A simple proof for Brouwer's fixed point theorem

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Abstract

Using only basic tools from calculus, we give a relatively simple proof for Brouwer's fixed point theorem.

In this note we will give a relatively simple proof of the following fixed point theorem, known as Brouwer's fixed point theorem in the literature:

Theorem 1: *Let $D := \{x \in \mathbb{R}^n : |x| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$. Then every continuous function $f : D \to D$ has at least one fixed point, i.e. there exists some $x_* \in D$ with $f(x_*) = x_*$.*

The restriction to the ball D is not essential. Once the result is proved for the domain D, it can be generalised directly to any other domain homeomorphic to D, for example the unit cube $C := [0,1]^n$. In contrast to the other fixed point theorems such as the contraction mapping principle, Brouwer's fixed point theorem does not imply the uniqueness of the fixed point. In fact, easy examples such as the identity mapping show that even infinitely many fixed points may be possible. We also want to point out a generalisation from $\mathbb{R}^n$ to Banach spaces is possible, the so-called Schauder fixed point theorem. Instead of continuous mappings on then has to consider completely continuous mappings. The Schauder fixed point theorem is used by Gilbarg and Trudinger in [1, chapter 11.2] to develop the Leray-Schauder method. With this method one can solve the Dirichlet problem for quasilinear elliptic partial differential equations provided certain a priori estimates of solutions are given (see [1, Theorem 11.4]).

1. The proof

Before actually giving the proof of Brouwer's fixed point theorem, we first need some auxiliary lemmas.

Lemma 1: *Let $f, g \in C^2(D, \mathbb{R}^n)$ be two functions with $f = g$ in $D \cap U$ for some open neigborhood U of ∂D. Then we have*

$$\int_D J_f(x)dx = \int_D J_g(x)dx$$

for the Jacobi determinants J_f and J_g of f and g.

Proof:
Given a function $f : D \to \mathbb{R}^n$ we define a vector field $a : D \to \mathbb{R}^n$ by

$$a_k(x) := \det(f_{x_1}(x), \ldots, f_{x_{k-1}}(x), f(x), f_{x_{k+1}}(x), \ldots, f_{x_n}(x)) \quad \text{for } k = 1, \ldots, n\,.$$

In a similar way we define a vector field $b : D \to \mathbb{R}^n$ for the function g. Using the product rule of differentiation we compute the divergence of this vector field

$$\begin{aligned}
\operatorname{div} a(x) &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \det(f_{x_1}, \ldots, f_{x_{i-1}}, f, f_{x_{i+1}}, \ldots, f_{x_n}) \\
&= \sum_{j<i} \det(f_{x_1}, \ldots, f_{x_{j-1}}, f_{x_i x_j}, f_{x_{j+1}}, \ldots, f_{x_{i-1}}, f, f_{x_{i+1}}, \ldots, f_{x_n}) \\
&\quad + n \det(f_{x_1}, \ldots, f_{x_n}) \\
&\quad + \sum_{j>i} \det(f_{x_1}, \ldots, f_{x_{i-1}}, f, f_{x_{j+1}}, \ldots, f_{x_{j-1}}, f_{x_i x_j}, f_{x_{j+1}}, \ldots, f_{x_n}) \\
&= n \det(f_{x_1}, \ldots, f_{x_n}) = n\, J_f(x) \; .
\end{aligned}$$

In a similar way we have $\operatorname{div} b(x) = n\, J_g(x)$. From the assumption $f = g$ in $D \cap U$ for some open neighborhood of U of ∂D we conclude $a = b$ in $D \cap U$. Hence the divergence theorem yields

$$\int_D J_f(x)dx = \frac{1}{n} \int_D \operatorname{div} a(x)dx = \frac{1}{n} \int_D \operatorname{div} b(x)dx = \int_D J_g(x)dx$$

which proves the lemma. □

Remark: For this lemma we have used the following fact: For two vector fields $a, b \in C^1(D, \mathbb{R}^n)$ with $a = b$ in some open neighorhood of ∂D we have $\int_D \operatorname{div} a(x)dx = \int_D \operatorname{div} b(x)dx$. This is a consequence of the divergence theorem, however it can also be proven directly only using Fubini's theorem for iterated integrals and the fundamental theorem of calculus.

Next we need the following generalisation of Lemma 1.

Lemma 2: *Let $f, g \in C^2(D, \mathbb{R}^n)$ be two functions such that $f = g$ on ∂D. Then we have*

$$\int_D J_f(x)dx = \int_D J_g(x)dx \; .$$

Proof:
Let the two functions $f, g \in C^2(D, \mathbb{R}^n)$ with $f = g$ on ∂D be given. We now construct a family of functions $f_\varepsilon \in C^2(D, \mathbb{R}^n)$ for a parameter $\varepsilon > 0$ with the properties:

$$f_\varepsilon(x) = f(x) \quad \text{if } |x| < 1 - 2\varepsilon \quad , \quad f_\varepsilon(x) = g(x) \quad \text{if } |x| > 1 - \varepsilon \; .$$

Additionally, we assume that that f_ε and all its first derivatives are uniformly bounded independent of ε. Such a family can be constructed e.g. by

$$f_\varepsilon(x) = \varrho_\varepsilon(|x|) f(x) + (1 - \varrho_\varepsilon(|x|)) g(x)$$

with a suitable function $\varrho_\varepsilon \in C^2(\mathbb{R}, [0, 1])$. By Lemma 1 we now have

$$\int_D J_{f_\varepsilon}(x)dx = \int_D J_g(x)dx \; .$$

Noting that

$$\int_D J_g(x)dx = \int_{|x|<1-2\varepsilon} J_{f_\varepsilon}(x)dx + \int_{1-2\varepsilon<|x|<1} J_{f_\varepsilon}(x)dx = \int_{|x|<1-2\varepsilon} J_f(x)dx + \int_{1-2\varepsilon<|x|<1} J_{f_\varepsilon}(x)dx$$

we obtain for $\varepsilon \to 0$ the relation

$$\int_D J_f(x)dx = \int_D J_g(x)dx \ ,$$

using the uniform boundedness of f_ε and its derivatives. $\square$

Theorem 2: *Let $f \in C^2(D, \mathbb{R}^n)$ be a function such that $f(x) = x$ for $x \in \partial D$. Then there exists a point $x_* \in D$ with $f(x_*) = 0$.*

Proof:
If $f(x) \neq 0$ for all $x \in D$ then we could define a function $g(x) := \frac{f(x)}{|f(x)|}$ with the regularity $g \in C^2(D, \mathbb{R}^n)$. Note that $g(D) \subset \partial D$ which yields $J_g(x) \equiv 0$ and hence $\int_D J_g(x)dx = 0$. We now define $h(x) := x$ for $x \in D$ and note $g(x) = x = h(x)$ for $x \in \partial D$. From Lemma 2 we conclude $\int_D J_h(x)dx = 0$. On the other hand, we note that $J_h(x) \equiv 1$ and thus $\int_D J_h(x)dx = |D| > 0$, a contradiction. $\square$

Remark: A a direct consequence of this theorem we obtain the retraction lemma: There does not exist a function $f \in C^2(D, \mathbb{R}^n)$ with both $f(D) \subset \partial D$ and $f(x) = x$ for $x \in \partial D$.

We can now give the
Proof of Theorem 1(Brouwer's fixed point theorem):

1.) In the first step we prove the fixed point only for functions $f \in C^2(D, D)$. Assume that f does not have a fixed point. For $x \in D$ let $\lambda \in \mathbb{R}$ be a real number such that $f(x) + \lambda(x - f(x)) \in \partial D$, i.e. λ is solution of the quadratic equation

$$1 = |f(x) + \lambda(x - f(x))|^2 = |x - f(x)|^2\lambda^2 + 2f(x) \cdot (x - f(x))\lambda + |f(x)|^2 \ .$$

Because of $x \in D$, $f(x) \in D$ and $f(x) \neq x$ there always exist two such solutions $\lambda^{\pm}(x)$ with $\lambda^-(x) \leq 0$ and $\lambda^+(x) \geq 1$. We are interested in λ^+ and note that $\lambda^+(x) = 1$ for $x \in \partial D$. From $f \in C^2(D, \mathbb{R}^n)$ we conclude $\lambda^+(x) \in C^2(D, \mathbb{R})$. We define a function

$$F \in C^2(D, \mathbb{R}^n) \quad , \quad F(x) := f(x) + \lambda^+(x)(x - f(x)) \quad \text{for } x \in D$$

and note $F(D) \subset \partial D$ as well as $F(x) = x$ for $x \in \partial D$. However, by Theorem 2 such a function F cannot exist and we obtain a contradiction.

2.) To prove the fixed point theorem also for continuous functions $f \in C^0(D, D)$ we approximate it by a sequence $f^n \in C^2(D, \mathbb{R}^n)$ of functions converging uniformly in D to f. Because of $f(D) \subset D$ we may choose f^n such that $f^n(D) \subset D$ holds. By 1.) there exists a fixed point $x_n \in D$ of f^n. After extracting a convergent subsequence from x_n we have $x_n \to x_*$ for $n \to \infty$ and some $x_* \in D$. Because of the uniform convergence of f^n to f that x_* is a fixed point of f. $\square$

Using Brouwer's fixed point theorem, we can now generalise Theorem 2 to the following

Corollary 1: *Let $f \in C^0(D, \mathbb{R}^n)$ be a mapping with $f(\partial D) \subset \partial D$ and $f(x) \neq -x$ for all $x \in \partial D$. Then f has a zero, i.e. there exists some $x_* \in D$ with $f(x_*) = 0$.*

Proof: Assume to the contrary that $f(x) \neq 0$ in D. Then by Theorem 1 the function $g(x) := -\frac{f(x)}{|f(x)|}$ must have a fixed point x_0 with $g(x_0) = x_0$ and $x_0 \in \partial D$. This however is a contradiction to the assumption $f(x) \neq -x$ for all $x \in \partial D$. $\square$

References

[1] D.Gilbarg, N.S.Trudinger: *Elliptic Partial Differential Equations of Second Order.* Springer, Berlin Heidelberg New York, 1983.

[2] F.Sauvigny: *Partielle Differentialgleichungen der Geometrie und der Physik, Teil 1 und 2.* Springer Berlin Heidelberg, 2004, 2005.

[3] F.Sauvigny: *Partial Differential Equations, Vol. 1 and 2.* Springer Universitext, 2006.

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