Solutions to a Model for Interface Motion by Interface Diffusion

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Abstract

Existence of weak solutions is proved for a phase field model describing an interface in an elastically deformable solid, which moves by diffusion of atoms along the interface. The volume of the different regions separated by the interface is conserved, since no exchange of atoms across the interface occurs. The diffusion is only driven by reduction of the bulk free energy. The evolution of the order parameter in this model is governed by a degenerate parabolic fourth order equation. If a regularizing parameter in this equation tends to zero, then solutions tend to solutions of a sharp interface model for interface diffusion. The existence proof is valid only for a $1\frac{1}{2}$–dimensional situation.

1 Introduction

In this article we study a phase field model for the evolution of an interface in an elastically deformable solid, which moves by diffusion of atoms along this interface. We prove existence of weak solutions, however not for the full three-dimensional model, but for an initial-boundary value problem in $1\frac{1}{2}$–space dimensions. Our studies continue work on the formulation and mathematical investigation of phase field models for evolution of interphases in solids, which was started in [2] and continued in [3, 4].

The interface described by the model separates the body in two regions consisting of atoms of different types and having different elastic properties. No exchange of atoms across the interface occurs, the volumes of the different regions separated by the interface are therefore conserved in time. We call these regions phases. The diffusion of the atoms along the interface is only driven by bulk terms of the free energy, surface terms are neglected.

These properties of the model carry over from the properties of a related sharp interface model: The phases in the phase field model are characterized by an order parameter, whose evolution is governed by a non-uniformly parabolic partial differential equation of fourth order. This equation is formulated in [3] following ideas explained in [2, 3, 4], which suggest that when a certain regularizing parameter ν in this equation tends to zero, then solutions of the model equations converge to solutions of a sharp interface model for interface motion by interface diffusion. In this sharp interface model the normal speed is proportional to the value obtained by application of the surface Laplacian

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to the jump of the Eshelby tensor across the interface. Though several reasons suggest that this conjectured convergence behavior is valid, there is no rigorous proof available for a general situation. Yet, in Section 2 we state without proof a recently obtained convergence result, which supports the conjecture.

In this introduction we first state the initial-boundary value problem for the phase field model in three space dimensions. Since we do not have an existence result for this three dimensional problem, we subsequently reduce the problem to an initial-boundary value problem in one space dimension modelling the movement of a planar interface with flux of atoms linearly growing in a direction tangential to the interface. Such a time dependent flux can be generated by suitable boundary data and volume forces. Because of this tangential flux we speak of a $1\frac{1}{2}$–dimensional problem. We close the introduction by stating our existence result for this $1\frac{1}{2}$–dimensional problem in Theorem 1.3. This is the main result of this article. It is proved in Sections 3 – 6.

Before proving Theorem 1.3 we discuss in Section 2 the background of the model: We state the sharp interface model and the convergence result for $\nu \to 0$. In the formulation of the new phase field model it was an important guiding line that the second law of thermodynamics must be satisfied. We show at the end of Section 2 that this law is fulfilled. This property of the model is essential for the existence proof, since in Section 3 a-priori estimates are derived from it.

To formulate the initial-boundary value problem, let Ω be an open subset in $\mathbb{R}^3$. It represents the material points of a solid body. At the point $x \in \Omega$ at time t the material is in phase 1 or 2 if the value $S(t,x) \in \mathbb{R}$ of the order parameter S is near to zero or one. The other unknowns are the displacement $u(t,x) \in \mathbb{R}^3$ of the material point x at time t and the Cauchy stress tensor $T(t,x) \in \mathcal{S}^3$. Here $\mathcal{S}^3$ denotes the set of symmetric 3×3-matrices. The unknowns must satisfy the quasi-static equations

$$-\operatorname{div}_x T(t,x) = b(t,x), \tag{1.1}$$

$$T(t,x) = D\big(\varepsilon(\nabla_x u) - \bar{\varepsilon} S\big)(t,x), \tag{1.2}$$

$$S_t(t,x) = c \operatorname{div}_x \Big(\nabla_x \big(\psi_S(\varepsilon(\nabla_x u), S) - \nu \Delta_x S\big) |\nabla_x S|\Big)(t,x) \tag{1.3}$$

for $(t,x) \in (0,\infty) \times \Omega$, and the boundary and initial conditions

$$u(t,x) = \gamma(t,x), \qquad (t,x) \in [0,\infty) \times \partial\Omega, \tag{1.4}$$

$$\frac{\partial}{\partial n} S(t,x) = 0, \qquad (t,x) \in [0,\infty) \times \partial\Omega, \tag{1.5}$$

$$\frac{\partial}{\partial n}\big(\psi_S(\varepsilon, S) - \nu \Delta_x S\big)\, |\nabla_x S|(t,x) = 0, \qquad (t,x) \in [0,\infty) \times \partial\Omega, \tag{1.6}$$

$$S(0,x) = S_0(x), \qquad x \in \bar{\Omega}. \tag{1.7}$$

Here n is the unit outward normal vector, $\nabla_x u$ denotes the 3×3-matrix of first order derivatives of u, the deformation gradient, and

$$\varepsilon\left(\nabla_x u\right) = \frac{1}{2}\big(\nabla_x u + (\nabla_x u)^T\,\big)$$

is the strain tensor, where $(\nabla_x u)^T$ denotes the transposed matrix. $\bar{\varepsilon} \in \mathcal{S}^3$ is a given matrix, the transformation strain. The elasticity tensor $D : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, positive definite mapping. $\psi_S = \frac{\partial}{\partial S}\psi$ is the partial derivative of the free energy

$$\psi(\varepsilon, S) = \frac{1}{2}\big(D(\varepsilon - \bar{\varepsilon} S)\big) \cdot (\varepsilon - \bar{\varepsilon} S) + \hat{\psi}(S), \tag{1.8}$$

where for $\hat{\psi} : \mathbb{R} \to [0, \infty)$ we choose a double well potential with minima at points $S = 0$ and $S = 1$ and where the scalar product of two matrices is denoted by $A \cdot B = \sum a_{ij} b_{ij}$. Thus,

$$\psi_S(\varepsilon, S) = -T \cdot \bar{\varepsilon} + \hat{\psi}'(S). \tag{1.9}$$

Given are the positive constant c, the small positive constant ν, the volume force $b : [0, \infty) \times \Omega \to \mathbb{R}^3$ and the boundary and initial data $\gamma : [0, \infty) \times \partial\Omega \to \mathbb{R}^3$, $S_0 : \Omega \to \mathbb{R}$.

This completes the formulation of an initial-boundary value problem. The equations (1.1) and (1.2), which differ from the system of linear elasticity only by the term $\bar{\varepsilon} S$, determine the elastic properties of the two phases characterized by the values $S \approx 0$ or $S \approx 1$: In the first phase the material is stress free at the strain state $\varepsilon(\nabla_x u) = 0$, in the other phase at $\varepsilon(\nabla_x u) = \bar{\varepsilon}$. The elasticity tensor D has the same value at both phases, but it would be important for applications to study the case where D is a function of S with $D[0] \neq D[1]$. The evolution equation (1.3) for the order parameter S is non-uniformly parabolic because of the regularizing term $\operatorname{div}_x\big(\nabla_x(\nu \Delta S)\,|\nabla_x S|\big)$. A complete justification of the model (1.1) – (1.7) would require to show that this initial-boundary value problem has solutions and to show that if the parameter ν in the regularizing term tends to zero, then these solutions converge to solutions of the sharp interface model (2.1) – (2.5) stated in Section 2, i.e. to strengthen the convergence result stated in Theorem 2.2. We contribute to the justification by proving that solutions exist for the $1\frac{1}{2}$–dimensional problem stated in the next section.

In [11] a phase field model for interface motion by interface diffusion was formulated in another way: In the Cahn-Hilliard equation $S_t = -\operatorname{div}_x\big(m(S)\nabla_x(\nu \Delta_x S - \psi'(S))\big)$ a degenerate mobility function $m(S)$ was chosen with zeros at $S = 0$ and $S = 1$, implying that the mobility is different from zero only in a narrow band in the neighborhood of the interface. In [11] it was shown that for $\nu \to 0$ the solution S approaches the characteristic function of a region bounded by a front $\tilde{\Gamma}(t)$ moving with normal velocity s given by

$$s = -C \Delta_{\tilde{\Gamma}(t)} \kappa_{\tilde{\Gamma}(t)}\,, \tag{1.10}$$

where $\kappa_{\tilde{\Gamma}(t)}$ denotes the mean curvature of $\tilde{\Gamma}(t)$ and where $C = C'\nu$. This is the evolution equation for an interface moving by surface diffusion driven by surface free energy only, cf. the discussion in the next section. Since in this approach the curvature appears automatically, we believe that it can not be used when the diffusion is driven by the bulk free energy as in our case, and that equation (1.3) must be used instead. In [17] it was proved that solutions to the Cahn-Hilliard equation with degenerate mobility exist. For other related investigations we refer to [6, 10, 12, 26, 28] and the references cited therein.

Statement of the main result. In the sharp interface model (2.1) – (2.5) the normal speed of the interface determined by equation (2.3) is proportional to the value obtained by application of the surface Laplacian to the jump in the Eshelby tensor. Therefore, this model and also the regularized model (1.1) – (1.7) is not of interest in a strictly one-dimensional situation, where all unknowns only depend on the first component x_1 of $x \in \mathbb{R}^3$ and of t, since in this case the normal speed of a planar interface $\tilde{\Gamma}(t) = \{(h(t), x_2, x_3) \mid (x_2, x_3) \in \mathbb{R}^2\}$ would be equal to zero, hence $h(t) = \text{const}$. In this article we thus consider a $1\frac{1}{2}$–dimensional problem.

In this $1\frac{1}{2}$–dimensional problem we have $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid a < x_1 < d\}$. The volume force and boundary data can be split as

$$b(t,x) = \tilde{b}(t,x_1) + b_1(t,x_2,x_3), \tag{1.11}$$
$$\gamma(t,x) = \tilde{\gamma}(t,x_1) + \gamma_1(t,x_1,x_2,x_3), \tag{1.12}$$

where b_1, γ_1 are such that the elliptic boundary value problem

$$-\mathrm{div}_x\, \sigma(t,x) = b_1(t,x_2,x_3), \tag{1.13}$$
$$\sigma(t,x) = D(\varepsilon(\nabla_x v(t,x))), \tag{1.14}$$
$$v(t,x) = \gamma_1(t,x), \quad x \in \partial\Omega \tag{1.15}$$

of linear elasticity has a solution $x \mapsto (v(t,x), \sigma(t,x)) : \Omega \to \mathbb{R}^3 \times \mathcal{S}^3$ satisfying

$$\partial_{x_1} \sigma(t,x) \cdot \bar{\varepsilon} = 0, \quad \nabla_x(\partial^2_{x_2} + \partial^2_{x_3})\,(\sigma(t,x)\cdot\bar{\varepsilon}) = 0, \quad x \in \Omega. \tag{1.16}$$

It follows that $(\partial^2_{x_2} + \partial^2_{x_3})\,(\sigma\cdot\bar{\varepsilon})$ is independent of x. We thus define

$$r(t) = (\partial^2_{x_2} + \partial^2_{x_3})\,(\sigma(t,x)\cdot\bar{\varepsilon}) \in \mathbb{R}. \tag{1.17}$$

Examples for $b_1, \gamma_1, v, \sigma, \bar{\varepsilon}$ with theses properties can be readily constructed. In particular, examples can be given with $b_1 = 0$. Of course, if $r \neq 0$, then (v,σ) will be unbounded for $|(x_2,x_3)| \to \infty$. For the solution of (1.1) – (1.7) to the data given in (1.11), (1.12) we make the ansatz

$$(u,T,S)(t,x) = (\tilde{u},\tilde{T},\tilde{S})(t,x_1) + (v,\sigma,0)(x).$$

Noting that (1.9) and (1.16) imply

$$\psi_S(\varepsilon(\nabla_x u),S)_{x_1} = (-\tilde{T}\cdot\bar{\varepsilon} - \sigma\cdot\bar{\varepsilon} + \hat{\psi}'(S))_{x_1} = (-\tilde{T}\cdot\bar{\varepsilon} + \hat{\psi}'(S))_{x_1} = \psi_S(\varepsilon(\nabla_x\tilde{u}),S)_{x_1}\,,$$

we obtain by insertion of this ansatz into (1.1) – (1.7) an initial-boundary value problem for $(\tilde{u},\tilde{T},\tilde{S})$ in one space dimension. To formulate this initial-boundary value problem we simplify the notation and denote $(\tilde{u},\tilde{T},\tilde{S})$ and $\tilde{b}$ again by (u,T,S) and by b, respectively. We write x for x_1, let $\Omega = (a,d)$ be a bounded open interval, set $Q_{T_e} = (0,T_e)\times\Omega$, where T_e (time of existence) is a positive constant, and denote

$$\varepsilon(u_x) = \frac{1}{2}\big((u_x,0,0) + (u_x,0,0)^T\big) \in \mathcal{S}^3.$$

With these notations $(u,T,S) : Q_{T_e} \to \mathbb{R}^3 \times \mathcal{S}^3 \times \mathbb{R}$ must satisfy the equations

$$-T_{1x} = b, \tag{1.18}$$
$$T = D(\varepsilon(u_x) - \bar{\varepsilon}S), \tag{1.19}$$
$$S_t = c\big((\psi_S(\varepsilon(u_x),S) - \nu S_{xx})_x|S_x|\big)_x + cr(t)|S_x|\,, \tag{1.20}$$

and the boundary and initial conditions

$$u|_{[0,T_e]\times\partial\Omega} = 0, \tag{1.21}$$
$$S_x|_{[0,T_e]\times\partial\Omega} = 0, \tag{1.22}$$
$$\big((\psi_S - \nu S_{xx})_x|S_x|\big)|_{[0,T_e]\times\partial\Omega} = 0, \tag{1.23}$$
$$S|_{\{0\}\times\Omega} = S_0. \tag{1.24}$$

Here $T_1(t,x)$ denotes the first column of the matrix $T(t,x)$. In (1.21) we assumed that the function $\tilde{\gamma}$ from (1.12) vanishes. This is possible without restriction of generality because of the linearity of the equations (1.18), (1.19). Given are the data $b(t,x) \in \mathbb{R}^3$ and $S_0(x) \in \mathbb{R}$. Since $r(t) \in \mathbb{R}$ defined in (1.17) can be computed by solving the boundary value problem (1.13) – (1.15), also $r : [0,T_e) \to \mathbb{R}$ can be considered to be given.

Equations (1.18) – (1.20) constitute the initial-boundary value problem in one space dimension, which we study in this article. It models a planar interface propagating with speed $cr(t)$. To state the existence result for this problem we need some notations and definitions, which we introduce next.

For a subset $\mathcal{A}$ of Q_{T_e}, for a function $g : \mathcal{A} \to V$ with values in some set V and for $t \in [0,T_e]$ let

$$\mathcal{A}(t) = \{x \mid (t,x) \in \mathcal{A}\} \subseteq \mathbb{R} \quad \text{and} \quad g(t) : \mathcal{A}(t) \to V,\ g(t)(x) = g(t,x).$$

We show that the component S in a solution (u,T,S) of the initial-boundary value problem has the weak derivative S_{xxx}, however on a set $\mathcal{A} \subseteq Q_{T_e}$, for which $\mathcal{A}(t)$ is open for almost all t, but which itself is not open in $\mathbb{R}^2$. Such weak derivatives are more general then standard weak derivatives on open sets. We define these general derivatives as follows:

Definition 1.1 *Let $\mathcal{A} \subset Q_{T_e}$ such that $\mathcal{A}(t)$ is open for almost all $t \in [0,T_e]$, and let $\alpha \in \mathbb{N}_0$. We call $g : \mathcal{A} \to \mathbb{R}$ the α-th local weak L^2–derivative of $S \in L^2(Q_{T_e})$ with respect to x in $\mathcal{A}$, if for almost all $t \in [0,T_e]$ the function $g(t)$ belongs to $L^{2,\mathrm{loc}}(\mathcal{A}(t))$ and is the local weak derivative of S in the usual sense:*

$$g(t) = \partial_x^\alpha S(t)|_{\mathcal{A}(t)}, \tag{1.25}$$

and if moreover there exists a sequence $\{\mathcal{A}_n\}_n$ of measurable sets $\mathcal{A}_n \subset \mathcal{A}$ with $g|_{\mathcal{A}_n} \in L^2(\mathcal{A}_n)$ for all $n \in \mathbb{N}$, such that

$$\text{meas } \big(\mathcal{A} \setminus \bigcup_{n=1}^{\infty} \mathcal{A}_n\big) = 0.$$

Local weak derivatives in the sense of this definition are unique because of (1.25), and it is immediately seen that if $\mathcal{A}$ is open then the local weak derivative in the sense of this definition coincides with the ordinary local weak derivative. Therefore we use the same name and the same notation as for ordinary local weak derivatives.

For $S \in L^2(0,T_e;H^2(\Omega))$ let

$$\mathcal{A}^S = \{(t,x) \in Q_{T_e} \mid |S_x(t,x)| > 0\}.$$

Since by the Sobolev embedding theorem $S_x(t)$ is continuous for almost all t, it follows that $\mathcal{A}^S(t)$ is open for almost all t.

Definition 1.2 *Let $b \in L^\infty(0,T_e;L^2(\Omega))$, $r \in L^\infty(0,T_e)$ and $S_0 \in L^2(\Omega)$. A function (u,T,S) with*

$$u \in L^2(0,T_e;H^2(\Omega)), \qquad u(t) \in H_0^1(\Omega) \text{ a.e. in } (0,T_e), \tag{1.26}$$

$$T \in L^2(0,T_e;H^1(\Omega)), \tag{1.27}$$

$$S \in L^2(0,T_e;H^2(\Omega)) \cap L^\infty(Q_{T_e}), \qquad S_x(t) \in H_0^1(\Omega) \text{ a.e. in } (0,T_e), \tag{1.28}$$

is a weak solution of the problem (1.18) – (1.24), if (u, T, S) *solves* (1.18), (1.19) *weakly, if* S *has the local weak derivative* S_{xxx} *in* $\mathcal{A}^S$ *with* $|S_x|S_{xxx} \in L^1(\mathcal{A}^S)$ *and if*

$$(S, \varphi_t)_{Q_{T_e}} + c(\nu S_{xxx}|S_x|, \varphi_x)_{\mathcal{A}^S} + c\left(\Big(T \cdot \bar{\varepsilon} - \hat{\psi}'(S)\Big)_x |S_x|, \varphi_x\right)_{Q_{T_e}} + (cr|S_x|, \varphi)_{Q_{T_e}} = -(S_0, \varphi(0))_\Omega \tag{1.29}$$

holds for all $\varphi \in C_0^\infty((-\infty, T_e) \times \mathbb{R})$.

The main result of this article is

Theorem 1.3 *Assume that there exists a constant* $M > 0$ *such that the double well potential* $\hat{\psi} \in C^3(\mathbb{R}, [0, \infty))$ *satisfies*

$$\max\left\{\hat{\psi}'(S)^2, S^2\right\} \le M(\hat{\psi}(S) + 1). \tag{1.30}$$

Then to all $S_0 \in H^1(\Omega)$, $r \in L^\infty(0, T_e)$ *and* $b \in L^2(Q_{T_e})$ *with* $b_t \in L^2(Q_{T_e})$ *there exists a weak solution* (u, T, S) *of* (1.18) – (1.24), *which in addition to (1.26) – (1.29) satisfies*

$$u \in L^\infty(0, T_e; H^2(\Omega)), \qquad T \in L^\infty(0, T_e; H^1(\Omega)) \tag{1.31}$$

$$S \in L^\infty(0, T_e; H^1(\Omega)), \qquad S_t \in L^{\frac{4}{3}}(0, T_e; W^{-1,\frac{4}{3}}(\Omega)), \tag{1.32}$$

$$|S_x|S_{xxx} \in L^{\frac{4}{3}}(Q_{T_e}), \tag{1.33}$$

where we defined $|S_x|S_{xxx} = 0$ *on* $Q_{T_e} \setminus \mathcal{A}^S$ *.*

This theorem is proved in Sections 3 – 6. The obvious idea is to replace the degenerate parabolic equation (1.20) by the non-degenerate equation

$$S_t = c\Big((\psi_S - \nu S_{xx})_x \,(|S_x|_\kappa + \kappa) \Big)_x + cr|S_x|_\kappa\,, \tag{1.34}$$

where

$$|y|_\kappa = \frac{|y|^2}{\sqrt{|y|^2 + \kappa}}, \tag{1.35}$$

with a constant $\kappa > 0$, and to approximate a solution of (1.18) – (1.24) by a sequence of solutions $(u^\kappa, T^\kappa, S^\kappa)$ of an initial-boundary value problem to the equations (1.18), (1.19), (1.34) with $\kappa \to 0$. Yet, though (1.34) is non-degenerate parabolic, we can not show that the system (1.18), (1.19), (1.34) has classical solutions. Instead, we replace the term $|S_x|_\kappa$ in (1.34) by $|\widetilde{\hat{S}_x}|_\kappa$, where $\widetilde{\hat{S}_x}$ is obtained from the given function $\hat{S}_x$ by convolution with a mollifier. Since $\hat{S}_x$ is given, the resulting equation is linear in the terms with the highest order derivatives and has smooth coefficients. This allows to apply a classical theorem from the theory of fourth-order parabolic equations to obtain Hölder continuous solutions. We then derive suitable a-priori estimates and apply a standard approximation procedure to obtain weak solutions of (1.18), (1.19), (1.34). This construction is carried out in Section 3. The proof of the important Lemma 3.4 is based on an energy inequality implied by the second law of thermodynamics.

The a-priori estimates in Section 3 depend on κ and can therefore not be used to prove existence of solutions of the original initial-boundary value problem, since in such a proof the limit $\kappa \to 0$ must be studied. In Section 4 we thus derive an "energy estimate" for solutions of the system (1.18), (1.19), (1.34), which is employed in Section 5 to prove a-priori estimates for such solutions, which hold uniformly with respect to κ. Using these estimates and the Aubin–Lions Lemma we can then show in Section 6 that a suitable sequence of such solutions converges to a solution of the original initial-boundary value problem.

This complicated double approximation procedure is necessary, since in the derivation of the energy estimate in Section 4 we use the special form of the term $(|S_x|_\kappa + \kappa)S_{xxx}$ appearing in (1.34). Replacing $|S_x|_\kappa$ by $|\widetilde{\hat{S}_x}|_\kappa$ destroys this form. Because of this, we cannot prove this energy estimate directly for the sequence of solutions from the first approximation procedure but need to use solutions of (1.34) to derive uniform a-priori estimates.

To get the local weak derivative S_{xxx} in Section 6 we apply Egorov's Theorem to decompose the set $\hat{\mathcal{A}}_n = \{(t,x) \in Q_{T_e} \mid |S_x(t,x)| > \frac{1}{n}\}$ into a set $\mathcal{A}_n$, on which the sequence S_x^κ converges uniformly to S_x and thus satisfies $|S_x^\kappa| \geq \frac{1}{2n}$ for sufficiently small κ, and into the set $\hat{\mathcal{A}}_n \setminus \mathcal{A}_n$ of small measure. Using the uniform estimate $\int_{Q_{T_e}} (|S_x^\kappa|_\kappa + \kappa)|S_{xxx}^\kappa|^2 \, d(\tau,x) \leq C$ from Corollary 5.3, we can then show that S_{xxx}^κ converges in $L^2(\mathcal{A}_n)$ to S_{xxx}. In the last step we use that $\mathcal{A}^S$ differs from $\bigcup_{n=1}^\infty \mathcal{A}_n$ only by a set of measure zero.

We already mentioned the existence result [17] for the degenerate Cahn-Hilliard equation. Another degenerate parabolic equation, for which existence of solutions was studied in several articles is the thin film equation $S_t = -\mathrm{div}_x(m(S)\nabla_x \Delta_x S)$, for which $m(S)$ vanishes at zero. We refer to [7, 8, 9, 14] and the references therein. Yet, the mathematical properties of (1.3) containing the term $|\nabla_x S|$ differ essentially from the properties of these equations.

2 The sharp interface problem

It is possible to construct traveling wave solutions of (1.1) – (1.7), which converge for $\nu \to 0$ to solutions of a sharp interface model for interface motion by interface diffusion. Based on this construction we recently found a convergence result for a more general situation. This result is far from a proof, that solutions of (1.1) – (1.7) in general show this asymptotic behavior, but it supports the conjecture. To motivate our investigations we state this result without giving the proof, which is to be published. At the end of this section we show that the model equations (1.1) – (1.3) satisfy the second law of thermodynamics.

To state the sharp interface model, let the interface be given by a sufficiently smooth three-dimensional manifold $\tilde{\Gamma}$ in $[t_1,t_2] \times \Omega \subset \mathbb{R}^4$ such that for all $t \in [t_1,t_2]$

$$\tilde{\Gamma}(t) = \Big\{ x \in \Omega \mid (t,x) \in \tilde{\Gamma} \Big\}$$

is a two-dimensional manifold. The two different phases are characterized by the values of a discontinuous order parameter S, which has the constant values 0 and 1 in the regions separated by the phase interface, and which jumps along the interface. The sharp interface problem, which determines the unknown position of the interface and the unknown functions u, T, consists of the equations

$$
\begin{aligned}
-\mathrm{div}_x T &= b, && (2.1)\\
T &= D(\varepsilon(\nabla_x u) - \bar{\varepsilon} S), && (2.2)\\
s[S] &= -c\Delta_{\tilde{\Gamma}(t)}(n \cdot [C]n), && (2.3)\\
[u] &= 0, && (2.4)\\
[T]n &= 0, && (2.5)
\end{aligned}
$$

and of suitable initial and boundary conditions. (2.1) and (2.2) must hold on $([t_1, t_2] \times \Omega)\backslash\tilde{\Gamma}$, the jump conditions (2.3) – (2.5) are given on $\tilde{\Gamma}$. Here c is a positive constant, $n(t, x) \in \mathbb{R}^3$ is the unit normal vector to $\tilde{\Gamma}(t)$ at $x \in \tilde{\Gamma}(t)$ pointing into the region where $S = 1$, and $s(t, x) \in \mathbb{R}$ is the normal speed of $\tilde{\Gamma}(t)$ at $x \in \tilde{\Gamma}(t)$ in direction $n(t, x)$. Also, $\Delta_{\tilde{\Gamma}(t)}$ is the surface Laplacian on $\tilde{\Gamma}(t)$, and $[u], [T], [S], [C]$ denote the jumps of u, T, S and of the Eshelby tensor

$$
C(\nabla_x u, S) = \psi(\varepsilon(\nabla_x u), S)I - (\nabla_x u)^T T
$$

across $\tilde{\Gamma}$, where I is the 3×3–unit matrix and ψ is the free energy given in (1.8). We use the notation $(\nabla_x u)^T T$ to denote the matrix product.

The evolution law (2.3) describes motion of the interface $\tilde{\Gamma}(t)$ due to diffusion of atoms along the interface. The flux is given by $-c\nabla_{\tilde{\Gamma}(t)}(n\cdot[C]n)$ with the surface gradient $\nabla_{\tilde{\Gamma}(t)}$. There is no exchange of atoms between the phases, hence the volume $\int_\Omega S(x, t)dx$ of one of the phases is conserved in time. The evolution law is derived in the standard way by application of the second law of thermodynamics under the assumption that the free energy is given by $\Psi(t) = \int_\Omega \psi(\varepsilon, S)dx$ and thus contains only bulk terms: the Clausius-Duhem inequality must be satisfied, which for this free energy leads to the flux term given above. For this derivation we refer to [2], where the application of the second law of thermodynamics to an interface problem is discussed with mathematical rigor.

If one assumes more generally that the free energy is a sum of bulk and surface terms

$$
\Psi(t) = \alpha_1 \int_\Omega \psi(\varepsilon(\nabla_x u(t, x)), S(t, x))dx + \alpha_2 \int_{\tilde{\Gamma}(t)} d\sigma
$$

with $\alpha_1, \alpha_2 \geq 0$, then the evolution law obtained is

$$
s[S] = -c\Delta_{\tilde{\Gamma}(t)} \left(\alpha_1(n \cdot [C]n) + \alpha_2 \kappa_{\tilde{\Gamma}(t)}\right), \tag{2.6}
$$

where $\kappa_{\tilde{\Gamma}(t)}$ is the mean curvature of $\tilde{\Gamma}(t)$. For $\alpha_1 = 0$ the equation (1.10) derived by Mullins [24] results. For the derivation of (1.10) and more general equations we refer to [13, 15, 27] and the literature cited there. Existence, regularity and asymptotic behavior of a family of smooth hypersurfaces, whose evolution is governed by (1.10) or by an alternative evolution law proposed in [13] is studied in [18, 19, 20]. We mention that it is possible to generalize (1.3) to an equation regularizing the evolution law (2.6). This equation is given in [3].

To state the convergence result we need some preparations: Suppose that the functions u, T and the interface $\tilde{\Gamma}$ solve (2.1) – (2.5) and are sufficiently regular. Precisely, we assume that $\tilde{\Gamma}$ is a C^2-manifold such that the two-dimensional manifold $\tilde{\Gamma}(t)$ does not have a boundary for all $t \in [t_1, t_2]$ and such that the set $\bigcup_{t\in[t_1,t_2]} \tilde{\Gamma}(t)$ is compactly contained in Ω. The set of all $(t, x) \in [t_1, t_2] \times \Omega$ with $S(t, x) = 0$ is denoted by Γ, and Γ' is the set of all $(t, x) \in [t_1, t_2] \times \Omega$ with $S(t, x) = 1$. Therefore the sets Γ, Γ' and $\tilde{\Gamma}$ are pairwise disjoint and satisfy $\Gamma \cup \Gamma' \cup \tilde{\Gamma} = [t_1, t_2] \times \Omega$. We also assume that the functions u and T are two times continuously differentiable on Γ and on Γ' with two times continuously differentiable extensions from Γ to $\Gamma \cup \tilde{\Gamma}$ and from Γ' to $\Gamma' \cup \tilde{\Gamma}$.

These assumptions imply that there is $\delta > 0$ such that for all $t \in [t_1, t_2]$,

$$(x, \xi) \mapsto y = x + n(x)\xi \, : \, \tilde{\Gamma}(t) \times (-\delta, \delta) \to \Omega$$

defines a new coordinate system in a neighborhood of $\tilde{\Gamma}(t)$. Let

$$\mathcal{U} = \Big\{ (t, x + n(x)\xi) \mid (t, x) \in \tilde{\Gamma}, \ |\xi| < \delta \Big\} \subseteq [t_1, t_2] \times \Omega,$$

and choose $\varphi \in C^\infty([t_1, t_2] \times \Omega)$ such that $\varphi = 0$ outside the set $\mathcal{U}$ and $\varphi = 1$ in a neighborhood of $\tilde{\Gamma}$. We use φ to decompose T and u in two terms:

$$T(t, y) = [T(t, x)] S(t, y) \varphi(t, y) + \sigma(t, y), \tag{2.7}$$

$$u(t, y) = [\nabla_x u(x, t)] n(x, t) \int_0^\xi S(t, x + n(t, x)\zeta) d\zeta \, \varphi(t, y) + v(t, y), \tag{2.8}$$

where for $(t, y) \in \mathcal{U}$ the pair $(x, \xi) \in \tilde{\Gamma}(t) \times \mathbb{R}$ is defined by $y = x + n(t, x)\xi$. Thus, outside of $\mathcal{U}$ we have $\sigma(t, y) = T(t, y)$, $v(t, y) = u(t, y)$, and our assumptions imply that σ is continuous and v is continuously differentiable at $\tilde{\Gamma}$. The theorem below shows that the regularizing effect of (1.1) – (1.3) mainly is to replace the jump terms on the right hand side of (2.7) and (2.8) by a smooth transition profile. This transition profile is obtained by constructing a traveling wave solution $(u(t, x), T(t, x), S_0(x - rt))$ of the one-dimensional version (1.18) – (1.20) of the equations (1.1) – (1.3). This ansatz leads to an ordinary differential equation for S_0, which after some computations can be reduced to equation (2.10) stated below. The general result is

Lemma 2.1 *Let $\hat{\psi} \in C^2(\mathbb{R}, [0, \infty))$ be a double well potential, which satisfies $\hat{\psi}(0) = \hat{\psi}(1)$, $\hat{\psi}'(0) = -\hat{\psi}'(1)$ and*

$$\hat{\psi}(S) - \hat{\psi}(0) + \frac{1}{2} \min_{x \in \tilde{\Gamma}(t)} \big(\bar{\varepsilon} \cdot [T(t, x)] \big) S(1 - S) > 0, \qquad \text{for } 0 < S < 1, \tag{2.9}$$

$$\hat{\psi}'(0) + \frac{1}{2} \min_{x \in \tilde{\Gamma}(t)} (\bar{\varepsilon} \cdot [T(t, x)]) > 0.$$

Then for all $x \in \tilde{\Gamma}(t)$ there are $a = a(t, x) < 0 < d = d(t, x)$ and a strictly increasing solution $\xi \mapsto S_0(\xi) = S_0(t, x, \xi) : [a, d] \to [0, 1]$ of

$$S_0' = \sqrt{2 \left(\hat{\psi}(S_0) - \hat{\psi}(0) + \frac{1}{2} \bar{\varepsilon} \cdot [T(t, x)] S_0 (1 - S_0) \right)}, \tag{2.10}$$

$$S_0(0) = \frac{1}{2}, \quad S_0(a) = 0, \quad S_0(d) = 1, \quad S_0'(a) = S_0'(d) = 0.$$

We extend the function S_0 to all of $\tilde{\Gamma} \times \mathbb{R}$ by setting

$$S_0(t,x,\xi) = \begin{cases} 0, & \xi < a(t,x), \\ 1, & \xi > d(t,x). \end{cases}$$

The symmetry conditions for $\hat{\psi}$ at $S = 0,1$ are imposed for simplicity. Without it (2.10) becomes slightly more complicated. A closer investigation of the solution (u,T) of (2.1), (2.2) shows that $\bar{\varepsilon} \cdot [T(t,x)] \le 0$ for all $(t,x) \in \tilde{\Gamma}$. Therefore condition (2.9) requires that $\hat{\psi}$ assumes one minimum at a point less than 0, the other minimum at a point greater than 1 and that $\hat{\psi}$ has a sufficiently large hump between 0 and 1. Every double well potential with these properties is allowed. An example for a potential satisfying all the conditions of this lemma is

$$\hat{\psi}(S) = k(S+l)^2(S-(1+l))^2$$

with $l > 0$ and $kl(1+l) > -\frac{1}{2}\min_{x\in\tilde{\Gamma}(t)}(\bar{\varepsilon}\cdot[T(t,x)])$. For this potential S_0 is an elliptic function. In Section 1 we assumed in contrast that the double well potential $\hat{\psi}$ has minima at $S = 0,1$. This is a normalizing condition, which we only imposed for simplicity. In fact, all the proofs in Sections 3 – 6 and therefore all results from Section 1 are valid without change for double well potentials with minima at arbitrary locations.

Theorem 2.2 *Set*

$$\begin{aligned}
\hat{S}_\nu(t,y) &= S_0(t,x,\xi/\nu^{\frac{1}{2}})\varphi(t,y) + S(t,y)(1-\varphi(t,y)), \\
\hat{T}_\nu(t,y) &= [T(t,x)]S_0(t,x,\xi/\nu^{\frac{1}{2}})\varphi(t,y) + \sigma(t,y), \\
\hat{u}_\nu(t,y) &= [\nabla_x u(t,x)]n(t,x)\int_0^{\xi/\nu^{\frac{1}{2}}} S_0(t,x,\zeta)d\zeta\,\varphi(t,y) + v(t,y),
\end{aligned}$$

where for $(t,y) \in \mathcal{U}$ the pair $(x,\xi) \in \tilde{\Gamma}(t) \times \mathbb{R}$ is determined from $x + n(t,x)\xi = y$. Then the function $(\hat{u}_\nu, \hat{T}_\nu, \hat{S}_\nu)$ satisfies the equations (1.1) – (1.3) *weakly in the sense of distributions up to an error of order $o(1)$ for $\nu \to 0$, i.e.* (1.3) *holds in the sense that*

$$\Big(\partial_t \hat{S}_\nu - c\,\mathrm{div}_x\Big(\nabla_x(\psi_S - \nu\Delta_x\hat{S}_\nu)|\nabla_x\hat{S}_\nu|\Big), \varphi\Big)_{(t_1,t_2)\times\Omega} = o(1)$$

for all $\varphi \in C_0^\infty((t_1,t_2)\times\Omega)$, and corresponding relations hold for (1.1), (1.2).

Comparison with (2.7), (2.8) shows that the jump function S in the solution of the sharp interface problem is replaced by the transition profile $S_0(t,x,\xi/\nu^{\frac{1}{2}})$, which scales with $\nu^{-\frac{1}{2}}$ for $\nu \to 0$.

Second law of thermodynamics. The second law requires that there exist a free energy ψ^* and a flux q such that the Clausius-Duhem inequality $\frac{\partial}{\partial t}\psi^* + \mathrm{div}_x\, q \le b \cdot u_t$ holds, cf. [5]. With ψ given in (1.8) we choose

$$\psi^*(\varepsilon, S, \nabla_x S) = \psi(\varepsilon, S) + \frac{\nu}{2}|\nabla_x S|^2, \tag{2.11}$$

$$\begin{aligned}
&q(u_t, S_t, \varepsilon, \nabla_x\varepsilon, S, \ldots, \nabla_x^3 S) \\
&\qquad = -T\cdot u_t - \nu S_t \cdot \nabla_x S - c(\psi_S - \nu\Delta_x S)\nabla_x(\psi_S - \nu\Delta_x S)\,|\nabla_x S|,
\end{aligned}$$

and apply (1.1) and the relation $\nabla_\varepsilon \psi \cdot \varepsilon_t = T \cdot u_t$, which holds by (1.2) and the symmetry of T, to obtain after a short computation

$$\begin{aligned}&\frac{\partial}{\partial t}\psi^* + \mathrm{div}_x q - b \cdot u_t \\ &\quad = (\psi_S - \nu\Delta_x S)S_t - c\ \mathrm{div}_x\big((\psi_S - \nu\Delta_x S)\,\nabla_x(\psi_S - \nu\Delta_x S)|\,\nabla_x S|\big).\end{aligned}$$

Insertion of (1.3) into this equation results in

$$\frac{\partial}{\partial t}\psi^* + \mathrm{div}_x q - b \cdot u_t = -c\,|\nabla_x(\psi_S - \nu\Delta_x S)|^2|\nabla_x S| \le 0,$$

which shows that the second law holds for the system (1.1) – (1.3).

3 Existence for the approximate problem

To construct approximate solutions to (1.18) – (1.24) we prove in this section that there exist weak solutions of the quasilinear, uniformly parabolic initial-boundary value problem in Q_{T_e} given by the equations

$$\begin{aligned}
-T_{1x} &= b, && (3.1)\\
T &= D(\varepsilon(u_x) - \bar{\varepsilon}S), && (3.2)\\
S_t &= c\Big(\big((\psi_S(\varepsilon(u_x), S) - \nu S_{xx}\big)_x(|S_x|_\kappa + \kappa)\Big)_x + cr|S_x|_\kappa\,, && (3.3)\\
u &= 0, \qquad \text{on } [0, T_e] \times \partial\Omega, && (3.4)\\
S_x &= 0, \qquad \text{on } [0, T_e] \times \partial\Omega, && (3.5)\\
(\psi_S - \nu S_{xx})_x &= 0, \qquad \text{on } [0, T_e] \times \partial\Omega, && (3.6)\\
S(0, x) &= S_0(x), \quad x \in \Omega\,, && (3.7)
\end{aligned}$$

with a fixed positive parameter κ, obtained from (1.18) – (1.24) by replacing (1.20) with (1.34). By definition, $(u, T, S) \in L^2(0, T_e; H^1(\Omega)^3)$ with $S_{xxx} \in L^2(Q_{T_e})$ is a weak solution of (3.1) – (3.7) if (3.1), (3.2), (3.4) (3.5) are satisfied weakly and if for all $\varphi \in C_0^\infty((-\infty, T_e) \times \mathbb{R})$

$$\begin{aligned}
-(S, \varphi_t)_{Q_{T_e}} &= (S_0, \varphi(0))_\Omega \\
&+ c\big((|S_x|_\kappa + \kappa)(\nu S_{xxx} - \psi_{S,x}), \varphi_x\big)_{Q_{T_e}} + c(r|S_x|_\kappa\,, \varphi)_{Q_{T_e}}\,. && (3.8)
\end{aligned}$$

We start with a technical result used throughout the following sections.

Lemma 3.1 *For every $y \in \mathbb{R}$ and $\kappa > 0$ we have*

$$|y|_\kappa \le |y| \le |y|_\kappa + \kappa. \tag{3.9}$$

Proof. The inequality sign on the left is obvious from the definition of $|y|_\kappa$ in (1.35). To prove the second inequality note that

$$|y|\sqrt{y^2 + \kappa^2} \le \Big(\sqrt{y^2 + \kappa^2}\Big)^2 = y^2 + \kappa^2 \le y^2 + \kappa\sqrt{y^2 + \kappa^2} = (|y|_\kappa + \kappa)\sqrt{y^2 + \kappa^2}.$$

Division of both sides of this estimate by $\sqrt{y^2 + \kappa^2}$ yields the stated inequality.

In the following we denote the $L^2(\Omega)$–norm by $\|\cdot\|$. If v is a function defined on Q_{T_e} then, by our convention, $v(t)$ is defined on Ω. If no confusion is possible, we sometimes drop the argument t and write v to denote $v(t)$. The main result of this section is

Theorem 3.2 *Assume that* $S_0 \in H^1(\Omega)$, $r \in L^\infty(0,T_e)$ *and* $b \in L^2(Q_{T_e})$ *with* $b_t \in L^2(Q_{T_e})$. *Then there is a constant* $\bar{C}$ *independent of* κ *and a weak solution* (u,T,S) *of (3.1) – (3.7) with* $S \in L^2(0,T_e;H^3(\Omega)) \subseteq L^2(0,T_e;C^{2+\alpha}(\bar{\Omega}))$, $\alpha < \frac{1}{2}$, *and with* $S_t \in L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))$, *such that*

$$\|u\|_{L^\infty(0,T_e;H^2(\Omega))} + \|T\|_{L^\infty(0,T_e;H^1(\Omega))} \leq \bar{C}, \tag{3.10}$$

$$\|u\|_{L^\infty(Q_{T_e})} + \|T\|_{L^\infty(Q_{T_e})} \leq \bar{C}, \tag{3.11}$$

$$\|S\|_{L^\infty(0,T_e;H^1(\Omega))} \leq \bar{C}, \tag{3.12}$$

$$\|S\|_{L^\infty(Q_{T_e})} \leq \bar{C}. \tag{3.13}$$

The proof is given in several lemmas, in which we construct a sequence of approximate solutions and derive uniform a-priori bounds for it. These bounds allow to select a subsequence, which converges to a solution of (3.1) – (3.7). The function (u^n,T^n,S^n) in this sequence is constructed as solution of the semilinear initial-boundary value problem obtained by replacing S in the term $(|S_x|_\kappa + \kappa)$ of (3.3) by the function $\widetilde{S^{n-1}}$, where tilde denotes a smoothing operation, and by inserting for the data (b,r,S_0) a sequence (b^n,r^n,S_0^n) of functions with higher regularity. We thus have to study the semilinear problem, which consists of the equations

$$-T_{1x} = b, \tag{3.14}$$

$$T = D(\varepsilon(u_x) - \bar{\varepsilon}S), \tag{3.15}$$

$$S_t = c\Big((|\widetilde{\hat{S}_x}|_\kappa + \kappa)(\psi_S(\varepsilon(u_x),S) - \nu S_{xx})_x \Big)_x + cr|S_x|_\kappa\,, \tag{3.16}$$

and of the boundary and initial conditions (3.4) – (3.7). Here $\hat{S} \in L^2(0,T_e;H^2(\Omega))$ is a given function and

$$\widetilde{\hat{S}_x}(t,x) = (\chi_\eta * S_x)(t,x) = \int_{Q_{T_e}} \chi_\eta(t-\tau,x-y)\,\hat{S}_x(\tau,y)\,d(\tau,y)\,, \tag{3.17}$$

with the standard mollifier $\chi_\eta \in C_0^\infty(\{x \in \mathbb{R}^2 \mid |x| \leq \eta\})$.

Lemma 3.3 *Let* $0 < \alpha < 1$. *To every* $\hat{S} \in L^2(Q_{T_e})$, $b \in C^{\frac{\alpha}{4},\alpha}(\bar{Q}_{T_e})$, $r \in C^{\frac{\alpha}{4}}([0,T_e])$ *and* $S_0 \in C^{4+\alpha}(\bar{\Omega})$ *there is a unique solution* (u,T,S) *of the initial-boundary value problem* (3.14) – (3.16), (3.4) – (3.7). *This solution belongs to the space*

$$L^\infty(0,T_e;C^{2+\alpha}(\bar{\Omega})) \times L^\infty(0,T_e;C^{1+\alpha}(\bar{\Omega})) \times C^{1+\frac{\alpha}{4},4+\alpha}(\bar{Q}_{T_e})$$

and satisfies $S_{xxt} \in L^2(Q_{T_e})$.

Proof. We reduce the system (3.14) – (3.16) to a single equation. To this end note that for every $t \in [0,T_e]$ and for given $S(t)$ the equations (3.14), (3.15), (3.4) form a boundary value problem for $(u(t),T(t))$. It is shown in [3] that the solution is given by

$$u(t,x) = u^*\left(\int_a^x S(t,y)dy - \frac{x-a}{d-a}\int_a^d S(t,y)dy \right) + w(t,x), \tag{3.18}$$

$$T(t,x) = D(\varepsilon^* - \bar{\varepsilon})S(t,x) - \frac{D\varepsilon^*}{d-a}\int_a^d S(t,y)dy + \sigma(t,x), \tag{3.19}$$

where $u^* \in \mathbb{R}^3, \varepsilon^* \in \mathcal{S}^3$ are suitable constants only depending on $\bar{\varepsilon}$ and D and where $(w(t), \sigma(t)) : \Omega \to \mathbb{R}^3 \times \mathcal{S}^3$ is the solution to the elliptic boundary value problem

$$
\begin{aligned}
-\sigma_{1x}(t) &= b(t), && (3.20)\\
\sigma(t) &= D\varepsilon(w_x(t)), && (3.21)\\
w(t)|_{\partial\Omega} &= 0. && (3.22)
\end{aligned}
$$

Noting (1.9), we obtain from (3.19) that

$$
\psi_S(\varepsilon(u_x), S) = \hat{\psi}'(S) - T \cdot \bar{\varepsilon} = \hat{\psi}'(S) - c_1 S + \frac{\bar{\varepsilon} \cdot D\varepsilon^*}{d-a} \int_a^d S dy - \sigma \qquad (3.23)
$$

with $c_1 = \bar{\varepsilon} \cdot D(\varepsilon^* - \bar{\varepsilon})$. Insertion of this equation into (3.16) yields the initial-boundary value problem

$$
\begin{aligned}
S_t &= -c\Big(\widetilde{(|\hat{S}_x|_\kappa + \kappa)}(\nu S_{xxx} + (c_1 - \hat{\psi}''(S))S_x + \sigma_x)\Big)_x + cr|S_x|_\kappa\,, && (3.24)\\
S_x &= 0, \qquad (\nu S_{xx} - \psi_S)_x = 0, \qquad \text{on } (0, T_e) \times \partial\Omega, && (3.25)\\
S(0, x) &= \tilde{S}_0(x), \qquad x \in \Omega. && (3.26)
\end{aligned}
$$

Since (3.24) is a semilinear, strictly parabolic equation for S with Hölder continuous coefficients, we can use a theorem in [22, p. 616] or in [16] to assert that for any given $\hat{S} \in L^2(0, T_e; H^2(\Omega))$ there is a unique classical solution $S \in C^{1+\alpha/4, 4+\alpha}(\bar{Q}_{T_e})$ of this initial-boundary value problem with $S_{xxt} \in L^2(Q_{T_e})$. This function S and the functions u and T given by (3.18), (3.19) solve (3.14) – (3.16), (3.4) – (3.7).

In the following lemmas we assume that (u, T, S) is the solution of (3.14) – (3.16), (3.4) – (3.7) given by Lemma 3.3 to data having the regularity stated in the lemma and satisfying the estimates

$$
\|b\|_{L^2(Q_{T_e})} \le K, \quad \|b_t\|_{L^2(Q_{T_e})} \le K, \quad \bar{r} = \|r\|_{L^\infty(0,T_e)} \le K, \quad \|S_0\|_{H^1(\Omega)} \le K. \qquad (3.27)
$$

Lemma 3.4 *There is a constant $\bar{C}$ independent of η and κ but depending on K, such that for every $\hat{S} \in L^2(0, T_e; H^2(\Omega))$, all (b, r, S_0) satisfying (3.27) and for any $t \in [0, T_e]$*

$$
\begin{aligned}
\|S(t)\|_{H^1(\Omega)} &\le \bar{C}, && (3.28)\\
\|S\|_{L^\infty(Q_{T_e})} &\le \bar{C}, && (3.29)\\
\|u(t)\|_{H^2(\Omega)} + \|T(t)\|_{H^1(\Omega)} &\le \bar{C}, && (3.30)\\
\|u\|_{L^\infty(Q_{T_e})} + \|T\|_{L^\infty(Q_{T_e})} &\le \bar{C}. && (3.31)
\end{aligned}
$$

Proof. The derivation of the following estimate for the free energy ψ^* defined in (2.11) and (1.8) is based on a variation of the second law of thermodynamics.

The definition (1.8) of ψ, equation (3.15) and the symmetry of T imply $\nabla_\varepsilon \psi \cdot \varepsilon_t = T \cdot \varepsilon_t = T \cdot u_{xt}$. Thus, if we note (3.14), (3.16) and integrate by parts three times, where we use the boundary conditions (3.4), (3.5), (3.6), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \psi^*(\varepsilon, S, S_x)(t,x)dx - \int_\Omega b(t,x)u_t(t,x)dx \\
&= \int_\Omega (\nabla_\varepsilon \psi \cdot \varepsilon_t + \psi_S S_t + \nu S_x S_{xt})\, dx - \int_\Omega b u_t dx \\
&= \int_\Omega (\psi_S - \nu S_{xx}) S_t \, dx \\
&= c\int_\Omega (\psi_S - \nu S_{xx})\Big(\big((\psi_S - \nu S_{xx})_x(\widetilde{|\hat{S}_x|_\kappa} + \kappa)\big)_x + r|S_x|_\kappa\Big)dx \\
&= -c\int_\Omega ((\psi_S - \nu S_{xx})_x)^2(\widetilde{|\hat{S}_x|_\kappa} + \kappa) - r(\psi_S - \nu S_{xx})|S_x|_\kappa dx.
\end{aligned} \tag{3.32}
$$

Integration with respect to t yields

$$
\begin{aligned}
&\int_\Omega \psi^*(\varepsilon, S, S_x)(t,x)dx - \int_\Omega \psi^*(\varepsilon, S, S_x)(0,x)dx \\
&\le \int_{Q_t} b u_t \, d(\tau, x) + c\int_{Q_t} r(\psi_S - \nu S_{xx})|S_x|_\kappa \, d(\tau,x).
\end{aligned} \tag{3.33}
$$

For brevity we wrote $\varepsilon = \varepsilon(u_x)$. To estimate the second term on the right hand side of this inequality note first that the boundary condition (3.5) implies

$$
\int_\Omega S_{xx}|S_x|_\kappa \, dx = \int_\Omega \left(\int_0^{S_x} |y|_\kappa \, dy\right)_x dx = 0. \tag{3.34}
$$

Note also that by (3.27) and the Sobolev embedding theorem there is a constant C such that

$$
\|b(t)\| \le C, \tag{3.35}
$$

for all $t \in [0, T_e]$. Using this estimate, we obtain from elliptic regularity theory for the solution (w, σ) of the elliptic problem (3.20) – (3.22)

$$
\|w(t)\|_{L^\infty(\Omega)} + \|\sigma(t)\|_{L^\infty(\Omega)} \le C_1(\|w(t)\|_{H^2(\Omega)} + \|\sigma(t)\|_{H^1(\Omega)}) \le C_2\|b(t)\| \le C, \tag{3.36}
$$

for all $0 \le t \le T_e$. Equation (3.23) and (3.36) imply

$$
|\psi_S(\varepsilon(u_x), S)| \le C(1 + \|S\|_{L^1(\Omega)} + |S| + |\hat{\psi}'(S)|).
$$

Using this estimate, (3.34), assumption (1.30) and (3.9) we compute

$$
\begin{aligned}
\left| cr\int_\Omega (\psi_S - \nu S_{xx})|S_x|_\kappa \, dx \right| &= \left| cr\int_\Omega \psi_S |S_x|_\kappa \, dx \right| \\
&\le \bar{r}C \int_\Omega \Big(1 + \|S\|_{L^1(\Omega)} + |S| + |\hat{\psi}'(S)|\Big)|S_x|_\kappa \, dx \\
&\le \bar{r}C\left(1 + \|S\|^2 + \|\hat{\psi}'(S)\|^2 + \|S_x\|^2\right) \\
&\le \bar{r}C\left(1 + \int_\Omega \hat{\psi}(S)dx + \|S_x\|^2\right) \\
&\le \bar{r}C\left(1 + \int_\Omega \psi^*(\varepsilon, S, S_x)dx\right).
\end{aligned} \tag{3.37}
$$

To estimate the first term on the right-hand side of (3.33) we write it in the form

$$\int_{Q_t} bu_t\, d(\tau,x) = \int_0^t \left(\frac{d}{dt} \int_\Omega bu\, dx - \int_\Omega b_t u\, dx \right) d\tau = \int_\Omega bu\, dx \Big|_0^t - \int_{Q_t} b_t u\, d(\tau,x)\,. \quad (3.38)$$

Equation (3.18), the bound for S_0 in (3.27) and (3.36) give

$$\|u(0)\| \le C, \tag{3.39}$$

and (3.35) and (3.39) yield

$$\left| \int_\Omega b(0,x)u(0,x)dx \right| \le C. \tag{3.40}$$

We next use that $u(t)$ vanishes at the boundary and that the definition of $\varepsilon(u_x)$ implies $|\varepsilon(u_x)|^2 \ge \frac{1}{2}|u_x|^2$ to conclude from Poincaré's inequality and from (3.27), (3.35) for every $\mu > 0$

$$\left| \int_\Omega bu\, dx \right| \le \|b\|\ \|u\| \le C\|b\|\ \|u_x\| \le \frac{C^2}{2\mu}\|b\|^2 + \frac{\mu}{2}\|u_x\|^2 \le C_\mu + \mu\|\varepsilon\|^2, \tag{3.41}$$

$$\left| \int_{Q_t} b_t u dx d\tau \right| \le \int_0^t \|b_t\|\ \|u\| d\tau \le C \int_0^t \left(\|b_t\|^2 + \|u_x\|^2 \right) d\tau \le C + C\int_0^t \|\varepsilon\|^2 d\tau. \tag{3.42}$$

By (3.37), (3.38), (3.40) – (3.42) we can estimate the right hand side of (3.33) as

$$\begin{aligned} &\left| \int_{Q_t} bu_t\, d(\tau,x) + c\int_{Q_t} r(\psi_S - \nu S_{xx})|S_x|_\kappa d(\tau,x) \right| \\ &\qquad \le C_\mu + \mu\|\varepsilon\|^2 + C\int_0^t \left(\|\varepsilon\|^2 + \bar r + \bar r \int_\Omega \psi^*(\varepsilon,S,S_x)dx \right) d\tau. \end{aligned} \tag{3.43}$$

From the definition of ψ^* in (2.11) we see that the bound for S_0 in (3.27) and (3.39) imply

$$\left| \int_\Omega \psi^*(\varepsilon,S,S_x)(0,x)dx \right| \le C. \tag{3.44}$$

Combination of (3.33) with (3.43) and (3.44)) results in

$$\int_\Omega \psi^*(\varepsilon,S,S_x)(t,x)dx \le C_\mu + \mu\|\varepsilon\|^2 + C\int_0^t \left(\|\varepsilon\|^2 + \bar r + \bar r\int_\Omega \psi^*(\varepsilon,S,S_x)dx \right) d\tau. \tag{3.45}$$

In order to absorb the term $\mu\|\varepsilon\|^2$ in the right hand side we use assumption (1.30) to find

$$\begin{aligned} \|\varepsilon\|^2 &\le 2\|\bar\varepsilon S\|^2 + 2\|\varepsilon - \bar\varepsilon S\|^2 \\ &\le C\int_\Omega M(\hat\psi(S)+1) + \frac{1}{2}(D(\varepsilon - \bar\varepsilon S))\cdot(\varepsilon - \bar\varepsilon S)\, dx \\ &\le C\int_\Omega \psi^*(\varepsilon,S,S_x)\, dx\,. \end{aligned}$$

We insert this estimate into (3.45) and choose μ sufficiently small to obtain

$$\int_\Omega \psi^*(\varepsilon, S, S_x)\, dx \le C\left(1 + \bar{r}t + \int_0^t (1+\bar{r})\int_\Omega \psi^*(\varepsilon, S, S_x) dx d\tau\right).$$

Applying Gronwall's inequality in the integral form we conclude from this inequality that there is C_{T_e} such that for every $t \in [0, T_e]$

$$\int_\Omega \psi^*(\varepsilon, S, S_x)(t, x)\, dx \le C_{T_e}\,. \tag{3.46}$$

(3.28) follows from this estimate, since (1.30) and (2.11) imply

$$|S|^2 + \frac{\nu}{2}|S_x|^2 \le (M+1)(\psi^*(\varepsilon, S, S_x) + 1).$$

The inequality (3.29) is an immediate consequence of (3.28) and the Sobolev embedding theorem, and (3.30) results from (3.18), (3.19) together with the estimates for S and σ in (3.28), (3.36). Finally, (3.31) is a consequence of (3.30) and the Sobolev embedding theorem. The proof is complete.

Lemma 3.5 *There is a constant C, independent of η but dependent on κ and K, such that for any $\hat{S}$ with $\|\hat{S}\|_{L^\infty(0,T_e;H^1(\Omega))} \le \bar{C}$, for all data (b, r, S_0) satisfying (3.27) and for any $t \in [0, T_e]$ there hold*

$$\int_{Q_t} (\widetilde{|\hat{S}_x|_\kappa} + \kappa)|S_{xxx}|^2 d(\tau, y) \;\; \le \;\; C, \tag{3.47}$$

$$\int_{Q_t} |S_{xx}|^2 d(\tau, y) \;\; \le \;\; C. \tag{3.48}$$

Proof. (3.48) follows immediately from (3.28) and (3.47) by the interpolation inequality. To prove (3.47) we multiply (3.24) by $-S_{xx}$, integrate with respect to x, integrate by parts using the boundary conditions (3.5), (3.6) and note (3.34) to arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|S_x\|^2 + c\nu \int_\Omega (\widetilde{|\hat{S}_x|_\kappa} + \kappa)|S_{xxx}|^2 dx \\
&= c\int_\Omega (\widetilde{|\hat{S}_x|_\kappa} + \kappa)\psi_{Sx} S_{xxx}\, dx - cr\int_\Omega |S_x|_\kappa S_{xx}\, dx \\
&= c\int_\Omega (\widetilde{|\hat{S}_x|_\kappa} + \kappa)\psi_{Sx} S_{xxx}\, dx.
\end{aligned} \tag{3.49}$$

Since by (3.29) the function S is bounded, we conclude from (3.23) and (3.36) that there is a suitable constant C such that

$$|(\psi_S)_x(\varepsilon, S)| = |\hat{\psi}''(S)S_x - c_1 S_x - \sigma_x| \le C(|S_x| + 1). \tag{3.50}$$

Observing (3.50) we obtain by integration of (3.49) with respect to t for $\mu > 0$

$$\begin{aligned}
&\frac{1}{2}\|S_x\|^2 + c\nu \int_{Q_t} (\widetilde{|\hat{S}_x|_\kappa} + \kappa)|S_{xxx}|^2 d(\tau, x) \\
&\le \;\; \frac{1}{2}\|S_{0x}\|^2 + C\int_{Q_t} (\widetilde{|\hat{S}_x|_\kappa} + \kappa)(|S_x| + 1)|S_{xxx}| d(\tau, x) \\
&\le \;\; \frac{1}{2}\|S_{0x}\|^2 + \mu\int_{Q_t} (\widetilde{|\hat{S}_x|_\kappa} + \kappa)|S_{xxx}|^2 d(\tau, x) \\
&\qquad + C_\mu \int_{Q_t} (\widetilde{|\hat{S}_x|_\kappa} + \kappa)(|S_x|^2 + 1) d(\tau, x).
\end{aligned} \tag{3.51}$$

Using that (3.17) implies $\|\widetilde{\hat{S}_x}\|_{L^2(Q_{T_e})} \le \|\hat{S}_x\|_{L^2(Q_{T_e})} \le C\|\hat{S}_x\|_{L^\infty(0,T_e;L^2(\Omega)} \le \bar{C}$, we obtain from Hölder's inequality

$$\begin{aligned}\left|C_\mu \int_{Q_t} (|\widetilde{\hat{S}_x}|_\kappa + \kappa)(|S_x|^2+1)\, d(\tau,x)\right| &\le C_\mu(\|\,|\widetilde{\hat{S}_x}|_\kappa\,\|_{L^2(Q_t)}+1)(\|S_x\|^2_{L^4(Q_t)}+1)\\ &\le C_\mu(\bar{C}+1)(\|S_x\|^2_{L^4(Q_t)}+1). \qquad (3.52)\end{aligned}$$

Since $S_x|_{\partial\Omega}=0$, the Gagliardo-Nirenberg inequality implies

$$\|S_x\|_{L^4(\Omega)} \le C\|S_{xxx}\|^{\frac{1}{8}}\|S_x\|^{\frac{7}{8}},$$

cf. [1, 25]. By (3.28) we have $\|S_x\| \le \bar{C}$, whence Hölder's and Young's inequalities give for $t \le T_e$

$$\begin{aligned}C_\mu(\bar{C}+1)\left(\int_0^t \|S_x\|^4_{L^4(\Omega)}d\tau\right)^{\frac{1}{2}} &\le C_\mu(\bar{C}+1)\left(\int_0^t \|S_{xxx}\|^{\frac{1}{2}}\|S_x\|^{\frac{7}{2}}d\tau\right)^{\frac{1}{2}}\\ &\le C_\mu(\bar{C}+1)\left(\int_0^t \|S_{xxx}\|^2 d\tau\right)^{\frac{1}{8}}\left(t\bar{C}^{\frac{14}{3}}\right)^{\frac{3}{8}}\\ &\le \eta\|S_{xxx}\|^2_{L^2(Q_t)} + C_{\mu\eta}(\bar{C}+1)^{\frac{15}{7}}\bar{C}^{\frac{8}{7}}. \qquad (3.53)\end{aligned}$$

We combine the inequalities (3.51) – (3.53) and obtain (3.47), if we choose $\mu = c\nu/2$ and $\eta = c\nu\kappa/4$.

Lemma 3.6 *To every $\bar{C}$ there is a constant C, independent of η but dependent on κ and K, such that for any $\hat{S}$ with $\|\hat{S}\|_{L^\infty(0,T_e;H^1(\Omega))} \le \bar{C}$, for all data (b,r,S_0) satisfying* (3.27) *and for any $t \in [0,T_e]$*

$$\|S_t\|_{L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))} \le C. \qquad (3.54)$$

We omit the proof of this lemma, since it is similar to the proof of (5.9) in Section 5.

Lemma 3.7 (Aubin - Lions) *Let B_0, B_1 be reflexive Banach spaces and let B be a Banach space such that*

$$B_0 \subset\subset B \subset B_1\,,$$

where $\subset\subset$ denotes compact embedding. Define

$$W = \left\{ f \mid f \in L^{p_0}(0,T_e;B_0),\ f' = \frac{df}{dt} \in L^{p_1}(0,T_e;B_1) \right\}$$

with T_e being a given positive number and $1 < p_0, p_1 < +\infty$. Then the embedding of W in $L^{p_0}(0,T_e;B)$ is compact.

A proof of this lemma can be found in [23, p. 57], for example.

Proof of Theorem 3.2. To construct a sequence of approximate solutions of (3.1) – (3.7) we choose a sequence of functions $(b^n, r^n, S_0^n) \in C^{\frac{\alpha}{4},\alpha}(\bar{Q}_{T_e}) \times C^{\frac{\alpha}{4}}([0,T_e]) \times C^{4+\alpha}(\bar{\Omega})$ satisfying (3.27) with a fixed constant K independent of n, such that

$$\|b^n - b\|_{L^2(Q_{T_e})} + \|r^n - r\|_{L^2(0,T_e)} + \|S_0^n - S^0\|_{H^1(\Omega)} \to 0 \qquad (3.55)$$

for $n \to \infty$. Here (b, r, S_0) are the data given in Theorem 3.2. Now set $S^0 \equiv S_0$. If S^n is known, let $(u^{n+1}, T^{n+1}, S^{n+1})$ be the solution of (3.14) – (3.16), (3.4) – (3.7) with $\hat{S}$ in (3.16) replaced by S^n and with (b^n, r^n, S_0^n) inserted for (b, r, S_0). In (3.17) we set $\eta = \frac{1}{n+1}$. By Lemma 3.4 there is a constant $\bar{C}$ independent of n and of κ such that

$$\|S^n\|_{L^\infty(0,T_e;H^1(\Omega))} + \|u^n\|_{L^\infty(0,T_e;H^2(\Omega))} + \|T^n\|_{L^\infty(0,T_e;H^1(\Omega))} \le \bar{C}. \tag{3.56}$$

Lemmas 3.5 and 3.6 then show that there is a constant C, also independent of n, such that

$$\|S^n\|_{L^2(0,T_e;H^3(\Omega))} + \|S_t^n\|_{L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))} \le C. \tag{3.57}$$

We apply the Aubin-Lions lemma with $p_0 = 2$, $p_1 = \frac{4}{3}$,

$$B_0 = H^3(\Omega), \quad B = C^{2+\alpha}(\bar{\Omega}), \quad B_1 = W^{-1,\frac{4}{3}}(\Omega),$$

where we choose $0 < \alpha < \frac{1}{2}$. The space $H^3(\Omega)$ is compactly embedded in $C^{2+\alpha}(\bar{\Omega})$ and the spaces $L^2(0,T_e;H^3(\Omega))$, $L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))$ are both reflexive. Since the sequence S^n is bounded in $L^2(0,T_e;H^3(\Omega))$ and S_t^n is bounded in $L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))$, by (3.57), this lemma shows that there is a subsequence, still denoted by S^n, and a function $S \in L^2(0,T_e;C^{2+\alpha}(\bar{\Omega}))$ such that for $n \to \infty$

$$\|S^n - S\|_{L^2(0,T_e;C^{2+\alpha}(\bar{\Omega}))} \to 0, \tag{3.58}$$

$$\|S^n(t) - S(t)\| \to 0, \quad \text{for almost all } 0 < t < T_e\,. \tag{3.59}$$

(3.58) and (3.18), (3.19) together with (3.55) and (3.20) – (3.22) imply that there is $(u, T) \in L^2(0,T_e;H^2(\Omega) \times H^1(\Omega))$ such that

$$\|u^n - u\|_{L^2(0,T_e;H^2(\Omega))} + \|T^n - T\|_{L^2(0,T_e;H^1(\Omega))} \to 0. \tag{3.60}$$

From (3.56) – (3.59) it follows for the limit function in a well known way that

$$S \in L^\infty(0,T_e;H^1(\Omega)) \cap L^2(0,T_e;H^3(\Omega)), \quad S_t \in L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega)), \tag{3.61}$$

with

$$\|S\|_{L^\infty(0,T_e;H^1(\Omega))} \le \bar{C}. \tag{3.62}$$

As a consequence of this estimate and the Sobolev embedding theorem we also have $S \in L^\infty(Q_{T_e})$ with

$$\|S\|_{L^\infty(Q_{T_e})} \le \bar{C}. \tag{3.63}$$

By (3.57) we can choose the subsequence such that

$$S_{xxx}^n \rightharpoonup S_{xxx} \quad \text{in } L^2(Q_{T_e}). \tag{3.64}$$

Similarly, (3.60) and (3.56) imply

$$\begin{gathered} u \in L^\infty(0,T_e;H^2(\Omega)), \qquad T \in L^\infty(0,T_e;H^1(\Omega)), \\ \|u\|_{L^\infty(0,T_e;H^2(\Omega))} + \|T\|_{L^\infty(0,T_e;H^1(\Omega))} \le \bar{C}. \end{gathered} \tag{3.65}$$

In order to show that (u, T, S) is a solution of (3.1) – (3.7) we derive several convergence estimates. Noting that the convolution operator in (3.17) satisfies $\|\tilde{v}\|_{L^2(Q_{T_e})} \leq \|v\|_{L^2(Q_{T_e})}$ and that $\eta = \frac{1}{n+1} \to 0$, we infer from (3.58) that

$$\begin{aligned} \|\widetilde{S_x^n} - S_x\|_{L^2(Q_{T_e})} &\leq \|\widetilde{S_x^n} - \widetilde{S_x}\|_{L^2(Q_{T_e})} + \|\widetilde{S_x} - S_x\|_{L^2(Q_{T_e})} \\ &\leq \|S_x^n - S_x\|_{L^2(Q_{T_e})} + \|\widetilde{S_x} - S_x\|_{L^2(Q_{T_e})} \to 0. \end{aligned} \tag{3.66}$$

Because $v \mapsto |v|_\kappa$ is a continuous mapping on $L^2(Q_{T_e})$ we conclude from (3.58) and from this estimate that

$$\| \, |S_x^n|_\kappa - |S|_\kappa \|_{L^2(Q_{T_e})} \to 0, \qquad \| \, |\widetilde{S_x^n}|_\kappa - |S_x|_\kappa \, \|_{L^2(Q_{T_e})} \to 0. \tag{3.67}$$

Together with (3.64) we obtain

$$(|\widetilde{S_x^n}|_\kappa + \kappa) S_{xxx}^{n+1} \rightharpoonup (|S_x|_\kappa + \kappa) S_{xxx} \,, \quad \text{weakly in } L^1(Q_{T_e}). \tag{3.68}$$

To show convergence of $\psi_S(\varepsilon^n, S^n)_x$ we write

$$\hat{\psi}''(S^n) S_x^n - \hat{\psi}''(S) S_x = \hat{\psi}'''(\xi)(S^n - S) S_x^n + \hat{\psi}''(S)(S_x^n - S_x) = I_1 + I_2 \,,$$

where $\xi(t, x)$ is a suitable number between $S^n(t, x)$ and $S(t, x)$. Since S^n is uniformly bounded in $L^\infty(Q_{T_e})$, by (3.29), and S belongs to $L^\infty(Q_{T_e})$, by (3.63), we conclude that also $\hat{\psi}'''(\xi)$ is uniformly bounded on Q_{T_e} with respect to n, hence (3.56) and (3.58) yield

$$\begin{aligned} \|I_1\|_{L^2(Q_{T_e})}^2 &\leq C \int_0^{T_e} \|S^n - S\|_{L^\infty(\Omega)}^2 \|S_x^n\|^2 d\tau \\ &\leq C \|S^n - S\|_{L^2(0,T_e;L^\infty(\Omega))}^2 \|S_x^n\|_{L^\infty(0,T_e;L^2(\Omega))}^2 \to 0. \end{aligned} \tag{3.69}$$

Moreover, (3.29) and (3.58) imply $\|I_2\|_{L^2(Q_{T_e})} \to 0$ for $n \to \infty$. Since $\psi_S(\varepsilon, S)_x = \hat{\psi}''(S) S_x - T_x \cdot \bar{\varepsilon}$, these relations and (3.60), (3.67) yield

$$(|\widetilde{S_x^n}|_\kappa + \kappa) \psi_S(\varepsilon^{n+1}, S^{n+1})_x \to (|S_x|_\kappa + \kappa) \psi_S(\varepsilon, S)_x \,, \quad \text{in } L^1(Q_{T_e}). \tag{3.70}$$

Finally, from (3.67) and (3.27), (3.55) we deduce

$$\begin{aligned} &\|r^n |S_x^n|_\kappa - r |S_x|_\kappa\|_{L^2(Q_{T_e})} \\ &\leq \|r^n\|_{L^\infty(0,T_e)} \| \, |S_x^n|_\kappa - |S_x|_\kappa \|_{L^2(Q_{T_e})} + \|r^n - r\|_{L^2(0,T_e)} \| \, |S_x| \, \|_{L^\infty(0,T_e;L^2(\Omega))} \\ &\to 0. \end{aligned} \tag{3.71}$$

Since (u^n, T^n, S^n) is a strong solution of (3.14) – (3.16), (3.4) – (3.7), we conclude from (3.55), (3.58), (3.60), and from (3.68), (3.70), (3.71) that (u, T, S) satisfies (3.1), (3.2), (3.4), (3.5) and (3.8). Hence, it is a weak solution of (3.1) – (3.7). Relation (3.61) implies that S belongs to the function spaces given in Theorem 3.2, the estimates (3.10), (3.12), (3.13) follow from (3.65), (3.62), (3.63), and the inequality (3.11) is a consequence of (3.10) and the Sobolev embedding theorem. This completes the proof of Theorem 3.2.

4 Energy inequality

The estimate (3.12) for first order derivatives of the component S in weak solutions of (3.1) – (3.7) is independent of $\kappa > 0$. Later we also need estimates for higher derivatives, which are independent of κ. Their proof in Section 5 is based on an energy inequality, in fact the limit version of the integrated form of (3.49), which we show to hold in this section.

Lemma 4.1 *The weak solution* (u, T, S) *of* (3.1) – (3.7) *given in Theorem 3.2 satisfies for almost all* $0 < t < T_e$

$$\frac{1}{2}\|S_x(t)\|^2 + c\nu \int_{Q_t} (|S_x|_\kappa + \kappa)|S_{xxx}|^2 \, d(\tau, x) \\ \leq \frac{1}{2}\|S_{0x}\|^2 + c \int_{Q_t} (|S_x|_\kappa + \kappa)\psi_{Sx} S_{xxx} \, d(\tau, x). \tag{4.1}$$

Proof. Let (u^n, T^n, S^n) be the sequence of functions constructed in the proof of Theorem 3.2. By definition, S^n satisfies the equation (3.49) with $\hat{S}$ replaced by S^{n-1}. Integration of this equation yields

$$\frac{1}{2}\|S_x^n(t)\|^2 + c\nu \int_{Q_t} (|\widetilde{S_x^{n-1}}|_\kappa + \kappa)|S_{xxx}^n|^2 d(\tau, x) \\ = \frac{1}{2}\|S_{0x}\|^2 + c \int_{Q_t} (|\widetilde{S_x^{n-1}}|_\kappa + \kappa)(\psi_S^n)_x S_{xxx}^n \, d(\tau, x), \tag{4.2}$$

where $(\psi_S^n)_x = \hat{\psi}''(S^n)S_x^n - c_1 S_x^n - \sigma_x$, by (3.23). Let LH_n and RH_n denote the left hand side and the right hand side of this equation. We show that

$$\frac{1}{2}\|S_x(t)\|^2 + c\nu \int_{Q_t} (|\widetilde{S_x}|_\kappa + \kappa)|S_{xxx}|^2 d(\tau, x) \leq \liminf_{n\to\infty} LH_n \, , \tag{4.3}$$

$$\liminf_{n\to\infty} RH_n = \lim_{n\to\infty} RH_n = \int_{Q_t} (|S_x|_\kappa + \kappa)\psi_{Sx} S_{xxx} \, d(\tau, x). \tag{4.4}$$

Since $\liminf_{n\to\infty} LH_n = \liminf_{n\to\infty} RH_n$, by (4.2), Lemma 4.1 is an immediate consequence of these relations.

To prove (4.3) note that (3.59), (3.64) and the lower semi-continuity of the L^2–norm with respect to weak convergence imply

$$\|S_x(t)\|^2 = \lim_{n\to\infty} \|S_x^n(t)\|^2, \qquad \|S_{xxx}\|^2_{L^2(Q_t)} \leq \liminf_{n\to\infty} \|S_{xxx}^n\|^2_{L^2(Q_t)} \, , \tag{4.5}$$

for almost all $0 < t < T_e$. Moreover, the estimate

$$\int_{Q_t} (|\widetilde{S_x^{n-1}}|_\kappa + \kappa)|S_{xxx}^n|^2 d(\tau, x) \leq C,$$

which follows from (3.47) and the definition of the sequence S^n, implies that there is $\chi \in L^2(Q_{T_e})$ and a subsequence such that

$$(|\widetilde{S_x^{n-1}}|_\kappa)^{\frac{1}{2}} S_{xxx}^n \rightharpoonup \chi, \quad \text{weakly in } L^2(Q_{T_e}).$$

Again using the weak lower semi-continuity of the L^2–norm, we infer from this relation and from (4.5) that

$$\frac{1}{2}\|S_x(t)\|^2 + c\nu \int_{Q_t} |\chi|^2 + \kappa |S_{xxx}|^2 d(\tau,x) \le \liminf_{n\to\infty} LH_n \,. \tag{4.6}$$

For the proof of (4.3) it remains to identify the function χ. To this end we use the elementary inequality $|\sqrt{|x|} - \sqrt{|y|}\,| \le \sqrt{|x-y|}$, $x, y \in \mathbb{R}$, and the uniform Lipschitz continuity of the function $y \to |y|_\kappa$ to compute

$$\left| (|\widetilde{S_x^{n-1}}|_\kappa)^{\frac{1}{2}} - (|S_x|_\kappa)^{\frac{1}{2}} \right|^4 \le \left| |\widetilde{S_x^{n-1}}|_\kappa - |S_x|_\kappa \right|^2 \le C \left| \widetilde{S_x^{n-1}} - S_x \right|^2 .$$

We integrate and note (3.66) to conclude

$$\|(|\widetilde{S_x^{n-1}}|_\kappa)^{\frac{1}{2}} - (|S_x|_\kappa)^{\frac{1}{2}}\|^4_{L^4(Q_{T_e})} \le C \|\widetilde{S_x^{n-1}} - S_x\|^2_{L^2(Q_{T_e})} \to 0$$

for $n \to \infty$. Consequently we have that

$$(|\widetilde{S_x^{n-1}}|_\kappa)^{\frac{1}{2}} \to (|S_x|_\kappa)^{\frac{1}{2}}, \tag{4.7}$$

strongly in $L^4(Q_{T_e})$ as $n \to \infty$. On the other hand, by (3.64) we have $S^n_{xxx} \rightharpoonup S_{xxx}$ in $L^2(Q_{T_e})$, which implies for $\varphi, \psi \in L^4(Q_{T_e})$ that

$$\int_{Q_{T_e}} (S^n_{xxx}\varphi)\psi \, d(\tau,x) \to \int_{Q_{T_e}} (S_{xxx}\varphi)\psi \, d(\tau,x).$$

This means that $S^n_{xxx}\varphi \rightharpoonup S^n_{xxx}$, weakly in the dual space $L^{\frac{4}{3}}(Q_{T_e})$ of $L^4(Q_{T_e})$, which together with (4.7) yields

$$\int_{Q_{T_e}} \left(|\widetilde{S_x^{n-1}}|_\kappa \right)^{\frac{1}{2}} S^n_{xxx}\varphi \, d(\tau,x) \to \int_{Q_{T_e}} (|S_x|_\kappa)^{\frac{1}{2}} S_{xxx}\varphi \, d(\tau,x).$$

Thus, $(|\widetilde{S_x^{n-1}}|_\kappa)^{\frac{1}{2}} S^n_{xxx} \rightharpoonup (|S_x|_\kappa)^{\frac{1}{2}} S_{xxx}$ weakly in $L^{\frac{4}{3}}(Q_{T_e})$. Since $(|\widetilde{S_x^{n-1}}|_\kappa)^{\frac{1}{2}} S^n_{xxx} \rightharpoonup \chi$ in $L^2(Q_{T_e})$ implies $(|\widetilde{S_x^{n-1}}|_\kappa)^{\frac{1}{2}} S^n_{xxx} \rightharpoonup \chi$ in $L^{\frac{4}{3}}(Q_{T_e})$, it follows that

$$\chi = (|S_x|_\kappa)^{\frac{1}{2}} S_{xxx}.$$

Insertion of this equation into (4.6) yields (4.3).

To prepare the proof of (4.4) we derive three convergence relations. Note first that the Gagliardo-Nirenberg inequality

$$\|f\|_{L^\infty(\Omega)} \le C \|f_x\|^{\frac{1}{2}} \|f\|^{\frac{1}{2}},$$

holds for every $f \in H^1_0(\Omega)$. We can therefore apply it to the function S_x. Noting (3.58), which yields that $S^n_x \to S_x$ strongly in $L^2(0, T_e; H^1(\Omega))$, we obtain for $n \to \infty$

$$\int_0^t \|S^n_x - S_x\|^4_{L^\infty(\Omega)} d\tau \le C \int_0^t \|S^n_{xx} - S_{xx}\|^2 d\tau \to 0. \tag{4.8}$$

Using that $|(|y|_\kappa)'| \leq C$ we conclude from this estimate

$$\int_0^t \| \, |S_x^n|_\kappa - |S_x|_\kappa \|_{L^\infty(\Omega)}^4 d\tau \leq C \int_0^t \|S_x^n - S_x\|_{L^\infty(\Omega)}^4 d\tau \to 0. \tag{4.9}$$

Furthermore, by virtue of (3.29) we obtain the third convergence relation

$$\int_0^t \|S^n - S\|_{L^\infty(\Omega)}^4 d\tau \leq C \int_0^t \|S^n - S\|_{L^\infty(\Omega)}^2 d\tau \to 0. \tag{4.10}$$

Now we can prove (4.4). To this end we insert $(\psi_S^n)_x = (\hat{\psi}_S^n)_x - (c_1 S_x^n + \sigma_x)_x$ into the left hand side of this relation and consider first the term with $(\hat{\psi}_S^n)_x$. Write

$$\begin{aligned} \int_{Q_t} |\widetilde{S_x^{n-1}}|_\kappa (\hat{\psi}_S^n)_x S_{xxx}^n d(\tau, x) &= \int_{Q_t} (|\widetilde{S_x^{n-1}}|_\kappa - |S_x|_\kappa)(\hat{\psi}_S^n)_x S_{xxx}^n d(\tau, x) \\ &\quad + \int_{Q_t} |S_x|_\kappa (\hat{\psi}''(S^n) - \hat{\psi}''(S)) S_x^n S_{xxx}^n d(\tau, x) \\ &\quad + \int_{Q_t} |S_x|_\kappa \hat{\psi}''(S)(S_x^n - S_x) S_{xxx}^n d(\tau, x) \\ &\quad + \int_{Q_t} |S_x|_\kappa \hat{\psi}''(S) S_x S_{xxx}^n \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.11}$$

We shall prove that each term on the right-hand side of (4.11) converges. For I_1 the Hölder and triangle inequalities and the estimates (3.28), (3.47), (4.9) imply

$$\begin{aligned} |I_1| &\leq \int_0^t \| \, |\widetilde{S_x^{n-1}}|_\kappa - |S_x|_\kappa \, \|_{L^\infty(\Omega)} \|S_x^n\| \, \|S_{xxx}^n\| d\tau \\ &\leq C \left(\int_0^t \| \, |\widetilde{S_x^{n-1}}|_\kappa - |S_x|_\kappa \, \|_{L^\infty(\Omega)}^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|S_{xxx}^n\|^2 d\tau \right)^{\frac{1}{2}} \\ &\leq C \left(\int_0^t \| \, |\widetilde{S_x^{n-1}}|_\kappa - |S_x|_\kappa \, \|_{L^\infty(\Omega)}^2 d\tau \right)^{\frac{1}{2}} \to 0, \end{aligned} \tag{4.12}$$

as $n \to \infty$. From the Gagliardo-Nirenberg inequality in the form

$$\|S_x\|_{L^\infty(\Omega)} \leq C \|S_{xxx}\|^{\frac{1}{4}} \|S_x\|^{\frac{3}{4}}$$

and from (3.28), (3.47) we infer

$$\begin{aligned} \int_0^t \|S_x^n |S_x|_\kappa\|_{L^\infty(\Omega)}^4 d\tau &\leq \int_0^t \|S_x^n\|_{L^\infty(\Omega)}^4 \| \, |S_x|_\kappa\|_{L^\infty(\Omega)}^4 d\tau \\ &\leq C \int_0^t \|S_{xxx}\| \|S_{xxx}^n\| d\tau \\ &\leq C \int_0^t \left(\|S_{xxx}\|^2 + \|S_{xxx}^n\|^2 \right) d\tau \leq C. \end{aligned}$$

By assumption $\hat{\psi}'''(S)$ is continuous. Consequently, the last estimate together with the mean value theorem, (3.29) and (4.10) yield

$$
\begin{aligned}
|I_2| &\leq \int_0^t \|\hat{\psi}''(S^n) - \hat{\psi}''(S)\|_{L^\infty(\Omega)} \| \, |S_x|_\kappa S_x^n\| \, \|S_{xxx}^n\| d\tau \\
&\leq C \int_0^t \|S^n - S\|_{L^\infty(\Omega)} \| \, |S_x|_\kappa S_x^n\| \, \|S_{xxx}^n\| d\tau \\
&\leq C \left(\int_0^t \|S^n - S\|_{L^\infty(\Omega)}^4 d\tau \right)^{\frac{1}{4}} \left(\int_0^t \| \, |S_x|_\kappa S_x^n\|_{L^\infty(\Omega)}^4 d\tau \right)^{\frac{1}{4}} \left(\int_0^t \|S_{xxx}^n\|^2 d\tau \right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^t \|S^n - S\|_{L^\infty(\Omega)}^4 d\tau \right)^{\frac{1}{4}} \to 0, \qquad (4.13)
\end{aligned}
$$

as $n \to \infty$. Proceeding similarly as in the derivation of (4.12) we obtain from (4.8) that

$$
|I_3| \leq C \int_0^t \|S_x^n - S_x\|_{L^\infty(\Omega)} \| \, S_x\| \, \|S_{xxx}^n\| d\tau \to 0, \qquad (4.14)
$$

as $n \to \infty$. Finally, the Sobolev embedding theorem and (3.29) yield

$$
\| \, |S_x|_\kappa \psi''(S) S_x\|_{L^2(Q_{T_e})} \leq C\|S_x^2\|_{L^2(Q_{T_e})} = C\|S_x\|_{L^4(Q_{T_e})}^2 \leq C \int_0^t \|S_x\|_{H^1(\Omega)}^2 d\tau \leq C,
$$

from which we know that $|S_x|_\kappa \psi''(S) S_x$ is an element of $L^2(Q_{T_e})$ and can be regarded as a test function. Since S^n converges weakly in $L^2(0, T_e; H^3(\Omega))$, one obtains for $n \to \infty$ that

$$
I_4 \to \int_{Q_t} |S_x|_\kappa \hat{\psi}''(S) S_x S_{xxx} d(\tau, x). \qquad (4.15)
$$

Combination of (4.11) – (4.15) yields

$$
\int_{Q_t} |\widetilde{S_x^{n-1}}|_\kappa (\hat{\psi}_S^n)_x S_{xxx}^n \, d(\tau, x) \to \int_{Q_t} |S_x|_\kappa (\hat{\psi}_S)_x S_{xxx} \, d(\tau, x).
$$

In a similar but simpler way we obtain

$$
\begin{aligned}
\int_{Q_t} |\widetilde{S_x^{n-1}}|_\kappa (c_1 S_x^n + \sigma_x) S_{xxx}^n \, d(\tau, x) &\to \int_{Q_t} |S_x|_\kappa (c_1 S_x + \sigma_x) S_{xxx} \, d(\tau, x) \\
\int_{Q_t} \kappa (\psi_S^n)_x S_{xxx}^n \, d(\tau, x) &\to \int_{Q_t} \kappa (\psi_S)_x S_{xxx} \, d(\tau, x).
\end{aligned}
$$

The last three relations imply (4.4). This completes the proof of Lemma 4.1.

5 A Priori Estimates independent of κ

In this section we establish a-priori estimates for solutions $(u^\kappa, T^\kappa, S^\kappa)$ of the regularized problem (3.1) – (3.7), which hold independently of κ. Since we consider the limit $\kappa \to 0$, we assume that

$$
0 < \kappa \leq 1. \qquad (5.1)
$$

Let $(u^\kappa, T^\kappa, S^\kappa)$ be the weak solution of (3.1) – (3.7) given by Theorem 3.2 to fixed data $S_0 \in H^1(\Omega)$, $r \in L^\infty(0, T_e)$ and $b \in L^2(Q_{T_e})$ with $b_t \in L^2(Q_{T_e})$.

Lemma 5.1 *There is a constant C independent of κ such that for any $t \in [0, T_e]$ there hold*

$$\int_{Q_t} (|S_x^\kappa|_\kappa + \kappa)|S_{xxx}^\kappa|^2 d(\tau, y) \leq C\left(1 + \int_0^t \|S_{xx}^\kappa\|^{\frac{1}{2}} d\tau\right), \tag{5.2}$$

$$\| \, |S_x^\kappa|_\kappa S_{xxx}^\kappa \|_{L^{\frac{4}{3}}(Q_t)} \leq C\left(1 + \int_0^t \|S_{xx}^\kappa\|^{\frac{1}{2}} d\tau\right)^{\frac{1}{2}}. \tag{5.3}$$

Proof. To prove (5.2) we use (3.50) in the energy inequality (4.1) and apply (3.12) and Young's inequality to obtain

$$\begin{aligned}
&\frac{1}{2}\|S_x^\kappa\|^2 + c\nu \int_{Q_t} (|S_x^\kappa|_\kappa + \kappa)|S_{xxx}^\kappa|^2 d(\tau, x) \\
&\leq C\left(\int_{Q_t} (|S_x^\kappa|_\kappa + \kappa)(|S_x^\kappa| + 1)|S_{xxx}^\kappa| d(\tau, x) + 1\right) \\
&\leq \mu \int_{Q_t} (|S_x^\kappa|_\kappa + \kappa)|S_{xxx}^\kappa|^2 d(\tau, x) + C_\mu \left(\int_{Q_t} (|S_x^\kappa|_\kappa + \kappa)|S_x^\kappa|^2 d(\tau, x) + 1\right).
\end{aligned} \tag{5.4}$$

The last term on the the right-hand side is estimated using Hölder's inequality and (3.9), (3.12). This yields

$$\begin{aligned}
\left|C_\mu \int_{Q_t} (|S_x^\kappa|_\kappa + \kappa)|S_x^\kappa|^2 d(\tau, x)\right| &\leq C_\mu \int_0^t (\| \, |S_x^\kappa|_\kappa \, \| + 1)\|S_x^\kappa\|_{L^4(\Omega)}^2 d\tau \\
&\leq C_\mu \int_0^t (\|S_x^\kappa\| + 1)\|S_x^\kappa\|_{L^4(\Omega)}^2 d\tau \\
&\leq C_\mu \int_0^t \|S_x^\kappa\|_{L^4(\Omega)}^2 d\tau.
\end{aligned} \tag{5.5}$$

Noting that $S_x^\kappa|_{\partial\Omega} = 0$, we have the Gagliardo-Nirenberg inequality in the form

$$\|S_x^\kappa\|_{L^4(\Omega)} \leq C\|S_{xx}^\kappa\|^{\frac{1}{4}}\|S_x^\kappa\|^{\frac{3}{4}}, \tag{5.6}$$

hence, again using (3.12),

$$C_\mu\|S_x^\kappa\|_{L^4(\Omega)}^2 \leq C_\mu\|S_{xx}^\kappa\|^{\frac{1}{2}}\|S_x^\kappa\|^{\frac{3}{2}} \leq C_\mu\|S_{xx}^\kappa\|^{\frac{1}{2}}. \tag{5.7}$$

The inequality (5.2) results if we choose $\mu = \frac{1}{2}c\nu$ in (5.4) and estimate the right hand side of (5.4) with (5.5) and (5.7).

To verify (5.3) let $2 > p \geq 1, q = \frac{2}{p}$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Hölder's inequality yields

$$\begin{aligned}
&\int_{Q_t} (|S_x^\kappa|_\kappa |S_{xxx}^\kappa|)^p \, d(\tau, x) \\
&= \int_{Q_t} (|S_x^\kappa|_\kappa)^{\frac{p}{2}} \left((|S_x^\kappa|_\kappa)^{\frac{p}{2}} \, |S_{xxx}^\kappa|^p\right) d(\tau, x) \\
&\leq \left(\int_{Q_t} (|S_x^\kappa|_\kappa)^{\frac{pq'}{2}} \, d(\tau, x)\right)^{\frac{1}{q'}} \left(\int_{Q_t} (|S_x^\kappa|_\kappa)^{\frac{pq}{2}} \, |S_{xxx}^\kappa|^{pq} d(\tau, x)\right)^{\frac{1}{q}} \\
&\leq \left(\int_{Q_t} (|S_x^\kappa|_\kappa)^{\frac{p}{2-p}} \, d(\tau, x)\right)^{\frac{2-p}{2}} \left(\int_{Q_t} |S_x^\kappa|_\kappa |S_{xxx}^\kappa|^2 d(\tau, x)\right)^{\frac{p}{2}}.
\end{aligned} \tag{5.8}$$

The estimates (3.9), (3.12) and (5.2) show that the right hand side of (5.8) is bounded for $\frac{p}{2-p} \le 2$, i.e. $p \le \frac{4}{3}$. This proves (5.3) and completes the proof of the lemma.

Lemma 5.2 *There is a constant C independent of κ such that*

$$\|\partial_t S^\kappa\|_{L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))} \le C, \tag{5.9}$$

$$\|S^\kappa_{xx}\|_{L^2(Q_{T_e})} \le C. \tag{5.10}$$

Proof. Because of the boundary condition (3.6) we obtain by partial integration

$$\big((\psi_S - \nu S^\kappa_{xx})_x \,, S^\kappa_x\big)_{Q_{T_e}} = (\psi_S, S^\kappa_{xx})_{Q_{T_e}} + \nu \|S^\kappa_{xx}\|^2_{L^2(Q_{T_e})}. \tag{5.11}$$

Since (3.11) and (3.13) imply

$$\|\psi_S\|_{L^\infty(Q_{T_e})} = \|\hat{\psi}'(S^\kappa) - T^\kappa \cdot \bar{\varepsilon}\|_{L^\infty(Q_{T_e})} \le C,$$

it follows that

$$|(\psi_S, S^\kappa_{xx})_{Q_{T_e}}| \le \frac{\nu}{4}\|S^\kappa_{xx}\|^2_{L^2(Q_{T_e})} + C_\nu. \tag{5.12}$$

To estimate the left hand side of (5.11) we use the Gagliardo-Nirenberg inequality

$$\|S^\kappa_x\|_{L^{\frac{8}{3}}(\Omega)} \le C\|S^\kappa_{xx}\|^{\frac{1}{8}}\|S^\kappa_x\|^{\frac{7}{8}},$$

and obtain together with (3.12)

$$\|(S^\kappa_x)^2\|_{L^{\frac{4}{3}}(Q_{T_e})} = \left(\int_0^{T_e} \|S^\kappa_x\|^{\frac{8}{3}}_{L^{\frac{8}{3}}(\Omega)} d\tau\right)^{\frac{3}{4}} \le C\left(\int_0^{T_e} \|S^\kappa_{xx}\|^{\frac{1}{3}} d\tau\right)^{\frac{3}{4}}.$$

We employ this estimate and (3.50), (3.9), (5.3), (3.12), (5.2) to compute

$$\begin{aligned}
&\|(\psi_S - \nu S^\kappa_{xx})_x\, S^\kappa_x\|_{L^{\frac{4}{3}}(Q_{T_e})} \le \|(\psi_S - \nu S^\kappa_{xx})_x(|S^\kappa_x|_\kappa + \kappa)\|_{L^{\frac{4}{3}}(Q_{T_e})} \\
&\quad \le C\|(|S^\kappa_x|+1)(|S^\kappa_x|_\kappa + \kappa)\|_{L^{\frac{4}{3}}(Q_{T_e})} + \nu\|S^\kappa_{xxx}|S^\kappa_x|_\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})} + \nu\|\kappa S^\kappa_{xxx}\|_{L^{\frac{4}{3}}(Q_{T_e})} \\
&\quad \le C\|(S^\kappa_x)^2\|_{L^{\frac{4}{3}}(Q_{T_e})} + C\|S^\kappa_x\|_{L^{\frac{4}{3}}(Q_{T_e})} + C + C\left(1 + \int_0^{T_e} \|S^\kappa_{xx}\|^{\frac{1}{2}} d\tau\right)^{\frac{1}{2}} \\
&\qquad\qquad + C\|\sqrt{\kappa} S^\kappa_{xxx}\|_{L^2(Q_{T_e})} \\
&\quad \le C\left(\int_0^{T_e} \|S^\kappa_{xx}\|^{\frac{1}{3}} d\tau\right)^{\frac{3}{4}} + C + \left(\int_0^{T_e} \|S^\kappa_{xx}\|^{\frac{1}{2}} d\tau\right)^{\frac{1}{2}} \\
&\quad \le C_\nu + \frac{\nu}{4}\int_0^{T_e} \|S^\kappa_{xx}\|^2 d\tau = C_\nu + \frac{\nu}{4}\|S^\kappa_{xx}\|^2_{L^2(Q_{T_e})}.
\end{aligned} \tag{5.13}$$

We also used Hölder's and Young's inequalities. Combination of this estimate with (5.11) and (5.12) yields (5.10).

To prove (5.9) we infer from (3.8) and (3.12), (5.13) that for all $\varphi \in C_0^\infty(Q_{T_e})$

$$\begin{aligned}
|(S_t^\kappa, \varphi)_{Q_{T_e}}| &= \left| c\big((\psi_S - \nu S_{xx}^\kappa)_x (|S_x^\kappa|_\kappa + \kappa), \varphi_x\big)_{Q_{T_e}} - c(r|S_x^\kappa|_\kappa, \varphi)_{Q_{T_e}} \right| \\
&\le c\|(\psi_S - \nu S_{xx}^\kappa)_x (|S_x^\kappa|_\kappa + \kappa)\|_{L^{\frac{4}{3}}(Q_{T_e})} \|\varphi_x\|_{L^4(Q_{T_e})} \\
&\qquad + c\bar{r} \| \, |S_x^\kappa|_\kappa \|_{L^{\frac{4}{3}}(Q_{T_e})} \|\varphi\|_{L^4(Q_{T_e})} \\
&\le C \left(1 + \bar{r} + \|S_{xx}^\kappa\|^2_{L^2(Q_{T_e})}\right) \|\varphi\|_{L^4(0,T_e;W^{1,4}(Q_{T_e}))}, \qquad (5.14)
\end{aligned}$$

where we used the notation $\bar{r} = \|r\|_{L^\infty(0,T_e)}$. This means that

$$\|S_t^\kappa\|_{L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))} \le C\left(1 + \bar{r} + \|S_{xx}^\kappa\|^2_{L^2(Q_{T_e})}\right), \qquad (5.15)$$

which together with (5.10) implies (5.9). The proof is complete.

We combine Lemmas 5.1 and 5.2 to obtain in an obvious way

Corollary 5.3 *There is a constant C, independent of κ, such that for any $t \in [0, T_e]$*

$$\begin{aligned}
\int_{Q_t} (|S_x^\kappa|_\kappa + \kappa)|S_{xxx}^\kappa|^2 d(\tau, y) &\le C, \qquad (5.16) \\
\| \, |S_x^\kappa|_\kappa S_{xxx}^\kappa \|_{L^{\frac{4}{3}}(Q_t)} &\le C. \qquad (5.17)
\end{aligned}$$

6 Existence of solutions to the phase field model

In this section we use the a-priori estimates established in the previous sections to study the convergence of the solutions $(u^\kappa, T^\kappa, S^\kappa)$ of the regularized problem (3.1) – (3.7) for $\kappa \to 0$, thereby proving Theorem 1.3. Besides Lemma 3.7 we need another well known result, which we state first: *

Theorem 6.1 (Egorov) *Let (Γ, Σ, μ) be a measure space with $\mu(\Gamma) < \infty$, let $f, f^1, f^2, f^3, \cdots$ be real valued, measurable functions on Γ, and assume that $f^j(x) \to f(x)$ as $j \to \infty$ for almost every $x \in \Gamma$.*

Then, for every $\varepsilon > 0$ there is a subset $M_\varepsilon \subset \Gamma$ with $\mu(M_\varepsilon) > \mu(\Gamma) - \varepsilon$, such that $f^j(x)$ converges to $f(x)$ uniformly on M_ε. That is, for every $\delta > 0$ there is an N_δ such that when $j > N_\delta$ we have that for every $x \in M_\varepsilon$

$$|f^j(x) - f(x)| < \delta.$$

A proof of Theorem 6.1 can be found in [21, p. 16], for example.

We start by deriving some results about strong and pointwise convergence for the family of weak solutions $(u^\kappa, T^\kappa, S^\kappa)$ of (3.1) – (3.7) constructed in Theorem 3.2:

Lemma 6.2 *Let $0 < \alpha < \frac{1}{2}$. There is a sequence $\kappa_n \to 0$ and a function $S \in L^2(0, T_e; C^{1+\alpha}(\bar{\Omega}))$ with*

$$S \in L^2(0,T_e;H^2(\Omega)) \cap L^\infty(0,T_e;H^1(\Omega)) \cap L^\infty(Q_{T_e}), \quad S_t \in L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega)),$$

such that the sequence S^{κ_n}, still denoted by S^κ, satisfies

$$\|S^\kappa - S\|_{L^2(0,T_e;C^{1+\alpha}(\bar\Omega))} \to 0, \tag{6.1}$$

$$\|S^\kappa(t) - S(t)\|_{C^{1+\alpha}(\bar\Omega)}, \to 0, \quad a.e.\ in\ [0,T_e], \tag{6.2}$$

$$\|\,|S_x^\kappa|_\kappa - |S_x|\,\|_{L^2(0,T_e;L^\infty(\Omega))} \to 0. \tag{6.3}$$

Proof. The estimates (3.12), (5.9), (5.10) imply that the functions S^κ are uniformly bounded in $L^2(0,T_e;H^2(\Omega))$ and that the time derivatives $\frac{\partial}{\partial t}S^\kappa$ are uniformly bounded in $L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))$ for $\kappa \to 0$. Based on these estimates we apply Lemma 3.7. We choose $p_0 = 2$, $p_1 = \frac{4}{3}$ and

$$B_0 = H^2(\Omega), \quad B = C^{1+\alpha}(\bar\Omega), \quad B_1 = W^{-1,\frac{4}{3}}(\Omega).$$

The spaces $H^2(\Omega)$ and $W^{-1,\frac{4}{3}}(\Omega)$ are reflexive and the Sobolev embedding theorem implies that $H^2(\Omega)$ is compactly embedded in $C^{1+\alpha}(\bar\Omega)$ for $0<\alpha<\frac{1}{2}$. From Lemma 3.7 we thus conclude that there is a subsequence, which we still denote by S^κ, which converges strongly in $L^{p_0}(0,T_e;B) = L^2(0,T_e;C^{1+\alpha}(\bar\Omega))$ to a function S for $\kappa\to 0$. Clearly, we can assure that $\|S^\kappa(t)-S(t)\|_{C^{1+\alpha}(\bar\Omega)} \to 0$ for almost all t in $[0,T_e]$ by going over to another subsequence, if necessary. This proves (6.1) and (6.2). We have $S \in L^2(0,T_e;H^2(\Omega))$ and $S_t \in L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega))$, since the sequences S^κ and S_t^κ are uniformly bounded in these spaces. $S \in L^\infty(0,T_e;H^1(\Omega)) \subseteq L^\infty(Q_{T_e})$ is implied by (3.12) together with (6.2). From inequality (3.9) we obtain $0 \le |S_x^\kappa(t,x)| - |S_x^\kappa(t,x)|_\kappa \le \kappa$, whence

$$\|\,|S_x|(t) - |S_x^\kappa|_\kappa(t)\,\|_{L^\infty(\Omega)} \le \|S_x(t) - S_x^\kappa(t)\|_{L^\infty(\Omega)} + \kappa. \tag{6.4}$$

Relation (6.3) is implied by this estimate and by (6.1).

We also need a convergence result for third derivatives: Let

$$\mathcal{A}^S = \{(t,x)\in Q_{T_e} \mid S_x(t,x) \neq 0\}.$$

Since $S_x(t) \in C^\alpha(\Omega)$ for almost all $t\in[0,T_e]$, the set $\mathcal{A}^S(t) = \{x\in\Omega \mid S_x(t,x)\neq 0\}$ is open.

Lemma 6.3 *The limit function S has the local weak L^2–derivative S_{xxx} on $\mathcal{A}^S$ in the sense of Definition 1.1. Moreover, there exists a subsequence S^κ such that*

$$|S_x^\kappa|_\kappa S_{xxx}^\kappa \rightharpoonup \chi, \quad weakly\ in\ L^{\frac{4}{3}}(Q_{T_e}), \tag{6.5}$$

where the function $\chi \in L^{\frac{4}{3}}(Q_{T_e})$ is given by

$$\chi(t,x) = \begin{cases} 0, & if\ S_x(t,x) = 0 \\ |S_x|S_{xxx}, & if\ S_x(t,x)\neq 0. \end{cases} \tag{6.6}$$

Proof. To show that S has the local weak derivative S_{xxx} on $\mathcal{A}^S$ we first construct the family of sets $\{\mathcal{A}_n\}_n$ appearing in Definition 1.1. Remember that the measurable function $t \to |S_x^\kappa - S_x|_{C^\alpha(\bar\Omega)}$ converges to zero for almost all $t\in[0,T_e]$, by (6.2). Thus, from Theorem 5.1 we see that there is a sequence $\{M_n\}_n$ of measurable subsets of $[0, T_e]$ with

$$\text{meas}\,([0, T_e] \setminus M_n) \le \frac{1}{n}, \quad M_n \subset M_{n+1}, \tag{6.7}$$

such that $|S_x^\kappa(t) - S_x(t)|_{C^\alpha(\bar{\Omega})}$ converges to zero uniformly with respect to $t \in M_n$. This means that $S_x(t) \in C^\alpha(\bar{\Omega})$ for all $t \in M_n$ and that S_x^κ converges to S_x uniformly on the set $M_n \times \Omega$. Let

$$\hat{\mathcal{A}}_n = \{(t, x) \in Q_{T_e} \mid |S_x(t, x)| > \frac{1}{n}\}, \quad \mathcal{A}_n = \hat{\mathcal{A}}_n \cap (M_n \times \Omega). \tag{6.8}$$

We have

$$\mathcal{A}_n \subset \mathcal{A}_{n+1}\,, \tag{6.9}$$

by (6.7), and

$$\bigcup_{n=1}^{\infty} \mathcal{A}_n = \Big(\bigcup_{n=1}^{\infty} \hat{\mathcal{A}}_n\Big) \setminus \bigcap_{n=1}^{\infty} \Big(([0, T_e] \setminus M_n) \times \Omega\Big) = \mathcal{A}^S \setminus (N \times \Omega), \tag{6.10}$$

with the set

$$N = \bigcap_{n=1}^{\infty} \big([0, T_e] \setminus M_n\big) \subset [0, T_e]$$

of Lebesgue measure zero. Here we used that $\bigcup_{n=1}^{\infty} \hat{\mathcal{A}}_n = \mathcal{A}^S$. Equation (6.10) yields

$$\mathcal{A}^S(t) = \bigcup_{n=1}^{\infty} \mathcal{A}_n(t) \tag{6.11}$$

for all $t \in [0, T_e] \setminus N$. Because $S_x(t)$ is continuous for all $t \in M_n$, it follows from (6.8) that

$$\hat{\mathcal{A}}_n(t) = \{x \in \Omega \mid (t, x) \in \hat{\mathcal{A}}_n\}$$

is open for all $t \in M_n$. Again by (6.8) and by (6.11), this means that $\mathcal{A}_n(t)$ and $\mathcal{A}^S(t)$ are open subsets of Ω for all $t \in [0, T_e]$.

We next show that S_{xxx} exists on $\mathcal{A}_n$. Since S_x^κ converges uniformly on $\mathcal{A}_n$, there is κ_0 such that for all $0 < \kappa < \kappa_0$ and all $(t, x) \in \mathcal{A}_n$

$$|S_x^\kappa(t, x)| > \frac{1}{2n}.$$

Recalling (5.16) and (3.9) we obtain

$$C \ge \int_{Q_{T_e}} (|S_x^\kappa|_\kappa + \kappa)|S_{xxx}^\kappa|^2 d(\tau, x) \ge \int_{Q_{T_e}} |S_x^\kappa|\, |S_{xxx}^\kappa|^2 d(\tau, x) \ge \int_{\mathcal{A}_n} \frac{1}{2n} |S_{xxx}^\kappa|^2 d(\tau, x),$$

hence

$$\|S_{xxx}^\kappa\|_{L^2(\mathcal{A}_n)} \le \sqrt{2nC}.$$

Thus, we can select a subsequence, again denoted by S^κ, such that

$$S_{xxx}^\kappa \rightharpoonup g_n, \quad \text{weakly in } L^2(\mathcal{A}_n). \tag{6.12}$$

To prove that $g_n(t) = S_{xxx}(t)$ on $\mathcal{A}_n(t)$ for almost all t, note that since $\mathcal{A}_n$ is measurable and since $\mathcal{A}_n(t)$ is open for all $t \in [0, T_e]$ we can define the space

$$L^2_{\mathcal{A}_n}(0, T_e; H^3_0(\Omega)) = \{v \in L^2(\mathcal{A}_n) \mid v(t) \in H^3_0(\mathcal{A}_n(t)) \text{ for all } t \in [0, T_e]\}.$$

This space is separable as closed subspace of the separable space $L^2(0, T_e; H^3(\Omega))$. Let $\mathcal{K}$ be a countable dense subset of $L^2_{\mathcal{A}_n}(0, T_e; H^3_0(\Omega))$. For $\varphi \in \mathcal{K}$, $t \in (0, T_e)$ and $h > 0$ such that $t + h \le T_e$ we obtain

$$\int_t^{t+h} \int_\Omega g_n(\tau, x)\varphi(\tau, x)\, dx d\tau = \lim_{\kappa \to 0} \int_t^{t+h} \int_{\mathcal{A}_n(\tau)} S^\kappa_{xxx}(\tau, x)\varphi(\tau, x)\, dx d\tau$$
$$= -\lim_{\kappa \to 0} \int_t^{t+h} \int_{\mathcal{A}_n(\tau)} S^\kappa(\tau, x)\varphi_{xxx}(\tau, x)\, dx d\tau = -\int_t^{t+h} \int_\Omega S(\tau, x)\varphi_{xxx}(\tau, x) dx d\tau.$$

We multiply this equation with $\frac{1}{h}$ and take the limit $h \to 0$ on both sides. By a result from integration theory these limits exist for almost all $t \in (0, T_e)$ and are equal to the inner integrals, whence

$$\int_\Omega g_n(t)\varphi(t)\, dx = \int_\Omega S(t)\varphi_{xxx}(t) dx, \tag{6.13}$$

for almost all t. Since countable unions of countable sets are countable, it follows that there is a subset $L \subseteq [0, T_e]$ of Lebesgue measure zero, such that for all $\varphi \in \mathcal{K}$ the equation (6.13) holds for $t \in [0, T_e] \setminus L$. From the density of $\mathcal{K}$ in $L^2_{\mathcal{A}_n}(0, T_e, H^3_0(\Omega))$ it follows by measure theory that $\{v(t) \mid v \in \mathcal{K}\}$ is a dense subset of $H^3_0(\mathcal{A}_n(t))$ for almost all $t \in [0, T_e]$. Consequently, (6.13) holds with $\varphi(t)$ replaced by an arbitrary function $\varphi \in H^3_0(\mathcal{A}_n(t))$ for almost all t, and this means that $S(t) \in H^3(\mathcal{A}_n(t))$ and

$$S_{xxx}(t) = g_n(t) \in L^2(\mathcal{A}_n(t)) \tag{6.14}$$

for almost all $t \in M_n$.

We next construct a function g on $\mathcal{A}^S$ and show that it is the third x-derivative of S on $\mathcal{A}^S$. Since N in (6.10) has measure zero, it suffices to define g on the set $\bigcup_{n=1}^\infty \mathcal{A}_n$. Using (6.9) we set

$$g|_{\mathcal{A}_n} = g_n$$

for all $n \in \mathbb{N}$. This condition can be satisfied since (6.14) and the uniqueness of weak derivatives imply $g_m|_{\mathcal{A}_n} = g_n$ for $m \ge n$. To verify that $g(t) \in L^{2,\text{loc}}(\mathcal{A}^S(t))$, let K be a compact subset of $\mathcal{A}^S(t) = \bigcup_{n=1}^\infty \mathcal{A}_n(t)$. Then $\{\mathcal{A}_n(t)\}$ is an open covering of the compact set K, and therefore finitely many of the sets $\mathcal{A}_n(t)$ suffice to cover K; whence $K \subseteq \mathcal{A}_n(t)$ for sufficiently large n, by (6.9). Relation (6.14) now implies $g(t)|_K = g_n(t)|_K \in L^2(K)$, thence $g(t) \in L^{2,\text{loc}}(\mathcal{A}^S(t))$. Next we consider $\varphi \in C_0^\infty(\mathcal{A}^S(t))$. Since $\operatorname{supp}\varphi \subseteq \mathcal{A}^S(t)$ is compact, we conclude as above that $\operatorname{supp}\varphi \subseteq \mathcal{A}_n(t)$ for sufficiently large n. Consequently (6.14) yields

$$(g(t), \varphi)_{\mathcal{A}^S(t)} = (g_n(t), \varphi)_{\mathcal{A}_n(t)} = -(S(t), \varphi_{xxx})_{\mathcal{A}_n(t)} = -(S(t), \varphi_{xxx})_{\mathcal{A}^S(t)},$$

and so we have $S(t) \in H^{3,\text{loc}}(\mathcal{A}^S(t))$ with $g(t) = S_{xxx}(t)$ for almost all $t \in [0, T_e]$. Since meas $(\mathcal{A}^S \setminus \mathcal{A}_n) \to 0$ for $n \to \infty$, by (6.10), and since $g|_{\mathcal{A}_n} = g_n \in L^2(\mathcal{A}_n)$, by (6.12), the conditions of Definition 1.1 are satisfied, hence S has the local weak L^2–derivative S_{xxx}

on $\mathcal{A}^S$.

To show that the relations (6.5) and (6.6) hold, we use (5.17), which implies that a subsequence exists, still denoted by S^κ, such that $|S_x^\kappa|_\kappa S_{xxx}^\kappa$ converges weakly in $L^{4/3}(Q_{T_e})$. Let χ be the limit function. To identify this function remember (6.4), which together with the uniform convergence of S_x^κ on $\mathcal{A}_n$ to S_x implies that $|S_x^\kappa|_\kappa$ converges uniformly on $\mathcal{A}_n$ to $|S_x|$. From (6.12) we thus conclude

$$|S_x^\kappa|_\kappa S_{xxx}^\kappa \rightharpoonup |S_x| S_{xxx} \,,$$

weakly in $L^2(\mathcal{A}_n)$. This implies that $|S_x^\kappa|_\kappa S_{xxx}^\kappa$ also converges to $|S_x|S_{xxx}$ weakly in $L^{\frac{4}{3}}(\mathcal{A}_n)$. Since weak limits are unique, we obtain that $\chi = |S_x|S_{xxx}$ on $\mathcal{A}_n$. This holds for every n, hence $\chi = |S_x|S_{xxx}$ almost everywhere in $\mathcal{A}^S$. We finally observe that Hölder's inequality, (5.16) and (6.3) yield

$$\begin{aligned}
&\int_{\{|S_x|\le\delta\}} \Big| |S_x^\kappa|_\kappa S_{xxx}^\kappa \Big|^{\frac{4}{3}} d(\tau,x) \\
&\le \left(\int_{\{|S_x|\le\delta\}} |S_x^\kappa|_\kappa^2 \, d(\tau,x) \right)^{\frac{1}{3}} \left(\int_{\{|S_x|\le\delta\}} |S_x^\kappa|_\kappa |S_{xxx}^\kappa|^2 \, d(\tau,x) \right)^{\frac{2}{3}} \\
&\le C \Big(\| \, |S_x^\kappa|_\kappa - |S_x| \, \|_{L^2(Q_{T_e})} + \|S_x\|_{L^2(\{|S_x|\le\delta\})} \Big)^{\frac{2}{3}} \\
&\le C(\delta + \delta \text{ meas } Q_{T_e})^{\frac{2}{3}},
\end{aligned}$$

for all κ sufficiently small. Since $|S_x^\kappa|_\kappa S_{xxx}^\kappa$ converges to χ weakly in $L^{\frac{4}{3}}(\{|S_x| \le \delta\})$, this estimate implies that

$$\|\chi\|_{L^{\frac{4}{3}}(\{S_x=0\})} \le \|\chi\|_{L^{\frac{4}{3}}(\{|S_x|\le\delta\})} \le C(1 + \text{meas } Q_{T_e})^{\frac{1}{2}} \delta^{\frac{1}{2}}.$$

This holds for all $\delta > 0$, whence $\chi = 0$ on the set $\{(t,x) \mid S_x(t,x) = 0\}$. The proof of Lemma 6.3 is complete.

Proof of Theorem 1.3. Let S^κ be the subsequence and S be the limit function from Lemma 6.2, let $(u^\kappa, T^\kappa, S^\kappa)$ be the corresponding sequence of weak solutions to (3.1) – (3.7) constructed in Theorem 3.2. The relations (3.18), (3.19) and (6.1) imply that a function $(u, T) \in L^2(0, T_e; H^2(\Omega) \times H^1(\Omega))$ exists such that

$$\|u^\kappa - u\|_{L^2(0,T_e;H^2(\Omega))} + \|T^\kappa - T\|_{L^2(0,T_e;H^1(\Omega))} \to 0 \tag{6.15}$$

for $\kappa \to 0$. We show that (u, T, S) is a weak solution of (1.18) – (1.24) and satisfies (1.31) – (1.33). To this end observe that the relations (1.28) and (1.32) for S follow from Lemma 6.2 and from (3.5), (3.13). Relation (1.33) is implied by Lemma 6.3, the relations (1.26), (1.27) are consequences of (6.15) and (3.4), and (1.31) results from (3.10). The partial differential equations (3.1), (3.2) together with (6.1), (6.15) imply that (u, T, S) satisfies (1.18), (1.19). To prove that (1.29) holds we study the convergence of the terms in the equation (3.8) satisfied by $(u^\kappa, T^\kappa, S^\kappa)$. Note that (6.1), (6.3) and (6.5), (6.6) yield for $\varphi \in C_0^\infty((-\infty, T_e) \times \mathbb{R})$ that

$$\begin{aligned}
(S^\kappa, \varphi_t)_{Q_{T_e}} &\to (S, \varphi_t)_{Q_{T_e}} \,, && (6.16) \\
c(r|S_x^\kappa|_\kappa, \varphi)_{Q_{T_e}} &\to c(r|S_x|, \varphi)_{Q_{T_e}} \,, && (6.17) \\
(|S_x^\kappa|_\kappa S_{xxx}^\kappa, \varphi_x)_{Q_{T_e}} &\to (|S_x|S_{xxx}, \varphi_x)_{\mathbb{P}_S} \,, && (6.18)
\end{aligned}$$

as $\kappa \to 0$. The inequality (5.16) yields

$$|(\kappa S^{\kappa}_{xxx}\,, \varphi_x)_{Q_{T_e}}| \le \kappa^{\frac{1}{2}} \|\kappa^{\frac{1}{2}} S^{\kappa}_{xxx}\|_{L^2(Q_{T_e})}\, \|\varphi_x\|_{L^\infty(Q_{T_e})} \le C\kappa^{\frac{1}{2}} \to 0. \tag{6.19}$$

For the term containing $\psi_S(\varepsilon^\kappa, S^\kappa)_x = \hat{\psi}''(S^\kappa)S^\kappa_x - T^\kappa_x \cdot \bar{\varepsilon}$ we argue as in (3.69) and obtain from (3.12), (3.13), (6.1), (6.3), (6.15) that

$$\psi_S(\varepsilon^\kappa, S^\kappa)_x\,(|S^\kappa_x|_\kappa + \kappa) \to \psi_S(\varepsilon, S)_x|S_x|, \quad \text{in } L^1(Q_{T_e}), \tag{6.20}$$

Inserting (6.16) – (6.20) into equation (3.8) shows that (u, T, S) solves (1.29) for all $\varphi \in C^\infty_0((-\infty, T_e) \times \mathbb{R})$. Therefore (u, T, S) is a weak solution of the problem (1.18) – (1.24) having the regularity properties stated in Theorem 1.3. The proof of this theorem is complete.

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