Instructive examples of smooth, complex differentiable and complex analytic mappings into locally convex spaces

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Abstract

For each $k \in \mathbb{N}$, we describe a mapping $f_k \colon \mathbb{C} \to E_k$ into a suitable non-complete complex locally convex space E_k such that f_k is k times continuously complex differentiable (i.e., a $C^k_{\mathbb{C}}$-map) but not $C^{k+1}_{\mathbb{C}}$ and hence not complex analytic. As a preliminary, we prove that the functions $h_{k,z} \colon \mathbb{N} \to \mathbb{C}$, $h_{k,z}(n) := n^k z^n$ are linearly independent in $\mathbb{C}^{\mathbb{N}}$ for $k \in \mathbb{Z}$, $z \in \mathbb{C}^\times$. We also describe a complex analytic map from ℓ^1 to a suitable complete complex locally convex space E which is unbounded on each non-empty open subset of ℓ^1. Finally, we present a smooth map $\mathbb{R} \to E$ into a non-complete locally convex space which is not real analytic although it is given locally by its Taylor series around each point.

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Introduction

It can be advantageous to perform infinite-dimensional differential calculus in general locally convex spaces, without completeness conditions. First of all, the theory becomes clearer and more transparent if completeness conditions are stated explicitly as hypotheses for those results which really depend on them, but omitted otherwise. Secondly, it simplifies practical applications if completeness properties only need to be checked when they are really needed. Therefore, various authors have defined and discussed C^k-maps (and analytic maps) between locally convex spaces without completeness hypotheses (see [8], [12, Chapter 1], [13] and [18]; cf. [3]).

In this article, we compile examples which illustrate the differences between various differentiability and analyticity properties of vector-valued functions, in particular differences which depend on non-completeness of the range space. Primarily, we consider continuous mappings $f\colon U \to F$, where $U \subseteq \mathbb{C}$ is open and F a complex locally convex space. Let us call such a map $C^1_{\mathbb{C}}$ if the complex derivative $f^{(1)}(z) = f'(z) = \frac{df}{dz}(z)$ exists for each $z \in U$, and $f'\colon U \to F$ is continuous. As usual, we say that f is $C^k_{\mathbb{C}}$ if it has continuous complex derivatives $f^{(j)}\colon U \to F$ for all $j \in \mathbb{N}_0$ such that $j \leq k$ (where $f^{(j)} := (f^{(j-1)})'$). Finally, call f *complex analytic* if it is of the form $f(z) = \sum_{n=0}^{\infty}(z-z_0)^n a_n$ close to each given point $z_0 \in U$, for suitable elements $a_n \in F$. The weakest relevant completeness property of F is *Mackey completeness*, which (among many others) can be defined by each of the following equivalent conditions:[1]

M1 The Riemann integral $\int_0^1 \gamma(t)\,dt$ exists in F for each smooth curve $\gamma\colon \mathbb{R} \to F$.

M2 $\sum_{n=1}^{\infty} t_n x_n$ converges in F for each bounded sequence $(x_n)_{n\in\mathbb{N}}$ in F and each sequence $(t_n)_{n\in\mathbb{N}}$ of scalars such that $\sum_{n=1}^{\infty} |t_n| < \infty$.

If F is Mackey complete, the following properties are known to be equivalent:[2]

(a) $f\colon \mathbb{C} \supseteq U \to F$ is complex analytic;

(b) f is $C^\infty_{\mathbb{C}}$;

(c) f is $C^1_{\mathbb{C}}$;

(d) f is weakly analytic, i.e. $\lambda \circ f\colon U \to \mathbb{C}$ is complex analytic for each continuous linear functional $\lambda\colon F \to \mathbb{C}$;

(e) $\int_{\partial\Delta} f(\zeta)\,d\zeta = 0$ for each triangle $\Delta \subseteq U$;

(f) $f(z) = \frac{1}{2\pi i}\int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z}\,d\zeta$ for each $z_0 \in U$, $r > 0$ such that $z_0 + r\mathbb{D} \subseteq U$, and each z in the interior of $z_0 + r\mathbb{D}$ (where $\mathbb{D} := \{z \in \mathbb{C}\colon |z| \leq 1\}$).

Here (a) and (b) remain equivalent if F fails to be Mackey complete (see [13, Chapter II, Theorem 2.1] or [2, Propositions 7.4 and 7.7]) and also (d), (e) and (f) remain equivalent (because f may be considered as a map into

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[1] See [20, Theorem 2.14], also [19, p. 119].

[2] See Theorems 2.1, 2.2 and 5.5 in [13, Chapter II], or simply replace sequential completeness with Mackey completeness in [3, Theorem 3.1] and its proof.

the completion of F for this purpose; see also [13, Chapter II, Theorem 2.2]). However, $C^1_{\mathbb{C}}$-maps need not be $C^2_{\mathbb{C}}$ (and hence need not be complex analytic) in this case, as an example in Hervé's book [16, p. 60] shows (for which only a partial proof is provided there).

Our first goal is to give examples which distinguish between the properties (a)–(f) in a more refined way. Thus, we describe functions satisfying (d)–(f) but which are not $C^1_{\mathbb{C}}$, and also $C^k_{\mathbb{C}}$-maps which are not $C^{k+1}_{\mathbb{C}}$, for each $k \in \mathbb{N}$ (Theorem 1.1). In particular, the latter functions are $C^1_{\mathbb{C}}$ but not complex analytic, like Hervé's example.

We mention that similar functions have been recorded in unpublished parts of the thesis [13] (Chapter II, Example 2.3), but also there a crucial step of the proof is left to the reader. Furthermore, our discussion is based on a different argument: At its heart is the linear independence of the functions $\mathbb{N} \to \mathbb{C}$, $n \mapsto n^k z^n$ in the space $\mathbb{C}^{\mathbb{N}}$ of complex sequences, for $k \in \mathbb{Z}$ and $z \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ (see Proposition 1.2).

This result (which we could not locate in the literature) forms the technical backbone of the article. As a byproduct, it entails refined results concerning completeness properties of free locally convex spaces over subsets of $\mathbb{C}$ (see Proposition 6.2), which go beyond the known general facts concerning free locally convex spaces (as in [25]).

Examples concerning real analyticity are given as well, as are examples concerning maps $f\colon U \to F$, where E and F are locally convex spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $U \subseteq E$ an open set. Various differentiability and analyticity properties of such maps f are regularly used in applications of infinite-dimensional calculus, notably in infinite-dimensional Lie theory:

(i) Given $k \in \mathbb{N}_0 \cup \{\infty\}$, f is called a $C^k_{\mathbb{K}}$-*map* in the sense of Keller's C^k_c-theory if f is continuous, the iterated directional (real or complex) derivatives

$$d^j f(x, v_1, \ldots, v_j) := (D_{v_j} \cdots D_{v_1} f)(x)$$

exist in F for all $j \in \mathbb{N}$ such that $j \leq k$, $x \in U$ and $v_1, \ldots, v_j \in E$, and the maps $d^j f\colon U \times E^j \to F$ so defined are continuous (see, e.g., [18], [8] and [12]). As usual, $C^\infty_{\mathbb{R}}$-maps are also called *smooth*.

(ii) If $\mathbb{K} = \mathbb{C}$, f is called *complex analytic* if f is continuous and for each $x \in U$, there exists a 0-neighbourhood $Y \subseteq U - x$ and continuous, complex homogeneous polynomials $p_n\colon E \to F$ of degree n such that

$$f(x+y) \;=\; \sum_{n=0}^{\infty} p_n(y) \tag{1}$$

for all $y \in Y$, with pointwise convergence (see [3]).

(iii) If $\mathbb{K} = \mathbb{R}$, following [23], [8] and [12], the map $f\colon U \to F$ is called *real analytic* if it extends to a complex analytic $F_{\mathbb{C}}$-valued map on an open neighbourhood of U in the complexification $E_{\mathbb{C}}$.

It is known that complex analytic maps coincide with $C^{\infty}_{\mathbb{C}}$-maps (see [2, Propositions 7.4 and 7.7] or [12, Chapter 1]), and furthermore compositions of composable $C^k_{\mathbb{K}}$-maps (resp., $\mathbb{K}$-analytic maps) are $C^k_{\mathbb{K}}$ (resp., $\mathbb{K}$-analytic) for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ (see [2, Proposition 4.5] or [12, Chapter 1]; cf. [8]). Occasionally, authors use a different notion of real analytic maps:

(iv) In [3], f is called real analytic if it is continuous and locally of the form (1) with continuous, real homogeneous polynomials p_n.

This approach is not useful in general because one is not able to prove (without extra assumptions) that compositions of such maps are again of the same type. Finally:

(v) Bourbaki [5] defines analytic maps from open subsets of normed spaces to quasi-complete locally convex spaces in a way (not recalled here) which forces such maps to be locally bounded (cf. [5, 3.3.1 (iv)]).[3,4]

It is known that Bourbaki's notion of complex analyticity coincides with the one from (ii) for maps between Banach spaces (cf. [3, Proposition 5.1]). It is also known that real analyticity as in (iv) coincides with the one from (iii) if E is a Fréchet space and F Mackey complete (cf. [3, Theorem 7.1] for the crucial case where F is sequentially complete). Nonetheless, the concepts differ in general. We illustrate the differences with simple examples:

In Proposition 5.2 (announced in [9, §1.10]), we describe a smooth map

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[3] Local boundedness follows from the definition of $\mathcal{H}_R(E_1, \ldots, E_n; F)$ in [5, 3.1.1].

[4] The booklet [5] is still of relevance for Lie theory as it provides the framework for the influential and frequently cited volume [6]. Also, it is a rare source of information on analytic maps on open subsets of infinite-dimensional normed spaces over valued fields.

$f\colon \mathbb{R} \to F$ to some (non-Mackey complete) real locally convex space F which is locally given by its Taylor series around each point (and thus real analytic in the inadequate sense of (iv)), but not real analytic in the sense of (iii).

In Section 3, we provide an example of a map $f\colon \ell^1 \to \mathbb{C}^{\mathbb{N}}$ on the space ℓ^1 of absolutely summable complex sequences which is complex analytic in the sense of (ii) but unbounded on each non-empty open subset of ℓ^1. As a consequence, f is not complex analytic in Bourbaki's sense (as in (v)). In particular, this means that the equivalence of (i) and (ii) in [5, 3.3.1] is false (which asserts that complex analytic maps in Bourbaki's sense coincide with complex differentiable maps).

For complex analytic maps in the sense of (ii), *ample* boundedness can be used as an adequate substitute for ordinary boundedness (see [13, Chapter II, §6, notably Theorem 6.1]; cf. also [3, Theorem 6.1 (i)]).

General conventions. We write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If $(E, \|.\|)$ is a normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ (e.g., $(E, \|.\|) = (\mathbb{K}, |.|)$), we write $B^E_r(x) := \{y \in E\colon |y - x| < r\}$ and $\overline{B}^E_r(x) := \{y \in E\colon |y - x| \leq r\}$ for $x \in E$ and $r > 0$. Given a vector space E over a field $\mathbb{K}$ and a subset $M \subseteq E$, we write $\operatorname{span}_{\mathbb{K}}(M)$ for the vector subspace of E spanned by M.

1 Examples of $C^k_{\mathbb{C}}$-maps which are not $C^{k+1}_{\mathbb{C}}$

For $k \in \mathbb{Z}$ and $z \in \mathbb{C}^\times$, we define

$$h_{k,z}\colon \mathbb{N} \to \mathbb{C}\,, \quad h_{k,z}(n) := n^k z^n\,. \tag{2}$$

Let $U \subseteq \mathbb{C}$ be a non-empty open subset and $M \subseteq \mathbb{C}^\times$ be a superset of e^U (for example, $U = \mathbb{C}$, $M = \mathbb{C}^\times$). For $k \in \mathbb{N}_0$, we let $E_k \subseteq \mathbb{C}^{\mathbb{N}}$ be the vector subspace spanned by the functions $h_{j,z}$ with $z \in M$ and $j \in \mathbb{N}_0$ such that $j \leq k$. We give E_k the topology induced by the direct product $\mathbb{C}^{\mathbb{N}}$ and define

$$f_k\colon U \to E_k\,, \quad f_k(z) := (e^{nz})_{n \in \mathbb{N}}\,.$$

Theorem 1.1 *For each $k \in \mathbb{N}$, the map $f_k\colon U \to E_k$ is $C^k_{\mathbb{C}}$ but not $C^{k+1}_{\mathbb{C}}$ (and hence not complex analytic). Furthermore, $f_0\colon U \to E_0$ is weakly analytic but not $C^1_{\mathbb{C}}$ (and hence not complex analytic).*

The following fact, proved in Section 2, is crucial for our proof of Theorem 1.1:

Proposition 1.2 *The functions $h_{k,z}$ (for $k \in \mathbb{Z}$, $z \in \mathbb{C}^\times$) are linearly independent in $\mathbb{C}^\mathbb{N}$.* $\square$

Also the following simple lemma is useful for the proof of Theorem 1.1 and later arguments. It is a variant of [2, Lemma 10.1].

Lemma 1.3 *Let E be a locally convex space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $E_0 \subseteq E$ be a vector subspace equipped with a locally convex vector topology making the inclusion map $\iota\colon E_0 \to E$ continuous. Let $k \in \mathbb{N}_0$ and $f\colon U \to E$ be a map on an open set $U \subseteq \mathbb{K}$, such that $f(U) \subseteq E_0$. Then the following holds:*

(a) *If the corestriction $f|^{E_0}\colon U \to E_0$ is $C^k_\mathbb{K}$, then f is $C^k_\mathbb{K}$ and $f^{(j)}(U) \subseteq E_0$ for all $j \in \mathbb{N}$ such that $j \leq k$.*

(b) *If E_0 carries the topology induced by E, then $f|^{E_0}\colon U \to E_0$ is $C^k_\mathbb{K}$ if and only if f is $C^k_\mathbb{K}$ and $f^{(j)}(U) \subseteq E_0$ for all $j \in \mathbb{N}$ such that $j \leq k$.*

Proof. (a) By the Chain Rule, $f = \iota \circ f|^{E_0}$ is $C^k_\mathbb{K}$ with $f^{(j)}(x) = \iota \circ f^{(j)}(x)$, from which the assertion follows.

(b) In view of (a), we only need to prove sufficiency of the described condition. We proceed by induction on k. The case $k = 0$ being trivial, assume that $k \geq 1$. If f satisfies the described condition, then $\frac{f|^{E_0}(y) - f|^{E_0}(x)}{y-x} = \frac{f(y)-f(x)}{y-x} \to f'(x) \in E_0$ as $y \to x$, showing that $f|^{E_0}$ is differentiable with $(f|^{E_0})' = f'|^{E_0}$. Since $f'|^{E_0}$ is $C^{k-1}_\mathbb{K}$ by induction, we see that f is $C^k_\mathbb{K}$. $\square$

Proof of Theorem 1.1. Consider the map $f\colon U \to \mathbb{C}^\mathbb{N}$, $f(z) := (e^{zn})_{n\in\mathbb{N}}$. The n-th component $U \to \mathbb{C}$, $z \mapsto e^{zn}$ of f being $C^\infty_\mathbb{C}$ for each n, also the map f into the direct product $\mathbb{C}^\mathbb{N}$ is $C^\infty_\mathbb{C}$ (see [12, Chapter 1] or [2, Lemma 10.2]), with $f^{(j)}(z) = (n^j e^{nz})_{n\in\mathbb{N}}$ and thus $f^{(j)}(z) = h_{j,e^z}$. If $j \leq k$, then $h_{j,e^z} \in E_k$ and thus $f^{(j)}(U) \subseteq E_k$. Thus $f_k = f|^{E_k}$ is $C^k_\mathbb{C}$, by Lemma 1.3 (b). However, $f^{(k+1)}(z) = h_{k+1,e^z} \notin E_k$ by Definition of E_k and Proposition 1.2. Hence $f_k = f|^{E_k}$ is not $C^{k+1}_\mathbb{C}$, by Lemma 1.3 (b). If $k = 0$, then f_0 is not $C^1_\mathbb{C}$ by the preceding. However, $\int_{\partial\Delta} f_0(\zeta)\, d\zeta = \int_{\partial\Delta} f(\zeta)\, d\zeta = 0$ for each triangle $\Delta \subseteq U$ (by "(b)⇒(e)" in the introduction), because f is $C^\infty_\mathbb{C}$. Hence f_0 satisfies property (e) from the introduction and hence f_0 is weakly analytic. $\square$

2 Proof of Proposition 1.2

Step 1: To prove Proposition 1.2, we only need to show that $(h_{k,z})_{k\in\mathbb{N}_0, z\in\mathbb{C}^\times}$ is a linearly independent family of functions in $\mathbb{C}^\mathbb{N}$. This follows from the fact that the multiplication operator $\mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$, $f \mapsto h_{\ell,1} \cdot f$ with $(h_{\ell,1} \cdot f)(n) = n^{\ell} f(n)$ is a linear automorphism and $h_{\ell,0} \cdot h_{k,z} = h_{k+\ell,z}$, for all $\ell, k \in \mathbb{Z}$.

Step 2: We show that, for fixed $z \in \mathbb{C}^{\times}$, the family $(h_{k,z})_{k\in\mathbb{N}_0}$ is linearly independent. Since $h_{0,z}^{-1} \cdot h_{k,z} = h_{k,1}$ and the multiplication operator $\mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$, $f \mapsto h_{0,z}^{-1} \cdot f$ with $(h_{0,z}^{-1} \cdot f)(n) = \frac{f(n)}{h_{0,z}(n)}$ is an automorphism of the complex vector space $\mathbb{C}^{\mathbb{N}}$, we may assume that $z = 1$. However, it is well known that the functions $h_{k,1} \colon \mathbb{N} \to \mathbb{C}$, $n \mapsto n^k$ are linearly independent for $k \in \mathbb{N}_0$, because the Vandermonde matrix $(n^{m-1})_{n,m=1}^N$ is invertible for each $N \in \mathbb{N}$.

Step 3: For fixed $k \in \mathbb{Z}$, the family $(h_{k,z})_{z\in\mathbb{C}^{\times}}$ is linearly independent. In fact, since $h_{-k,1} h_{k,z} = h_{0,z}$ and the multiplication operator $\mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$, $f \mapsto h_{-k,1} \cdot f$ is an automorphism, it suffices to show this for $k = 0$. But the functions $h_{0,z} \colon \mathbb{N} \to \mathbb{C}$ (for $z \in \mathbb{C}^{\times}$) are linearly independent (as is well known), because the Vandermonde matrix $(z_n^{m-1})_{n,m=1}^N$ is invertible for all $N \in \mathbb{N}$ and distinct elements $z_1, \ldots, z_N \in \mathbb{C}^{\times}$ and hence also $(z_n^m)_{n,m=1}^N$ is invertible.

The following lemma will be used to complete the proof of Proposition 1.2.

Lemma 2.1 *Let $F \subseteq \mathbb{R}$ be a finite, non-empty subset such that the elements e^{it} for $t \in F$ are pairwise distinct. If*

$$\lim_{n\to\infty} \sum_{t\in F} \alpha_t \, e^{itn} \;=\; 0 \tag{3}$$

for certain $\alpha_t \in \mathbb{C}$, then $\alpha_t = 0$ for all $t \in F$.

Proof. The proof of Lemma 2.1 is by induction on the cardinality $m := |F|$. For $m = 1$, the assertion is trivial. Now let $m > 1$ and suppose that the assertion holds for all sets of cardinality strictly less than m.

After multiplication of (3) by e^{irn} with suitable $r \in \mathbb{R}$ and replacing F with $F + r$, we may assume that $F \cap \pi\mathbb{Q} = \emptyset$. Assuming this, there exists a non-empty subset $B \subseteq F$ such that $B' := B \cup \{2\pi\}$ is a $\mathbb{Q}$-basis of the rational vector subspace $\operatorname{span}_{\mathbb{Q}}(\{2\pi\} \cup F)$ of $\mathbb{R}$. Thus

$$t = \sum_{s\in B'} q_{t,s} s \qquad \text{for } t \in F, \tag{4}$$

for certain $q_{t,s} \in \mathbb{Q}$.

There exists $N \in \mathbb{N}$ such that $q_{t,s}N \in \mathbb{Z}$ for all $t \in F$ and $s \in B'$. Let $\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$ be the circle group. Then the map

$$\phi \colon \mathbb{Z} \to \mathbb{T}^B\,, \qquad n \mapsto (e^{isn})_{s\in B}$$

is a homomorphism of groups, and has dense image by Kronecker's Approximation Theorem (in the form [17, 26.19 (c) (iv)]). By Weil's Lemma, also $\phi(\mathbb{N})$ is dense in $\mathbb{T}^B$ (cf. [17, Theorem 9.1]). Hence, given elements $w_s \in \mathbb{T}$ for $s \in B$, there exists a sequence $n_1 < n_2 < \cdots$ of positive integers such that

$$\lim_{\nu\to\infty} \phi(n_\nu) \;=\; (w_s)_{s\in B} \tag{5}$$

in $\mathbb{T}^B$. Substitute (4) into (3). Using the subsequence $n = Nn_\nu$ to form the limit in (3), we deduce with (5) that

$$\sum_{s\in B} \alpha_s w_s^N + \sum_{t\in F\setminus B} \alpha_t \prod_{s\in B} w_s^{Nq_{t,s}} \;=\; 0\,, \tag{6}$$

exploiting that $e^{i2\pi q_{t,2\pi}Nn_\nu} = 1$ for each $t \in F \setminus B$ and $\nu \in \mathbb{N}$. Since (6) holds for all $(w_s)_{s\in B} \in \mathbb{T}^B$ and $\mathbb{T}^B$ is the set of zeros of a non-zero real-valued polynomial on $\mathbb{C}^B$, we deduce with the algebraic "density property" (as in [4, §6.6]) that the polynomial

$$p := \sum_{s\in B} \alpha_s X_s^N + \sum_{t\in F\setminus B} \alpha_t \prod_{s\in B} X_s^{Nq_{t,s}}$$

(in the indeterminates X_s, $s \in B$) vanishes identically on $\mathbb{C}^B$ and thus $p = 0$ (alternatively, we might use that $\mathbb{T}^B$ is a totally real submanifold of $\mathbb{C}^B$).

Case 1. If $\prod_{s\in B} X_s^{Nq_{t,s}} \neq X_s^N$ for all $t \in F \setminus B$ and $s \in B$, then $p = 0$ forces that $\alpha_s = 0$ for all $s \in B$. Now (3) turns into $\lim_{n\to\infty} \sum_{t\in F\setminus B} \alpha_t e^{itn} = 0$; using the inductive hypothesis, we deduce that also $\alpha_t = 0$ for all $t \in F \setminus B$.

Case 2. Otherwise, there exist $s_0 \in B$ and $t_0 \in F\setminus B$ such that $\prod_{r\in B} X_r^{Nq_{t_0,r}} = X_{s_0}^N$. Then $q_{t_0,r} = \delta_{r,s_0}$ for $r \in B$ (using Kronecker's delta) and thus $t_0 = s_0 + q_{t_0,2\pi}2\pi$. We now define an equivalence relation on F via $t \sim s$ if and only if $t - s \in \pi\mathbb{Q}$, and choose a system $R \subseteq F$ of representatives for the equivalence classes, such that $B \subseteq R$. Since $t_0 \sim s_0$, we have $|R| < |F| = m$. For each $t \in R$, there is a finite subset $F_t \subseteq \pi\mathbb{Q}$ such that $t + F_t$ is the equivalence class of t. Then (3) can be rewritten as

$$\lim_{n\to\infty} \sum_{t\in R} \Big(\sum_{s\in F_t} \alpha_{t+s} e^{isn}\Big) e^{itn} \;=\; 0\,. \tag{7}$$

As a consequence of (4), we have $NF_t \subseteq 2\pi\mathbb{Z}$ with N as above. For each $k \in \mathbb{Z}$, choosing $n = k + N\nu$ in (7), we find that

$$\lim_{\nu\to\infty} \sum_{t\in R} \Big(\sum_{s\in F_t} \alpha_{t+s} e^{i(t+s)k} \Big) e^{itN\nu} \;=\; 0\,. \tag{8}$$

Note that if $t_1, t_2 \in R$ with $t_1 \neq t_2$, then $t_1 - t_2 \not\in \pi\mathbb{Q}$ and hence also $Nt_1 - Nt_2 \not\in \pi\mathbb{Q}$. Therefore the numbers e^{itN} for $t \in R$ are pairwise distinct. Applying the inductive hypothesis to (8), we deduce that

$$\sum_{s\in F_t} \alpha_{t+s} e^{i(t+s)k} \;=\; 0\,. \tag{9}$$

The functions $\mathbb{N} \to \mathbb{C}$, $k \mapsto e^{-i(t+s)k}$ for $s \in F_t$ being linearly independent (see, e.g., Step 3 in the proof of Proposition 1.2), we deduce from (9) that $\alpha_{t+s} = 0$ for all $s \in F_t$. Since $t \in R$ was arbitrary and $\bigcup_{t\in R}(t + F_t) = F$, we see that $\alpha_t = 0$ for all $t \in F$. This completes the proof of Lemma 2.1. $\square$

Proof of Proposition 1.2, completed. If Proposition 1.2 was wrong, we could find a minimal number $L \in \mathbb{N}$ such that $\sum_{j=1}^{L} \sum_{k=0}^{m} \alpha_{k,j} h_{k,z_j} = 0$ for some $m \in \mathbb{N}$, distinct elements $z_1, \ldots, z_L \in \mathbb{C}^\times$ and certain $\alpha_{k,j} \in \mathbb{C}$, not all of which are zero. Thus

$$(\forall n \in \mathbb{N}) \qquad \sum_{j=1}^{L} (\alpha_{0,j} n^0 + \cdots + \alpha_{m,j} n^m)(z_j)^n = 0\,. \tag{10}$$

Then $L \geq 2$, by Step 2. We write $z_j = r_j e^{iy_j}$ for $j \in \{1, \ldots, L\}$, with $r_j := |z_j| > 0$ and $y_j \in \mathbb{R}$. Reordering if necessary, we may assume that $R := r_1 = r_2 = \cdots = r_\ell$ and $r_{\ell+1}, \ldots, r_L < R$, for some $\ell \in \{1, \ldots, L\}$. Then $e^{iy_1}, \ldots, e^{iy_\ell}$ are pairwise distinct. Rewriting (10), we see that

$$\sum_{j=1}^{\ell} (\alpha_{0,j} n^0 + \cdots + \alpha_{m,j} n^m) e^{iy_j n} = -\sum_{j=\ell+1}^{L} (\alpha_{0,j} n^0 + \cdots + \alpha_{m,j} n^m) e^{iy_j n} \big(\tfrac{r_j}{R}\big)^n \tag{11}$$

and hence

$$\begin{aligned} & n^{-m} \sum_{j=1}^{\ell} (\alpha_{0,j} n^0 + \cdots + \alpha_{m,j} n^m) e^{iy_j n} \\ & \quad = \; -n^{-m} \sum_{j=\ell+1}^{L} (\alpha_{0,j} n^0 + \cdots + \alpha_{m,j} n^m) e^{iy_j n} \big(\tfrac{r_j}{R}\big)^n \end{aligned} \tag{12}$$

for each $n \in \mathbb{N}$. Since the right hand side of (12) tends to 0 as $n \to \infty$, so does its left hand side:

$$\sum_{j=1}^{\ell} \alpha_{m,j} e^{iy_j n} + \sum_{j=1}^{\ell} \sum_{k=0}^{m-1} \alpha_{k,j} n^{k-m} e^{iy_j n} \;\to\; 0 \quad \text{as } n \to \infty. \tag{13}$$

Because the double sum in (13) tends to 0 as $n \to \infty$, we infer that

$$\lim_{n\to\infty} \sum_{j=1}^{\ell} \alpha_{m,j} e^{iy_j n} \;=\; 0\,.$$

Hence $\alpha_{m,j} = 0$ for all $j \in \{1, \ldots, \ell\}$, by Lemma 2.1. Therefore (11) takes the form

$$\sum_{j=1}^{\ell} (\alpha_{0,j} n^0 + \cdots + \alpha_{m-1,j} n^{m-1}) e^{iy_j n} = -\sum_{j=\ell+1}^{L} (\alpha_{0,j} n^0 + \cdots + \alpha_{m,j} n^m) e^{iy_j n} \big(\tfrac{r_j}{R}\big)^n$$

for each $n \in \mathbb{N}$. After multiplication with $n^{-(m-1)}$, we get a formula analogous to (12) and can repeat the preceding reasoning to deduce that $\alpha_{m-1,j} = 0$ for all $j \in \{1, \ldots, \ell\}$. Proceeding in this way, we see that also for $k \in \{m-2, m-3, \ldots, 0\}$, we have $\alpha_{k,j} = 0$ for all $j \in \{1, \ldots, \ell\}$. In particular, $\alpha_{k,1} = 0$ for all $k \in \{0, 1, \ldots, m\}$. This contradicts the minimality of L. $\square$

3 Example of a complex analytic map which is not locally bounded

Let $E := \ell^1(\mathbb{N}, \mathbb{C})$ be the space of absolutely summable complex sequences with its usual norm $\|.\|_1$. We define

$$g\colon E \to \mathbb{C}\,, \quad g(x) := \sum_{k=1}^{\infty} 2^k\, (x_k)^{2k}$$

for $x = (x_k)_{k\in\mathbb{N}} \in E$ and $f\colon E \to \mathbb{C}^{\mathbb{N}}$, $f(x) := (g(nx))_{n\in\mathbb{N}}$. We now show, using the notion of complex analyticity described in (ii) in the introduction:

Proposition 3.1 *The map $f\colon E \to \mathbb{C}^{\mathbb{N}}$ is complex analytic. It is unbounded on each non-empty open subset of E.*

As a preliminary, we discuss g.

Lemma 3.2 *g is complex analytic. It is unbounded on $\overline{B}^E_2(x)$ for all $x \in E$.*

Proof. Since the partial sums $g_n \colon E \to \mathbb{C}$, $x \mapsto \sum_{k=1}^n 2^k\,(x_k)^{2k}$ are polynomials in the point evaluations $x \mapsto x_k$ (which are continuous linear functionals) and hence complex analytic, g will be complex analytic if we can show that each $x \in E$ has an open neighbourhood U such that $(g_n|_U)_{n\in\mathbb{N}}$ converges uniformly (see [3, Proposition 6.5]). There is $m \in \mathbb{N}$ such that $|x_k| < \frac{1}{4}$ for all $k \geq m$. Set $U := B^E_{1/4}(x)$. Then $|y_k| < \frac{1}{2}$ for all $y \in U$ and thus $\sum_{k=m}^\infty \sup_{y\in U} |2^k\,(y_k)^{2k}| \leq \sum_{k=m}^\infty 2^{-k} < \infty$, which entails uniform convergence on U.

Given $x \in E$ and $N \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $|2x_m| < 1$ and $2^m \geq N + |g(x)| + 1$. Set $y := (y_k)_{k\in\mathbb{N}}$, where $y_k := x_k$ if $k \neq m$, and $y_m := 1$. Then $\|y - x\|_1 \leq 2$ and $|g(y)| = |g(x) + 2^m - 2^m x_m^{2m}| \geq 2^m - |g(x)| - 2^m |x_m|^{2m} \geq N$. Thus g is unbounded on $\overline{B}^E_2(x)$. □

Proof of Proposition 3.1. Given $x \in E$ and a neighbourhood U of x in E, there exists $n \in \mathbb{N}$ such that $\overline{B}^E_{2/n}(x) \subseteq U$. Then $\overline{B}^E_2(nx) = n\overline{B}^E_{2/n}(x) \subseteq nU$, whence $g(nU) \subseteq \mathbb{C}$ is unbounded, by Lemma 3.2. Therefore the projection of $f(U)$ on the n-th component is unbounded and thus $f(U)$ is unbounded. □

4 Mappings to products and real analyticity

We describe a map $f = (f_n)_{n\in\mathbb{N}} \colon \mathbb{R} \to \mathbb{R}^\mathbb{N}$ which is not real analytic although each of its components $f_n \colon \mathbb{R} \to \mathbb{R}$ is real analytic (see also [13, Chapter II, Example 6.8] for a very similar example).

Example 4.1 For each $n \in \mathbb{N}$, the map $f_n \colon \mathbb{R} \to \mathbb{R}$, $f_n(t) := \frac{1}{1+(nt)^2}$ is real analytic and its Taylor series around 0 has radius of convergence $\frac{1}{n}$. It follows that the Taylor series around 0 of the smooth map

$$f \colon \mathbb{R} \to \mathbb{R}^\mathbb{N}, \quad f(t) := (f_n(t))_{n\in\mathbb{N}}$$

has radius of convergence 0, and thus f is not real analytic.

5 Example of a map which is not real analytic although it admits Taylor expansions

We describe a smooth map $f\colon \mathbb{R} \to E$ to a suitable real locally convex space E which is given locally by its Taylor series around each point, but which does not admit a complex analytic extension and hence fails to be real analytic. We observe first that such a pathology cannot occur if E is Mackey complete. Real analyticity is understood as in (iii) in the introduction.

Lemma 5.1 *Let E be a Fréchet space, F be a Mackey complete locally convex space and $f\colon U \to F$ be a smooth map on an open subset $U \subseteq E$ which is locally given by its Taylor series around each point* (*with pointwise convergence*). *Then f is real analytic.*

Proof. The hypothesis means that f is a real analytic map in the sense of [3]. If F is sequentially complete, [3, Theorem 7.1] provides a complex analytic extension for f into $F_{\mathbb{C}}$ (because E is a Fréchet space), and thus f is real analytic in the desired sense.

If F is merely Mackey complete, we know from the preceding that f is real analytic as a map into the completion $\widetilde{F}$ of F. Let $g\colon V \to \widetilde{F}_{\mathbb{C}}$ be a complex analytic extension of f to an open neighbourhood $V \subseteq E_{\mathbb{C}}$ of U. Given $x \in U$, let $W_x \subseteq E_{\mathbb{C}}$ be a balanced, open 0-neighbourhood with $x + W \subseteq V$. Then $g(x+w) = \sum_{n=0}^{\infty} \frac{\delta_x^n f(w)}{n!}$ for each $w \in W_x$, where $\delta_x^n f(w) := d^n f(x, w, \ldots, w)$. Given $w \in W_x$, there exists $t > 1$ such that $tx \in W_x$. Then $\sum_{n=0}^{\infty} t^n \frac{\delta_x^n f(w)}{n!} = \sum_{n=0}^{\infty} \frac{\delta_x^n f(tw)}{n!}$ converges in $\widetilde{F}_{\mathbb{C}}$, whence $(\frac{\delta_x^n f(tw)}{n!})_{n\in\mathbb{N}_0}$ is a bounded sequence in $F_{\mathbb{C}}$. Since $\sum_{n=0}^{\infty} t^{-n} < \infty$, the second characterization (M2) of Mackey completeness in the introduction shows that $\sum_{n=0}^{\infty} \frac{\delta_x^n f(w)}{n!} = \sum_{n=0}^{\infty} t^{-n} \frac{\delta_x^n f(tw)}{n!}$ converges in $F_{\mathbb{C}}$. Thus $g(x+w) \in F_{\mathbb{C}}$. So, after replacing V by $\bigcup_{x\in U}(x+W_x)$, we may assume that $g(V) \subseteq F_{\mathbb{C}}$. Then $g\colon V \to F_{\mathbb{C}}$ is complex analytic (see [12, Proposition 1.5.18]) and thus $f\colon U \to F$ is real analytic. $\square$

Let E be the space of all sequences $x = (x_n)_{n\in\mathbb{N}}$ of real numbers which have polynomial growth, i.e., there exists $m \in \mathbb{N}$ such that the sequence $(|x_n| n^{-m})_{n\in\mathbb{N}}$ is bounded. We equip E with the topology induced by $\mathbb{R}^{\mathbb{N}}$.

Proposition 5.2 *For E as before, the map*

$$f\colon \mathbb{R} \to E\,, \quad f(t) := (\sin(nt))_{n\in\mathbb{N}}$$

is smooth and $f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!}(t-t_0)^k$ *in* E*, for all* $t, t_0 \in \mathbb{R}$*. However,* f *is not real analytic.*

Proof. The map $g = (g_n)_{n\in\mathbb{N}}\colon \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$, $g(t) := f(t)$ is smooth, since all components $g_n\colon \mathbb{R} \to \mathbb{R}$, $g_n(t) = \sin(nt)$ are smooth. We have $g^{(2k)}(t) = (g_n^{(2k)}(t))_{n\in\mathbb{N}} = (n^{2k}(-1)^k \sin(nt))_{n\in\mathbb{N}} \in E$ and $g^{(2k+1)}(t) = (g_n^{(2k+1)}(t))_{n\in\mathbb{N}} = (n^{2k+1}(-1)^k \cos(nt))_{n\in\mathbb{N}} \in E$ for each $k \in \mathbb{N}_0$ and $t \in \mathbb{R}$, whence $f = g|^E$ is smooth (by Lemma 1.3 (b)).

Given $t_0 \in \mathbb{R}$, we have $g_n(t) = \sum_{k=0}^{\infty} \frac{g_n^{(k)}(t_0)}{k!}(t-t_0)^k$ for each $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Since E is equipped with the topology induced by $\mathbb{R}^{\mathbb{N}}$, it follows that $f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!}(t-t_0)^k$ for all $t, t_0 \in \mathbb{R}$ and thus f is given globally by its Taylor series around each point. The map g is real analytic, because

$$h\colon \mathbb{C} \to \mathbb{C}^{\mathbb{N}}, \quad h(z) = (\sin(nz))_{n\in\mathbb{N}}$$

is a complex analytic extension of g. If f was complex analytic, then we would have $h(V) \subseteq E_{\mathbb{C}}$ for some open neighbourhood V of $\mathbb{R}$ in $\mathbb{C}$. Then $it \in V$ for $t > 0$ sufficiently small. However

$$|\sin(int)| \;=\; |\sinh(nt)| \;\geq\; \frac{1}{4}e^{nt}$$

for large n and hence the sequence $h(it) = (\sin(int))_{n\in\mathbb{N}}$ does not have polynomial growth. Thus $h(it) \notin E_{\mathbb{C}}$, a contradiction. □

6 Consequences for free locally convex spaces

Given $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and a completely regular topological space M, there exists a Hausdorff locally convex topological $\mathbb{K}$-vector space $L(M, \mathbb{K})$ and a continuous map $\eta\colon M \to L(M, \mathbb{K})$ with the following properties (see [22]):

(a) Algebraically, $(L(M, \mathbb{K}), \eta)$ is the free $\mathbb{K}$-vector space ($\cong \mathbb{K}^{(M)}$) over the set M;

(b) For each continuous map $\alpha\colon M \to E$ to a locally convex topological $\mathbb{K}$-vector space E, there exists a unique continuous $\mathbb{K}$-linear map $\overline{\alpha}\colon L(M, \mathbb{K}) \to E$ such that $\overline{\alpha} \circ \eta = \alpha$.

$(L(M,\mathbb{K}),\eta)$ is determined by these properties up to canonical isomorphism; it is called the *free locally convex topological* $\mathbb{K}$*-vector space over* M.

6.1 It is easy to see that $L(M,\mathbb{C})$ is the complexification of $L(M,\mathbb{R})$ (by checking the universal property for $L(M,\mathbb{R})_{\mathbb{C}}$). Hence $L(M,\mathbb{R})$ is complete (resp., sequentially complete, resp., Mackey complete) if and only if so is $L(M,\mathbb{C}) = L(M,\mathbb{R})_{\mathbb{C}}$.

Proposition 6.2 *If* $M \subseteq \mathbb{C}$ *is a subset with non-empty interior* M^0*, then neither* $L(M,\mathbb{R})$ *nor* $L(M,\mathbb{C})$ *is Mackey complete.*

Proof. Since $\mathbb{C}$ is homeomorphic to the disk $B_1^{\mathbb{C}}(1)$, after replacing M with a homeomorphic copy we may assume that $0 \notin M$. By §6.1, we only need to show that $L(M,\mathbb{C})$ is not Mackey complete. Let $U \subseteq \mathbb{C}$ be a non-empty open set with compact closure $\overline{U}$, such that $e^{\overline{U}} \subseteq M$. Set $E := \operatorname{span}_{\mathbb{C}}\{h_{0,z} \colon z \in M\}$, with $h_{0,z}$ as in (2). Write E_0 for E, equipped with the topology induced by the direct product $\mathbb{C}^{\mathbb{N}}$. Let $\mathcal{O}$ be the finest locally convex topology on E such that $\eta \colon M \to E$, $z \mapsto h_{0,z}$ is continuous. Since the topology on E_0 is Hausdorff and makes η continuous, it follows that $\iota \colon (E,\mathcal{O}) \to E_0$, $x \mapsto x$ is continuous and $\mathcal{O}$ is Hausdorff. Since $(h_{0,z})_{z\in M}$ is a basis for E by Proposition 1.2, it follows that $(E,\mathcal{O})$ is the free complex locally convex space $L(M,\mathbb{C})$ over M (together with η). Now consider

$$g \colon \overline{U} \to E\,, \quad g(z) \;:=\; h_{0,e^z} \;=\; (e^{nz})_{n\in\mathbb{N}}\,.$$

Then $g = \eta \circ \exp|_{\overline{U}}$ is continuous, entailing that $g(\overline{U})$ is compact and hence bounded in $(E,\mathcal{O})$. The restrictions $\lambda_n := \pi_n|_E \to \mathbb{C}$ of the projections $\pi_n \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}$, $(x_k)_{k\in\mathbb{N}} \mapsto x_n$ are continuous linear on $(E,\mathcal{O})$ and separate points. Furthermore, $\lambda_n \circ g|_U \colon U \to \mathbb{C}$, $z \mapsto e^{nz}$ is complex analytic for each $n \in \mathbb{N}$. Hence, if $(E,\mathcal{O})$ was Mackey complete, then $g|_U \colon U \to E$ would be complex analytic (by [14, Theorem 1]). But then also the map $f_0 = \iota \circ g|_U \colon U \to E_0$ considered in Theorem 1.1 would be complex analytic, which it is not: contradiction. Hence $L(M,\mathbb{C})$ is not Mackey complete. $\square$

Remark 6.3 In the literature, one finds various results concerning $L(M,\mathbb{R})$ and its completion, which can be realized as a certain space of measures (see [25], also [7]). A result from [25] is of particular relevance:

$L(M,\mathbb{R})$ *is complete if and only if* M *is Dieudonné complete*[5] *and its compact subsets are finite.*

Hence $L(M,\mathbb{R})$ and $L(M,\mathbb{C})$ are non-complete in the situation of Proposition 6.2. Our proposition provides the additional information that $L(M,\mathbb{R})$ and $L(M,\mathbb{C})$ are not sequentially complete either, nor Mackey complete.

Let us close with some observations concerning the free (not necessarily locally convex !) topological $\mathbb{K}$-vector space $V(M,\mathbb{K})$ over a completely regular topological space M (obtained by replacing the topology on $L(M,\mathbb{K})$ with the finest vector topology making η continuous).

To start with, let $M \subseteq \mathbb{C}$ be a compact set with non-empty interior. Let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. Then $V(M,\mathbb{K})$ is complete, by [1]. If $V(M,\mathbb{K})$ was locally convex, we would have $L(M,\mathbb{K}) = V(M,\mathbb{K})$ and so $L(M,\mathbb{K})$ would be complete, contrary to Proposition 6.2. We conclude: $V(M,\mathbb{K})$ *is not locally convex.*

This argument can be generalized further. To this end, recall that a Hausdorff topological space M is said to be a k_ω*-space* if there exists a sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets of M with union M (a "k_ω-sequence") such that $M = \varinjlim K_n$ as a topological space[6] (see, e.g., [11] and the references therein for further information). For instance, every σ-compact locally compact topological space is a k_ω-space. Each k_ω-space is normal (by [15, Proposition 4.3 (i)]) and hence completely regular, ensuring that both $L(M,\mathbb{K})$ and $V(M,\mathbb{K})$ are defined. We show:

Proposition 6.4 *If* M *is a non-discrete* k_ω*-space, then neither* $V(M,\mathbb{R})$ *nor* $V(M,\mathbb{C})$ *is locally convex.*

Proof. Let $K_1 \subseteq K_2 \subseteq \cdots$ be a k_ω-sequence for M. If each K_n was finite, then K_n would be discrete and hence also $M = \varinjlim K_n$ would be discrete, contradicting the hypothesis. Therefore some K_n is infinite, whence $L(M,\mathbb{R})$ (and hence also $L(M,\mathbb{C})$) is not complete, by Uspenskiĭ's result recalled in Remark 6.3. Since $V(M,\mathbb{K})$ is complete by the next lemma, we see that it cannot coincide with $L(M,\mathbb{R})$ and hence cannot be locally convex. □

Lemma 6.5 *Let* M *be a* k_ω*-space and* $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$*. Then* $V(M,\mathbb{K})$ *is a* k_ω*-space and hence complete.*

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[5] That is, the largest uniformity on M compatible with its topology is complete.

[6] Thus, a set $A \subseteq M$ is closed if and only if $A \cap K_n$ is closed in K_n for each $n \in \mathbb{N}$.

Proof. Since each abelian topological group which is a k_ω-space is complete [24], we only need to show that $V(M,\mathbb{K})$ is a k_ω-space. To this end, it is convenient to identify M via η with a subset of $V(M,\mathbb{K})$. We pick a k_ω-sequence $(K_n)_{n\in\mathbb{N}}$ for M. Define $L_n := \overline{B}^{\mathbb{K}}_n(0)\cdot(K_n+\cdots+K_n)$ (with 2^n summands) for $n\in\mathbb{N}$. Then $L_1\subseteq L_2\subseteq\cdots$ is a sequence of compact subsets of $V(M,\mathbb{K})$, with union $V(M,\mathbb{K})$. The topology $\mathcal{O}$ making $V(M,\mathbb{K})$ the direct limit topological space $\varinjlim L_n$ is finer than the original topology and makes $V(M,\mathbb{K})$ a k_ω-space. We now write W for $V(M,\mathbb{K})$, equipped with the topology $\mathcal{O}$. Because the inclusion map $\iota\colon M\to W$ restricts to a continuous map on K_n for each n (since we can pass over L_n), we see that ι is continuous (as $M=\varinjlim K_n$). To complete the proof, it only remains to show that $\mathcal{O}$ is a vector topology. Since $W\times W=\varinjlim\,(K_n\times K_n)$ (see [10, Proposition 3.3]), the addition map $\alpha\colon W\times W\to W$ will be continuous if $\alpha|_{K_n\times K_n}$ is continuous for each n. Since $K_n+K_n\subseteq K_{n+1}$ and W induces the same topology on K_n and on K_{n+1} as $V(M,\mathbb{K})$, continuity of $\alpha|_{K_n\times K_n}$ follows from the continuity of the addition map $V(M,\mathbb{K})\times V(M,\mathbb{K})\to V(M,\mathbb{K})$. Likewise, since $\mathbb{K}\times W=\varinjlim\overline{B}^{\mathbb{K}}_n(0)\times K_n$ and $\overline{B}^{\mathbb{K}}_n(0)K_n\subseteq K_{n^2}$, we deduce from the continuity of the scalar multiplication map $\mathbb{K}\times V(M,\mathbb{K})\to V(M,\mathbb{K})$ that also the scalar multiplication $\mathbb{K}\times W\to W$ is continuous. □

Similar arguments show that the free topological group and the free abelian topological group over a k_ω-space are k_ω-spaces (see [21]).

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