tag{7}$$

Clearly, (7) is only well defined for those $t \in (0, +\infty)$ for which the term under the root in the denominator is positive. We will later determine for which t this is the case. Integrating (7) we can now define

$$p(t) := \int_r^t \frac{c - hs^n}{\sqrt{s^{2n-2} - (c - hs^n)^2}} ds \tag{8}$$

with $p(r) = 0$.

2.) Let us first study the case $h = 0$. The denominator of (7) has exactly one zero $a > 0$ given as solution of $a^{n-1} = c$ and $p'(t)$ is defined for all $t \in (a, +\infty)$. For the integral (8) to be defined, we need to have that $r \in (a, +\infty)$, which is equivalent to $c < r^{n-1}$. For example, we can set $c := \frac{1}{2}r^{n-1}$. The function $p(t)$ is now defined for all $t \in [r, +\infty)$ and also $p'(t) > 0$ for all $t \in [r, +\infty)$. The claim of the proposition now follows with $t_0 = R$.

3.) In case $h > 0$, the denominator of (7) has precisely two positive zeros $0 < a < b$ given as solutions of the equations

$$ha^n + a^{n-1} = c \quad , \quad hb^n - b^{n-1} = c \, .$$

Now $p'(t)$ is defined for all $t \in (a, b)$ and formally we have $p'(a) = +\infty$, $p'(b) = -\infty$. Note that for

$$t_0 := \left(c\,h^{-1}\right)^{\frac{1}{n}} \in (a, b)$$

we have

$$p'(t_0) = 0 \quad , \quad p'(t) > 0 \quad \text{for } t \in (a, t_0) \quad \text{and} \quad p'(t) < 0 \quad \text{for } t \in (t_0, b) \; ,$$

as desired. Now for the integral (8) to be defined, we need to have $a < r < t_0$, which is equivalent to restricting the parameter c such that

$$hr^n < c < hr^n + r^{n-1} \, . \tag{9}$$

We then obtain $p \in C^2([r, b), \mathbb{R})$.

4.) We will now show the inequality

$$p'(t_0 - s) > |p'(t_0 + s)| \quad \text{for all } s \in (0, t_0 - a) \, . \tag{10}$$

Together with $p(r) = 0$ this will yield $p(t) > 0$ for all $t \in (r, r + 2(t_0 - r)]$. In fact, after some computation (10) turns out to be equivalent to

$$q(t_0 - s) + q(t_0 + s) > 0 \quad \text{for } s \in (0, t_0 - a)$$

for the function $q(t) := (c - ht^n)t^{1-n} = ct^{1-n} - ht$. This however is a direct consequence of the inequality

$$c(t_0 + s)^{1-n} + c(t_0 - s)^{1-n} > 2ht_0$$

which holds for all $s \in (0, t_0)$, proving (10).

5.) We now set

$$R' = R'(c) := r + 2(t_0 - r) = 2t_0 - r = 2\left(ch^{-1}\right)^{\frac{1}{n}} - r < b \, .$$

From 4.) we conclude the positivity $p(t) > 0$ for all $t \in (r, R']$. Keeping in mind the restriction (9) on c we obtain the limit

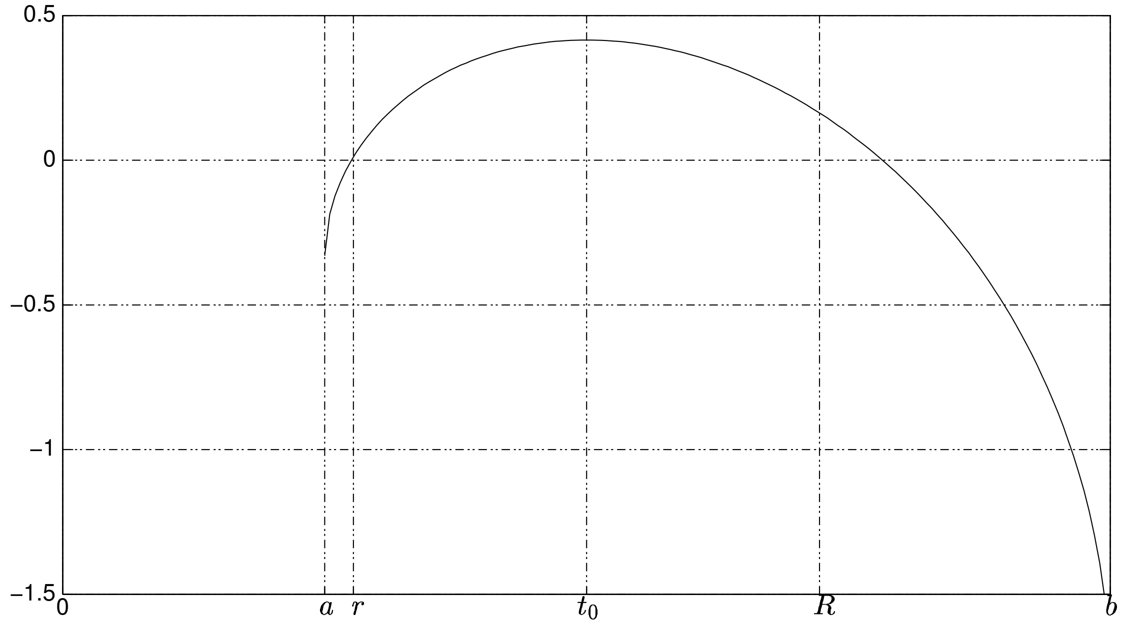
$$R'(c) \to 2\left(r^n + h^{-1}r^{n-1}\right)^{\frac{1}{n}} - r = 2r\left(1 + h^{-1}r^{-1}\right)^{\frac{1}{n}} - r$$

if we let $c \to hr^n + r^{n-1}$. This proves the claim of the proposition whenever

$$R < 2r\left(1 + h^{-1}r^{-1}\right)^{\frac{1}{n}} - r$$

is satisfied. An easy computation, however, asserts that this inequality is indeed equivalent to assumption (5) . □

The following picture shows the graph of the function $p(t)$ for $n = 2$, $h = \frac{1}{3}$, $a = 1$ and $b = 4$.



Remarks:

a) For $h = 0$ and $n = 2$ the function $p(t)$ has the explicite form $p(t) = c\,\mathrm{arcosh}(t/c)$, the well known catenary. If either $h > 0$ or $n \geq 3$ the function $p(t)$ can only be represented by the elliptic integral given in the proof of Proposition 1.

b) In the case $h = 0$ we obtain the n-dimensional catenoid, a rotationally symmetric minimal surface. The generating function is defined for all $t \in [r, +\infty)$. In case $n = 2$ we have $p(t) \to \infty$ as $t \to \infty$. However, for $n \geq 3$ the function $p(t)$ is uniformly bounded by some constant.

c) In case $h > 0$, the maximal domain of definition of the function $p(t)$ is the interval (a, b). In case $n = 2$ one can show that the length $b - a$ of this interval is given by $b - a = \frac{1}{h}$, in particular the length does not depend on the parameter c. This is no longer the case for dimension $n \geq 3$ where $b - a$ depends on both h and c.

We can now show the following a priori estimates of the height and boundary gradient.

Theorem 3: *Assumptions:*

a) Let the bounded $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^n$ satisfy a uniform exterior sphere condition of radius $r > 0$ and be included in the annulus $\{x \in \mathbb{R}^n \mid r < |x| < r + d\}$ for some constant $d > 0$.

b) Let the prescribed mean curvature $H = H(x, z) \in C^{1+\alpha}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfy the smallness assumption $|H(x, z)| \leq h$ for some constant

$$h < \frac{2(2r)^{n-1}}{(2r + d)^n - (2r)^n}.$$

c) Let $f \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ be a solution of problem (1) for zero boundary values.

Then there exists a constant $C = C(h, r, d)$ such that f satisfies the estimates

$$||f||_{C^0(\Omega)} \le C \quad \textit{and} \quad \sup_{\partial\Omega} |\nabla f(x)| \le C\,.$$

Proof:

1.) We first show the C^0-estimate. Since $\Omega \subset \{x \in \mathbb{R}^n : r < |x| < r + d\}$ the rotationally symmetric graph $\eta(x) := p(|x|)$ with constant mean curvature $-h$ is well defined for all $x \in \overline{\Omega}$. Here, $p(t)$ is the function defined by Proposition 1 for $R := r + d$. Noting $\eta(x) \ge 0$ and $f(x) = 0$ on $\partial\Omega$, the maximum principle yields $f(x) \le \eta(x)$ in Ω. Similary, we obtain $f(x) \ge -\eta(x)$. Combining these estimates we have

$$||f||_{C^0(\Omega)} = \sup_{\Omega} |f(x)| \le \sup_{\Omega} |\eta(x)| \le \sup_{r \le t \le r+d} |p(t)| = p(t_0) =: C_1\,.$$

Here, t_0 defined by Proposition 1 is the argument for which the function p achieves its maximum. Note that p only depends on r, d and h and hence $C_1 = C_1(r, d, h)$.

2.) Given some point $x_0 \in \partial\Omega$ we show the boundary gradient estimate at x_0. Since Ω satisfies a uniform exterior sphere condition of radius r, we may assume that

$$\Omega \cap B_r(0) = \emptyset \quad \text{and} \quad x_0 \in \partial B_r(0) \cap \partial\Omega$$

holds after a suitable translation. We define the annulus $U := \{x \in \mathbb{R}^n : r < |x| < t_0\}$ and consider the graph

$$\eta \in C^2(\overline{U}, \mathbb{R}) \quad , \quad \eta(x) := p(|x|) \quad \text{for } x \in \overline{U}\,.$$

From $f(x) = 0$ on $\partial\Omega$ together with $f(x) \le p(t_0) = \eta(x)$ for $|x| = t_0$ we conclude $f(x) \le \eta(x)$ on $\partial(\Omega \cap U)$. The maximum principle gives $f(x) \le \eta(x)$ in $\Omega \cap U$. Similarly we can also conclude that $f(x) \ge -\eta(x)$ in $\Omega \cap U$. From $x_0 \in \partial(\Omega \cap U)$ and $f(x_0) = \eta(x_0)$ we obtain

$$|\nabla f(x_0)| = \left|\frac{\partial}{\partial\nu} f(x_0)\right| \le \left|\frac{\partial}{\partial\nu}\eta(x_0)\right| = |p'(r)| =: C_2\,,$$

where ν is the outward normal to $\partial\Omega$ at x_0. □

Remark: Note that for this result we do not need the monotonocity assumption $H_z \ge 0$. However, we will need this assumption in the next section to prove a global gradient estimate.

2. Global gradient estimate and the proof of Theorem 1

In the previous section we have shown a C^0-estimate together with a boundary gradient estimate, thus we can assume

$$|f(x)| \le M \quad \text{in } \overline{\Omega} \tag{11}$$

for a given solution $f \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ of problem (1). It now remains to establish a global gradient estimate. In case $H = H(x)$, i.e. the prescribed mean curvature does not depend on f, such a global estimate can be found in [4, Theorem 15.2]. Using a differential equation for the normal, we will now demonstrate that such a global estimate also holds in case $H = H(x, z)$, if the monotonocity assumption $H_z \ge 0$ holds. Our argumentation is similar to [1], where the more general case of prescribed anisotropic mean curvature is studied. We first define the upper unit normal vector

$$N = N(x) := \frac{1}{\sqrt{1 + |\nabla f|^2}}(-\nabla f, 1) \quad \text{for } x \in \overline{\Omega}\,.$$

The following parameter invariant differential equation for the normal

$$\triangle N + (\mathrm{tr}(S^2) - n\nabla H \cdot N)N = -n\nabla H \tag{12}$$

was derived in [2, Corollary 4.3]. This equation is a generalisation of [13, Satz 1], where the case $n = 2$ and conformal parameters was studied. Here, $\triangle$ denotes the Laplace-Beltrami operator and $\mathrm{tr}(S^2) = \sum_{i=1}^{n} \kappa_i^2$ where κ_i are the principal curvatures of the surface. We now want to derive a lower bound for the function $\xi(x) := N_{n+1}(x) = N(x) \cdot e_{n+1} > 0$. Once this is achieved, we can use $\xi(x) = (1 + |\nabla f(x)|^2)^{-\frac{1}{2}}$ to derive an upper bound for $|\nabla f|$. Multiplying equation (12) by e_{n+1} we obtain for ξ the equation

$$\triangle \xi + (\mathrm{tr}(S^2) - n\nabla H \cdot N)\xi = -nH_z \; .$$

Then $\mathrm{tr}(S^2) \geq 0$ together with the assumption $H_z \geq 0$ will give the differential inequality

$$\triangle \xi + c\xi \leq 0 \quad \text{in } \Omega \tag{13}$$

with the constant

$$c := -n \sup_{\Omega \times [-M,M]} |\nabla H(x,z)| \leq 0 \; .$$

We now use a product trick $\xi(x) = \psi(x)\, \tilde{\xi}(x)$ for $x \in \overline{\Omega}$ where $\psi \in C^{2+\alpha}(\overline{\Omega}, (0,+\infty))$ is some positive function to be choosen later. Using (13) then $\tilde{\xi}$ must satisfy the differential inequality

$$\psi \triangle \tilde{\xi} + \sum_{i=1}^{n} \tilde{a}_i(x) \tilde{\xi}_{x_i} + \tilde{c}(x) \tilde{\xi} \leq 0 \tag{14}$$

for some coefficients $\tilde{a}_i$ and $\tilde{c} = \triangle \psi + c\psi$. We want the function $\tilde{\xi}$ to achieve its minimum on $\partial\Omega$. By the maximum principle this is guaranteed if $\tilde{c} \geq 0$. Therefore, we show

Proposition 2: *Let $f \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ be a solution of (1) satisfying (11). Let the prescribed mean curvature $H \in C^{1+\alpha}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfy $H_z \geq 0$ and*

$$|H| + |\nabla H| \leq h \quad \text{in } \Omega \times [-M, M]$$

with a constant $h \geq 0$. Then there exists some constant $\lambda = \lambda(h)$ such that the function $\psi(x) := e^{\lambda f(x)}$ for $x \in \overline{\Omega}$ satisfies the differential inequality

$$\triangle \psi + c\psi \geq 0 \quad \text{in } \widehat{\Omega} \tag{15}$$

where $\widehat{\Omega} := \{x \in \Omega \, : \, |\nabla f(x)| \geq 1\}$.

Proof:
We first note $\nabla \psi = \lambda \psi \nabla f$. Using the definition of the Laplace-Beltrami operator, a straightforward computation gives

$$\begin{aligned} \triangle \psi &= \lambda \psi \Big(\triangle f + \lambda \frac{|\nabla f|^2}{1 + |\nabla f|^2} \Big) \\ &= \lambda \psi \Big(\frac{nH(x,f)}{\sqrt{1 + |\nabla f|^2}} + \lambda \frac{|\nabla f|^2}{1 + |\nabla f|^2} \Big) \; . \end{aligned}$$

Here, we have used $\triangle f = nH(1 + |\nabla f|^2)^{-\frac{1}{2}}$, being the $n + 1$-th component of the parameter invariant mean curvature equation $\triangle X = nHN$. We now obtain at each point $x \in \widehat{\Omega}$ the following estimate

$$\triangle \psi + c\psi = \psi \Big(\lambda^2 \frac{|\nabla f|^2}{1 + |\nabla f|^2} + \lambda \frac{nH}{\sqrt{1 + |\nabla f|^2}} + c \Big) \geq \psi \Big(\frac{1}{2} \lambda^2 - nh\lambda - nh \Big) \; .$$

By solving a quadratic inequality we may choose $\lambda = \lambda(h)$ large enough such that $\triangle\psi + c\psi \geq 0$ holds in $\widehat{\Omega}$. □

As a consequence we obtain the following gradient estimate.

Corollary 2: *Given a $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^n$ let $f \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$ be a solution of (1) satisfying (11). Let the prescribed mean curvature $H \in C^{1+\alpha}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ be given satisfying $H_z \geq 0$ and*

$$|H| + |\nabla H| \leq h \quad \text{in } \Omega \times [-M, M] .$$

Then there exists a constant $C = C(M, h) \geq 1$ such that

$$\sup_{x\in\Omega} |\nabla f(x)| \leq C\Big(2 + \sup_{x\in\partial\Omega} |\nabla f(x)|\Big) .$$

Proof:
We consider the last component of the normal

$$\xi(x) := N(x) \cdot e_{n+1} = \frac{1}{\sqrt{1 + |\nabla f(x)|^2}} > 0 .$$

By (14) together with Proposition 2 there exists some constant $\lambda = \lambda(h) > 0$ such that for $\tilde{\xi}(x) = \xi(x)e^{-\lambda f(x)} > 0$ we have the differential inequality

$$\psi\triangle\tilde{\xi} + \sum_{i=1}^n \tilde{a}_i(x)\tilde{\xi}_{x_i} \leq 0 \quad \text{in } \widehat{\Omega} := \{x \in \Omega : |\nabla f(x)| \geq 1\} .$$

We now choose a point $x_0 \in \overline{\Omega}$ where $|\nabla f|$ achieves its maximum within $\overline{\Omega}$. If $x_0 \notin \widehat{\Omega}$ then we have $|\nabla f(x)| \leq |\nabla f(x_0)| \leq 1$ and we are done. Otherwise the maximum principle yields

$$\xi(x_0) \geq e^{-\lambda M}\tilde{\xi}(x_0) \geq e^{-\lambda M} \inf_{\partial\widehat{\Omega}} \tilde{\xi}(x) \geq e^{-2\lambda M} \inf_{\partial\widehat{\Omega}} \xi(x)$$

which gives

$$|\nabla f(x_0)| \leq \sqrt{1 + |\nabla f(x_0)|^2} \leq e^{2M\lambda} \sup_{\partial\widehat{\Omega}} \sqrt{1 + |\nabla f(x)|^2} \leq e^{2M\lambda}(1 + \sup_{\partial\widehat{\Omega}} |\nabla f(x)|) .$$

We now note that $\partial\widehat{\Omega} \subset \partial\Omega \cup \{x \in \Omega : |\nabla f(x)| = 1\}$ which yields

$$\sup_{\partial\widehat{\Omega}} |\nabla f(x)| \leq 1 + \sup_{\partial\Omega} |\nabla f(x)| .$$

The desired estimate now follows for $C := e^{2\lambda M}$. □

We can finally give the

Proof of Theorem 1:
For $t \in [0, 1]$ consider the family of Dirichlet problems

$$f \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}) \quad , \quad \operatorname{div} \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} = t\, nH(x, f) \quad \text{in } \Omega \quad \text{and} \quad f = 0 \quad \text{on } \partial\Omega . \tag{16}$$

Let f be such a solution for some $t \in [0, 1]$. By Theorem 3 together with Corollary 2 we have the estimate

$$||f||_{C^1(\Omega)} \leq C$$

with some constant C independet of t. The Leray-Schauder method [4, Theorem 13.8] yields a solution of the Dirichlet problem (16) for each $t \in [0,1]$. For $t = 1$ we obtain the desired solution of (1). $\square$

Proof of Corollary 1:
Corollary 1 is obtained as the limit case of Theorem 1 by increasing the radius r of the exterior sphere condition to infinity. First, since $\overline{\Omega}$ is bounded and included within the strip $\{x \in \mathbb{R}^n \ : \ 0 < x_1 < d\}$, after a suitable translation it will also be included within the annulus $\{x \in \mathbb{R}^n \ : \ r < |x| < r + d\}$ for sufficiently large $r > 0$. To show which smallness condition on h is required in order to apply Theorem 1 we have to compute the limit

$$\lim_{r \to \infty} \frac{2(2r)^{n-1}}{(2r+d)^n - (2r)^n} \, . \tag{17}$$

To do this, we calculate

$$\lim_{r \to \infty} \frac{(2r+d)^n - (2r)^n}{2(2r)^{n-1}} = \lim_{r \to \infty} \frac{(2r)^n + n(2r)^{n-1}d + O(r^{n-2}) - (2r)^n}{2(2r)^{n-1}} = \frac{nd}{2} \, .$$

We see that the limit in (17) is equal to $\frac{2}{nd}$ and hence the smallness condition $h < \frac{2}{nd}$ is required. Alternatively we could prove Corollary 1 also directly, by proving an analogue result to Theorem 3 for convex domains. Instead of using the nodoid we would then use a cylinder as barrier whose axis is lying in the $x_1, \ldots, x_n$ hyperplane. Note that the cylinder $\{x \in \mathbb{R}^{n+1} \ : x_1^2 + \cdots + x_n^2 = (\frac{d}{2})^2\}$ has constant mean curvature $h = \frac{2}{nd}$, corresponding to the smallness condition from above. $\square$

Remarks:

a) Using the methods from [1], it is also possible to generalise Theorem 1 and Corollary 1 to the case of prescribed anisotropic mean curvature, i.e. $H = H(X, N)$ depends on both the point in space X and the normal N of the graph.

b) The results can also be generalised in another direction: Define the boundary part

$$\Gamma_+ := \Big\{ x \in \partial\Omega \ : \ |H(x,z)| \leq \frac{n-1}{n} \hat{H}(x) \text{ for all } z \in \mathbb{R} \Big\}$$

where $\hat{H}(x)$ is the mean curvature of $\partial\Omega$ at x w.r.t. the inner normal. Now choose a subset $\Gamma \subset \Gamma_+$ such that $\operatorname{dist}(\Gamma, \partial\Omega \backslash \Gamma_+) > 0$. On Γ we can use the standard boundary gradient estimate (see [4, Corollary 14.8]) and prescribe $C^{2+\alpha}$ boundary values g there. Our boundary gradient estimate of Theorem 3, requiring zero boundary values, is then only needed on $\partial\Omega \backslash \Gamma$. This way, Theorem 1 and Corollary 1 also hold for Dirichlet boundary values $g \in C^{2+\alpha}(\partial\Omega, \mathbb{R})$ with $g(x) = 0$ on $\partial\Omega \backslash \Gamma$ and $|g(x)| \leq \varepsilon$, where $\varepsilon = \varepsilon(\Omega, \Gamma, H) > 0$ is some constant determined by the height of the nodoid constructed in Proposition 1.

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