A Weighted L^q-Approach to Oseen Flow Around a Rotating Body

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Abstract

We study time-periodic Oseen flows past a rotating body in $\mathbb{R}^3$ proving weighted *a priori* estimates in L^q-spaces using Muckenhoupt weights. After a time-dependent change of coordinates the problem is reduced to a stationary Oseen equation with the additional terms $(\omega \times x) \cdot \nabla u$ and $-\omega \wedge u$ in the equation of momentum where ω denotes the angular velocity. Due to the asymmetry of Oseen flow and to describe its wake we use anisotropic Muckenhoupt weights, a weighted theory of Littlewood-Paley decomposition and of maximal operators as well as one-sided univariate weights, one-sided maximal operators and a new version of Jones' factorization theorem.

Mathematics Subject Classification: Primary 76 D 05; Secondary 35 Q 30
Key Words: Littlewood-Paley theory; maximal operators; rotating obstacles; stationary Oseen flow; anisotropic Muckenhoupt weights; one-sided weights; one-sided maximal operators

1 Introduction

We consider a three-dimensional rigid body $K \subset\subset \mathbb{R}^3$ rotating with angular velocity $\omega = \widetilde{\omega}(0,0,1)^T$, $\widetilde{\omega} \neq 0$, and assume that the complement $\mathbb{R}^3 \setminus K$

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is filled with a viscous incompressible fluid modelled by the Navier-Stokes equations. Then we will analyze the viscous flow either past the rotating body K with velocity $u_\infty = ke_3 \neq 0$ at infinity or around a rotating body K which is moving in the direction of its axis of rotation. Given the coefficient of viscosity $\nu > 0$ and an external force $\widetilde{f} = \widetilde{f}(y,t)$, we are looking for the velocity $v = v(y,t)$ and the pressure $q = q(y,t)$ solving the nonlinear system

$$\begin{aligned} v_t - \nu\Delta v + v\cdot\nabla v + \nabla q &= \widetilde{f} && \text{in} \quad \Omega(t),\ t>0 \\ \operatorname{div} v &= 0 && \text{in} \quad \Omega(t),\ t>0 \\ v(y,t) &= \omega\wedge y && \text{on} \quad \partial\Omega(t),\ t>0 \\ v(y,t) &\to u_\infty \neq 0 && \text{as} \quad |y|\to\infty. \end{aligned} \tag{1.1}$$

Here the time-dependent exterior domain $\Omega(t)$ is given - due to the rotation with angular velocity ω - by

$$\Omega(t) = O_\omega(t)\Omega$$

where $\Omega \subset \mathbb{R}^3$ is a fixed exterior domain and $O_\omega(t)$ denotes the orthogonal matrix

$$O_\omega(t) = \begin{pmatrix} \cos\widetilde{\omega}t & -\sin\widetilde{\omega}t & 0 \\ \sin\widetilde{\omega}t & \cos\widetilde{\omega}t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.2}$$

Introducing the change of variables and the new functions

$$x = O_\omega(t)^T y \quad \text{and} \quad u(x,t) = O^T_{\omega(t)}(v(y,t) - u_\infty), \quad p(x,t) = q(y,t), \tag{1.3}$$

respectively, as well as the force term $f(x,t) = O(t)^T\widetilde{f}(y,t)$ we arrive at the modified Navier-Stokes system

$$\begin{aligned} &u_t - \nu\Delta u + u\cdot\nabla u + k\partial_3 u \\ &\qquad -(\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p = f \quad \text{in} \quad \Omega\times(0,\infty) \\ &\qquad \operatorname{div} u = 0 \quad \text{in} \quad \Omega\times(0,\infty) \\ &\qquad u(x,t) \to 0 \quad \text{as} \quad |x|\to\infty \end{aligned} \tag{1.4}$$

with boundary condition $u(x,t) = \omega\wedge x - u_\infty$ on $\partial\Omega$ in the exterior time-independent domain Ω.

Due to the new coordinate system attached to the rotating body the nonlinear system (1.4) contains two new linear terms, the classical Coriolis force term $\omega \wedge u$ (up to a multiplicative constant) and the term $(\omega \wedge x) \cdot \nabla u$ which is *not* subordinate to the Laplacian in unbounded domains. Linearizing (1.4) in u at $u \equiv 0$ and considering only the stationary problem we arrive at the modified Oseen system

$$\begin{aligned} -\nu \Delta u + k\partial_3 u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p &= f \quad \text{in} \quad \Omega \\ \operatorname{div} u &= 0 \quad \text{in} \quad \Omega \\ u &\to 0 \quad \text{at} \quad \infty \end{aligned} \tag{1.5}$$

together with the boundary condition $u(x,t) = \omega \wedge x - u_\infty$ on $\partial\Omega$. Note that there is no boundary condition in the case $\Omega = \mathbb{R}^3$.

The linear system (1.5) has been analyzed in classical L^q-spaces, $1 < q < \infty$, for the whole space case in [3], [4] proving the *a priori*-estimate

$$\begin{aligned} \|\nu \nabla^2 u\|_q + \|\nabla p\|_q &\leq c\|f\|_q, \\ \|k\partial_3 u\|_q + \|(\omega \wedge x) \cdot \nabla u + \omega \wedge u\|_q &\leq c(1 + \tfrac{k^4}{\nu^2 |\omega|^2})\|f\|_q \end{aligned} \tag{1.6}$$

with a constant $c > 0$ independent of ν, k and ω. For a discussion of weak solutions we refer to [14], [15]; the spectrum of the linear operator defined by (1.5) is considered in [8]. The corresponding case when $u_\infty = 0$ has recently been analyzed in [5]–[7], [11], [12], [19]–[21]. For a more comprehensive introduction including physical considerations and non-Newtonian fluids we refer to [9].

The aim of this paper is to generalize the *a priori*-estimate (1.6) to weighted L^q-spaces for the whole space $\mathbb{R}^3$. For this reason we introduce the weighted Lebesgue space

$$L^q_w(\mathbb{R}^3) = L^q_w = \Big\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^3) : \ \|u\|_{q,w} = \Big(\int_{\mathbb{R}^n} |u(x)|^q w(x)\, dx \Big)^{1/q} < \infty \Big\},$$

where $w \in L^1_{\mathrm{loc}}$ is a nonnegative weight function and should reflect the anisotropy of the flow and the existence of a wake region in the downstream direction $x_3 > 0$. Our tools will include Littlewood-Paley theory, singular integral operators, multiplier operators and maximal operators in weighted spaces so that we need weight functions satisfying Muckenhoupt type conditions. For a totally different approach using variational methods see [13].

Definition 1.1. Let $\mathcal{R}$ be a collection of bounded sets R in $\mathbb{R}^n$, each of positive Lebesgue measure $|R|$. A weight function $0 \leq w \in L^1_{\mathrm{loc}}$ belongs to the *Muckenhoupt class* $A_q(\mathcal{R}) = A_q(\mathbb{R}^n, \mathcal{R})$, $1 \le q < \infty$, if there exists a constant $C > 0$ such that

$$\sup_R \left(\frac{1}{|R|} \int_R w(x)\, dx \right) \left(\frac{1}{|R|} \int_R w^{-1/(q-1)}\, dx \right)^{q-1} \le C \quad \text{for any } R \in \mathcal{R}$$

if $1 < q < \infty$, and

$$\sup_{R \in \mathcal{R},\, R \ni x_0} \frac{1}{|R|} \int_R w(x)\, dx \le C w(x_0) \quad \text{for a.a. } x_0 \in \mathbb{R}^n,$$

if $q = 1$, respectively.

Due to the anisotropic nature of our problem we shall need a variant of the classical Muckenhoupt class $A_q(\mathcal{C}) = A_q(\mathbb{R}^3, \mathcal{C})$, where $\mathcal{C}$ is the set of all cubes $Q \subset \mathbb{R}^3$ with edges parallel to the coordinate axes. Namely, $\mathcal{C}$ is replaced by $\mathcal{J}$, the set of all bounded intervals (rectangles) in $\mathbb{R}^3$, leading to the class $A_q(\mathcal{J}) = A_q(\mathbb{R}^3, \mathcal{J})$. Obviously, $A_q(\mathbb{R}^3, \mathcal{J}) \subsetneq A_q(\mathbb{R}^3, \mathcal{C})$.

Moreover, to describe the anisotropy of the wake region more precisely by weights we have to introduce in addition to the weights on $\mathbb{R}^n$ *one-sided Muckenhoupt weights* and *one-sided maximal operators* on the real line, see Definition 1.2, Theorem 2.3 and Lemma 2.4 below.

Definition 1.2. (i) For every locally integrable function u on the real line let M^+u be defined by

$$M^+u(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |u(t)|\, dt.$$

Analogously,

$$M^-u(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |u(t)|\, dt.$$

(ii) A weight function $0 \le w \in L^1_{\mathrm{loc}}(\mathbb{R})$ lies in the *weight class* A_1^- if there exists a constant $c > 0$ such that $M^+w(x) \le cw(x)$ for almost all $x \in \mathbb{R}$. Analogously, $w \in A_1^+$ if and only if $M^-w(x) \le cw(x)$ for almost all $x \in \mathbb{R}$. The smallest constant $c \ge 0$ satisfying $M^\pm w(x) \le cw(x)$ for almost all $x \in \mathbb{R}$ is called the $A_1^\mp$-*constant of* w.

(iii) A weight function $0 \le w \in L^1_{\mathrm{loc}}$ belongs to the *one-sided Muckenhoupt class* A_q^+, $1 < q < \infty$, if there exists a constant $C > 0$ such that for all $x \in \mathbb{R}$

$$\sup_{h>0} \left(\frac{1}{h} \int_{x-h}^{x} w(t)\, dt \right) \left(\frac{1}{h} \int_x^{x+h} w(t)^{-1/(q-1)}\, dt \right)^{q-1} \le C.$$

The smallest constant $C \geq 0$ satisfying this estimate is called the A_q^+-*constant* of w. By analogy, we define the set of weights A_q^- and the A_q^--constant of a weight in A_q^-.

Now we are in a position to describe the most general weights considered in this paper. Note that these weights are independent of the angular variable θ in the cylindrical coordinate system $(r, \theta, x_3) \in [0, \infty) \times [0, 2\pi] \times \mathbb{R}$ attached to the axis of revolution $e_3 = (0, 0, 1)^T$. Hence we will write $w(x) = w(x_1, x_2, x_3) = w_r(x_3)$ for $r = |(x_1, x_2)|$, $x = (x_1, x_2, x_3)$.

Definition 1.3. For $1 \leq q < \infty$ let

$$\begin{aligned} \widetilde{A}_q^- = \widetilde{A}_q^-(\mathbb{R}^3) = \{w \in A_q(\mathbb{R}^3) : \ & w \text{ is } \theta - \text{independent for a.a. } r > 0, \\ & w(x_1, x_2, \cdot) = w_r(\cdot) \in A_q^-(\mathbb{R}) \\ & \text{with } A_q^-(\mathbb{R})\text{-constant essentially bounded in } r\}. \end{aligned} \tag{1.7}$$

Theorem 1.4. *Let the weight function* $0 \leq w \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ *be independent of the angular variable* θ *and satisfy the following condition depending on* $q \in (1, \infty)$*:*

$$\begin{aligned} 2 \leq q < \infty : \quad & w^\tau \in \widetilde{A}_{\tau q/2}^- \quad \textit{for some} \quad \tau \in [1, \infty) \\ 1 < q < 2 : \quad & w^\tau \in \widetilde{A}_{\tau q/2}^- \quad \textit{for some} \quad \tau \in \left(\tfrac{2}{q}, \tfrac{2}{2-q}\right]. \end{aligned} \tag{1.8}$$

(i) Given $f \in L^q_w(\mathbb{R}^3)^3$ *there exists a solution* $(u, p) \in L^1_{\mathrm{loc}}(\mathbb{R}^3)^3 \times L^1_{\mathrm{loc}}(\mathbb{R}^3)$ *of* (1.5) *satisfying the estimate*

$$\|\nu \nabla^2 u\|_{q,w} + \|\nabla p\|_{q,w} \leq c \|f\|_{q,w}, \tag{1.9}$$

with a constant $c = c(q, w) > 0$ *independent of* ν, k *and* ω.

(ii) Let $f \in L^{q_1}_{w_1}(\mathbb{R}^3)^3 \cap L^{q_2}_{w_2}(\mathbb{R}^3)^3$ *such that both* (q_1, w_1) *and* (q_2, w_2) *satisfy the conditions* (1.8)*, and let* $u_1, u_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^3)^3$ *together with corresponding pressure functions* $p_1, p_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ *be solutions of* (1.5) *satisfying* (1.9) *for* (q_1, w_1) *and* (q_2, w_2)*, respectively. Then there are* $\alpha, \beta \in \mathbb{R}$ *such that* u_1 *coincides with* u_2 *up to an affine linear field* $\alpha e_3 + \beta \omega \wedge x$, $\alpha, \beta \in \mathbb{R}$.

Corollary 1.5. *Let the weight function* $0 \leq w \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ *be independent of the angular variable* θ*. Moreover, let* w *satisfy the following condition depending on* $q \in (1, \infty)$*:*

$$\begin{aligned} 2 \leq q < \infty : \quad & w^\tau \in \widetilde{A}_{\tau q/2}^-(\mathcal{J}) \quad \textit{for some} \quad \tau \in [1, \infty) \\ 1 < q < 2 : \quad & w^\tau \in \widetilde{A}_{\tau q/2}^-(\mathcal{J}) \quad \textit{for some} \quad \tau \in \left(\tfrac{2}{q}, \tfrac{2}{2-q}\right] \end{aligned} \tag{1.10}$$

where the weight class $\widetilde{A}_\tau^-(\mathcal{J})$, $1 \le \tau < \infty$, *is defined by*

$$\widetilde{A}_\tau^-(\mathcal{J}) = \widetilde{A}_\tau^-(\mathbb{R}^3) \cap A_\tau(\mathcal{J}).$$

Given $f \in L^q_w(\mathbb{R}^3)^3$ *there exists a solution* $(u,p) \in L^1_{\text{loc}}(\mathbb{R}^3)^3 \times L^1_{\text{loc}}(\mathbb{R}^3)$ *of* (1.5) *satisfying the estimate*

$$\|k\partial_3 u\|_{q,w} + \|(\omega \wedge x)\cdot u - \omega \wedge u\|_{q,w} \le c\left(1 + \frac{k^5}{\nu^{5/2}|\omega|^{5/2}}\right)\|f\|_{q,w} \tag{1.11}$$

with a constant $c = c(q,w) > 0$ *independent of* ν, k *and* ω.

We remark that the ω-dependent term $1 + \frac{k^5}{\nu^{5/2}|\omega|^{5/2}}$ in (1.11) cannot be avoided in general; see [4] for an example in the space $L^2(\mathbb{R}^3)$.

As an example of anisotropic weight functions we consider

$$w(x) = \eta^\alpha_\beta(x) = (1+|x|)^\alpha(1+s(x))^\beta, \quad s(x) = |(x_1,x_2,x_3)| - x_3, \tag{1.12}$$

introduced in [2] to analyze the Oseen equations; see also [13]–[14].

Corollary 1.6. *The a priori estimate* (1.9) *holds for the anisotropic weights* $w = \eta^\alpha_\beta$, *see* (1.12), *provided that*

$$\begin{array}{lllll} 2 \le q < \infty & : -\frac{q}{2} < \alpha < \frac{q}{2}, & 0 \le \beta < \frac{q}{2} & \textit{and} & \alpha + \beta > -1 \\ 1 < q < 2 & : -\frac{q}{2} < \alpha < q-1, & 0 \le \beta < q-1 & \textit{and} & \alpha+\beta > -\frac{q}{2}. \end{array}$$

Note that the condition $\beta \ge 0$ will reflect the existence of a wake region in the downstream direction $x_3 > 0$ where the solution of the original nonlinear problem (1.1) will decay slower than in the upstream direction $x_3 < 0$.

2 Preliminaries

To prove Theorem 1.4 we need several properties of Muckenhoupt weights and of maximal operators. Recall that $\mathcal{J}$ stands for the set of all nondegenerate rectangles in $\mathbb{R}^n$ with edges parallel to the coordinate axes.

Proposition 2.1. *(1) Let* μ *be a nonnegative regular Borel measure such that the strong centered Hardy-Littlewood maximal operator*

$$\mathcal{M}_{\mathcal{J}}\mu(x) = \sup_{R\in\mathcal{J},\, R\ni x} \frac{1}{|R|}\int_R d\mu$$

is finite for almost all $x \in \mathbb{R}^n$*; here* R *runs through the collection* $\mathcal{J}$ *of rectangles containing additionally the point* x*, and* $|R|$ *denotes the Lebesgue measure of* R*. Then* $(\mathcal{M}_{\mathcal{J}}\mu)^\gamma \in A_1(\mathcal{J})$ *for all* $\gamma \in [0,1)$*.*

(2) For all $1 < q < \tau$ *we have* $A_1(\mathcal{J}) \subset A_q(\mathcal{J}) \subset A_\tau(\mathcal{J})$*.*

(3) Let $1 < q < \infty$ *and* $w \in A_q(\mathcal{J})$*. Then there are* $w_1, w_2 \in A_1(\mathcal{J})$ *such that*

$$w = \frac{w_1}{w_2^{q-1}}.$$

Conversely, given $w_1, w_2 \in A_1(\mathcal{J})$*, the weight* $w = w_1 w_2^{1-q}$ *belongs to* $A_q(\mathcal{J})$*.*

For the proofs see [10, Chapter IV, §6]. The claim (3) is a variant of Jones' factorization theorem, see [10, Chapter IV, Theorem 6.8].

For a rapidly decreasing function $u \in \mathcal{S}(\mathbb{R}^n)$ let

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx, \quad \xi \in \mathbb{R}^n,$$

be the Fourier transform of u. Its inverse will be denoted by $\mathcal{F}^{-1}$. Moreover, we define the centered Hardy-Littlewood maximal operator

$$\mathcal{M}u(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |u(y)|\, dy, \quad x \in \mathbb{R}^n,$$

for $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ where Q runs through the set of all closed cubes centered at x.

Theorem 2.2. *Let* $1 < q < \infty$ *and* $w \in A_q$*.*

(i) The operator $\mathcal{M}$*, defined e.g. on* $\mathcal{S}(\mathbb{R}^n)$*, is a bounded operator from* L^q_w *to* L^q_w*.*

(ii) Let $m \in C^n(\mathbb{R}^n \setminus \{0\})$ *satisfy the pointwise Hörmander-Mikhlin multiplier condition*

$$|\xi|^{|\alpha|}\, |D^\alpha m(\xi)| \le c_\alpha \quad \textit{for all} \quad \xi \in \mathbb{R}^n \setminus \{0\}$$

and all multiindices $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| \le n_1 \in \mathbb{N}$*, where* $n_1 \ge n/2$*. Then the multiplier operator* $u \mapsto \mathcal{F}^{-1}(m\hat{u})$*,* $u \in \mathcal{S}(\mathbb{R}^n)$*, can be extended to a bounded linear operator from* L^q_w *to* L^q_w*.*

(iii) Let m *be of class* C^n *in each "quadrant" of* $\mathbb{R}^n$ *and let a constant* $B \ge 0$ *exist such that* $\|m\|_\infty \le B$*,*

$$\sup_{x_{k+1},\dots,x_n} \int_{\mathcal{I}} \left| \frac{\partial^k m(x)}{\partial x_1 \cdots \partial x_k} \right| dx_1 \cdots dx_k \le B$$

for any dyadic interval $\mathcal{I}$ in $\mathbb{R}^k$, $1 \le k \le n$, and also for any permutation of the variables $x_1, \dots, x_k$ within $x_1, \dots, x_n$. If $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n, \mathcal{J})$, then m defines a bounded multiplier operator from $L^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$.

Proof. (i) See [10, Theorem IV 2.8], [18, Theorem 9]. (ii) See [10, Theorem IV 3.9] or [17, Theorem 4]. Note that the pointwise condition on m implies the integral condition in [10], [17]. For the proof of (iii) see [17]. ∎

Concerning *one-sided weights* and *one-sided maximal operators* on the real line, see Definition 1.2, we first recall the following duality property: $w \in A_q^+$ if and only if $w^{-q'/q} = w^{-1/(q-1)} \in A_{q'}^-$. Moreover we will need the following results:

Theorem 2.3 (Theorem 1 of [23]). *Let $1 < p < \infty$ and $p' = \frac{p}{p-1}$.*

(i) Let $w_1 \in A_1^+$, $w_2 \in A_1^-$. Then $\frac{w_1}{w_2^{p-1}} \in A_p^+$. Conversely, given $w \in A_p^+$ there exist $w_1 \in A_1^+$, $w_2 \in A_1^-$ such that $w = \frac{w_1}{w_2^{p-1}}$.

(ii) The operator M^+ is continuous from $L^p_w(\mathbb{R})$ to itself if and only if $w \in A_p^+$. Analogously, $M^- : L^p_w(\mathbb{R}) \to L^p_w(\mathbb{R})$ if and only if $w \in A_p^-$.

Obviously, $A_p \subset A_p^\pm$ where A_p denotes the usual Muckenhoupt class on the real line. Hence $|x|^\alpha, (1+|x|)^\alpha \in A_p^\pm$ if $-1 < \alpha < p-1$, $1 < p < \infty$. However, in view of the anisotropic weight $w = \eta_\beta^\alpha$ on $\mathbb{R}^3$, see (1.12), we have to consider also one-dimensional anisotropic weight functions such as

$$\widetilde{w}_{\alpha,\beta}(x) = \widetilde{w}_{\alpha,\beta}(x; r) = (r^2 + x^2)^{\alpha/2} (\sqrt{r^2 + x^2} - x)^\beta, \quad x \in \mathbb{R}, \, r > 0. \tag{2.1}$$

Lemma 2.4. (i) *For every $r > 0$ the univariate weight $\widetilde{w}_{\alpha,\beta}(x; r)$ lies in A_1^- if and only if $\beta \ge 0$, $\alpha \le \beta$ and $\alpha + \beta > -1$. Moreover, the A_1^--constant of $\widetilde{w}_{\alpha,\beta}$ is uniformly bounded in r.*

(ii) For every $r > 0$ the univariate weight

$$w_{\alpha,\beta}(x) = w_{\alpha,\beta}(x; r) = (1 + r^2 + x^2)^{\alpha/2} (1 + \sqrt{r^2 + x^2} - x)^\beta$$

lies in A_1^- with an A_1^--constant independent of $r > 0$ if and only if

$$\alpha \le 0 \le \beta \text{ and } \alpha + \beta > -1. \tag{2.2}$$

(iii) Let $1 < p < \infty$. Then for every $r > 0$

$$\begin{aligned} w_{\alpha,\beta}(\cdot\,; r) &\in A_p^+ \quad \text{for} \quad \alpha > -1, \quad \beta \le 0, \quad \alpha + \beta < p - 1 \\ w_{\alpha,\beta}(\cdot\,; r) &\in A_p^- \quad \text{for} \quad \alpha < p - 1, \quad \beta \ge 0, \quad \alpha + \beta > -1. \end{aligned} \tag{2.3}$$

Moreover, the $A_p^\pm$-constant is uniformly bounded in $r > 0$.

Proof. (i) A simple scaling argument shows that it suffices to look at the weight $\widetilde{w} = \widetilde{w}_{\alpha,\beta}$ in (2.1) for $r = 1$ only and that the A_1^--constant is independent of $r > 0$. We will consider three cases.

Case 1: $x > 0$. Then $\widetilde{w}(x) \sim (1+|x|)^{\alpha-\beta}$, i.e., there exists a constant $c > 0$ independent of $x > 0$ such that $\frac{1}{c}(1+|x|)^{\alpha-\beta} \le \widetilde{w}(x) \le c(1+|x|)^{\alpha-\beta}$ for all $x > 0$. Hence for all $h > 0$

$$\frac{1}{h}\int_x^{x+h} \widetilde{w}(t)\,dt \sim \frac{1}{h}\int_x^{x+h} (1+t)^{\alpha-\beta}\,dt.$$

If $\alpha - \beta > 0$, then the term on the right hand-side is strictly increasing to $+\infty$ as $h \to \infty$. Thus we are led to the condition $\alpha \le \beta$.

Now let $\alpha \le \beta$. Then for all $h > 0$

$$\frac{1}{h}\int_x^{x+h} (1+t)^{\alpha-\beta}\,dt \le \frac{1}{h}\int_x^{x+h} (1+x)^{\alpha-\beta}\,dt = (1+|x|)^{\alpha-\beta} \sim \widetilde{w}(x).$$

Case 2: $x < 0$ *and* $0 < h < |x|$. Then $\widetilde{w}(t) \sim (1+|t|)^{\alpha+\beta}$ for all $t \in (x, x+h)$. Assume that $\alpha + \beta = -1$ and let $h = |x|$. Then

$$\frac{1}{|x|}\int_x^0 (1+|t|)^{-1}\,dt = \frac{\log(1+|x|)}{|x|}$$

is not bounded by $c\widetilde{w}(x) = c/|x|$ uniformly in $x < 0$ for any constant $c > 0$. Analogously, if $\alpha+\beta < -1$, then for $h = |x|$ we see that $\frac{1}{|x|}\int_x^0 (1+|t|)^{\alpha+\beta}\,dt \sim \frac{1}{|x|}$ is not bounded by $c\widetilde{w}(x) = c(1+|x|)^{\alpha+\beta}$ uniformly in $x < 0$. Hence in the following we have to assume that $\alpha + \beta > -1$. We shall consider two subcases: $h > 0$ small with respect to $|x|$ and h comparable with $|x|$. If $0 < h < \frac{|x|}{2}$, then

$$\frac{1}{h}\int_x^{x+h} (1+|t|)^{\alpha+\beta}\,dt \sim \frac{1}{h}\int_x^{x+h} (1+|x|)^{\alpha+\beta}\,dt = (1+|x|)^{\alpha+\beta} \sim \widetilde{w}(x).$$

For the second subcase assume that $\frac{|x|}{2} < h < |x|$. Then we are led to the integral

$$\begin{aligned} &\frac{1}{|x|}\int_x^{x+h} (1+|t|)^{\alpha+\beta}\,dt \\ &\le \frac{1}{|x|}\int_x^{0} (1+|t|)^{\alpha+\beta}\,dt \;\sim\; \begin{cases} \frac{(1+|x|)^{\alpha+\beta+1}}{|x|}, & |x| > 1 \\ 1, & |x| < 1 \end{cases} \;\sim\; \widetilde{w}(x). \end{aligned}$$

Case 3: $x<0$ and $h>|x|$. In this case we have to consider the sum

$$\frac{1}{h}\int_x^0 \widetilde{w}\,dt+\frac{1}{h}\int_0^{x+h}\widetilde{w}\,dt\le\frac{1}{|x|}\int_x^0\widetilde{w}\,dt+\frac{c}{h}\int_0^{x+h}(1+t)^{\alpha-\beta}\,dt=:I_1+I_2,$$

where the first integral I_1 is bounded by $c\widetilde{w}(x)$ uniformly in $x<0$, see *Case 2*, and where for $|x|<1$ the second integral I_2 is bounded by $c\sim\widetilde{w}(x)$. Therefore, let $|x|>1$ in the following. If $\alpha-\beta\le-1$, then the condition $\alpha+\beta>-1$ implies that $\beta>0$; moreover, I_2 is easily shown to be bounded by $c\widetilde{w}(x)\sim(1+|x|)^{\alpha+\beta}$ uniformly in $x<0$ and $h>|x|$.

Now consider the case $\alpha-\beta>-1$. We shall investigate three possibilities of the position of h with respect to $|x|$. If $h=2|x|$, then

$$\frac{1}{|x|}\int_0^{|x|}(1+t)^{\alpha-\beta}\,dt=\frac{c}{|x|}\big((1+|x|)^{\alpha-\beta+1}-1\big).$$

Since $\frac{1}{|x|}=o\big(|x|^{\alpha+\beta}\big)=o(\widetilde{w}(x))$ by the condition that $\alpha+\beta>-1$, the assertion $I_2\le c\widetilde{w}(x)\sim|x|^{\alpha+\beta}$ necessarily implies that $|x|^{\alpha-\beta}\le c|x|^{\alpha+\beta}$ for $|x|>1$. Thus β must be nonnegative.

Next, if $|x|<h<2|x|$, then, since $\alpha-\beta\le\alpha+\beta$ and $\alpha+\beta>-1$,

$$I_2\le\frac{c}{|x|}\int_0^{|x|}(1+t)^{\alpha-\beta}\,dt\le c|x|^{\alpha+\beta}\sim\widetilde{w}(x).$$

Finally, if $h>2|x|>2$, then

$$I_2\le\frac{c}{h}(1+x+h)^{\alpha-\beta+1}\le ch^{\alpha-\beta}\le c|x|^{\alpha+\beta}\sim\widetilde{w}(x)$$

since $\alpha\le\beta$ (see *Case 1*). Summarizing the previous cases and estimates we have proved that there exists $c>0$ such that $M^+\widetilde{w}(x)\le c\widetilde{w}(x)$ for a.a. $x\in\mathbb{R}$, and that this results holds if and only if $\beta\ge0,\ \alpha\le\beta$, and $\alpha+\beta>-1$.

(ii) To verify the necessity of (2.2) let $r=1$ and $w=w_{\alpha,\beta}$. For $x>0$ when $(1+\sqrt{r^2+x^2}-x)^\beta\sim1$, we have to estimate

$$\frac{1}{h}\int_x^{x+h}w(t)\,dt\sim\frac{1}{h}\int_x^{x+h}(1+t)^\alpha\,dt$$

by $cw(x)\sim(1+x)^\alpha$. As in *Case 1* of Part (i) (with $\beta=0$) we get the necessary condition $\alpha\le0$.

Let $x < 0$. Again we shall distinguish according to the size of h with respect to $|x|$. If $0 < h < |x|$, than $w(t) \sim (1+|t|)^{\alpha+\beta}$ for all $t \in (x, x+h)$, and

$$\frac{1}{h}\int_x^{x+h} w(t)\,dt \sim \frac{1}{h}\int_x^{x+h} (1+|t|)^{\alpha+\beta}\,dt$$

is bounded by $cw(x) \sim (1+|x|)^{\alpha+\beta}$ only when $\alpha+\beta > -1$; cf. *Case 2* of Part (i). Finally, when $x < 0$ and $h > |x|$, say $h = 2|x| > 2$, and when $\alpha > -1$, then

$$\frac{1}{h}\int_x^{x+h} w(t)\,dt \sim \frac{1}{h}\int_x^{0} (1+|t|)^{\alpha+\beta}\,dt + \frac{1}{h}\int_0^{x+h} (1+t)^{\alpha}\,dt \le cw(x) + c|x|^{\alpha},$$

which is bounded by $cw(x) \sim (1+|x|)^{\alpha+\beta}$ only if $\beta \ge 0$. However, if $\alpha \le -1$, then the condition $\alpha+\beta > -1$ implies that even $\beta > 0$. Hence the conditions (2.2) are necessary to prove that $w \in A_1^-$.

We shall prove that conditions (2.2) are sufficient for $w_{\alpha,\beta}(x;r) \in A_1^-$ with an A_1^--constant independent of $r > 0$. Let us assume that (2.2) holds and let first $0 < r < 1$. Then

$$\begin{aligned} w(t) &\sim (1+|t|)^{\alpha} \cdot \begin{cases} 1, & t > 0 \\ (1+|t|)^{\beta}, & t < 0 \end{cases} \\ &\sim (1+|t|)^{\alpha+\beta/2} \cdot \begin{cases} (1+|t|)^{-\beta/2}, & t > 0 \\ (1+|t|)^{\beta/2}, & t < 0 \end{cases} \sim \widetilde{w}_{\alpha',\beta'}(t;r) \end{aligned}$$

where $\alpha' = \alpha + \beta/2$, $\beta' = \beta/2$. Since the assumptions (2.2) on α, β imply that α', β' satisfy the assumptions in (i), $w \in A_1^-$ with an A_1^--constant independent of $0 < r < 1$.

Next let $r \ge 1$. An elementary calculation shows that

$$w(t) \sim \begin{cases} \widetilde{w}_{\alpha,\beta}(t;r), & t < r^2 \\ \widetilde{w}_{\alpha,0}(t;r), & t > r^2 \end{cases}.$$

Then we will consider three cases.

Case 1: $x < r^2$ and $x+h < r^2$. In this case, by Part (i),

$$\frac{1}{h}\int_x^{x+h} w(t)\,dt \sim \frac{1}{h}\int_x^{x+h} \widetilde{w}_{\alpha,\beta}(t;r)\,dt \le c\widetilde{w}_{\alpha,\beta}(x;r) \sim cw(x)$$

with $c > 0$ independent of $r > 1$.

Case 2: $x > r^2$ and $x+h > r^2$. Now

$$\frac{1}{h}\int_x^{x+h} w(t)\,dt \sim \frac{1}{h}\int_x^{x+h} \widetilde{w}_{\alpha,0}(t;r)\,dt \le c\widetilde{w}_{\alpha,0}(x;r) \sim cw(x)$$

due to *Case 1* in Part (i).

Case 3: $x < r^2$ but $x+h > r^2$. Then

$$\frac{1}{h}\int_x^{x+h} w(t)\,dt \sim \frac{1}{h}\int_x^{r^2} \widetilde{w}_{\alpha,\beta}(t;r)\,dt + \frac{1}{h}\int_{r^2}^{x+h} \widetilde{w}_{\alpha,0}(t;r)\,dt$$

By Part (i), the first integral on the right hand side is bounded by $\frac{r^2-x}{h}\widetilde{w}_{\alpha,\beta}(x;r) \le \widetilde{w}_{\alpha,\beta}(x;r) \le cw(x)$. Hence it suffices to prove that

$$\frac{1}{h}\int_{r^2}^{x+h} \widetilde{w}_{\alpha,0}(t;r)\,dt \le cw(x).$$

If $|x| \le r^2$, then Part (i) implies that

$$\frac{1}{h}\int_{r^2}^{x+h} \widetilde{w}_{\alpha,0}(t;r)\,dt \le \frac{x+h-r^2}{h}\widetilde{w}_{\alpha,0}(r^2;r) \le \widetilde{w}_{\alpha,0}(r^2;r) \le cr^{2\alpha}$$

where $r^{2\alpha} \le (r+|x|)^\alpha \le cw(x)$ since $\alpha \le 0 \le \beta$.

If $x < -r^2$, then $w(x) \sim |x|^{\alpha+\beta}$, and a simple scaling argument and the condition $\beta \ge 0$ allow to reduce the problem to the case $r = 1$. Actually it suffices to show the existence of $c > 0$ such that

$$J := \int_1^{x+h} t^\alpha\,dt \le ch|x|^{\alpha+\beta} \quad \text{when} \quad x \le -1,\ x+h \ge 1.$$

If $\alpha < -1$, then J is bounded by $\frac{1}{|\alpha+1|} \le c|x|^{\alpha+\beta+1} \le ch|x|^{\alpha+\beta}$, since $\alpha+\beta > -1$ and $h > |x| > 1$. In the case $\alpha = -1$ the integral J equals

$$\log(x+h) \sim \log h + \frac{x}{h} \le c\big(1 + h^{\min(\beta,1)}\big) \le ch|x|^{\beta-1},$$

since $\beta > -1-\alpha = 0$. Finally, for $\alpha > -1$, we may bound J by $c(x+h)^{\alpha+1}$. If $1 < |x| < h < 2|x|$, this term is bounded by $c|x| \le ch|x|^\alpha \le ch|x|^{\alpha+\beta}$. In the remaining case when $h > 2|x|$, we get that $(x+h)^{\alpha+1} \le ch^{\alpha+1} \le ch|x|^{\alpha+\beta}$, since $\alpha \le 0 \le \beta$.

Now (ii) is completely proved.

(iii) By Theorem 2.3 (i) and Part (ii) of this Lemma

$$w(x) = \frac{(1+r^2+x^2)^{\alpha_1/2}}{(1+r^2+x^2)^{\alpha_2(p-1)/2}(1+\sqrt{r^2+x^2}-x)^{\beta_2(p-1)}} \in A_p^+$$

for all $\alpha_1, \alpha_2, \beta_2$ satisfying $-1 < \alpha_1 \le 0$, $\alpha_2 \le 0 \le \beta_2$ and $\alpha_2 + \beta_2 > -1$. Hence, with $\alpha = \alpha_1 - \alpha_2(p-1)$, $\beta = -\beta_2(p-1)$, we get that $w = w_{\alpha,\beta}(\cdot\,; r) \in A_p^+$ for all α, β satisfying $\alpha > -1$, $\beta \le 0$, and $\alpha + \beta < p - 1$. By analogy,

$$w(x) = \frac{(1+r^2+x^2)^{\alpha_1/2}(1+\sqrt{r^2+x^2}-x)^{\beta_1}}{(1+r^2+x^2)^{\alpha_2(p-1)/2}} \in A_p^-$$

for all $\alpha_1, \alpha_2, \beta_1$ satisfying $\alpha_1 \le 0 \le \beta_1$, $\alpha_1 + \beta_1 > -1$, $-1 < \alpha_2 \le 0$. Hence $w = w_{\alpha,\beta}(\cdot\,; r) \in A_p^-$ for all α, β such that $\beta \ge 0$, $\alpha < p-1$ and $\alpha + \beta > -1$. Moreover, in both cases the $A_p^\pm$-constant of the weight is uniformly bounded in $r > 0$. ■

Note that the univariate weights $\widetilde{w}_{\alpha,\beta}$ and $w_{\alpha,\beta}$ mainly differ for large $x > 0$. While $\widetilde{w}_{0,\beta}$ decays as $\left(\frac{1}{x}\right)^\beta$ as $x \to \infty$ for every fixed $r > 0$, the weight $w_{0,\beta}$ is bounded below by 1 as $x \to \infty$. The reason to consider the weights $w_{\alpha,\beta}$ rather than $\widetilde{w}_{\alpha,\beta}$ is based on the use of the anisotropic weights η_β^α on $\mathbb{R}^3$, see Corollary 1.5, when fixing $r = |(x_1, x_2)|$, $x_1, x_2 \in \mathbb{R}$, so that $\eta_\beta^\alpha(x_1, x_2, x_3) = w_{\alpha,\beta}(x_3; r)$.

Due to the geometry of the problem we introduce cylindrical coordinates $(r, x_3, \theta) \in (0,\infty) \times \mathbb{R} \times [0, 2\pi)$ and write $u(x_1, x_2, x_3) = u(r, x_3, \theta)$. Then the term $(e_3 \wedge x) \cdot \nabla u = -x_2\partial_1 u + x_1\partial_2 u$ may be rewritten in the form $(e_3 \wedge x) \cdot \nabla u = \partial_\theta u$ using the angular derivative ∂_θ applied to $u(r, x_3, \theta)$. Working first of all formally or in the space $\mathcal{S}'(\mathbb{R}^3)$ of tempered distributions we apply the Fourier transform $\mathcal{F} = \widehat{\ }$ to (1.5). With the Fourier variable $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $s = |\xi|$ we get from (1.5)

$$(\nu s^2 + ik\xi_3)\widehat{u} - \widetilde{\omega}(\partial_\varphi \widehat{u} - e_3 \wedge \widehat{u}) + i\xi\widehat{p} = \widehat{f}, \quad i\xi \cdot \widehat{u} = 0. \tag{2.4}$$

Here $(e_3 \wedge \xi) \cdot \nabla_\xi = -\xi_2\partial/\partial\xi_1 + \xi_1\partial/\partial\xi_2 = \partial_\varphi$ is the angular derivative in Fourier space when using cylindrical coordinates $(s, \xi_3, \phi,) \in \mathbb{R}_+ \times \mathbb{R} \times [0, 2\pi)$. Since $i\xi \cdot \widehat{u} = 0$ implies $i\xi \cdot \big(\partial_\varphi \widehat{u} - \omega \times \widehat{u}\big) = 0$, the unknown pressure p is given by $-|\xi|^2\widehat{p} = i\xi \cdot \widehat{f}$, i.e.,

$$\widehat{\nabla p}(\xi) = i\xi \cdot \widehat{p} = \frac{(\xi \cdot \widehat{f})\widehat{f}}{|\xi|^2}\,.$$

Then the Hörmander-Mikhlin multiplier theorem on weighted L^q-spaces (Theorem 2.2 (ii)) yields for every weight $w \in A_q(\mathbb{R}^3, \mathcal{C})$ the estimate

$$\|\nabla p\|_{q,w} \le c\|f\|_{q,w} \tag{2.5}$$

where $c = c(q, w) > 0$; in particular $\nabla p \in L^q_w$.

Hence u may be considered as a (solenoidal) solution of the reduced problem

$$-\nu\Delta u + k\partial_3 u - \widetilde{\omega}(\partial_\theta u - e_3 \wedge u) = F := f - \nabla p \quad \text{in} \quad \mathbb{R}^3, \tag{2.6}$$

or—in Fourier space—

$$(\nu s^2 + ik\xi_3)\widehat{u} - \widetilde{\omega}(\partial_\varphi \widehat{u} - e_3 \wedge \widehat{u}) = \widehat{F}.$$

As shown in [Fa2] this inhomogeneous linear differential equation of first order with respect to φ has the unique 2π-periodic solution

$$\begin{aligned} \widehat{u}(\xi) &= \frac{1}{1 - e^{-2\pi(\nu|\xi|^2 + ik\xi_3)/\widetilde{\omega}}} \int_0^{2\pi/\widetilde{\omega}} e^{-(\nu|\xi|^2 + ik\xi_3)t} O_\omega^T(t) \mathcal{F}F(O_\omega(t)\xi)\, dt, \\ &= \int_0^\infty e^{-\nu|\xi|^2 t} O_\omega^T(t)(\mathcal{F}F(O_\omega(t) \cdot -kte_3))(\xi)\, dt. \end{aligned} \tag{2.7}$$

Finally note that $e^{-\nu|\xi|^2 t}$ is the Fourier transform of the heat kernel $E_t(x) = (4\pi\nu t)^{-3/2} e^{-|x|^2/4\nu t}$ yielding

$$u(x) = \int_0^\infty E_t * O_\omega^T(t) F(O_\omega(t) \cdot -kte_3)(x)\, dt. \tag{2.8}$$

Since $F = f - \nabla p$ is solenoidal, the identity $i\xi \cdot \widehat{F} = 0$ easily implies that also u is solenoidal.

The main ingredients of the proof of Theorem 1.4 are a weighted version of Littlewood-Paley theory and a decomposition of the integral operator

$$\begin{aligned} Tf(x) &= \int_0^\infty \widehat{\psi}_{\nu t}(\xi) O_\omega^T(t) \mathcal{F}f(O_\omega(t) \cdot -kte_3)(\xi)\, \frac{dt}{t} \\ &= \int_0^\infty \widehat{\psi}_t(\xi) O_{\omega/\nu}^T(t) \mathcal{F}f\Big(O_{\omega/\nu}(t) \cdot -\frac{k}{\nu} te_3\Big)(\xi)\, \frac{dt}{t}, \end{aligned} \tag{2.9}$$

where

$$\widehat{\psi}(\xi) = \frac{1}{(2\pi)^{3/2}} |\xi|^2 e^{-|\xi|^2} \quad \text{and} \quad \widehat{\psi}_t(\xi) = \widehat{\psi}(\sqrt{t}\xi), \quad t > 0, \tag{2.10}$$

are the Fourier transforms of the function $\psi = -\Delta E_1 \in \mathcal{S}(\mathbb{R}^3)$ and of $\psi_t(x) = t^{-3/2}\psi(x/\sqrt{t})$, $t > 0$, resp. Note that due to Theorem 1.4 it suffices to find an estimate of $\|\Delta u\|_{q,w}$ in order to estimate all second order derivatives $\partial_j\partial_k u$ of u.

To decompose $\widehat{\psi}_t$ choose $\widetilde{\chi} \in C_0^\infty\big(\frac{1}{2}, 2\big)$ satisfying $0 \le \widetilde{\chi} \le 1$ and $\sum_{j=-\infty}^{\infty} \widetilde{\chi}(2^{-j}s) = 1$ for all $s > 0$. Then define χ_j, $j \in \mathbb{Z}$, by its Fourier transform

$$\widehat{\chi}_j(\xi) = \widetilde{\chi}(2^{-j}|\xi|), \quad \xi \in \mathbb{R}^n,$$

yielding $\sum_{j=-\infty}^{\infty} \widehat{\chi}_j = 1$ on $\mathbb{R}^n \setminus \{0\}$ and

$$\operatorname{supp} \widehat{\chi}_j \subset A(2^{j-1}, 2^{j+1}) := \{\xi \in \mathbb{R}^3 : 2^{j-1} \le |\xi| \le 2^{j+1}\}. \tag{2.11}$$

Using χ_j, we define for $j \in \mathbb{Z}$

$$\psi^j = \frac{1}{(2\pi)^{3/2}} \chi_j * \psi \quad \big(\widehat{\psi} = \widehat{\chi}_j \cdot \widehat{\psi}\,\big). \tag{2.12}$$

Obviously, $\sum_{j=-\infty}^{\infty} \psi^j = \psi$ on $\mathbb{R}^3$. Finally, in view of (2.9), (2.12), we define the linear operators

$$\begin{aligned} T_j f(x) &= \int_0^\infty \widehat{\psi}^j_{\nu t}(\xi) O_\omega^T(t) \mathcal{F} f(O_\omega(t)\cdot - kte_3)(\xi) \frac{dt}{t} \\ &= \int_0^\infty \widehat{\psi}^j_t(\xi) O_{\omega/\nu}^T(t) \mathcal{F} f\Big(O_{\omega/\nu}(t)\cdot - \frac{k}{\nu} te_3\Big)(\xi) \frac{dt}{t}\,. \end{aligned} \tag{2.13}$$

Since formally $T = \sum_{j=-\infty}^{\infty} T_j$, we have to prove that this infinite series converges even in the operator norm on L^q_w.

For later use we cite the following lemma, see [6].

Lemma 2.5. *The functions ψ^j, ψ^j_t, $j \in \mathbb{Z}$, $t > 0$, have the following properties:*

(i) $\operatorname{supp} \widehat{\psi}^j_t \subset A\left(\frac{2^{j-1}}{\sqrt{t}}, \frac{2^{j+1}}{\sqrt{t}}\right)$.

(ii) For $m > \frac{3}{2}$ let $h(x) = (1 + |x|^2)^{-m}$ and $h_t(x) = t^{-3/2} h\big(\frac{x}{\sqrt{t}}\big)$, $t > 0$. Then there exists a constant $c > 0$ independent of $j \in \mathbb{Z}$ such that

$$\begin{aligned} |\psi^j(x)| &\le c 2^{-2|j|}\, h_{2^{-2j}}(x), \quad x \in \mathbb{R}^3, \\ \|\psi^j\|_1 &\le c 2^{-2|j|}\,. \end{aligned} \tag{2.14}$$

To introduce a weighted Littlewood-Paley decomposition of L^q_w choose $\widetilde{\varphi} \in C_0^\infty(\frac{1}{2}, 2)$ such that $0 \le \widetilde{\varphi} \le 1$ and $\int_0^\infty \widetilde{\varphi}(s)^2 \frac{ds}{s} = \frac{1}{2}$. Then define $\varphi \in \mathcal{S}(\mathbb{R}^3)$ by its Fourier transform $\widehat{\varphi}(\xi) = \widetilde{\varphi}(|\xi|)$ yielding for every $s > 0$

$$\widehat{\varphi}_s(\xi) = \widetilde{\varphi}(\sqrt{s}|\xi|), \quad \operatorname{supp} \widehat{\varphi}_s \subset A\Big(\frac{1}{2\sqrt{2}}, \frac{2}{\sqrt{2}}\Big) \tag{2.15}$$

and the normalization $\int_0^\infty \widehat{\varphi}_s(\xi)^2 \frac{ds}{s} = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Theorem 2.6. *Let $1 < q < \infty$ and $w \in A_q(\mathbb{R}^3)$. Then there are constants $c_1, c_2 > 0$ depending on q, w and φ such that for all $f \in L^q_w$*

$$c_1 \|f\|_{q,w} \le \left\| \left(\int_0^\infty |\varphi_s * f(\cdot)|^2 \frac{ds}{s} \right)^{1/2} \right\|_{q,w} \le c_2 \|f\|_{q,w} \tag{2.16}$$

where $\varphi_s \in \mathcal{S}(\mathbb{R}^n)$ is defined by (2.15).

Proof. See [22, Proposition 1.9, Theorem 1.10], and also [17], [24]. ∎

3 Proofs

As a preliminary version of Theorem 1.4 we prove the following proposition. The extension to more general weights based on complex interpolation of L^q_w-spaces will be postponed to the end of Section 3.

Proposition 3.1. *Let the weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ be independent of the angle θ and define $w_r(x_3) := w(x_1, x_2, x_3)$ for fixed $r = |(x_1, x_2)| > 0$. Assume that*

$$\begin{array}{lll} w \in \widetilde{A}^-_{q/2} & & \text{if} \quad q > 2, \\ w \in \widetilde{A}^-_1 \quad \text{or} \quad \frac{1}{w} \in \widetilde{A}^+_1 & & \text{if} \quad q = 2, \\ w^{2/(2-q)} \in \widetilde{A}^-_{q/(2-q)} & & \text{if} \quad 1 < q < 2. \end{array} \tag{3.1}$$

Then the linear operator T defined by (2.9) satisfies the estimate

$$\|Tf\|_{q,w} \le c\|f\|_{q,w} \quad \text{for all} \quad f \in L^q_w \tag{3.2}$$

with a constant $c = c(q, w) > 0$ independent of f.

Proof. Step 1. First we consider the case $q > 2$, $w \in \widetilde{A}^-_{q/2} \subset A_q$, and define the sublinear operator $\mathcal{M}^j$, a modified maximal operator, by

$$\mathcal{M}^j g(x) = \sup_{s>0} \int_{A_s} \left(|\psi^j_t| * |g|\right)\left(O^T_{\omega/\nu}(t)x + \frac{k}{\nu}te_3\right)\frac{dt}{t}\,, \tag{3.3}$$

where $A_s = [\frac{s}{16}, 16s]$. Then we will prove the preliminary estimate

$$\|T_j f\|_{q,w} \leq c\|\psi^j\|_1^{1/2}\|\mathcal{M}^j\|^{1/2}_{L_v^{(q/2)'}}\|f\|_{q,w}\,, \quad j \in \mathbb{Z}, \tag{3.4}$$

where v denotes the θ-independent weight

$$v = w^{-\left(\frac{q}{2}\right)'/\left(\frac{q}{2}\right)} = w^{-\frac{2}{q-2}} \in \widetilde{A}^+_{(q/2)'} = \widetilde{A}^+_{q/(q-2)}. \tag{3.5}$$

To prove (3.4) we use the Littlewood-Paley decomposition of L^q_w, see (2.16), applied to $T_j f$. By a duality argument we find some function $0 \leq g \in L_v^{(q/2)'} = \left(L_w^{(q/2)}\right)^*$ with $\|g\|_{(q/2)',v} = 1$ such that

$$\left\|\int_0^\infty |\varphi_s * T_j f(\cdot)|^2\,\frac{ds}{s}\right\|_{q/2,w} = \int_0^\infty \int_{\mathbb{R}^3} |\varphi_s * T_j f(x)|^2 g(x)\,dx\,\frac{ds}{s}\,. \tag{3.6}$$

To estimate the right-hand side of (3.6) note that

$$\varphi_s * T_j f(x) = \int_0^\infty O^T_{\omega/\nu}(t)(\varphi_s * \psi^j_t * f)\left(O_{\omega/\nu}(t)x - \frac{k}{\nu}te_3\right)\frac{dt}{t}\,,$$

where $\varphi_s * \psi^j_t = 0$ unless $t \in A(s,j) := [2^{2j-4}s, 2^{2j+4}s]$. Since $\int_{t\in A(s,j)} \frac{dt}{t} = \log 2^8$ for every $j \in \mathbb{Z}$, $s > 0$, we get by the inequality of Cauchy-Schwarz and the associativity of convolutions that

$$\begin{aligned}|\varphi_s * T_j f(x)|^2 &\leq c\int_{A(s,j)} \left|\left(\psi^j_t * (\varphi_s * f)\right)\left(O_{\omega/\nu}(t)x - \frac{k}{\nu}te_3\right)\right|^2\frac{dt}{t}\\ &\leq c\|\psi^j\|_1 \int_{A(s,j)} \left(|\psi^j_t| * |\varphi_s * f|^2\right)\left(O_{\omega/\nu}(t)x - \frac{k}{\nu}te_3\right)\frac{dt}{t}\,;\end{aligned}$$

here we used the estimate $|(\psi^j_t * (\varphi_s * f))(y)|^2 \leq \|\psi^j_t\|_1(|\psi^j_t| * |\varphi_s * f|^2)(y)$

and the identity $\|\psi_t^j\|_1 = \|\psi^j\|_1$, see (2.14). Thus

$$\begin{aligned}
&|T_j f\|_{q,w}^2 \\
&\le c\|\psi^j\|_1 \int_0^\infty \int_{A(s,j)} \int_{\mathbb{R}^3} (|\psi_t^j| * |\varphi_s * f|^2)\Big(O_{\omega/\nu}(t)x - \frac{k}{\nu}te_3\Big)\, g(x)\, dx \frac{dt}{t}\frac{ds}{s} \\
&\le c\|\psi^j\|_1 \int_0^\infty \int_{A(s,j)} \int_{\mathbb{R}^3} (|\psi_t^j| * |\varphi_s * f|^2)(x)\, g\Big(O_{\omega/\nu}^T(t)x + \frac{k}{\nu}te_3\Big)\, dx \frac{dt}{t}\frac{ds}{s} \\
&\le c\|\psi^j\|_1 \int_{\mathbb{R}^3} \int_0^\infty |\varphi_s * f|^2(x) \int_{A(s,j)} \big(|\psi_t^j| * g\big)\Big(O_{\omega/\nu}^T(t)x + \frac{k}{\nu}te_3\Big)\, \frac{dt}{t}\frac{ds}{s} dx,
\end{aligned} \tag{3.7}$$

since ψ_t^j is radially symmetric. By definition of $\mathcal{M}^j$ the innermost integral is bounded by $\mathcal{M}^j g(x)$ uniformly in $s > 0$. Hence we may proceed in (3.7) using Hölder's inequality as follows:

$$\begin{aligned}
\|T_j f\|_{q,w}^2 &\le c\|\psi^j\|_1 \int_{\mathbb{R}^3} \Big(\int_0^\infty |\varphi_s * f|^2(x)\, \frac{ds}{s}\Big)\, \mathcal{M}^j g(x)\, dx \\
&\le c\|\psi^j\|_1 \, \Big\| \int_0^\infty |\varphi_s * f|^2(x)\, \frac{ds}{s} \Big\|_{q/2,w} \|\mathcal{M}^j g\|_{(q/2)',v}.
\end{aligned} \tag{3.8}$$

Now (2.16) and the normalization $\|g\|_{(q/2)',v} = 1$ complete the proof of (3.4).

Step 2. We estimate $\|\mathcal{M}^j g\|_{(q/2)',v}$. For functions γ depending on θ, x_3 only let $\mathcal{M}_{\mathrm{hel}}$ denote the "helical" maximal operator

$$\mathcal{M}_{\mathrm{hel}}\gamma(\theta, x_3) = \sup_{s>0} \frac{1}{s} \int_{A_s} |\gamma|\Big(\theta - \frac{\omega}{\nu}t, x_3 + \frac{k}{\nu}t\Big)\, dt,$$

where $A_s = \big[\frac{s}{16}, 16s\big]$. Then, writing $p := \big(\frac{q}{2}\big)'$, we claim that

$$\mathcal{M}^j g(x) \le c2^{-2|j|}\mathcal{M}(\mathcal{M}_{\mathrm{hel}}g)(x) \quad \text{for a.a. } x \in \mathbb{R}^n, \tag{3.9}$$

$$\|\mathcal{M}^j g\|_{p,v} \le c2^{-2|j|}\|g\|_{p,v}, \tag{3.10}$$

where in (3.9) $\mathcal{M}_{\mathrm{hel}}g$ is considered as $\mathcal{M}_{\mathrm{hel}}g(r,\cdot,\cdot)$ for almost all $r > 0$.

To prove (3.9) we use the pointwise estimate $|\psi_t^j(x)| \le c2^{-2|j|}h_{t2^{-2j}}(x)$, see Lemma 2.5 (ii). Hence

$$\mathcal{M}^j g(x) \le c2^{-2|j|} \sup_{s>0} \int_{A_s} (h_{t2^{-2j}} * |g|)\Big(O_{\omega/\nu}^T(t)x + \frac{k}{\nu}te_3\Big)\, \frac{dt}{t}.$$

Moreover, there exists a constant $c > 0$ independent of $s > 0$, $j \in \mathbb{Z}$, such that $h_{t2^{-2j}} \le c h_{s2^{-2j}}$ for all $t \in A_s$. Consequently,

$$\begin{aligned}\mathcal{M}^j g(x) &\le c2^{-2|j|} \sup_{s>0} h_{s2^{-2j}} * \int_{A_s} |g|\Big(O^T_{\omega/\nu}(t)x + \frac{k}{\nu}te_3\Big)\,\frac{dt}{t}\\ &\le c2^{-2|j|} \sup_{t>0} h_t * \mathcal{M}_{\rm hel} g(x).\end{aligned}$$

Since h is nonnegative, radially decreasing, and $\|h_t\|_1 = \|h\|_1 = c_0 > 0$ for all $t > 0$, a well-known convolution estimate, see [25], II §2.1, yields the pointwise estimate (3.9).

Step 3. Note that up to now we have not yet used any specific properties of the weight $v \in A_p$. To estimate $\mathcal{M}_{\rm hel} g$ we shall work with a suitable one-sided maximal operator since our weight belongs to a Muckenhoupt class in $\mathbb{R}^3$ but a problem occurs when the weight is considered with respect to x_3 only. This naturally corresponds to the physical circumstances of the problem, where in the Oseen case the wake should appear. To estimate $\mathcal{M}_{\rm hel} g$ we write $g_r(\theta, x_3) = g(r, \theta, x_3) = g(x)$ and $v_r(x_3) = v(x)$, $r = |(x_1, x_2)| > 0$, for the θ-independent weight v. Then by the 2π-periodicity of g_r and v_r with respect to θ we get for almost all $r > 0$

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^{2\pi} \mathcal{M}_{\rm hel} g_r(\theta, x_3)^p v_r(x_3)\,d\theta\,dx_3\\ &\le \int_{\mathbb{R}}\int_0^{2\pi} \Big| \sup_{s>0} \frac{1}{s}\int_0^{16s} |g_r|\Big(\theta - \frac{\omega}{k}\Big(x_3 + \frac{k}{\nu}t\Big), x_3 + \frac{k}{\nu}t\Big)\,dt\Big|^p v_r(x_3)\,d\theta\,dx_3\\ &= \int_{\mathbb{R}}\int_0^{2\pi} \Big| \sup_{s>0} \frac{1}{s}\int_0^{16s} \gamma_{r,\theta}\Big(x_3 + \frac{k}{\nu}t\Big)\,dt\Big|^p d\theta\, v_r(x_3)\,dx_3\\ &= 16 \int_0^{2\pi}\int_{\mathbb{R}} \big|M^+\gamma_{r,\theta}(x_3)\big|^p v_r(x_3)\,dx_3\,d\theta\end{aligned}$$

where $\gamma_{r,\theta}(y_3) = |g_r|(\theta - \frac{\omega}{k}y_3, y_3)$ and M^+ denotes the one-sided maximal operator, see Definition 1.2. Since $w_r \in A^-_{q/2}$, by (3.5) and Theorem 2.3 (i) $v_r = w_r^{-(q/2)'/(q/2)} \in A^+_{(q/2)'} = A^+_p$ with an A^+_p-constant uniformly bounded in $r > 0$. Then Theorem 2.3 (ii) yields the estimate

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^{2\pi} \mathcal{M}_{\rm hel} g_r(\theta, x_3)^p v_r(x_3)\,d\theta\,dx_3\\ &\quad\le c\int_0^{2\pi}\int_{\mathbb{R}} |\gamma_{r,\theta}(x_3)|^p v_r(x_3)\,dx_3\,d\theta = c\|g_r\|^p_{L^p(\mathbb{R}\times(0,2\pi),\, v_r(x_3))},\end{aligned}$$

where $c > 0$ is independent of k, ν. Integrating with respect to $r\,dr$, $r \in (0, \infty)$, Fubini's theorem allows to consider an extension of $\mathcal{M}_{\text{hel}}$ to a bounded operator from $L^p_v(\mathbb{R}^3)$ to itself with an operator norm bounded uniformly in k, ν. Moreover, $\mathcal{M} : L^p_v(\mathbb{R}^3) \to L^p_v(\mathbb{R}^3)$ is bounded by Theorem 2.3 (ii). Hence, (3.9) implies (3.10), and by (3.4) as well as Lemma 2.5 (ii) we get the estimate

$$\|T_j f\|_{q,w} \leq c 2^{-2|j|} \|f\|_{q,w}$$

for all $f \in L^q_w(\mathbb{R}^3)$ with a constant $c > 0$ independent of $j \in \mathbb{Z}$. Summarizing the previous inequalities we proved (3.2) for $q > 2$.

Step 4. Now let $q = 2$, $w \in \widetilde{A}_1^-$. In this case the Littlewood-Paley decomposition of $T_j f$ in L^2_w implies that

$$\|T_j f\|_{2,w}^2 \leq c \int_0^\infty \int_{\mathbb{R}^n} |\varphi_s * T_j f|^2(x)\, g(x)\, dx\, \frac{ds}{s}\,,$$

where

$$g \in L^\infty_v,\ v = \frac{1}{w} \quad \text{and} \quad \|g\|_{\infty,v} = \operatorname{ess\,sup}_{\mathbb{R}^3} |g\, v| = 1.$$

By the same reasoning as before we arrive at (3.4), i.e.,

$$\|T_j f\|_{2,w} \leq c 2^{-|j|} \|\mathcal{M}^j g\|_{\infty,v}^{1/2}\, \|f\|_{2,w}, \tag{3.11}$$

and at (3.9). Concerning $\mathcal{M}_{\text{hel}}$ we use the pointwise estimate $g_r(\theta, x_3) \leq w_r(x_3)$ for a.a. $\theta \in (0, 2\pi)$, $x_3 \in \mathbb{R}$, and get that

$$\mathcal{M}_{\text{hel}} g_r(\theta, x_3) \leq \sup_{s>0} \frac{1}{s} \int_0^{16s} w_r\big(x_3 + \frac{k}{\nu} t\big)\, dt \leq 16\, M^+ w_r(x_3) \leq c w_r(x_3)$$

with a constant $c > 0$ independent of $r > 0$. Since w is an $A_1(\mathbb{R}^3)$-weight, (3.9) implies that

$$\mathcal{M}^j g(x) \leq c 2^{-2|j|} \mathcal{M} w(x) \leq c 2^{-2|j|} w(x)$$

and consequently that $\|\mathcal{M}^j g\|_{\infty,v} \leq c 2^{-2|j|}$ with a constant $c > 0$ independent of $j \in \mathbb{Z}$. Hence $\|T_j f\|_{2,w} \leq c 2^{-2|j|}$ proving (3.2) when $q = 2$.

Step 5. The remaining estimates are proved by duality arguments. Obviously the dual operator to T is defined by

$$T^* f(x) = \int_0^\infty (-\Delta) O_\omega(t) E_t * f(O_\omega^T(t) x + k t e_3)\, dt,$$

which has the same structure as K, but with an "opposite orientation". Hence T^* is bounded on L^q_w for $q \geq 2$ and all weights $w \in \widetilde{A}^+_{q/2}$. Now let $1 < q < 2$ and $w^{2/(2-q)} \in \widetilde{A}^-_{q/(2-q)} = \widetilde{A}^-_{(q'/2)'}$. Then by simple duality arguments $w' = w^{-q'/q} \in \widetilde{A}^+_{(q'/2)}$ and

$$|\langle Tf, g\rangle| = |\langle f, T^* g\rangle| \leq \|f\|_{q,w}\|T^* g\|_{q',w'} \leq c\|f\|_{q,w}\|g\|_{q',w'}.$$

Finally let $q = 2$ and $\frac{1}{w} \in \widetilde{A}^+_1$. As before,

$$|\langle Tf, g\rangle| \leq \|f\|_{2,w}\|T^* g\|_{2,1/w} \leq c\|f\|_{2,w}\|g\|_{2,1/w}.$$

Now Proposition 3.1 is completely proved. ■

Lemma 3.2 ([1]). *Let $1 \leq p_1, p_2 < \infty$, let $0 < w_1, w_2$ be weight functions, $\delta \in (0,1)$, and*

$$\frac{1}{p} = \frac{1-\delta}{p_1} + \frac{\delta}{p_2}, \quad w^{\frac{1}{p}} = w_1^{\frac{1-\delta}{p_1}} \cdot w_2^{\frac{\delta}{p_2}}.$$

Then

$$\left[L^{p_1}_{w_1}, L^{p_2}_{w_2}\right]_\delta = L^p_w$$

in the sense of complex interpolation.

In the following we shall derive an anisotropic variant of Jones's factorization theorem tailored to our situation, when we need to work with one-sided Muckenhoupt weights with respect to x_3, satisfying the usual Muckenhoupt condition in three dimensions.

Lemma 3.3 (Anisotropic Version of Jones' Factorization Theorem). *Suppose that $w \in \widetilde{A}^-_q$. Then there exist weights $w_1 \in \widetilde{A}^-_1$ and $w_2 \in \widetilde{A}^+_1$ such that*

$$w = \frac{w_1}{w_2^{q-1}}.$$

Here $\widetilde{A}^+_1$ is defined by analogy with $\widetilde{A}^-_1$, cf. Definition 1.2, by assuming for $w_2 \in \widetilde{A}^+_1$ that $(w_2)_r \in A^+_1$ with A^+_1-constant uniformly bounded in $r > 0$. An analogous result holds for $w \in \widetilde{A}^+_q$.

Proof. Let $q \geq 2$. Given $w \in \widetilde{A}^-_q$ we consider the operator T defined by

$$\begin{aligned} Tf = {} & \left(w^{-1/q}\mathcal{M}(f^{q/q'}w^{1/q})\right)^{q'/q} + w^{1/q}\mathcal{M}(fw^{-1/q}) \\ & + \left(w^{-1/q}M^+_1(f_r^{q/q'}w_r^{1/q})\right)^{q'/q} + w^{1/q}M^-_1(f_r w_r^{-1/q}) \end{aligned}$$

where $r = |(x_1, x_2)|$. Then for all $f \in L^q(\mathbb{R}^3)$

$$
\begin{aligned}
\|Tf\|_q^q \leq c \Big\{ & \int_{\mathbb{R}^3} w^{-q'/q} \big(\mathcal{M}(f^{q/q'} w^{1/q})\big)^{q'} \, dx + \int_{\mathbb{R}^3} w \big(\mathcal{M}(f w^{-1/q})\big)^q \, dx \\
& + \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}} w_r^{-q'/q} \big(M_1^+(f_r^{q/q'} w_r^{1/q})\big)^{q'} \, dx_3 \Big) \, d(x_1, x_2) \\
& + \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}} w_r \big(M_1^+(f_r w_r^{-1/q})\big)^q \, dx_3 \Big) \, d(x_1, x_2) \Big\} \\
\leq {} & A^q \|f\|_q^q,
\end{aligned}
$$

with a constant $A = A(q, w) > 0$.

Let us fix a nonnegative θ-independent function $f \in L^q(\mathbb{R}^3)$ with $\|f\|_q = 1$ and define

$$
\eta = \sum_{k=1}^{\infty} (2A)^{-k} T^k(f),
$$

where $T^k(f) = T(T^{k-1}(f))$. Obviously Tf and therefore also η are θ-independent. Moreover, $\eta \in L^q(\mathbb{R}^3)$ and $\|\eta\|_q \leq \sum_{k=1}^{\infty} 2^{-k} = 1$. In particular, $\eta(x) < \infty$ for a.a. $x \in \mathbb{R}^3$, $\eta_r(\cdot) \in L^q(\mathbb{R})$ for a.a. $(x_1, x_2) \in \mathbb{R}^2$ and $\eta_r(x_3) < \infty$ for a.a. $x_3 \in \mathbb{R}$. Since T is subadditive and positivity-preserving, we get the pointwise inequality

$$
T\eta \leq \sum_{k=1}^{\infty} (2A)^{-k} T^{k+1}(f) = \sum_{k=2}^{\infty} (2A)^{1-k} T^k(f) \leq (2A)\eta.
$$

Now let $w_1 := w^{1/q} \eta^{q/q'}$ and $w_2 := w^{-1/q} \eta$ such that $w = w_1 / w_2^{q-1}$. Then

$$
\begin{aligned}
\mathcal{M}(w_1) &\leq w^{1/q} (T\eta)^{q/q'} \leq w^{1/q} \eta^{q/q'} (2A)^{q/q'} = (2A)^{q/q'} w_1 \\
M_1^+()((w_1)_r) &\leq w^{1/q} (T\eta)^{q/q'} \leq w^{1/q} \eta^{q/q'} (2A)^{q/q'} = (2A)^{q/q'} (w_1)_r \\
\mathcal{M}(w_2) &\leq w^{-1/q} T(\eta) \quad \leq w^{-1/q} \eta \, 2A = 2A w_2 \\
M_1^-((w_2)_r) &\leq w^{-1/q} T(\eta) \quad \leq w^{-1/q} \eta \, 2A = 2A (w_2)_r
\end{aligned}
$$

proving that $w_1 \in \widetilde{A}_1^-$, $w_2 \in \widetilde{A}_1^+$.

The case $1 \leq q < 2$ follows by a simple duality argument, since $w \in \widetilde{A}_q^-$ is equivalent to $w^{-q'/q} \in \widetilde{A}_{q'}^+$. ■

Proof of Theorem 1.4 (i). Let $q \in (1,\infty)$ and $w \in A_q$ such that the L^q_w-estimate of ∇p holds, see (2.5). Hence it suffices to consider u defined by (2.7)–(2.8). We consider arbitrary $q_1, q_2 \in (1,\infty)$ and $\delta \in (0,1)$ with

$$1 < q_1 < q < q_2 < \infty, \quad q_1 \le 2 \le q_2 \quad \text{and} \quad \frac{1}{q} = \frac{1-\delta}{q_1} + \frac{\delta}{q_2}, \tag{3.12}$$

and assume that $w^\tau \in \widetilde{A}^-_{\tau q/2}$ with $\tau = \frac{2}{2-q(1-\delta)} \in [1,\infty)$. By Lemma 3.3 there exist weights $u \in \widetilde{A}^-_1$, $v \in \widetilde{A}^+_1$ such that

$$w^\tau = \frac{u}{v^{\tau q/2 - 1}} = \frac{u}{v^{\frac{q}{2-q(1-\delta)}-1}}.$$

Then we define the weights w_1, w_2 by

$$w_1^{2/(2-q_1)} = \frac{u}{v^{\frac{2(q_1-1)}{2-q_1}}} \quad \text{and} \quad w_2 = \frac{u}{v^{\frac{q_2-2}{2}}}$$

yielding

$$w_1^{2/(2-q_1)} \in \widetilde{A}^-_{q_1/(2-q_1)}, \quad w_2 \in \widetilde{A}^-_{q_2/2}.$$

Since, due to an elementary calculation, $w = w_1^{q(1-\delta)/q_1} w_2^{q\delta/q_2}$, Lemma 3.3 and Proposition 3.1 prove that T is bounded on $L^q_w(\mathbb{R}^3)$. Since $u_1 \in \widetilde{A}^-_1$, $v_1 \in \widetilde{A}^+_1$ are arbitrary, we proved the boundedness of T on L^q_w for arbitrary w if

$$w^\tau \in \widetilde{A}^-_{\tau q/2}, \quad \tau = \frac{2}{2-q(1-\delta)} \in [1,\infty).$$

Now we have to find all admissible τ subject to the restrictions given by (3.12). For this reason consider the easier term

$$s = 2\left(1 - \frac{1}{\tau}\right) = q(1-\delta) = q\,\frac{\frac{1}{q} - \frac{1}{q_2}}{\frac{1}{q_1} - \frac{1}{q_2}}.$$

First Case $1 < q < 2$, in which $1 < q_1 < q$ and $q_2 \ge 2$. Due to monotonicity properties of s as a function of $\frac{1}{q_1}$ and of $\frac{1}{q_2}$ it suffices to check s at the corners of the rectangle $(\frac{1}{q}, 1) \times (0, \frac{1}{2}]$. The corresponding function values are $q, 1$ and $2-q$. Hence the range of s equals the interval $(2-q, q)$ yielding for $\tau = \frac{2}{2-s}$ the condition

$$\frac{2}{q} < \tau < \frac{2}{2-q}.$$

Note that the limiting value $\tau = \frac{2}{2-q}$ is allowed due to Proposition 3.1. Finally the condition $w^\tau \in \widetilde{A}^-_{\tau q/2}$, $\frac{2}{q} < \tau \leq \frac{2}{2-q}$, easily implies that $w \in A_q$: By Lemma 3.3 there exist $u_1 \in \widetilde{A}^-_1, v_1 \in \widetilde{A}^+_1$ such that

$$w = \frac{u_1^{\frac{1}{\tau}}}{v_1^{\frac{q}{2}-\frac{1}{\tau}}}, \tag{3.13}$$

where $u_1^{\frac{1}{\tau}} \in \widetilde{A}^-_1$ and $\frac{q}{2} - \frac{1}{\tau} \leq q-1$ yielding $v_1^{(\frac{q}{2}-\frac{1}{\tau})/(q-1)} \in \widetilde{A}^+_1$.

Second Case $q > 2$, in which $1 < q_1 \leq 2$ and $q_2 > q$. In this case the values of s at the corners of the rectangle $[\frac{1}{2}, 1) \times (0, \frac{1}{q})$ in the $\left(\frac{1}{q_1}, \frac{1}{q_2}\right)$-plane are $0, 1$ and 2. Hence

$$1 < \tau < \infty,$$

and we observe that $\tau = 1$ is admissible due to Proposition 3.1. Finally note that the condition $w^\tau \in A_{\tau q/2}$ implies also $w \in \widetilde{A}^-_q$: There exist $u_1 \in \widetilde{A}^-_1, v_1 \in \widetilde{A}^+_1$ such that w satisfies (3.13), where again $\frac{q}{2} - \frac{1}{\tau} + 1 \leq q$ for all $\tau \in (1, \infty)$.

Third Case $q = 2$. In this case it suffices to interpolate between $L^2_{w_1}$ and $L^2_{w_2}$, where $w_1 \in \widetilde{A}^-_1$ and $\frac{1}{w_2} \in \widetilde{A}^+_1$, see Proposition 3.1. Then T is bounded on L^2_w for all

$$w = \frac{w_1^{1-\delta}}{w_2^\delta}, \ 0 < \delta < 1.$$

Then $w^{1/(1-\delta)} = w_1/w_2^{\delta/(1-\delta)}$, or with $\tau = \frac{1}{1-\delta} \in (1, \infty)$,

$$w^\tau = \frac{w_1}{w_2^{\tau-1}} \in \widetilde{A}^-_\tau = \widetilde{A}^-_{\tau q/2}.$$

(ii) Note that $L^{q_i}_{w_i}(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$, $i = 1, 2$; indeed, $w_i \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\int_{|x|\geq 1} w_i(x)|x|^{-nq_i}\,dx < \infty$, see [10, IV.3 (3.2)]. Since the equation (1.5) is linear, it suffices to consider $f = 0$ and a solution $u \in S'(\mathbb{R}^n)^n$ of (1.8). In the proof of [6], Theorem 1.1 (2), (3), it was shown that under these assumptions u is a polynomial and that $u(x) = \alpha\omega + \beta\omega \wedge x + \gamma(x_1, x_2, -2x_3)^T$, $\alpha, \beta, \gamma \in \mathbb{R}$ ($u(x) = \beta(-x_2, x_1)$ if $n = 2$). ∎

Proof of Corollary 1.5. Considering *a priori* estimates for $\frac{\partial u}{\partial x_3}$ we use the representation (2.7) of u. In order to analyze the dependence of the following estimates on the parameters k, ν and $\widetilde{\omega}$ let

$$k' = k/\widetilde{\omega},\ \nu' = \nu/\widetilde{\omega} \quad \text{and} \quad D(\xi) = 1 - e^{-2\pi(\nu'|\xi|^2 + ik'\xi_3)}.$$

Then for $f \in \mathcal{S}(\mathbb{R}^3)^3$ we get the identity

$$\widehat{k\partial_3 u(\xi)} = \frac{ik'\xi_3}{D(\xi)} \int_0^{2\pi} e^{-(\nu'|\xi|^2 + ik'\xi_3)t}\, O_{e_3}^T(t)\, \widehat{F}(O_{e_3}(t)\xi)\, dt, \tag{3.14}$$

where $F = f - \nabla p$, see (2.6). Choose a cut-off function $\eta \in C_0^\infty(B_1(0))$ with $\eta(\xi) = 1$ for $\xi \in B_{1/2}(0)$ and define the multiplier functions

$$m_0(\xi) = \frac{ik'\xi_3\eta_{\nu'}(\xi)}{D(\xi)}, \quad m_1(\xi) = \frac{k'}{\sqrt{\nu'}} \frac{1 - \eta_{\nu'}(\xi)}{D(\xi)},$$

where $\eta_{\nu'}(\xi) = \eta(\sqrt{\nu'}\xi)$, as well as

$$\mu_{0,t}(\xi) = e^{-(\nu'|\xi|^2 + ik'\xi_3)t}, \quad \mu_{1,t}(\xi) = i\xi_3\sqrt{\nu'}e^{-(\nu'|\xi|^2 + ik'\xi_3)t}, \quad t \in (0, 2\pi).$$

Then we get

$$\widehat{k\partial_3 u(\xi)} = m_0(\xi)\widehat{I}_0(\xi) + m_1(\xi)\widehat{I}_1(\xi),$$

where $I_0(x), I_1(x)$ are defined by their Fourier transforms

$$\widehat{I}_0(\xi) = \int_0^{2\pi} \mu_{0,t}(\xi)\, O_{e_3}^T(t)\, \widehat{F}(O_{e_3}(t)\cdot)(\xi)\, dt,$$
$$\widehat{I}_1(\xi) = \int_0^{2\pi} \mu_{1,t}(\xi)\, O_{e_3}^T(t)\, \widehat{F}(O_{e_3}(t)\cdot)(\xi)\, dt.$$

Concerning the multiplier function $\mu_{0,t}$ we note that e.g.

$$\begin{aligned} \left|\xi_3 \tfrac{\partial \mu_{0,t}}{\partial \xi_3}\right| &= \left|(-2\nu' t\xi_3^2 - ik't\xi_3)e^{-(\nu'|\xi|^2 + ik'\xi_3)t}\right| \\ &\le C\big(\nu' t|\xi|^2 + \tfrac{k'}{\sqrt{\nu'}}\sqrt{\nu' t}|\xi_3|\big)e^{-\nu'|\xi|^2 t} \\ &\le C\big(1 + \tfrac{k'}{\sqrt{\nu'}}\big) \end{aligned}$$

with a constant $C > 0$ independent of $\xi \neq 0$, $t \in (0, 2\pi)$, $k' > 0$ and $\nu' > 0$. Then it is easily seen that $\mu_{0,t}$, $\mu_{1,t}$ satisfy the pointwise multiplier estimates

$$\sup_{t\in(0,2\pi)} \max_\alpha \sup_{\xi\neq 0} \big(|\xi^\alpha D_\xi^\alpha \mu_{0,t}(\xi)| \ + \ \sqrt{t}|\xi^\alpha D_\xi^\alpha \mu_{1,t}(\xi)|\big) \le C\Big(1 + \frac{k}{\sqrt{\nu|\omega|}}\Big)$$

uniformly in $k' > 0$ and $\nu' > 0$, where $\alpha \in \mathbb{N}_0^3$ runs through the set of all multi-indices $\alpha \in \{0,1\}^3$. Hence Theorem 2.2 (iii) and (2.5) show that

$$\|I_0\|_{q,w} \leq c\Big(1 + \frac{k}{\sqrt{\nu|\omega|}}\Big) \int_0^{2\pi} \|F(O_{e_3}(t)\cdot)\|_{q,w}\, dt \leq c\Big(1 + \frac{k}{\sqrt{\nu|\omega|}}\Big)\|f\|_{q,w},$$

$$\|I_1\|_{q,w} \leq c\Big(1 + \frac{k}{\sqrt{\nu|\omega|}}\Big) \int_0^{2\pi} \frac{1}{\sqrt{t}}\|F(O_{e_3}(t)\cdot)\|_{q,w}\, dt \leq c\Big(1 + \frac{k}{\sqrt{\nu|\omega|}}\Big)\|f\|_{q,w},$$

where $c > 0$ is independent of k, ω and ν. Moreover, a lengthy, but elementary calculation proves that m_0, m_1 satisfy the pointwise estimates

$$\max_{j=0,1} \max_{\alpha} \sup_{\xi \neq 0} |\xi^\alpha D_\xi^\alpha m_j(\xi)| \leq C\Big(1 + \frac{k^4}{\nu^2|\omega|^2}\Big)$$

with $c > 0$ independent of ν, ω, k; for details see [3]. Now another application of Theorem 2.2 (iii) yields the estimate

$$\|k\partial_3 u\|_{q,w} \leq c\left(1 + \frac{k^5}{\nu^{5/2}|\omega|^{5/2}}\right)\|f\|_{q,w}$$

for $f \in S(\mathbb{R}^3)^3$, with a constant $c > 0$ independent of f, k, ν and ω. Since $S(\mathbb{R}^3)$ is dense in $L^q_w(\mathbb{R}^3)$, this result extends to all $f \in L^q_w$; for its proof we refer to [3]. However, note that we did not estimate $\widehat{F}(O_\omega(t)\cdot\ - kte_3)\xi)$ in $L^q(\Omega)$ as in [3]; instead we have to deal with $\widehat{F}(O_{e_3}(t)\cdot)$, and the shift operator is estimated with the help of multipliers.

Now Corollary 1.5 is completely proved. ■

Proof of Corollary 1.6. We have to check for which α, β the weight $w(x) = \eta^\alpha_\beta(x) = (1 + |x|)^\alpha (1 + s(x))^\beta$ satisfies the conditions needed in Theorem 1.4. From [2] and [16, Theorem 5.2] we know that $w = \eta^\alpha_\beta \in A_p$, $1 < p < \infty$, if and only if $-1 < \beta < p - 1$ and $-3 < \alpha + \beta < 3(p - 1)$; moreover, by Lemma 2.4 (iii) we have to satisfy the conditions $\alpha < p - 1$, $\beta \geq 0$, $\alpha + \beta > -1$ to get that $w_r(\cdot) \in A_p^-$.

Let $q > 2$. Then in view of (1.8) and (2.3) we have to analyze the convex set

$$\begin{aligned} \mathcal{C} \;=\; & \big\{(\alpha,\beta);\ \alpha < \tfrac{q}{2} - \tfrac{1}{\tau},\ \beta \geq 0,\ \alpha + \beta > -\tfrac{1}{\tau},\ -\tfrac{1}{\tau} < \beta < \tfrac{q}{2} - \tfrac{1}{\tau}, \\ & -\tfrac{3}{\tau} < \alpha + \beta < \tfrac{3q}{2} - \tfrac{3}{\tau} \quad \text{for some } \tau \in [1,\infty)\big\}. \end{aligned}$$

Obviously the conditions $\beta > -\frac{1}{\tau}$ and $-\frac{3}{\tau} < \alpha + \beta < \frac{3q}{2} - \frac{3}{\tau}$ are redundant since $\frac{q}{2} - \frac{1}{\tau}$ is positive; moreover, the conditions $\alpha + \beta > -\frac{1}{\tau}$ and $\beta < \frac{q}{2} - \frac{1}{\tau}$ yield $\alpha > -\frac{q}{2}$. We will see that

$$\mathcal{C} = \big\{(\alpha, \beta);\ -\frac{q}{2} < \alpha < \frac{q}{2},\ 0 \le \beta < \frac{q}{2},\ \alpha + \beta > -1\big\}.$$

Indeed, it suffices to consider pairs (α, β) with $\alpha < 0$. If moreover $\alpha + \beta < 0$, we find $\tau_0 > 1$ such that $\alpha + \beta = -\frac{1}{\tau_0}$. Then $\beta = -\frac{1}{\tau_0} - \alpha < -\frac{1}{\tau_0} + \frac{q}{2}$ and $\alpha < 0 < \frac{q}{2} - \frac{1}{\tau_0}$; consequently $(\alpha, \beta) \in \mathcal{C}$. If $\alpha + \beta \ge 0$, we may choose τ sufficiently large to show that $(\alpha, \beta) \in \mathcal{C}$.

Now consider the case $1 < q < 2$. As in the previous case we have to analyze the set $\mathcal{C}$ where now τ runs in the interval $\big(\frac{2}{q}, \frac{2}{2-q}\big]$. Since $\tau > \frac{2}{q}$, the same conditions as before are redundant; moreover, $\alpha > -\frac{q}{2}$. Then we will show show that

$$\mathcal{C} = \big\{(\alpha, \beta); -\frac{q}{2} < \alpha < q - 1,\ 0 \le \beta < q - 1,\ \alpha + \beta > -\frac{q}{2}\big\}.$$

Indeed, if e.g. $\alpha < 0$ and $\alpha + \beta \le \frac{q}{2} - 1 < 0$, then there exists $\tau_0 \in \big(\frac{2}{q}, \frac{2-q}{2}\big]$ such that $\alpha + \beta = -\frac{1}{\tau_0}$, $\beta = -\frac{1}{\tau_0} - \alpha < -\frac{1}{\tau_0} + \frac{q}{2}$ and $\alpha < 0 < \frac{q}{2} - \frac{1}{\tau_0}$; however, when $\alpha + \beta > \frac{q}{2} - 1$, we may choose $\tau = \frac{2}{2-q}$ to see that $(\alpha, \beta) \in \mathcal{C}$.

■

Acknowledgment: The research was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan no. AV0Z10190503, by the Grant Agency of the Academy of Sciences No. IAA100190505, and by the joint research project of DAAD (D/04/25763) and the Academy of Sciences of the Czech Republic (D-CZ 3/05-06).

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