Applications of hypocontinuous bilinear maps in infinite-dimensional differential calculus

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Abstract

Paradigms of bilinear maps $\beta \colon E_1 \times E_2 \to F$ between locally convex spaces (like evaluation or composition) are not continuous, but merely hypocontinuous. We describe situations where, nonetheless, compositions of β with Keller C_c^n-maps (on suitable domains) are C_c^n. Our main applications concern holomorphic families of operators, and the foundations of locally convex Poisson vector spaces.

Introduction

If $\beta \colon E_1 \times E_2 \to F$ is a continuous bilinear map between locally convex spaces, then β is smooth and hence $\beta \circ f \colon U \to F$ is smooth for each smooth map $f \colon U \to E_1 \times E_2$ on an open subset U of a locally convex space.

Unfortunately, many bilinear mappings of interest are discontinuous. For example, it is known that the evaluation map $E' \times E \to \mathbb{R}$, $(\lambda, x) \mapsto \lambda(x)$ is discontinuous for each locally convex vector topology on E', if E is a non-normable locally convex space (cf. [25]). Hence also the composition map $L(F, G) \times L(E, F) \to L(E, G)$, $(A, B) \mapsto A \circ B$ is discontinuous, for any non-normable locally convex space F, locally convex spaces $E, G \neq \{0\}$, and any locally convex vector topologies on $L(F, G)$, $L(E, F)$ and $L(E, G)$ such that the maps $F \to L(E, F)$, $y \mapsto y \otimes \lambda$ and $L(E, G) \to G$, $A \mapsto A(x)$ are continuous for some $\lambda \in E'$ and some $x \in E$ with $\lambda(x) \neq 0$, where $(y \otimes \lambda)(z) := \lambda(z)y$ (see Remark 1.21).

Nonetheless, both evaluation and composition do show a certain weakened continuity property, namely *hypocontinuity*. So far, hypocontinuity arguments have been used in differential calculus on Fréchet spaces in some isolated cases (cf. [11], [16] and [34]). In this article, we distill a simple, but useful general principle from these arguments (which is a variant of a result from [32]). Let us call a Hausdorff topological space X a k^∞-*space* if X^n is a k-space for each $n \in \mathbb{N}$. Our observation (recorded in Theorem 2.5) is the following:

If a bilinear map $\beta \colon E_1 \times E_2 \to F$ is hypocontinuous with respect to compact subsets of E_1 or E_2 and $f \colon U \to E_1 \times E_2$ is a C^n-map on an open subset U of a locally convex space X which is a k^∞-space, then $\beta \circ f \colon U \to F$ is a C^n-map.

As a byproduct, we obtain an affirmative solution to an old open problem by Serge Lang (see Corollary 2.6). Our main applications concern two areas.

Application 1: Holomorphic families of operators.

In Section 3, we apply our results to holomorphic families of operators, i.e., holomorphic maps $U \to L(E, F)$ on an open set $U \subseteq \mathbb{C}$ (or $U \subseteq X$ for a suitable locally convex space X). We obtain generalizations (and simpler proofs) for various results formulated in [6].

Application 2: Locally convex Poisson vector spaces.

Finite-dimensional Poisson vector spaces are encountered naturally in finite-dimensional Lie theory as the dual spaces $\mathfrak{g}^*$ of finite-dimensional Lie algebras. They give rise to a distribution on $\mathfrak{g}^*$ whose maximal integral manifolds are the coadjoint orbits of the corresponding simply connected Lie group G. These are known to play an important role in the representation theory of G (by Kirillov's orbit philosophy).

The study of infinite-dimensional Poisson vector spaces (and Poisson manifolds) only began recently with works of Odzijewicz and Ratiu concerning the Banach case (see [29], [30]). On Neeb's initiative, a setting of locally convex Poisson vector spaces (which need not be Banach spaces) has recently been developed ([18], [28]). In Section 4, we preview this framework and explain how hypocontinuity arguments can be used to overcome the analytic problems arising beyond the Banach case. In particular, hypocontinuity is the crucial tool needed to define the Poisson bracket and Hamiltonian vector fields. In Section 5, we describe situations where the Poisson bracket is hypocontinuous or even continuous. In the most relevant case, we also prove continuity of the linear map which takes a smooth function to the corresponding Hamiltonian vector field (Section 6).

Having thus secured the foundations, natural next steps will be the investigation of co-adjoint orbits for examples of non-Banach, infinite-dimensional Lie groups and geometric quantization in this context. These will be undertaken in [18], [28] and later research.

Let us remark that an alternative path leading to Theorem 2.5 is to prove, in a first step, smoothness of hypocontinuous bilinear maps (with respect to compact sets) in an alternative sense, replacing "continuity" by "continuity on each compact subset" in the definition of a C^n-map (see [32, Theorem 4.1]).[1] The second step is to observe that such C^n-maps coincide with ordinary C^n-maps (i.e., Keller C_c^n-maps) if the domain is a k^∞-space.

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1   Preliminaries and basic facts

Throughout this article, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathbb{D} := \{z \in \mathbb{K} \colon |z| \leq 1\}$. As the default, the letters E, E_1, E_2, F and G denote locally convex topological $\mathbb{K}$-vector spaces. When speaking of linear or bilinear maps, we mean $\mathbb{K}$-linear (resp., $\mathbb{K}$-bilinear) maps. A subset $U \subseteq E$ is

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[1] Compare also [31] for the use of Kelleyfications in differential calculus.

called *balanced* if $\mathbb{D}U \subseteq U$. The locally convex spaces considered need not be Hausdorff, but whenever they serve as the domain or range of a differentiable map, we tacitly assume the Hausdorff property. We are working in a setting of infinite-dimensional differential calculus known as Keller's C_c^n-theory (see, e.g., [10] or [19] for streamlined introductions).

Definition 1.1 Let E and F be locally convex spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $U \subseteq E$ be open and $f: U \to F$ be a map. We say that f is $C_{\mathbb{K}}^0$ if f is continuous. The map f is called $C_{\mathbb{K}}^1$ if it is continuous, the limit

$$df(x, y) \;=\; \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}$$

exists for all $x \in U$ and all $y \in E$ (with $0 \neq t \in \mathbb{K}$ sufficiently small), and the map $df: U \times E \to F$ is continuous. Given $n \in \mathbb{N}$, we say that f is $C_{\mathbb{K}}^{n+1}$ if f is $C_{\mathbb{K}}^1$ and $df: U \times E \to F$ is $C_{\mathbb{K}}^n$. We say that f is $C_{\mathbb{K}}^\infty$ if f is $C_{\mathbb{K}}^n$ for each $n \in \mathbb{N}_0$. If $\mathbb{K}$ is understood, we simply write C^n instead of $C_{\mathbb{K}}^n$, for $n \in \mathbb{N}_0 \cup \{\infty\}$.

If $f: E \supseteq U \to F$ is $C_{\mathbb{K}}^1$, then $f'(x) := df(x, \bullet): E \to F$ is a continuous $\mathbb{K}$-linear map, for each $x \in U$. It is known that compositions of composable $C_{\mathbb{K}}^n$-maps are $C_{\mathbb{K}}^n$. Furthermore, continuous linear (or multilinear) maps are $C_{\mathbb{K}}^\infty$ (see [19, Chapter 1], or [10] for all of this).

Remark 1.2 Keller's C_c^n-theory is used as the basis of infinite-dimensional Lie theory by many authors (see [13], [14], [19], [26], [27], [34]). Others use the "convenient calculus" [23].

For some purposes, it is useful to impose certain completeness properties on the locally convex space F involved. These are, in decreasing order of strength: Completeness (every Cauchy net converges); quasi-completeness (every bounded Cauchy net converges); sequential completeness (every Cauchy sequence converges); and Mackey completeness (every Mackey-Cauchy sequence converges, or equivalently: the Riemann integral $\int_0^1 \gamma(t)\, dt$ exists in F, for each smooth curve $\gamma: \mathbb{R} \to F$; see [23, Theorem 2.14] for further information).

Remark 1.3 If $\mathbb{K} = \mathbb{C}$, then a map $f: E \supseteq U \to F$ is $C_{\mathbb{C}}^\infty$ if and only if it is *complex analytic* in the usual sense (as in [4]), i.e., f is continuous and for each $x \in U$, there exists a 0-neighbourhood $Y \subseteq U - x$ and continuous homogeneous polynomials $p_n: E \to F$ of degree n such that $f(x + y) = \bigcup_{n=0}^\infty p_n(y)$ for all $y \in Y$. Such maps are also called holomorphic. If F is Mackey complete, then f is $C_{\mathbb{C}}^1$ if and only if it is $C_{\mathbb{C}}^\infty$ (see [3, Propositions 7.4 and 7.7] or [19, Chapter 1] for all of this; cf. [10]). For suitable non-Mackey complete F, there are $C_{\mathbb{C}}^n$-maps $\mathbb{C} \to F$ for all $n \in \mathbb{N}$ which are not $C_{\mathbb{C}}^{n+1}$ ([15], [20]).

Remark 1.4 If $f: U \to F$ is a map from an open subset of $\mathbb{K}$ to a locally convex space, then f is $C_{\mathbb{K}}^1$ in the above sense if and only if the (real, resp. complex) derivative $f^{(1)}(x) = f'(x) = \frac{df}{dx}(x)$ exists for each $x \in U$, and $f': U \to F$ is continuous. Likewise, f is $C_{\mathbb{K}}^n$ if it has continuous derivatives $f^{(k)}: U \to F$ for all $k \in \mathbb{N}_0$ such that $k \leq n$ (where $f^{(k)} := (f^{(k-1)})'$). This is easy to see (and spelled out in [19, Chapter 1]). If $\mathbb{K} = \mathbb{C}$ here, then complex analyticity of f simply means that f can be expressed in the form $f(z) = \sum_{n=0}^\infty (z - z_0)^n a_n$ close to each given point $z_0 \in U$, for suitable elements $a_n \in F$.

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Remark 1.5 Consider a map $f\colon U \to F$ from an open set $U \subseteq \mathbb{C}$ to a Mackey complete locally convex space F. Replacing sequential completeness with Mackey completeness in [4, Theorem 3.1] and its proof, one finds that also each of the following conditions is equivalent to f being a $C_{\mathbb{C}}^{\infty}$-map (see also [20, Chapter II], notably Theorems 2.2, 2.3 and 5.5):[2]

(a) f is weakly holomorphic, i.e. $\lambda \circ f\colon U \to \mathbb{C}$ is holomorphic for each $\lambda \in F'$;

(b) $\int_{\partial \Delta} f(\zeta)\, d\zeta = 0$ for each triangle $\Delta \subseteq U$;

(c) $f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta$ for each $z_0 \in U$, $r > 0$ such that $z_0 + r\mathbb{D} \subseteq U$, and each z in the interior of the disk $z_0 + r\mathbb{D}$.

Definition 1.6 Given locally convex spaces E and F, we let $L(E,F)$ be the vector space of all continuous linear maps $A\colon E \to F$. If $\mathcal{S}$ is a set of bounded subsets of E, we write $L(E,F)_{\mathcal{S}}$ for $L(E,F)$, equipped with the topology of uniform convergence on the sets $M \in \mathcal{S}$. Finite intersections of sets of the form

$$\lfloor M, U \rfloor \ := \ \{A \in L(E,F)\colon A(M) \subseteq U\}$$

(for $M \in \mathcal{S}$ and $U \subseteq F$ a 0-neighbourhood) form a basis for the filter of 0-neighbourhoods of this vector topology. See [5, Chapter III, §3] for further information. Given $M \subseteq E$ and $N \subseteq E'$, we write $M^{\circ} := \lfloor M, \mathbb{D} \rfloor \subseteq E'$ and $^{\circ}N := \{x \in E\colon (\forall \lambda \in N)\ \lambda(x) \in \mathbb{D}\}$ for the polar in E' (resp., in E).

Remark 1.7 If F is Hausdorff and $F \neq \{0\}$, then $L(E,F)_{\mathcal{S}}$ is Hausdorff if and only $\bigcup_{M \in \mathcal{S}} M$ is total in E, i.e., it spans a dense vector subspace.

In fact, totality of $\bigcup \mathcal{S}$ is sufficient for the Hausdorff property by Proposition 3 in [5, Chapter III, §3, no. 2]. If $V := \mathrm{span}_{\mathbb{K}}(\bigcup \mathcal{S})$ is not dense in E, the Hahn-Banach Theorem provides a linear functional $0 \neq \lambda \in E'$ such that $\lambda|_V = 0$. We pick $0 \neq y \in F$. Then $y \otimes \lambda \in \lfloor M, U \rfloor$ for each $M \in \mathcal{S}$ and 0-neighbourhood $U \subseteq F$, whence $y \otimes \lambda \in W$ for each 0-neighbourhood $W \subseteq L(E,F)_{\mathcal{S}}$. Since $y \otimes \lambda \neq 0$, $L(E,F)_{\mathcal{S}}$ is not Hausdorff.

Proposition 1.8 *Given a separately continuous bilinear map $\beta\colon E_1 \times E_2 \to F$ and a set $\mathcal{S}$ of bounded subsets of E_2, consider the following conditions:*

(a) *For each $M \in \mathcal{S}$ and each 0-neighbourhood $W \subseteq F$, there exists a 0-neighbourhood $V \subseteq E_1$ such that $\beta(V \times M) \subseteq W$.*

(b) *The mapping $\beta^{\vee}\colon E_1 \to L(E_2, F)_{\mathcal{S}}$, $x \mapsto \beta(x, \bullet)$ is continuous.*

(c) *$\beta|_{E_1 \times M}\colon E_1 \times M \to F$ is continuous, for each $M \in \mathcal{S}$.*

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[2]Considering f as a map into the completion of F, we see that (a), (b) and (c) remain equivalent if F is not Mackey complete. But (a)–(c) do not imply that f is $C_{\mathbb{C}}^1$ ([20, Example II.2.3], [15, Theorem 1.1]).

Then (a) *and* (b) *are equivalent, and* (a) *implies* (c)*. If*

$$(\forall M \in \mathcal{S})\,(\exists N \in \mathcal{S})\quad \mathbb{D}M \subseteq N\,,\tag{1}$$

then all of (a)–(c) *are equivalent.*

Definition 1.9 A bilinear map β which is separately continuous and satisfies the equivalent conditions (a) and (b) of Proposition 1.8 is called $\mathcal{S}$-*hypocontinuous* (in the second argument), or simply *hypocontinuous* if $\mathcal{S}$ is clear from the context. Hypocontinuity in the first argument with respect to a set of bounded subsets of E_1 is defined analogously.

Proof of Proposition 1.8. For the equivalence (a)$\Leftrightarrow$(b) and the implication (b)$\Rightarrow$(c), see Proposition 3 and 4 in [5, Chapter III, §5, no. 3], respectively.

We now show that (c)$\Rightarrow$(a) if (1) is satisfied. Given $M \in \mathcal{S}$ and 0-neighbourhood $W \subseteq F$, by hypothesis we can find $N \in \mathcal{S}$ such that $\mathbb{D}M \subseteq N$. By continuity of $\beta|_{E_1 \times N}$, there exist 0-neighbourhoods V in E_1 and U in E_2 such that $\beta(V \times (N \cap U)) \subseteq W$. Since M is bounded, $M \subseteq nU$ for some $n \in \mathbb{N}$. Then $\frac{1}{n}M \subseteq N \cap U$. Using that β is bilinear, we obtain $\beta((\frac{1}{n}V) \times M) = \beta(V \times (\frac{1}{n}M)) \subseteq \beta(V \times (N \cap U)) \subseteq W$. $\qquad\square$

Remark 1.10 By Proposition 1.8 (b), $\mathcal{S}$-hypocontinuity of $\beta\colon E_1 \times E_2 \to F$ only depends on the topology on $L(E_2, F)_{\mathcal{S}}$, not on $\mathcal{S}$ itself. Given $\mathcal{S}$, define $\mathcal{S}' := \{\mathbb{D}M\colon M \in \mathcal{S}\}$. Then the topologies on $L(E_2, F)_{\mathcal{S}}$ and $L(E_2, F)_{\mathcal{S}'}$ coincide (as is clear), and hence β is $\mathcal{S}$-hypocontinuous if and only if β is $\mathcal{S}'$-hypocontinuous. After replacing $\mathcal{S}$ with $\mathcal{S}'$, we can therefore always assume that (1) is satisfied, whenever this is convenient.

Each continuous bilinear map is hypocontinuous (as condition (a) in Proposition 1.8 is easy to check), but the converse is false. The next proposition compiles various useful facts.

Proposition 1.11 *Let* $\beta\colon E_1 \times E_2 \to F$ *be an* $\mathcal{S}$-*hypocontinuous bilinear map, for some set* $\mathcal{S}$ *of bounded subsets of* E_2*. Then the following holds.*

(a) $\beta(B \times M)$ *is bounded in* F*, for each bounded subset* $B \subseteq E_1$ *and each* $M \in \mathcal{S}$*.*

(b) *Assume that, for each convergent sequence* $(y_n)_{n \in \mathbb{N}}$ *in* E_2*, with limit* y*, there exists* $M \in \mathcal{S}$ *such that* $\{y_n\colon n \in \mathbb{N}\} \cup \{y\} \subseteq M$*. Then* β *is sequentially continuous.*

The condition described in (b) *is satisfied, for example, if* $\mathcal{S}$ *is the set of all bounded subsets of* E_2*, or the set of all compact subsets of* E_2*.*

Proof. (a) See Proposition 4 in [5, Chapter III, §5, no. 3].
 (b) See [22, p. 157, Remark following §40, 1., (5)]. $\qquad\square$

In many cases, separately continuous bilinear maps are automatically hypocontinuous. To make this precise, we recall that a subset B of a locally convex space E is called a *barrel* if it is closed, convex, balanced and absorbing. The space E is called *barrelled* if every barrel is a 0-neighbourhood. See Proposition 6 in [5, Chapter III, §5, no. 3] for the following fact:

Proposition 1.12 *If $\beta\colon E_1 \times E_2 \to F$ is a separately continuous bilinear map and E_1 is barrelled, then β is hypocontinuous with respect to any set $\mathcal{S}$ of bounded subsets of E_2.* $\square$

A simple fact will be useful.

Lemma 1.13 *Let M be a topological space, F be a locally convex space, and $BC(M, F)$ be the space of bounded F-valued continuous functions on M, equipped with the topology of uniform convergence. Then the evaluation map $\mu\colon BC(M, F) \times M \to F$, $\mu(f, x) := f(x)$ is continuous.*

Proof. Let (f_α, x_α) be a convergent net in $BC(M, F) \times X$, convergent to (f, x), say. Then $\mu(f_\alpha, x_\alpha) - \mu(f, x) = (f_\alpha(x_\alpha) - f(x_\alpha)) + (f(x_\alpha) - f(x))$, where $f_\alpha(x_\alpha) - f(x_\alpha) \to 0$ as $f_\alpha \to f$ uniformly and $f(x_\alpha) - f(x) \to 0$ as f is continuous. $\square$

We now turn to two paradigmatic bilinear maps, namely evaluation and composition.

Proposition 1.14 *Let E and F be locally convex spaces and $\mathcal{S}$ be a set of bounded subsets of E which covers E, i.e., $\bigcup_{M \in \mathcal{S}} M = E$. Then the evaluation map*

$$\varepsilon\colon L(E, F)_\mathcal{S} \times E \to F, \quad \varepsilon(A, x) := A(x)$$

is hypocontinuous in the second argument with respect to $\mathcal{S}$. If E is barrelled, then ε is also hypocontinuous in the first argument, with respect to any locally convex topology $\mathcal{O}$ on $L(E, F)$ which is finer than the topology of pointwise convergence, and any set $\mathcal{T}$ of bounded subsets of $(L(E, F), \mathcal{O})$.

Proof. By Remark 1.10, we may assume that $\mathcal{S}$ satisfies (1). Given $A \in L(E, F)$, we have $\varepsilon(A, \bullet) = A$, whence ε is continuous in the second argument. It is also continuous in the first argument, as the topology on $L(E, F)_\mathcal{S}$ is finer than the topology of pointwise convergence, by the hypothesis on $\mathcal{S}$. Let $M \in \mathcal{S}$ now. As $L(E, F)$ is equipped with the topology of uniform convergence on the sets in $\mathcal{S}$, the restriction map $\rho\colon L(E, F) \to BC(M, F)$, $A \mapsto A|_M$ is continuous. By Lemma 1.13, the evaluation map $\mu\colon BC(M, F) \times M \to F$ is continuous. Now $\varepsilon|_{L(E,F) \times M} = \mu \circ (\rho \times \mathrm{id}_M)$ shows that $\varepsilon|_{L(E,F) \times M}$ is continuous. Since we assume (1), the implication "(c)$\Rightarrow$(a)" in Proposition 1.8 shows that ε is $\mathcal{S}$-hypocontinuous.

Since $\mathcal{O}$ is finer than the topology of pointwise convergence, the map ε remains separately continuous in the situation described at the end of the proposition. Hence, if E is barrelled, Proposition 1.12 ensures hypocontinuity with respect to $\mathcal{T}$. $\square$

While it was sufficient so far to consider an individual set $\mathcal{S}$ of bounded subsets of a given locally convex space, we now frequently wish to select such a set $\mathcal{S}$ simultaneously for each space. The following definition captures the situations of interest.

Definition 1.15 A *bounded set functor* is a functor $\mathcal{S}$ from the category of locally convex spaces to the category of sets, with the following properties:

(a) $\mathcal{S}(E)$ is a set of bounded subsets of E, for each locally convex space E.

(b) If $A\colon E \to F$ is a continuous linear map, then $A(M) \in \mathcal{S}(F)$ for each $M \in \mathcal{S}(E)$, and $\mathcal{S}(A)\colon \mathcal{S}(E) \to \mathcal{S}(F)$ is the map taking $M \in \mathcal{S}(E)$ to its image $A(M)$ under A.

Given a bounded set functor $\mathcal{S}$ and locally convex spaces E and F, we write $L(E, F)_{\mathcal{S}}$ as a shorthand for $L(E, F)_{\mathcal{S}(E)}$. Also, an $\mathcal{S}(E_2)$-hypocontinuous bilinear map $\beta\colon E_1 \times E_2 \to F$ will simply be called $\mathcal{S}$-hypocontinuous in the second argument.

Example 1.16 Bounded set functors are obtained if $\mathcal{S}(E)$ denotes the set of all bounded, (quasi-)compact, or finite subsets of E, respectively. We then write b, c, resp., p for $\mathcal{S}$.

Further examples abound: For instance, we can let $\mathcal{S}(E)$ be the set of precompact subsets of E, or the set of metrizable compact subsets (if only Hausdorff spaces are considered).

Remark 1.17 If $\mathcal{S}$ is a bounded set functor, and $A\colon E \to F$ a continuous linear map between locally convex spaces, then also its adjoint $A'\colon F'_{\mathcal{S}} \to E'_{\mathcal{S}}$, $\lambda \mapsto \lambda \circ A$ is continuous, because $A'(\lfloor A(M), U \rfloor) \subseteq \lfloor M, U \rfloor$ for each $M \in \mathcal{S}(E)$ and 0-neighbourhood $U \subseteq \mathbb{K}$.

The double use of $f'(x)$ (for differentials) and A' (for adjoints) should not cause confusion.

Definition 1.18 If $\mathcal{S}(E)$ contains all finite subsets of E, then $\eta_E(x)\colon E'_{\mathcal{S}} \to \mathbb{K}$, $\lambda \mapsto \lambda(x)$ is continuous for each $x \in E$ and we obtain a linear map $\eta_E\colon E \to (E'_{\mathcal{S}})'$, called the *evaluation homomorphism*. We say that E is *$\mathcal{S}$-reflexive* if $\eta_E\colon E \to (E'_{\mathcal{S}})'_{\mathcal{S}}$ is an isomorphism of topological vector spaces. If $\mathcal{S} = b$, we simply speak of a *reflexive* space; if $\mathcal{S} = c$, we speak of a *Pontryagin reflexive* space. Occasionally, we call E'_b the *strong* dual of E.

See Proposition 9 in [5, Chapter III, §5, no. 5] for the following fact in the three cases described in Example 1.16. It might also be deduced from [22, §40, 5., (6)].

Proposition 1.19 *Let E, F, and G be locally convex spaces and $\mathcal{S}$ be a bounded set functor such that $\mathcal{S}(E)$ covers E and*

$$\forall M \in \mathcal{S}(L(E, F)_{\mathcal{S}}) \; \forall N \in \mathcal{S}(E) \; \exists K \in \mathcal{S}(F)\colon \quad \varepsilon(M \times N) \subseteq K\,, \tag{2}$$

where $\varepsilon\colon L(E, F) \times E \to F$, $(A, x) \mapsto A(x)$. Then the composition map

$$\Gamma\colon L(F, G)_{\mathcal{S}} \times L(E, F)_{\mathcal{S}} \to L(E, G)_{\mathcal{S}}\,, \quad \Gamma(\alpha, \beta) := \alpha \circ \beta$$

is $\mathcal{S}(L(E, F)_{\mathcal{S}})$-hypocontinuous in the second argument. $\qquad\square$

Remark 1.20 If $\mathcal{S} = b$, then condition (2) is satisfied by Proposition 1.11 (a). If $\mathcal{S} = p$, then $\varepsilon(M \times N)$ is finite and thus (2) holds. If $\mathcal{S} = c$, then $\varepsilon|_{M \times N}$ is continuous since $\varepsilon\colon L(E, F)_{\mathcal{S}} \times E \to F$ is $\mathcal{S}(E)$-hypocontinuous by Proposition 1.14. Hence $\varepsilon(M \times N)$ is compact (and thus (2) is satisfied).

Proof of Proposition 1.19. Γ *is continuous in the second argument*: Let $M \in \mathcal{S}(E)$, $U \subseteq G$ be a 0-neighbourhood, and $A \in L(F, G)$. Then $A^{-1}(U)$ is a 0-neighbourhood in F. For $B \in L(E, F)$, we have $A(B(M)) \subseteq U$ if and only if $B(M) \subseteq A^{-1}(U)$, showing that $\Gamma(A, \lfloor M, A^{-1}(U) \rfloor) \subseteq \lfloor M, U \rfloor$. Hence, $\Gamma(A, \bullet)$ being linear, it is continuous.

Continuity in the first argument: Let $U \subseteq G$ be a 0-neighbourhood, $B \in L(E, F)$ and $M \in \mathcal{S}(E)$. Then $B(M) \in \mathcal{S}(F)$ by Definition 1.15 (b) and $\Gamma(\lfloor B(M), U \rfloor, B) \subseteq \lfloor M, U \rfloor$.

To complete the proof, let $M \in \mathcal{S}(L(E, F)_{\mathcal{S}})$, $U \subseteq G$ be a 0-neighbourhood and $N \in \mathcal{S}(E)$. By (2), there exists $K \in \mathcal{S}(F)$ such that $\varepsilon(M \times N) \subseteq K$. Note that for all $B \in M$ and $A \in \lfloor K, U \rfloor$, we have $\Gamma(A, B).N = (A \circ B)(N) = A(B(N)) \subseteq A(K) \subseteq U$. Thus $\Gamma(\lfloor K, U \rfloor \times M) \subseteq \lfloor N, U \rfloor$. Since $\lfloor K, U \rfloor$ is a 0-neighbourhood in $L(F, G)_{\mathcal{S}}$, condition (a) of Proposition 1.8 is satisfied. □

Remark 1.21 Despite the hypocontinuity of the composition map Γ, it is discontinuous in the situations specified in the introduction. To see this, pick $\lambda \in E'$ and $x \in E$ as described in the introduction. Let $0 \neq z \in G$ and give F' the topology induced by $F' \to L(F, G)$, $\zeta \mapsto z \otimes \zeta$. There is $\mu \in G'$ such that $\mu(z) \neq 0$. If Γ was continuous, then also the following map would be continuous: $F' \times F \to \mathbb{K}$, $(\zeta, y) \mapsto \mu(\Gamma(z \otimes \zeta, y \otimes \lambda)(x)) = \mu(z)\lambda(x)\zeta(y)$. But this map is a non-zero multiple of the evaluation map and hence discontinuous [25].

2 Differentiability properties of compositions with hypocontinuous bilinear mappings

In this section, we introduce a new class of topological spaces ("k^∞-spaces"). We then discuss compositions of hypocontinuous bilinear maps with C^n-maps on open subsets of locally convex spaces which are k^∞-spaces.

Recall that a Hausdorff topological space X is called a *k-space* if, for every subset $A \subseteq X$, the set A is closed in X if and only if $A \cap K$ is closed in K for each compact subset $K \subseteq X$. Equivalently, a subset $U \subseteq X$ is open in X if and only if $U \cap K$ is open in K for each compact subset $K \subseteq X$. It is clear that closed subsets, as well as open subsets of k-spaces are again k-spaces when equipped with the induced topology. If X is a k-space, then a map $f \colon X \to Y$ to a topological space Y is continuous if and only if $f|_K \colon K \to Y$ is continuous for each compact subset $K \subseteq X$, as is easy to see.[3] This property is crucial for the following. It can also be interpreted as follows: $X = \varinjlim_K K$ as a topological space.

Definition 2.1 We say that a topological space X is a k^∞-*space* if it is Hausdorff and its n-fold power $X^n = X \times \cdots \times X$ is a k-space, for each $n \in \mathbb{N}$.

Example 2.2 It is well known (and easy to prove) that every metrizable topological space is a k-space. Finite powers of metrizable spaces being metrizable, we see: *Every metrizable topological space is a k^∞-space.*

---

[3]Given a closed set $A \subseteq Y$, the intersection $f^{-1}(A) \cap K = (f|_K)^{-1}(A)$ is closed in K in the latter case and thus $f^{-1}(A)$ is closed, whence f is continuous.

Example 2.3 A Hausdorff topological space X is called a k_ω-*space* if it is a k-space and *hemicompact*,[4] i.e., there exists a sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets of X such that $X = \bigcup_{n \in \mathbb{N}} K_n$ and each compact subset of X is contained in some K_n. Since finite products of k_ω-spaces are k_ω-spaces (see, e.g., [17, Proposition 4.2 (c)]), it follows that each k_ω-space is a k^∞-space. For an introduction to k_ω-spaces, the reader may consult [17].

Remark 2.4 We remark that E'_c is a k_ω-space (and hence a k^∞-space), for each metrizable locally convex space E (see [1, Corollary 4.7 and Proposition 5.5]). In particular, every Silva space E is a k_ω-space (and hence a k^∞-space), i.e., every locally convex direct limit $E = \varinjlim E_n$ of an ascending sequence $E_1 \subseteq E_2 \subseteq \cdots$ of Banach spaces, such that the inclusion maps $E_n \to E_{n+1}$ are compact operators (see [14, Example 9.4]).

Having set up the terminology, let us record a simple, but useful observation.

Theorem 2.5 *Let* $n \in \mathbb{N}_0 \cup \{\infty\}$. *If* $n = 0$, *let* $U = X$ *be a topological space. If* $n \geq 1$, *let* X *be a locally convex space and* $U \subseteq X$ *be an open subset. Let* $\beta \colon E_1 \times E_2 \to F$ *be a bilinear map and* $f \colon U \to E_1 \times E_2$ *be a* C^n-*map. Assume that at least one of* (a), (b) *holds:*

(a) X *is metrizable and* β *is sequentially continuous; or:*

(b) X *is a* k^∞-*space and* β *is hypocontinuous in the second argument with respect to a set* $\mathcal{S}$ *of bounded subsets of* E_2 *which contains all compact subsets of* E_2.

Then $\beta \circ f \colon U \to F$ *is* C^n.

Proof. It suffices to consider the case where $n < \infty$. The proof is by induction.

We assume (a) first. If $n = 0$, let $(x_k)_{k \in \mathbb{N}}$ be a convergent sequence in U, with limit x. Then $f(x_k) \to x$ by continuity of f and hence $\beta(f(x_k)) \to \beta(f(x))$, since β is sequentially continuous.

Now let $n \geq 1$ and assume that the assertion holds if n is replaced with $n-1$. Given $x \in U$ and $y \in X$, let $(t_k)_{k \in \mathbb{N}}$ be a sequence in $\mathbb{K} \setminus \{0\}$ such that $x + t_k y \in U$ for each $k \in \mathbb{N}$, and $\lim_{k \to \infty} t_k = 0$. Write $f = (f_1, f_2)$ with $f_j \colon U \to E_j$. Then

$$
\frac{\beta(f(x + t_k y)) - \beta(f(x))}{t_k}
$$
$$
= \; \beta\left(\frac{f_1(x + t_k y) - f_1(x)}{t_k}, f_2(x + t_k y)\right) + \beta\left(f_1(x), \frac{f_2(x + t_k y) - f_2(x)}{t_k}\right)
$$
$$
\to \; \beta(df_1(x, y), f_2(x)) + \beta(f_1(x), df_2(x, y)) \quad \text{as } k \to \infty,
$$

by continuity of f and sequential continuity of β. Hence the limit $d(\beta \circ f)(x, y) = \lim_{t \to 0} \frac{\beta(f(x+ty)) - \beta(f(x))}{t}$ exists, and is given by

---

[4]An equivalent definition runs as follows: A Hausdorff space X is a k_ω-space if and only if $X = \varinjlim K_n$ for an ascending sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of X with union X.

$$d(\beta \circ f)(x, y) \;=\; \beta(df_1(x, y), f_2(x)) + \beta(f_1(x), df_2(x, y)). \tag{3}$$

The mappings $g_1, g_2 \colon U \times X \to E_1 \times E_2$ defined via $g_1(x, y) := (df_1(x, y), f_2(x))$ and $g_2(x, y) := (f_1(x), df_2(x, y))$ are $C_{\mathbb{K}}^{n-1}$. Since

$$d(\beta \circ f) \;=\; \beta \circ g_1 + \beta \circ g_2 \tag{4}$$

by (3), we deduce from the inductive hypotheses that $d(\beta \circ f)$ is $C_{\mathbb{K}}^{n-1}$ and hence continuous. Thus $\beta \circ f$ is $C_{\mathbb{K}}^1$ with $d(\beta \circ f)$ a $C_{\mathbb{K}}^{n-1}$-map and hence $\beta \circ f$ is $C_{\mathbb{K}}^n$, which completes the inductive proof in the situation of (a).

In the situation of (b), let $K \subseteq U$ be compact. Then $f_2(K) \subseteq E_2$ is compact and hence $f_2(K) \in \mathcal{S}$, by hypothesis. Since $\beta|_{E_1 \times f_2(K)}$ is continuous by Proposition 1.8 (c), we see that $(\beta \circ f)|_K = \beta|_{E_1 \times f_2(K)} \circ f|_K$ is continuous. Since X and hence also its open subset U is a k-space, it follows that $\beta \circ f$ is continuous, settling the case $n = 0$.

Now let $n \geq 1$ and assume that the assertion holds if n is replaced with $n - 1$. Since β is sequentially continuous by Proposition 1.11 (b), we see as in case (a) that the directional derivative $d(\beta \circ f)(x, y)$ exists, for all $(x, y) \in U \times X$, and that $d(\beta \circ f)$ is given by (4). Since g_1 and g_2 are $C_{\mathbb{K}}^{n-1}$, the inductive hypothesis can be applied to the summands in (4). Thus $d(\beta \circ f)$ is $C_{\mathbb{K}}^{n-1}$, whence $\beta \circ f$ is $C_{\mathbb{K}}^1$ with $d(\beta \circ f)$ a $C_{\mathbb{K}}^{n-1}$-map, and so $\beta \circ f$ is $C_{\mathbb{K}}^n$. $\square$

Combining Proposition 1.19 and Theorem 2.5, as a first application we obtain an affirmative answer to an open question formulated by Serge Lang [24, p. 8, Remark].

Corollary 2.6 *Let U be an open subset of a Fréchet space, E, F and G be Fréchet spaces, and $f \colon U \to L(E, F)_b$, $g \colon U \to L(F, G)_b$ be continuous maps. Then also the mapping $U \to L(E, G)_b$, $x \mapsto g(x) \circ f(x)$ is continuous.* $\square$

3 Holomorphic families of operators

In this section, we compile conclusions from the previous results and some useful additional material. Specializing to the case $X := \mathbb{K} := \mathbb{C}$ and $n := \infty$, we obtain results concerning holomorphic families of operators, i.e., holomorphic maps $U \to L(E, F)_{\mathcal{S}}$, where U is an open subset of $\mathbb{C}$. Among other things, such holomorphic families are of interest for representation theory and cohomology (see [6] and [7]).

Proposition 3.1 *Let X, E, F and G be locally convex spaces over $\mathbb{K}$, such that X is a k^∞-space (e.g. $X = \mathbb{K} = \mathbb{C}$). Let $U \subseteq X$ be open, $n \in \mathbb{N}_0 \cup \{\infty\}$ and $f \colon U \to L(E, F)_{\mathcal{S}}$ as well as $g \colon U \to L(F, G)_{\mathcal{S}}$ be $C_{\mathbb{K}}^n$-maps, where $\mathcal{S} = b$ or $\mathcal{S} = c$. Then also the map $U \to L(E, G)_{\mathcal{S}}$, $z \mapsto g(z) \circ f(z)$ is $C_{\mathbb{K}}^n$.*

Proof. The composition map $L(F, G)_{\mathcal{S}} \times L(E, F)_{\mathcal{S}} \to L(E, G)_{\mathcal{S}}$ is hypocontinuous with respect to $\mathcal{S}(L(E, F)_{\mathcal{S}})$, by Proposition 1.19. Hence Theorem 2.5 (b) applies. $\square$

The remainder of this section is devoted to the proof of the following result. Here $\mathcal{S}$ is a bounded set functor such that $\mathcal{S}(E)$ covers E, for each locally convex space E.

Proposition 3.2 *Let E, F and X be locally convex spaces over $\mathbb{K}$. If $\eta_F\colon F \to (F'_{\mathcal{S}})'_{\mathcal{S}}$ is continuous, then $g\colon U \to L(F'_{\mathcal{S}}, E'_{\mathcal{S}})_{\mathcal{S}}$, $z \mapsto f(z)'$ is $C^n_{\mathbb{K}}$, for each $n \in \mathbb{N}_0 \cup \{\infty\}$ and $C^n_{\mathbb{K}}$-map $f\colon U \to L(E, F)_{\mathcal{S}}$ on an open subset $U \subseteq X$.*

The proof of Proposition 3.2 exploits the continuity of the formation of adjoints.

Proposition 3.3 *Let E, F be locally convex spaces and $\mathcal{S}$ be a bounded set functor such that $\mathcal{S}(F)$ covers F. If the evaluation homomorphism $\eta_F\colon F \to (F'_{\mathcal{S}})'_{\mathcal{S}}$ is continuous, then*

$$\Psi\colon L(E, F)_{\mathcal{S}} \to L(F'_{\mathcal{S}}, E'_{\mathcal{S}})_{\mathcal{S}}, \quad \alpha \mapsto \alpha' \tag{5}$$

is a continuous linear map.

Proof. After replacing $\mathcal{S}(V)$ with $\{r_1 M_1 \cup \cdots \cup r_n M_n \colon r_1, \ldots, r_n \in \mathbb{K}, M_1, \ldots, M_n \in \mathcal{S}(V)\}$ for each locally convex space V (which does not change $\mathcal{S}$-topologies), we may assume that $\mathcal{S}(E)$ is closed under finite unions and multiplication with scalars. Let $M \in \mathcal{S}(F'_{\mathcal{S}})$ and $U \subseteq E'_{\mathcal{S}}$ be a 0-neighbourhood; we have to show that $\Psi^{-1}(\lfloor M, U \rfloor)$ is a 0-neighbourhood in $L(E, F)_{\mathcal{S}}$. After shrinking U, without loss of generality $U = N^{\circ}$ for some $N \in \mathcal{S}(E)$ (by our special hypothesis concerning $\mathcal{S}(E)$). For $\alpha \in L(E, F)$, we have

$$
\begin{aligned}
\alpha' \in \lfloor M, U \rfloor \quad &\Leftrightarrow \quad (\forall \lambda \in M) \ \lambda \circ \alpha = \alpha'(\lambda) \in U = N^{\circ} \\
&\Leftrightarrow \quad (\forall \lambda \in M)\,(\forall x \in N) \ |\lambda(\alpha(x))| \leq 1 \\
&\Leftrightarrow \quad \alpha(N) \subseteq {}^{\circ}M \\
&\Leftrightarrow \quad \alpha \in \lfloor N, {}^{\circ}M \rfloor.
\end{aligned}
$$

Since $M \in \mathcal{S}(F'_{\mathcal{S}})$ and η_F is continuous, the polar ${}^{\circ}M = \eta_F^{-1}(M^{\circ})$ is a 0-neighbourhood in F. Thus $\lfloor N, {}^{\circ}M \rfloor = \Psi^{-1}(\lfloor M, U \rfloor)$ is a 0-neighbourhood in $L(E, F)_{\mathcal{S}}$. $\square$

Proof of Proposition 3.2. Since f is a $C^n_{\mathbb{K}}$-map and Ψ in Proposition 3.3 is continuous linear and hence a $C^{\infty}_{\mathbb{K}}$-map, also $g = \Psi \circ f$ is $C^n_{\mathbb{K}}$. $\square$

The locally convex spaces E such that $\eta_E\colon E \to (E'_b)'_b$ is continuous are known as "quasi-barrelled" spaces. They can characterized easily. Recall that a subset A of a locally convex space E is called *bornivorous* if it absorbs all bounded subsets of E. The space E is called *bornological* if every convex, balanced, bornivorous subset of E is a 0-neighbourhood. See Proposition 2 in [21, §11.2] (and the lines following it) for the following simple fact:

The evaluation homomorphism $\eta_E\colon E \to (E'_b)'_b$ is continuous if and only if each closed, convex, balanced subset $A \subseteq E$ which absorbs all bounded subsets of E (i.e., each bornivorous barrel A) is a 0-neighbourhood in E.

Thus $\eta_E\colon E \to (E'_b)'_b$ is continuous if E is bornological or barrelled. It is also known that $\eta_E\colon E \to (E'_c)'_c$ is continuous if E is a k-space (cf. [1, Corollary 5.12 and Proposition 5.5]).

We mention that Proposition 3.1 generalizes [6, Lemma 2.4], where $X = \mathbb{K} = \mathbb{C}$, $\mathcal{S} = b$, E is assumed to be a Montel space and E, F G are complete (see last line of [6, p. 637]).

The method of proof used in *loc. cit.* depends on completeness properties of $L(E, G)_b$, because the characterization of holomorphic functions via Cauchy integrals (as in Remark 1.5 (c)) requires Mackey completeness. Proposition 3.2 generalizes [6, Lemma 2.3], where $X = \mathbb{K} = \mathbb{C}$, $\mathcal{S} = b$ and continuity of η_F is presumed as well (penultimate sentence of their proof), and whose proof is valid whenever $L(F', E')_b$ is at least Mackey complete (since only weak holomorphicity of g is checked there).

4 Locally convex Poisson vector spaces

We now consider locally convex Poisson vector spaces in a framework which arose from [18]. Fundamental facts concerning such spaces will be proved, using hypocontinuity as a tool.

4.1 Throughout this section, we let $\mathcal{S}$ be a bounded set functor such that the following holds for each locally convex space E:

(a) $\mathcal{S}(E)$ contains all compact subsets of E; and:

(b) For each $M \in \mathcal{S}(E'_\mathcal{S})$ and $N \in \mathcal{S}(E)$, the set $\varepsilon(M \times N) \subseteq \mathbb{K}$ is bounded, where $\varepsilon \colon E' \times E \to \mathbb{K}$ is the evaluation map.

Since $\mathcal{S}(\mathbb{K})$ contains all compact sets and each bounded subset of $\mathbb{K}$ is contained in a compact set, condition (b) means that there exists $K \in \mathcal{S}(\mathbb{K})$ such that $\varepsilon(M \times N) \subseteq K$.

Definition 4.2 An $\mathcal{S}$-*reflexive locally convex Poisson vector space* is a locally convex space E which is $\mathcal{S}$-reflexive and a k^∞-space, together with an $\mathcal{S}$-hypocontinuous bilinear map $[.,.] \colon E'_\mathcal{S} \times E'_\mathcal{S} \to E'_\mathcal{S}$, $(\lambda, \eta) \mapsto [\lambda, \eta]$ which makes $E'_\mathcal{S}$ a Lie algebra.

Of course, we are mostly interested in the case where $[.,.]$ is continuous, but only $\mathcal{S}$-hypocontinuity is required for the basic results described below.

Remark 4.3 We mainly have two choices of $\mathcal{S}$ in mind.

(a) The case $\mathcal{S} = b$. If E is a Hilbert space, a reflexive Banach space, a nuclear Fréchet space, or the strong dual of a nuclear Fréchet space, then both reflexivity is satisfied and also the k^∞-property (by Example 2.2 and Remark 2.4).[5]

(b) If $\mathcal{S} = c$, then the scope widens considerably. For example, every Fréchet space E is both Pontryagin reflexive (see [2, Propositions 15.2 and 2.3]) and a k^∞-space (see Example 2.2), and the same holds for E'_c (see [1, Proposition 5.9] and Remark 2.4).

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[5]Let E be a nuclear Fréchet space. Then E is Pontryagin reflexive [2, Propositions 15.2 and 2.3]. By [2, Theorem 16.1] and [1, Proposition 5.9], also E'_c is nuclear and Pontryagin reflexive. Since E is metrizable and hence a k-space, E'_c is complete [1, Proposition 4.11]. We now see with [33, Proposition 50.2] that every closed, bounded subset of E (and E'_c) is compact. Hence $E'_b = E'_c$, $(E'_b)'_b = (E'_c)'_c$ and both E and $E'_b = E'_c$ are also reflexive. By Remark 2.4, $E'_b = E'_c$ is a k^∞-space.

Remark 4.4 The most typical examples of $\mathcal{S}$-reflexive locally convex Poisson vector spaces are dual spaces of topological Lie algebras. More precisely, let $\mathcal{S} = b$ or $\mathcal{S} = c$, and $(\mathfrak{g}, [.,.]_{\mathfrak{g}})$ be a locally convex topological Lie algebra. If $\mathfrak{g}$ is $\mathcal{S}$-reflexive and $\mathfrak{g}'_{\mathcal{S}}$ happens to be a k^{∞}-space, then $E := \mathfrak{g}'_{\mathcal{S}}$ is an $\mathcal{S}$-reflexive locally convex Poisson vector space with Lie bracket defined via $[\lambda, \mu] := [\eta_{\mathfrak{g}}^{-1}(\lambda), \eta_{\mathfrak{g}}^{-1}(\mu)]_{\mathfrak{g}}$ for $\lambda, \mu \in E' = (\mathfrak{g}'_{\mathcal{S}})'$, using the isomorphism $\eta_{\mathfrak{g}} \colon \mathfrak{g} \to (\mathfrak{g}'_{\mathcal{S}})'_{\mathcal{S}}$. Here are typical examples.

(a) If $\mathfrak{g}$ is a Banach-Lie algebra whose underlying Banach space is reflexive, then $\mathfrak{g}'_b$ is a *reflexive* locally convex Poisson vector space (i.e., w.r.t. $\mathcal{S} = b$); see Remark 4.3 (a).

(b) If $\mathfrak{g}$ is a Fréchet-Lie algebra (a topological Lie algebra which is a Fréchet space), then $\mathfrak{g}'_c$ is a *Pontryagin reflexive* locally convex Poisson vector space (i.e., with respect to $\mathcal{S} = c$), by Remark 4.3 (b).

(c) If $\mathfrak{g}$ is a Silva-Lie algebra, then $\mathfrak{g}$ is reflexive (hence also Pontryagin reflexive), and $\mathfrak{g}'_b = \mathfrak{g}'_c$ is a Fréchet-Schwartz space (see [9]) and hence a k^{∞}-space. Therefore $\mathfrak{g}'_b = \mathfrak{g}'_c$ is a reflexive and Pontryagin reflexive locally convex Poisson vector space.

If a topological group G is a projective limit $\varprojlim G_n$ of a projective sequence $\cdots \to G_2 \to G_1$ of finite-dimensional Lie groups, then $\mathfrak{g} := \varprojlim L(G_n) \cong \mathbb{R}^{\mathbb{N}}$ can be considered as the Lie algebra of G and coadjoint orbits of G in $\mathfrak{g}'$ can be studied [28], where $\mathfrak{g}'_c \ (= \mathfrak{g}'_b)$ is a Pontryagin reflexive (and reflexive) locally convex Poisson vector space, by (b).

If a group G is a union $\bigcup_{n \in \mathbb{N}} G_n$ of finite-dimensional Lie groups $G_1 \subseteq G_2 \subseteq \cdots$, then G can be made an infinite-dimensional Lie group with Lie algebra $\mathfrak{g} = \varinjlim L(G_n) \cong \mathbb{R}^{(\mathbb{N})}$ (see [13]), where $\mathfrak{g}'_c \ (= \mathfrak{g}'_b)$ is a Pontryagin reflexive (and reflexive) locally convex Poisson vector space, by (c). Again coadjoint orbits can be studied [18]. Manifold structures on them do not pose problems, since all homogeneous spaces of G are manifolds [13, Proposition 7.5].

Given a Lie algebra $(\mathfrak{g}, [.,.])$ and $x \in \mathfrak{g}$, we write $\mathrm{ad}_x := \mathrm{ad}(x) := [x, .] \colon \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x, y]$.

Definition 4.9 can be adapted to spaces which are not $\mathcal{S}$-reflexive, along the lines of [29], [30]:

Definition 4.5 A *locally convex Poisson vector space* with respect to $\mathcal{S}$ is a locally convex space E that is a k^{∞}-space and whose evaluation homomorphism $\eta_E \colon E \to (E'_{\mathcal{S}})'_{\mathcal{S}}$ is a topological embedding, together with an $\mathcal{S}$-hypocontinuous bilinear map $[.,.] \colon E'_{\mathcal{S}} \times E'_{\mathcal{S}} \to E'_{\mathcal{S}}$, $(\lambda, \eta) \mapsto [\lambda, \eta]$ which makes $E'_{\mathcal{S}}$ a Lie algebra, and such that

$$\eta_E(x) \circ \mathrm{ad}_{\lambda} \in \eta_E(E) \quad \text{for all } x \in E \text{ and } \lambda \in E'. \tag{6}$$

Identifying E with $\eta_E(E) \subseteq (E'_{\mathcal{S}})'_{\mathcal{S}}$, we can rewrite (6) as

$$(\mathrm{ad}_{\lambda})'(E) \subseteq E \quad \text{for all } \lambda \in E'. \tag{7}$$

Remark 4.6 Every $\mathcal{S}$-reflexive Poisson vector space $(E, [.,.])$ in the sense of Definition 4.2 also is a Poisson vector space with respect to $\mathcal{S}$, in the sense of Definition 4.5. In fact, since $[.,.]$ is separately continuous, the linear map $\mathrm{ad}_{\lambda} = [\lambda, .] \colon E'_{\mathcal{S}} \to E'_{\mathcal{S}}$ is continuous, for each $\lambda \in E'_{\mathcal{S}}$. Hence $\alpha \circ \mathrm{ad}_{\lambda} \in (E'_{\mathcal{S}})' = \eta_E(E)$ for each $\alpha \in (E'_{\mathcal{S}})'$, and thus (6) is satisfied.

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Remark 4.7 If $\mathcal{S} = b$, then $\eta_E\colon E \to (E'_b)'_b$ is a topological embedding if and only η_E is continuous, i.e., if and only if E is quasi-barrelled in the sense recalled in Section 3 (see [21, §11.2]). Most locally convex spaces of practical interest are bornological or barrelled and hence quasi-barrelled.

If $\mathcal{S} = c$, then η_E is a topological embedding *automatically* in the situation of Definition 4.5 as we assume that E is a k^∞-space (and hence a k-space). In fact, $\eta_E\colon E \to (E'_c)'_c$ is injective (by the Hahn-Banach theorem) for each locally convex space E, and open onto its image (cf. [1, Proposition 6.10] or [2, Lemma 14.3]). Hence $\eta_E\colon E \to (E'_c)'_c$ is an embedding if and only if it is continuous, which holds if E is a k-space (cf. [2, Lemma 14.4]).

Remark 4.8 Since reflexive Banach spaces are rather rare, the more complicated non-reflexive theory cannot be avoided in the study of Banach-Lie-Poisson vector spaces (as in [29], [30]). By contrast, typical non-Banach locally convex spaces are reflexive and hence fall within the simple, basic framework of Definition 4.2. And the class of Pontryagin reflexive spaces is even more comprehensive.

Definition 4.9 Let $(E, [.,.])$ be a locally convex Poisson vector space with respect to $\mathcal{S}$, and $U \subseteq E$ be open. Given $f, g \in C^\infty_\mathbb{K}(U, \mathbb{K})$, we define a function $\{f, g\}\colon U \to \mathbb{K}$ via

$$\{f, g\}(x) := \langle [f'(x), g'(x)], x \rangle \quad \text{for } x \in U, \tag{8}$$

where $\langle .,. \rangle\colon E' \times E \to \mathbb{K}$, $\langle \lambda, x \rangle := \lambda(x)$ is the evaluation map and $f'(x) = df(x,.)$.

Condition (6) in Definition 4.5 enables us to define a map $X_f\colon U \to E$ via

$$X_f(x) := \eta_E^{-1}\big(\eta_E(x) \circ \mathrm{ad}(f'(x))\big) \quad \text{for } x \in U, \tag{9}$$

where $\eta_E\colon E \to (E'_\mathcal{S})'_\mathcal{S}$ is the evaluation homomorphism.

Theorem 4.10 *Let $(E, [.,.])$ be a locally convex Poisson vector space with respect to $\mathcal{S}$ and $U \subseteq E$ be an open subset. Then*

(a) $\{f, g\} \in C^\infty_\mathbb{K}(U, \mathbb{K})$, *for all $f, g \in C^\infty_\mathbb{K}(U, \mathbb{K})$.*

(b) *For each $f \in C^\infty_\mathbb{K}(U, \mathbb{K})$, the map $X_f\colon U \to E$ is $C^\infty_\mathbb{K}$.*

The following fact will help us to prove Theorem 4.10.

Lemma 4.11 *Let E and F be locally convex spaces, $U \subseteq E$ be open and $f\colon U \to F$ be a $C^\infty_\mathbb{K}$-map. Then also the map $f'\colon U \to L(E, F)_\mathcal{S}$, $x \mapsto f'(x) = df(x, \bullet)$ is $C^\infty_\mathbb{K}$, for each set $\mathcal{S}$ of bounded subsets of E such that $L(E, F)_\mathcal{S}$ is Hausdorff.*

Proof. For $\mathcal{S} = b$, see [16]. The general case is a trivial consequence of the case $\mathcal{S} = b$. $\square$

Proof of Theorem 4.10. (a) The maps $f'\colon U \to L(E, \mathbb{K})_\mathcal{S} = E'_\mathcal{S}$ and $g'\colon U \to E'_\mathcal{S}$ are $C^\infty_\mathbb{K}$ by Lemma 4.11, E is a k^∞-space by hypothesis, and $[.,.]$ is $\mathcal{S}$-hypocontinuous. Hence

$h := [.,.] \circ (f', g') \colon U \to E'_{\mathcal{S}}$, $x \mapsto [f'(x), g'(x)]$ is $C^\infty_{\mathbb{K}}$, by Theorem 2.5 (b). The evaluation map $\varepsilon \colon E'_{\mathcal{S}} \times E \to \mathbb{K}$ is $\mathcal{S}$-hypocontinuous in the second argument by Proposition 1.14, and the inclusion map $\iota \colon U \to E$ is $C^\infty_{\mathbb{K}}$. Hence $\{f, g\} = \varepsilon \circ (h, \iota)$ is $C^\infty_{\mathbb{K}}$, by Theorem 2.5 (b).

(b) Since the bilinear map $[.,.] \colon E'_{\mathcal{S}} \times E'_{\mathcal{S}} \to E'_{\mathcal{S}}$ is $\mathcal{S}$-hypocontinuous, the linear map $[.,.]^\vee \colon E'_{\mathcal{S}} \to L(E'_{\mathcal{S}}, E'_{\mathcal{S}})_{\mathcal{S}}$, $\lambda \mapsto [\lambda, .] = \mathrm{ad}(\lambda)$ is continuous, by Proposition 1.8 (b). Hence $h \colon U \to L(E'_{\mathcal{S}}, E'_{\mathcal{S}})_{\mathcal{S}}$, $h(x) := \mathrm{ad}(f'(x))$ is $C^\infty_{\mathbb{K}}$, using Lemma 4.11. The composition map $\Gamma \colon (E'_{\mathcal{S}})'_{\mathcal{S}} \times L(E'_{\mathcal{S}}, E'_{\mathcal{S}})_{\mathcal{S}} \to (E'_{\mathcal{S}})'_{\mathcal{S}}$, $(\alpha, A) \mapsto \alpha \circ A$ is $\mathcal{S}$-hypocontinuous in the second argument, because condition (b) in §4.1 ensures that Proposition 1.19 can be applied. Then $V := \{A \in L(E'_{\mathcal{S}}, E'_{\mathcal{S}})_{\mathcal{S}} \colon (\forall x \in E)\, \eta_E(x) \circ A \in \eta_E(E)\}$ is a vector subspace of $L(E'_{\mathcal{S}}, E'_{\mathcal{S}})_{\mathcal{S}}$ and the bilinear map $\Theta \colon E \times V \to E$, $\Theta(x, A) := \eta_E^{-1}(\Gamma(\eta_E(x), A))$ is $\mathcal{S}$-hypocontinuous in its second argument, using that η_E is an isomorphism of topological vector spaces onto its image. The inclusion map $\iota \colon U \to E$, $x \mapsto x$ being $C^\infty_{\mathbb{K}}$, Theorem 2.5 shows that $X_f = \Theta \circ (\iota, h)$ is $C^\infty_{\mathbb{K}}$. $\qquad\square$

Definition 4.12 Let $(E, [.,.])$ be a locally convex Poisson vector space with respect to $\mathcal{S}$, and $U \subseteq E$ be an open subset.

(a) The map $\{.,.\} \colon C^\infty_{\mathbb{K}}(U, \mathbb{K}) \times C^\infty_{\mathbb{K}}(U, \mathbb{K}) \to C^\infty_{\mathbb{K}}(U, \mathbb{K})$ taking (f, g) to $\{f, g\}$ (as in Definition 4.9) is called the *Poisson bracket*.

(b) Given $f \in C^\infty_{\mathbb{K}}(U, \mathbb{K})$, the map $X_f \colon U \to E$ is a smooth vector field on U, by Theorem 4.10 (b). It is called the *Hamiltonian vector field* associated with f.

Remark 4.13 Basic differentiation rules entail that $\{.,.\}$ makes $C^\infty_{\mathbb{K}}(U, \mathbb{K})$ a Poisson algebra, i.e., $(C^\infty_{\mathbb{K}}(U, \mathbb{K}), \{.,.\})$ is a Lie algebra[6] and $\{f, .\} \colon C^\infty_{\mathbb{K}}(U, \mathbb{K}) \to C^\infty_{\mathbb{K}}(U, \mathbb{K})$, $g \mapsto \{f, g\}$ is a derivation for the commutative, associative $\mathbb{K}$-algebra $(C^\infty_{\mathbb{K}}(U, \mathbb{K}), \cdot)$, for each $f \in C^\infty_{\mathbb{K}}(U, \mathbb{K})$.

5   Continuity properties of the Poisson bracket

If E and F are locally convex spaces, $U \subseteq E$ is an open set and $n \in \mathbb{N}_0 \cup \{\infty\}$, then $C^n_{\mathbb{K}}(U, F)$ carries a natural topology (the "C^n-topology"), namely the initial topology with respect to the maps

$$C^\infty_{\mathbb{K}}(U, F) \to C(U \times E^k, F)_{c.o.} \quad f \mapsto d^k f$$

for $k \in \mathbb{N}_0$ such that $k \leq n$, where the right hand side is equipped with the compact-open topology, $d^0 f := f$ and $d^k f(x, y_1, \ldots, y_k) := (D_{y_k} \cdots D_{y_1} f)(x)$ is defined as an iterated directional derivative, if $k \geq 1$. Our goal is the following result:

Theorem 5.1 *Let $(E, [.,.])$ be a locally convex Poisson vector space with respect to $\mathcal{S} = c$. Let $U \subseteq E$ be open. Then the Poisson bracket*

$$\{.,.\} \colon C^\infty_{\mathbb{K}}(U, \mathbb{K}) \times C^\infty_{\mathbb{K}}(U, \mathbb{K}) \to C^\infty_{\mathbb{K}}(U, \mathbb{K})$$

---

[6]The Jacobi identity can be established as in the proof of [29, Theorem 4.2].

is hypocontinuous with respect to compact subsets of $C_{\mathbb{K}}^{\infty}(U, \mathbb{K})$. If $[.,.] \colon E_c' \times E_c' \to E_c'$ is continuous, then also the Poisson bracket is continuous.

Remark 5.2 The topology on spaces of smooth maps goes along well with the $\mathcal{S}$-topology on spaces of operators if $\mathcal{S} = c$, since it enables to control smooth functions and their differentials on compact sets. For $\mathcal{S} \neq c$ (notably, for $\mathcal{S} = b$), there is no clear connection between the topologies, and one cannot hope for an analogue of Theorem 5.1.

Various auxiliary results are needed to prove Theorem 5.1. With little risk of confusion with subsets of spaces of operators, given a 0-neighbourhood $W \subseteq F$ and a compact set $K \subseteq U$ we shall write $\lfloor K, W \rfloor := \{f \in C(U, F) \colon f(K) \subseteq W\}$.

Lemma 5.3 *Let E, F be locally convex spaces and $U \subseteq E$ be open. Then the linear map*

$$D \colon C_{\mathbb{K}}^{\infty}(U, F) \to C_{\mathbb{K}}^{\infty}(U, L(E, F)_c), \quad f \mapsto f'$$

is continuous.

Proof. The map D is linear and also $C^{\infty}(U, L(E, F)_c) \to C(U \times E^k, L(E, F)_c)$, $f \mapsto d^k f$ is linear, for each $k \in \mathbb{N}_0$. Hence

$$d^k \circ D \colon C^{\infty}(U, F) \to C(U \times E^k, L(E, F)_c)_{c.o.} \tag{10}$$

is linear, whence it will be continuous if it is continuous at 0. We pick a typical 0-neighbourhood in $C(U \times E^k, L(E, F)_c)_{c.o.}$, say $\lfloor K, V \rfloor$ with a compact subset $K \subseteq U \times E^k$ and a 0-neighbourhood $V \subseteq L(E, F)_c$. After shrinking V, we may assume that $V = \lfloor A, W \rfloor$ for some compact set $A \subseteq E$ and 0-neighbourhood $W \subseteq F$.

We now recall that for $f \in C_{\mathbb{K}}^{\infty}(U, F)$, we have

$$d^k(f')(x, y_1, \ldots, y_k) = d^{k+1} f(x, y_1, \ldots, y_k, \bullet) \colon E \to F \tag{11}$$

for all $k \in \mathbb{N}_0$, $x \in U$ and $y_1, \ldots, y_k \in E$ (see [16]). Since $\lfloor K \times A, W \rfloor$ is an open 0-neighbourhood in $C(U \times E^{k+1}, F)$ and the map $C^{\infty}(U, F) \to C(U \times E^{k+1}, F)_{c.o.}$, $f \mapsto d^{k+1} f$ is continuous, we see that the set Ω of all $f \in C^{\infty}(U, F)$ such that $d^{k+1} f \in \lfloor K \times A, W \rfloor$ is a 0-neighbourhood in $C^{\infty}(U, F)$. In view of (11), we have $d^k(f') \in \lfloor K, \lfloor A, W \rfloor \rfloor$ for each $f \in \Omega$. Hence $d^k \circ D$ from (10) is continuous at 0, as required. $\quad\square$

Lemma 5.4 *Let X be a Hausdorff topological space, F be a Hausdorff locally convex space, $K \subseteq X$ be compact and also $M \subseteq C(X, F)_{c.o.}$ be compact. Let $\mathrm{eval} \colon C(X, F) \times X \to F$, $(f, x) \mapsto f(x)$ be the evaluation map. Then $\mathrm{eval}(M \times K)$ is compact.*

Proof. The restriction map $\rho \colon C(X, F)_{c.o.} \to C(K, F)_{c.o.}$, $f \mapsto f|_K$ being continuous by [8, §3.2 (2)], the set $\rho(M)$ is compact in $C(K, F)_{c.o.}$. The evaluation map $\varepsilon \colon C(K, F) \times K \to F$, $(f, x) \mapsto f(x)$ is continuous by [8, Theorem 3.4.2]. Hence also $\mathrm{eval}(M \times K) = \varepsilon(\rho(M) \times K)$ is compact. $\quad\square$

Lemma 5.5 *Let E, F_1, F_2 and G be locally convex spaces, and $\beta\colon F_1 \times F_2 \to G$ be a bilinear map which is hypocontinuous with respect to compact subsets of F_2. Then*

$$C_{\mathbb{K}}^n(U, \beta)\colon C_{\mathbb{K}}^n(U, F_1) \times C_{\mathbb{K}}^n(U, F_2) \to C_{\mathbb{K}}^n(U, G)\,, \quad (f, g) \mapsto \beta \circ (f, g)$$

is hypocontinuous with respect to compact subsets of $C_{\mathbb{K}}^n(U, F_2)$, for each $n \in \mathbb{N}_0 \cup \{\infty\}$. If β is continuous, then also $C_{\mathbb{K}}^n(U, \beta)$ is continuous.

Proof. If β is continuous and hence smooth, then $C^n(U, \beta)$ is smooth and hence continuous, as a very special case of [12, Proposition 4.16].[7]

If β is hypocontinuous, it suffices to prove hypocontinuity of $C^n(U, \beta)$ for each finite n. To see this, let $i_{n,G}\colon\colon C^\infty(U, G) \to C^n(U, G)$ be the inclusion map for $n \in \mathbb{N}_0$, and define i_{n,F_1} and i_{n,F_2} analogously. Since the topology on $C^\infty(U, G)$ is initial with respect to the maps $i_{n,G}$, the restriction $C^\infty(U, \beta)|_Y$ to a subset $Y \subseteq C^\infty(U, F_1) \times C^\infty(U, F_2)$ is continuous if and only if $i_{n,G} \circ C^\infty(U, \beta)|_Y = C^n(U, \beta)|_{\overline{Y}}$ is continuous, where $\overline{Y} := (i_{n,F_1} \times i_{n,F_2})(Y)$. Applying this with $Y = \{f\} \times C^\infty(U, F_2)$, $Y = C^\infty(U, F_1) \times \{g\}$ and $Y = C^\infty(U, F_1) \times M$ with $M \subseteq C^\infty(U, F_2)$ compact, we see that hypocontinuity of each $C^n(U, \beta)$ implies hypocontinuity of $C^\infty(U, \beta)$ (using the characterization given in Proposition 1.8 (c)).

The case $n = 0$. Let $M \subseteq C(U, F_2)$ be compact and consider a typical 0-neighbourhood in $C(U, G)$, say $\lfloor K, W \rfloor$ with $K \subseteq U$ compact and a 0-neighbourhood $W \subseteq G$. By Lemma 5.4, the set $N := \mathrm{eval}(M \times K) \subseteq F_2$ is compact, where $\mathrm{eval}\colon C(U, F_2) \times U \to F_2$ is the evaluation map. Since β is hypocontinuous, there exists a 0-neighbourhood $V \subseteq F_1$ with $\beta(V \times N) \subseteq W$. Then $\beta \circ (\lfloor K, V \rfloor \times M) \subseteq \lfloor K, W \rfloor$ and hence $C(U, \beta)$ is hypocontinuous.

Induction step. Given $n \in \mathbb{N}$, assume that $C^{n-1}(U, \beta)$ is hypocontinuous in the second argument for each U and β. The topology on $C^n(U, G)$ being initial with respect to the linear maps $\lambda_1\colon C^n(U, G) \to C(U, G)_{c.o.}$, $f \mapsto f$ and $\lambda_2\colon C^n(U, G) \to C^{n-1}(U \times E, G)$, $f \mapsto df$ (by [12, Lemma A.1 (d)]), we only need to show that $\lambda_j \circ C^n(U, \beta)$ is hypocontinuous for $j \in \{1, 2\}$. We have $\lambda_1 \circ C^n(U, \beta) = C(U, \beta) \circ (i_1 \times i_2)$, where $i_j\colon C^n(U, F_j) \to C(U, F_j)$ is the inclusion map which is continuous and linear. Since $C(U, \beta)$ is hypocontinuous by the case $n = 0$, we readily deduce that $\lambda_1 \circ C^n(U, \beta)$ is hypocontinuous. The map $\delta_j\colon C^n(U, F_j) \to C^{n-1}(U \times E, F_j)$, $f \mapsto df$ is continuous linear and $\pi\colon U \times E \to U$, $(x, y) \mapsto x$ is smooth, whence $\rho_j\colon C^n(U, F_j) \to C^{n-1}(U \times E, F_j)$, $\rho_j(f) := f \circ \pi$ is continuous linear (cf. [12, Lemma 4.4]). By (4), we have

$$\lambda_2 \circ C^n(U, \beta) \;=\; C^{n-1}(U \times E, \beta) \circ (\delta_1 \times \rho_2) + C^{n-1}(U \times E, \beta) \circ (\rho_1 \times \delta_2)\,. \tag{12}$$

Since $C^{n-1}(U \times E, \beta)\colon C^{n-1}(U \times E, F_1) \times C^{n-1}(U \times E, F_2) \to C^{n-1}(U \times E, G)$ is hypocontinuous by the inductive hypothesis and each summand in (12) is a composition thereof with a direct product of continuous linear maps, we deduce that each summand and hence also $\lambda_2 \circ C^n(U, \beta)$ is hypocontinuous in the second argument. This completes the proof. $\square$

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[7]Note that the ordinary C^n-topology is used there, by [12, Proposition 4.19 (d) and Lemma A2].

Lemma 5.6 *Let E, F and G be locally convex spaces, $U \subseteq E$ be open and $\beta\colon E \times F \to G$ be a bilinear map which is hypocontinuous with respect to compact subsets of E. Then*

$$\beta_*\colon C^n_{\mathbb{K}}(U, F) \to C^n_{\mathbb{K}}(U, G), \quad (\beta_*(f))(x) := \beta(x, f(x)) \quad \text{for } f \in C^n_{\mathbb{K}}(U, F), \ x \in U$$

is a continuous linear map, for each $n \in \mathbb{N}_0 \cup \{\infty\}$.

Proof. It is clear that β_* is linear. We only need to prove the assertion for finite n, by an argument similar to that in the proof of Lemma 5.5. The proof is by induction.

The case $n = 0$. Suppose we are given a 0-neighbourhood in $C(U, G)$, say $\lfloor K, W \rfloor$ with $K \subseteq U$ compact and a 0-neighbourhood $W \subseteq G$. By Proposition 1.8 (a), there exists a 0-neighbourhood $V \subseteq F$ such that $\beta(K \times V) \subseteq W$. Then $\beta_*(\lfloor K, V \rfloor) \subseteq \lfloor K, W \rfloor$. Thus β_* is continuous at 0 and hence continuous, being linear.

Induction step. Let $n \in \mathbb{N}$ and assume that the assertion holds for $n - 1$ in place of n. Let us write $\beta_{*,n}\colon C^n(U, F) \to C^n(U, G)$, for added clarity. The topology on $C^n(U, G)$ being initial with respect to the linear maps $\lambda_1\colon C^n(U, G) \to C(U, G)_{c.o.}$, $f \mapsto f$ and $\lambda_2\colon C^n(U, G) \to C^{n-1}(U \times E, G)$, $f \mapsto df$, we only need to show that $\lambda_j \circ \beta_{*,n}$ is continuous for $j \in \{1, 2\}$. We have $\lambda_1 \circ \beta_{*,n} = \beta_{*,0} \circ i$, where $i\colon C^n(U, F) \to C(U, F)$ is the continuous linear inclusion map and $\beta_{*,0}$ is continuous by the above. To tackle $\lambda_2 \circ \beta_{*,n}$, note that

$$A\colon (E \times E) \times F \to G, \quad A((x, y), z) := \beta(y, z) \quad \text{and}$$
$$B\colon (E \times E) \times F \to G, \quad B((x, y), z) := \beta(x, z)$$

are hypocontinuous with respect to compact subsets of $E \times E$. Let $\pi_1\colon U \times E \to U$ be the projection on the first component. Then the map $p\colon C^n(U, F) \to C^{n-1}(U \times E, F)$, $f \mapsto f \circ \pi_1$ is continuous linear (cf. [12, Lemma 4.4]). By (4), we have

$$\lambda_2 \circ \beta_{*,n} = A_* \circ p + B_* \circ d, \tag{13}$$

where $d\colon C^n(U, F) \to C^{n-1}(U \times E, F)$, $f \mapsto df$ is continuous linear and also the maps $A_*\colon C^{n-1}(U \times E, F) \to C^{n-1}(U \times E, G)$ and $B_*\colon C^{n-1}(U \times E, F) \to C^{n-1}(U \times E, G)$ are continuous linear, by the inductive hypothesis. Hence $\lambda_2 \circ \beta_{*,n}$ is continuous linear. This completes the proof. $\square$

Proof of Theorem 5.1. By Lemma 5.3, the mapping $D\colon C^\infty(U, \mathbb{K}) \to C^\infty(U, E'_c)$, $f \mapsto f'$ is continuous and linear. By Lemma 5.5, the bilinear map

$$C^\infty(U, [.,.])\colon C^\infty(U, E') \times C^\infty(U, E') \to C^\infty(U, E'), \quad (f, g) \mapsto (x \mapsto [f(x), g(x)])$$

is hypocontinuous with respect to compact subsets of the second factor; and if $[.,.]$ is continuous, then also $C^\infty(U, [.,.])$. The evaluation map $\beta\colon E \times E'_c \to \mathbb{K}$, $\beta(x, \lambda) := \lambda(x)$ is hypocontinuous with respect to compact subsets of E by Proposition 1.14. Hence $\beta_*\colon C^\infty(U, E'_c) \to C^\infty(U, \mathbb{K})$, $f \mapsto \beta_*(f) = \beta \circ (\mathrm{id}_U, f)$ is continuous linear by Lemma 5.6. Since

$$\{.,.\} = \beta_* \circ C^\infty(U, [.,.]) \circ (D \times D)$$

by definition, we see that $\{.,.\}$ is a composition of continuous maps if $[.,.]$ is continuous, and hence continuous. In the general case, $\{.,.\}$ is a composition of a hypocontinuous bilinear map and continuous linear maps and hence hypocontinuous. $\square$

6 Continuity of the map taking f to X_f

In this section, we show continuity of the mapping which takes a smooth function to the corresponding Hamiltonian vector field, in the case $\mathcal{S} = c$.

Theorem 6.1 *Let $(E, [.,.])$ be a locally convex Poisson vector space with respect to $\mathcal{S} = c$. Let $U \subseteq E$ be an open subset. Then the map*

$$\Psi \colon C^\infty_{\mathbb{K}}(U, \mathbb{K}) \to C^\infty_{\mathbb{K}}(U, E), \quad f \mapsto X_f \tag{14}$$

is continuous and linear.

Proof. Let $\eta_E \colon E \to (E'_c)'_c$ be the evaluation homomorphism and $V := \{A \in L(E'_c, E'_c) \colon (\forall x \in E)\, \eta_E(x) \circ A \in \eta_E(E)\}$. Then V is a vector subspace of $L(E'_c, E'_c)$ and $\mathrm{ad}(E') \subseteq V$. The composition map $\Gamma \colon (E'_c)'_c \times L(E'_c, E'_c)_c \to (E'_c)'_c$, $(\alpha, A) \mapsto \alpha \circ A$ is hypocontinuous with respect to equicontinuous subsets of $(E'_c)'_c$, by Proposition 9 in [5, Chapter III, §5, no. 5]. If $K \subseteq E$ is compact, then the polar K° is a 0-neighbourhood in E'_c, entailing that $(K^\circ)^\circ \subseteq (E'_c)'$ is equicontinuous. Hence η_E takes compact subsets of E to equicontinuous subsets of $(E'_c)'$, and hence

$$\beta \colon E \times V \to E, \quad (x, A) \mapsto \eta_E^{-1}(\Gamma(\eta_E(x), A))$$

is hypocontinuous with respect to compact subsets of E. Using Lemma 5.6, we see that $\beta_* \colon C^\infty(U, V) \to C^\infty(U, E)$ is continuous linear. Also the map $D \colon C^\infty(U, \mathbb{K}) \to C^\infty(U, E'_c)$, $f \mapsto f'$ is continuous linear by Lemma 5.3. Furthermore, $\mathrm{ad} = [.,.]^\vee \colon E'_c \to L(E'_c, E'_c)_c$ is continuous linear since $[.,.]$ is hypocontinuous (see Proposition 1.8 (b)), entailing that also

$$C^\infty(U, \mathrm{ad}) \colon C^\infty(U, E'_c) \to C^\infty(U, L(E'_c, E'_c)_c), \quad f \mapsto \mathrm{ad} \circ f$$

is continuous linear (see, e.g., [12, Lemma 4.13]). Hence $\Psi = \beta_* \circ C^\infty(U, \mathrm{ad}) \circ D$ is continuous and linear. $\qquad\square$

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