The finite sections approach to the index formula for band-dominated operators

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Abstract

In [2] there is derived a formula which expresses the Fredholm index of a band-dominated operator on $l^p(\mathbb{Z})$ in terms of local indices of its limit operators. The proof makes thoroughly use of K-theory for C^*-algebras (which, of course, appears as a natural approach to index problems). The purpose of this short note is to develop a completely different approach to the index formula for band-dominated operators which is exclusively based on ideas and results from asymptotic numerical analysis.

1 Introduction

Let $1 < p < \infty$. We work on the Banach space $l^p(\mathbb{Z})$ of all two-sided infinite sequences $x : \mathbb{Z} \to \mathbb{C}$ for which

$$\|x\|_p = \left(\sum_{n \in \mathbb{Z}} |x(n)|^p \right)^{1/p} < \infty.$$

A bounded linear operator A on $l^p(\mathbb{Z})$ is a *band operator* if its matrix representation $(a_{ij})_{i,\, j \in \mathbb{Z}}$ with respect to the standard basis of $l^p(\mathbb{Z})$ is a band matrix, i.e., if there is a $k \in \mathbb{N}$ such that $a_{ij} = 0$ whenever $|i - j| > k$. By a *band-dominated operator* we mean the norm limit of a sequence of band operators.

We are interested in Fredholm properties of band-dominated operators. Recall that a bounded linear operator A on a Banach space X is called a Fredholm operator if its kernel $\{x \in X : Ax = 0\}$ and its cokernel $X/\{Ax : x \in X\}$ are finite-dimensional linear spaces, and that the difference

$$\operatorname{ind} A := \dim \ker A - \dim \operatorname{coker} A$$

is called the Fredholm index of A.

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The Fredholm properties of a band-dominated operator can be expressed in terms of its limit operators. To state these results we have to recall some more notations. For every $k \in \mathbb{Z}$, we consider the operator of shift by k,

$$U_k : l^p(\mathbb{Z}) \to l^p(\mathbb{Z}), \quad (U_k x)(m) := x(m-k).$$

Further, we let $\mathcal{H}$ stand for the set of all sequences $h : \mathbb{N} \to \mathbb{Z}$ which tend to infinity in the sense that given $C > 0$, there is an n_0 such that $|h(n)| > C$ for all $n \geq n_0$. An operator $A_h \in L(l^p(\mathbb{Z}))$ is called a *limit operator* of $A \in L(l^p(\mathbb{Z}))$ with respect to the sequence $h \in \mathcal{H}$ if the sequence $(U_{-h(n)} A U_{h(n)})_{n \in \mathbb{N}}$ tends strongly to A_h as $n \to \infty$ and if the adjoint sequence $(U_{-h(n)} A U_{h(n)})^* = (U_{-h(n)} A^* U_{h(n)})$ converges strongly as well. Notice that every operator can possess at most one limit operator with respect to a given sequence $h \in \mathcal{H}$. The set $\sigma_{op}(A)$ of all limit operators of a given operator A is called the *operator spectrum* of A.

The main result of [3] (see also [4] for a comprehensive account) reads as follows.

Theorem 1 *Let $A \in L(l^p(\mathbb{Z}))$ be a band-dominated operator. Then*
(a) *every sequence $h \in \mathcal{H}$ possesses a subsequence g such that the limit operator A_g of A with respect to g exists.*
(b) *the operator A is compact if and only if $\sigma_{op}(A) = \{0\}$.*
(c) *the operator A is Fredholm if and only if each of its limit operators is invertible and if the norms of their inverses are uniformly bounded.*
(d) *if A is a band operator, then A is Fredholm if and only if each of its limit operators is invertible.*

Next we recall the index formula for Fredholm band-dominated operators. Let P denote the projection operator

$$P : l^p(\mathbb{Z}) \to l^p(\mathbb{Z}), \quad (Px)(n) = \begin{cases} x(n) & \text{if} \quad n \geq 0 \\ 0 & \text{if} \quad n < 0, \end{cases}$$

the range of which we identify with $l^p(\mathbb{Z}^+)$ in the natural way. The complementary projection $I - P$ is denoted by Q. Let A be a band-dominated operator on $l^p(\mathbb{Z})$. Then the operators PAQ and QAP are compact (they are of finite rank if A is a band operator), and so are the operators $A - (PAP + Q)(P + QAQ)$ and $A - (P + QAQ)(PAP + Q)$. Hence, a band-dominated operator A is Fredholm if and only if both $PAP + Q$ and $P + QAQ$ are Fredholm operators, in which case the Fredholm index of A is equal to the sum of the Fredholm indices of $PAP + Q$ and $P + QAQ$. We call $\operatorname{ind}_+(A) := \operatorname{ind}(PAP + Q)$ and $\operatorname{ind}_-(A) := \operatorname{ind}(P + QAQ)$ the *plus-index* and the *minus-index* of A. Evidently,

$$\operatorname{ind} A = \operatorname{ind}_+(A) + \operatorname{ind}_-(A) \tag{1}$$

for every Fredholm band-dominated operator A. Notice also that the operator spectrum splits into

$$\sigma_{op}(A) = \sigma_+(A) \cup \sigma_-(A)$$

where $\sigma_+(A)$ and $\sigma_-(A)$ collect the limit operators of A which correspond to sequences tending to $+\infty$ and to $-\infty$, respectively.

Theorem 2 *Let $A \in L(l^p(\mathbb{Z}))$ be a Fredholm band-dominated operator. Then*
(a) for all $B_+ \in \sigma_+(A)$ and $B_- \in \sigma_-(A)$,

$$\operatorname{ind}_+(B_+) = \operatorname{ind}_+(A) \quad \text{and} \quad \operatorname{ind}_-(B_-) = \operatorname{ind}_-(A).$$

(b) all operators in $\sigma_+(A)$ have the same plus-index, and all operators in $\sigma_-(A)$ have the same minus-index.
(c) for arbitrarily chosen operators $B_+ \in \sigma_+(A)$ and $B_- \in \sigma_-(A)$,

$$\operatorname{ind} A = \operatorname{ind}_+(B_+) + \operatorname{ind}_-(B_-). \tag{2}$$

Of course, if A is Fredholm, then all operators in $\sigma_+(A)$ also have the same minus-index, which follows from (b), from the invertibility of all limit operators of A, and from (1). It is also evident that assertions (b) and (c) are immediate consequences of assertion (a).

Theorem 2 was proved in [2] in case $p = 2$. The (more or less straightforward) generalization to $p \in (1, \infty)$ was done in [6]. A natural approach to the computation of the Fredholm index is via K-theory for C^*-algebras. Indeed, the proof of Theorem 2 given in [2] is heavily based upon calculations of K-groups of C^*-algebras of band-dominated operators and of their related ideals.

The purpose of this note is to present a completely different proof of the index result in Theorem 2 which is exclusively based on ideas and results from asymptotic numerical analysis, especially from the theory of the finite sections method.

2 The finite sections method

For $n \in \mathbb{N}$, consider the projection operators

$$P_n : l^p(\mathbb{Z}^+) \to l^p(\mathbb{Z}^+), \quad (P_n x)(m) := \begin{cases} x(m) & \text{if } 0 \le m < n \\ 0 & \text{otherwise.} \end{cases}$$

Let $A \in L(l^p(\mathbb{Z}^+))$. The operators $P_n A P_n$ considered as acting on $\operatorname{im} P_n$ are called the *finite sections* of A. The sequence $(P_n A P_n)_{n \in \mathbb{N}}$ of the finite sections of A is called *stable* if the operators $P_n A P_n : \operatorname{im} P_n \to \operatorname{im} P_n$ are invertible for all sufficiently large n and if the norms of their inverses are uniformly bounded. If the sequence $(P_n A P_n)$ is stable, then the operator A is invertible, and the finite sections method applies to the solution of the equation $Au = f$ in the sense that there is an n_0 such that the equations $P_nAP_nu_n = P_nf$ possess unique solutions $u_n \in \operatorname{im} P_n$ for each $n \geq n_0$ and for each right hand side $f \in l^p(\mathbb{Z}^+)$ and that these solutions converge in the norm of $l^p(\mathbb{Z}^+)$ to the solution of the equation $Au = f$.

In case A is a band-dominated operator, the stability of the finite sections sequence (P_nAP_n) can be characterized in terms of limit operators of A. The following is the main result of [5]. It states not only a stability criterion for the full sequence (P_nAP_n) of the finite sections, but also for each of its subsequences. The notation $\sigma_{op,\eta}(A)$ refers to the set of all limit operators of A with respect to subsequences of a given strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$. Further we introduce the flip operator

$$J : l^p(\mathbb{Z}) \to l^p(\mathbb{Z}), \quad (Jx)(n) := x(-n-1).$$

Evidently, $JPJ = Q$.

Theorem 3 *Let $A \in L(l^p(\mathbb{Z}^+))$ be a band-dominated operator and $\eta : \mathbb{N} \to \mathbb{N}$ a strongly monotonically increasing sequence. Then the sequence $(P_{\eta(n)}AP_{\eta(n)})$ is stable if and only if the operator A and all operators*

$$JQA_hQJ \quad \text{with} \quad A_h \in \sigma_{op,\eta}(A)$$

are invertible on $l^p(\mathbb{Z}^+)$.

3 Compact and Fredholm sequences

In the following two sections, we let $p = 2$. This restriction is not really necessary; but we would like to refer to the results of [7] which were derived for $p = 2$. The case of general $p \in (1, \infty)$ will be discussed in Section 5. We will see there that the general result is indeed a simple consequence of the index formula for $p = 2$.

It is convenient to consider the sequence (P_nAP_n) of the finite sections of an operator $A \in l^2(\mathbb{Z}^+)$ as an element of a C^*-algebra, $\mathcal{F}$, which is defined as follows. The algebra $\mathcal{F}$ is the set of all bounded sequences (A_n) of operators $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$ provided with element-wise defined operations, an element-wise defined involution, and the norm $\|(A_n)\| := \sup \|A_n\|$. One easily checks that $\mathcal{F}$ is indeed a C^*-algebra and that the set $\mathcal{G}$ of all sequences in $\mathcal{F}$ which tend to zero in the operator norm forms a closed ideal of $\mathcal{F}$.

The relevance of the algebra $\mathcal{F}$ in our context becomes evident from the fact that a sequence $(A_n) \in \mathcal{F}$ is stable if and only if the coset $(A_n) + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{G}$. Moreover, the algebra $\mathcal{F}$ and its quotient algebra $\mathcal{F}/\mathcal{G}$ also provide a natural frame to formulate a Fredholm theory for sequences (A_n) of matrices. We are going to recall the basic notations from [1, 7].

A sequence (K_n) in $\mathcal{F}$ is a *sequence of rank one matrices* if every matrix K_n has range dimension less than or equal to one. The smallest closed ideal of $\mathcal{F}$ which contains all sequences of rank one matrices will be denoted by $\mathcal{K}$. The product of a sequence of rank one matrices with another sequence in $\mathcal{F}$ is again a sequence of rank one matrices. Hence, the set $\mathcal{K}_0$ of all finite sums of sequences of rank one matrices forms an (in general, non-closed) ideal of $\mathcal{F}$ the closure of which is just the ideal $\mathcal{K}$. Consequently, a sequence $(A_n) \in \mathcal{F}$ belongs to $\mathcal{K}$ if and only if, for every $\varepsilon > 0$, there is a sequence $(K_n) \in \mathcal{F}$ such that

$$\sup_n \|A_n - K_n\| < \varepsilon \quad \text{and} \quad \sup_n \dim \operatorname{im} K_n < \infty. \tag{3}$$

In particular, the ideal $\mathcal{K}$ encloses the ideal $\mathcal{G}$. We refer to the elements of $\mathcal{K}$ as *compact sequences*. The role of the ideal $\mathcal{K}$ in numerical analysis can be compared with the role of the ideal of the compact operators in operator theory.

An appropriate notion of the rank of a sequence in $\mathcal{F}$ can be introduced as follows. A sequence $\mathbf{A} \in \mathcal{F}$ is said to have *finite essential rank* if it is the sum of a sequence (G_n) in $\mathcal{G}$ and of a sequence (K_n) with $\sup_n \operatorname{rank} K_n < \infty$. If $\mathbf{A}$ is of finite essential rank, then there is a smallest integer $r \geq 0$ such that $\mathbf{A}$ can be written as $(G_n)+(K_n)$ with $(G_n) \in \mathcal{G}$ and $\sup_n \operatorname{rank} K_n \leq r$. We call this integer the *essential rank* of $\mathbf{A}$ and write $\operatorname{ess\,rank} \mathbf{A} = r$. If $\mathbf{A}$ is not of finite essential rank, then we set $\operatorname{ess\,rank} \mathbf{A} = \infty$. Thus, the sequences of essential rank 0 are just the sequences in $\mathcal{G}$. Clearly, the sequences of finite essential rank form an ideal of $\mathcal{F}$ which is dense in $\mathcal{K}$, and

$$\operatorname{ess\,rank}(\mathbf{A} + \mathbf{B}) \leq \operatorname{ess\,rank} \mathbf{A} + \operatorname{ess\,rank} \mathbf{B},$$

$$\operatorname{ess\,rank}(\mathbf{AB}) \leq \min\{\operatorname{ess\,rank} \mathbf{A},\, \operatorname{ess\,rank} \mathbf{B}\}$$

for arbitrary sequences $\mathbf{A}, \mathbf{B} \in \mathcal{F}$. Several equivalent descriptions of compact sequences can be found in [7], Section 4.

In correspondence with the ideal $\mathcal{K}$, one introduces an appropriate class of Fredholm sequences by calling a sequence $(A_n) \in \mathcal{F}$ *Fredholm* if it is invertible modulo the ideal $\mathcal{K}$ of the compact sequences. It is not hard to show that a sequence $(A_n) \in \mathcal{F}$ is Fredholm in this sense if and only if there are sequences $(B_n) \in \mathcal{F}$ and $(J_n) \in \mathcal{K}$ with $\sup_n \operatorname{rank} J_n < \infty$ such that

$$B_n A_n = I_n + J_n \quad \text{for all } n \in \mathbb{N} \tag{4}$$

(see Theorem 5.1 in [7]).

If the sequence $\mathbf{A} = (A_n) \in \mathcal{F}$ is Fredholm, then the sequence $\mathbf{AA}^* = (A_n A_n^*)$ is Fredholm, too. The smallest non-negative integer k for which there exist a sequence $(B_n) \in \mathcal{F}$ as well as a sequence $(J_n) \in \mathcal{K}$ of essential rank k such that

$$B_n A_n^* A_n = I_n + J_n \quad \text{for all } n \in \mathbb{N}$$

is called the *α-number* $\alpha(\mathbf{A})$ of the Fredholm sequence $\mathbf{A}$.

The *index* of a Fredholm sequence $\mathbf{A} \in \mathcal{F}$ is the integer

$$\mathrm{ind}\,(\mathbf{A}) := \alpha(\mathbf{A}) - \alpha(\mathbf{A}^*).$$

Observe that, in the case at hand, the index of a Fredholm sequence is always zero, which is a consequence of the fact that the operators A_n act on finite dimensional spaces. So the more interesting quantity associated with a Fredholm sequence seems to be its α-number. But we will see that, on the other hand, the vanishing of the index of a Fredholm sequence allows one to make use of the index as a conservation quantity. Some general properties and equivalent characterizations of Fredholm sequences can be found in Chapter 6 of [1] and Section 5 of [7].

It is not hard to see that the finite sections sequence (P_nAP_n) of a band-dominated operator A on $l^2(\mathbb{Z}^+)$ is Fredholm in the above sense if and only if A is a Fredholm operator (Theorem 5.7 in [7]). The adventure of computing the α-number of the sequence (P_nAP_n) was undertaken in [7], Theorem 5.8. Here is the result.

Theorem 4 *Let $A \in l^2(\mathbb{Z}^+)$ be a Fredholm band-dominated operator. Then (P_nAP_n) is a Fredholm sequence, and*

$$\alpha(P_nAP_n) = \dim\ker A + \max\{\dim\ker(QA_hQ + P) : A_h \in \sigma_+(A)\}. \qquad (5)$$

4 The index formula for $p = 2$

We are now in a position to settle the finite sections proof of the index formula for band-dominated operators. Its main ingredient is a subsequence version of Theorem 4 which is in Theorem 5 below. Given a strongly monotonically increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$, we consider the C^*-algebra $\mathcal{F}_\eta$ of all bounded sequences (A_n) of operators $A_n : \mathrm{im}\, P_{\eta(n)} \to \mathrm{im}\, P_{\eta(n)}$ and introduce the according ideals $\mathcal{G}_\eta$ of all zero sequences in $\mathcal{F}_\eta$ and $\mathcal{K}_\eta$ of all compact sequences in $\mathcal{F}_\eta$. Equivalently, the elements of $\mathcal{F}_\eta$, $\mathcal{G}_\eta$ and $\mathcal{F}_\eta$ are the restrictions of sequences in $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{K}$ onto the range of η. Again we call the sequences in $\mathcal{K}_\eta$ *compact*, whereas the sequences which are invertible modulo $\mathcal{K}_\eta$ are called *Fredholm*. The α-number of a Fredholm sequence in $\mathcal{F}_\eta$ is defined in complete analogy to the α-number of a Fredholm sequence in $\mathcal{F}$.

Theorem 5 *Let $A \in L(l^2(\mathbb{Z}^+))$ be a Fredholm band-dominated operator and $\eta : \mathbb{N} \to \mathbb{N}$ a strongly monotonically increasing sequence. Then $(P_{\eta(n)}AP_{\eta(n)})$ is a Fredholm sequence in $\mathcal{F}_\eta$, and*

$$\alpha(P_{\eta(n)}AP_{\eta(n)}) = \dim\ker A + \max\{\dim\ker(QA_hQ + P) : A_h \in \sigma_{op,\eta}(A)\}. \qquad (6)$$

This result can be proved in complete analogy to its "full" version Theorem 4, with only evident modifications needed. We omit the tedious repetition of the arguments from [7].

A situation of particular interest arises if the sequence η in Theorem 5 is chosen such that the limit operator A_η exists. In this case, the operator spectrum $\sigma_{op,\eta}(A)$ consists of exactly one operator, namely A_η.

Corollary 6 *Let $A \in L(l^2(\mathbb{Z}^+))$ be a Fredholm band-dominated operator, and let $h : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence for which the limit operator A_h exists. Then $(P_{h(n)}AP_{h(n)})$ is a Fredholm sequence in $\mathcal{F}_h$, and*

$$\alpha(P_{h(n)}AP_{h(n)}) = \dim\ker A + \dim\ker(QA_hQ + P). \tag{7}$$

If $A \in L(l^2(\mathbb{Z}^+))$ is a Fredholm band-dominated operator, then its adjoint has the same property, and if the limit operator of A with respect to the sequence h exists, then the limit operator $(A^*)_h$ exists as well, and $(A^*)_h = (A_h)^*$. Applying the preceding corollary to the sequence $(P_{h(n)}A^*P_{h(n)})$ yields

$$\begin{aligned} \alpha(P_{h(n)}A^*P_{h(n)}) &= \dim\ker A^* + \dim\ker(Q(A^*)_hQ + P) \\ &= \dim\ker A^* + \dim\ker(Q(A_h)^*Q + P). \end{aligned} \tag{8}$$

We have already mentioned that

$$\alpha(P_{h(n)}AP_{h(n)}) = \alpha(P_{h(n)}A^*P_{h(n)}).$$

Thus, subtracting (8) from (7) and taking into account that $\operatorname{ind} A = \dim\ker A - \dim\ker A^*$ for every Fredholm operator, we obtain

$$\operatorname{ind} A = -\operatorname{ind}(QA_hQ + P) \tag{9}$$

for every Fredholm band-dominated operator A on $l^2(\mathbb{Z}^+)$ and every strongly monotonically sequence h for which the limit operator A_h exists. Notice that the "ind" on the left-hand side of (9) refers to the Fredholm index of an operator on $l^2(\mathbb{Z}^+)$, whereas the "ind" on the right-hand side stands for the index of an operator acting on $l^2(\mathbb{Z})$. Of course, the index of a Fredholm operator $A \in L(l^2(\mathbb{Z}^+))$ coincides with the index of the associated operator $PAP+Q$ considered as acting on $l^2(\mathbb{Z})$. Thus, (9) can be written in the more symmetric form

$$\operatorname{ind}(PAP + Q) = -\operatorname{ind}(QA_hQ + P) \tag{10}$$

which holds for every Fredholm band-dominated operator A on $l^2(\mathbb{Z}^+)$ and every limit operator A_h in $\sigma_+(A)$.

Let now A be a Fredholm band-dominated operator on $l^2(\mathbb{Z}^-)$. Then JAJ is a Fredholm band-dominated operator on $l^2(\mathbb{Z}^+)$, and $\operatorname{ind} A = \operatorname{ind} JAJ$. Moreover, since $U_{-n}J = JU_n$ for every integer n, one has $U_{-n}JAJU_n = JU_nAU_{-n}J$, whence it follows that the limit operator A_h exists if and only if the limit operator $(JAJ)_{-h}$ exists and that $JA_hJ = (JAJ)_{-h}$ in this case. Inserting the operator JAJ into (9) one thus concludes

$$\begin{aligned} \operatorname{ind} JAJ &= -\operatorname{ind}(Q(JAJ)_h Q + P) \\ &= -\operatorname{ind}(JPA_{-h}PJ + JQJ) = -\operatorname{ind}(PA_{-h}P + Q) \end{aligned}$$

for each sequence h tending to $+\infty$ for which the limit operator $(JAJ)_h$ (equivalently, the limit operator A_{-h}) exists. Writing this identity in a symmetric form we get

$$\operatorname{ind}(QAQ + P) = -\operatorname{ind}(PA_hP + Q) \tag{11}$$

which holds for every Fredholm band-dominated operator A on $l^2(\mathbb{Z}^-)$ and every limit operator A_h in $\sigma_-(A)$.

Let now A be a Fredholm band-dominated operator on $l^2(\mathbb{Z})$. Then PAP and QAQ are Fredholm band-dominated operators on $l^2(\mathbb{Z}^+)$ and $l^2(\mathbb{Z}^-)$, respectively, and from (10) and (11) one gets

$$\begin{aligned} \operatorname{ind} A &= \operatorname{ind}(PAP + Q) + \operatorname{ind}(QAQ + P) \\ &= -\operatorname{ind}(QA_hQ + P) - \operatorname{ind}(PA_gP + Q) \end{aligned} \tag{12}$$

which holds for arbitrary limit operators $A_h \in \sigma_+(A)$ and $A_g \in \sigma_-(A)$. Since the limit operators A_h and A_g are invertible if A is Fredholm, one has

$$\operatorname{ind}(QA_hQ+P) = -\operatorname{ind}(PA_hP+Q) \quad \text{and} \quad \operatorname{ind}(PA_gP+Q) = -\operatorname{ind}(QA_gQ+P),$$

which, together with (12), implies the assertion in case $p = 2$. ■

5 The index formula for general $p \in (1, \infty)$

There are several ways to establish the index formula in case of a general $p \in (1, \infty)$. The simplest way is to apply a result of [6] which says that both the Fredholm property and the Fredholm index of a band-dominated operator A on $l^p(\mathbb{Z})$ are independent of $p \in (1, \infty)$. With this information it is easy to derive the general case from the case $p = 2$ (see [6] for details).

An alternative way is to create a Fredholm theory for matrix sequences acting on Banach spaces and to follow the above proof, but now starting with a general p from the beginning. This demanding program was pursued by Rogozhin and one of the authors in a series of papers [8, 9, 10]. The main obstacle is that the singular value decomposition of a matrix, which played a prominent role in the Hilbert space theory in [1, 7], is of less use for operators on l^p-spaces. In this more general setting, the singular values (which are purely algebraic quantities) have to be replaced by the approximation numbers (which depend on the underlying norm).

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