On the Spectrum of an Oseen–Type Operator Arising from Flow past a Rotating Body

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Abstract

We present the description of the spectrum of a linear perturbed Oseen–type operator which arises from equations of motion of a viscous incompressible fluid in the exterior of a rotating compact body. Considering the operator in the function space $L^2_\sigma(\Omega)$ we prove that the essential spectrum consists of an infinite set of overlapping parabolic regions in the left half–plane of the complex plane. Our approach is based on a reduction to invariant closed subspaces of $L^2_\sigma(\Omega)$ and on a Fourier series expansion with respect to an angular variable in a cylindrical coordinate system attached to the axis of rotation.

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1 Motivation and introduction

Suppose that $\mathcal{B}$ is a compact body in $\mathbb{R}^3$ which is rotating about the x_1–axis with a constant angular velocity $\omega > 0$. Denote by $\Omega(t)$ the exterior of $\mathcal{B}$ at time t and assume that $\Omega(t)$ is a domain with boundary of class $C^{1,1}$.

The flow of a viscous incompressible fluid in the exterior of the body $\mathcal{B}$ can be described by the Navier–Stokes equation and the equation of continuity in the space–time region $\{(\boldsymbol{x},t) \in \mathbb{R}^3 \times \mathcal{I};\ t \in \mathcal{I},\ \boldsymbol{x} \in \Omega(t)\}$ where $\mathcal{I}$ is an interval on the time axis. The disadvantage of this description is the variability of the spatial domain $\Omega(t)$. Therefore, many authors use a time–dependent transformation of spatial coordinates which in fact also represents the rotation about the x_1 axis such that the body $\mathcal{B}$ is fixed and its exterior is just $\Omega(0)$ in the new coordinate system. The system of equations after the transformation has the form

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} - \omega(\boldsymbol{e}_1 \times \boldsymbol{x})\cdot\nabla\boldsymbol{u} + \omega\boldsymbol{e}_1 \times \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u} = 0 \tag{1.2}$$

in $\Omega(0) \times I$, where $\boldsymbol{e}_1$ is the unit vector oriented in the direction of the x_1–axis. The condition of the adherence of the fluid to the body on the boundary, after the transformation, has the form

$$\boldsymbol{u}(\boldsymbol{x},t) = \omega\boldsymbol{e}_1 \times \boldsymbol{x}, \qquad \boldsymbol{x} \in \partial\Omega(0). \tag{1.3}$$

In order to simplify the notation, we shall write only Ω instead of $\Omega(0)$.

Among a series of results on qualitative properties of the system (1.1)–(1.3) and related linear problems, let us mention T. Hishida [15], [16], [17], G. P. Galdi [9], [10], R. Farwig, T. Hishida, D. Müller [5], R. Farwig [3], [4], Š. Nečasová [23], M. Geissert, H. Heck, M. Hieber [11], S. Kračmar, Š. Nečasová, P. Penel [19], R. Farwig, J. Neustupa [6] and R. Farwig, Š. Nečasová, J. Neustupa [7].

We shall use the usual function spaces and notation:

- $\boldsymbol{n}$ is the outer normal vector on $\partial\Omega$.
- $(.\,,\,.)_{0,2}$ and $\|\,.\,\|_{0,2}$ are the scalar product and the norm in $L^2(\Omega)^3$, respectively.
- $W_0^{1,2}(\Omega)$ is the subspace of the Sobolev space $W^{1,2}(\Omega)$ consisting of functions vanishing on $\partial\Omega$ in the sense of traces. As is well-known, $W_0^{1,2}(\Omega)$ equals the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,2}(\Omega)$.
- $\|\,.\,\|_{k,2}$ denotes the norm in $W^{k,2}(\Omega)^3$, $k \in \mathbb{N}$.
- $C_{0,\sigma}^\infty(\Omega)$ denotes the space of all divergence–free functions from $C_0^\infty(\Omega)^3$.
- $L^2_\sigma(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)^3$. The space $L^2_\sigma(\Omega)$ can be characterized as the space of all divergence–free (in the sense of distributions) vector functions $\boldsymbol{u}$ from $L^2(\Omega)^3$ such that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ in the sense of traces ([8], pp. 111–115).
- Π_σ denotes the orthogonal projection of $L^2(\Omega)^3$ onto $L^2_\sigma(\Omega)$.

Suppose that $\boldsymbol{U}^*$ is a steady strong solution of the problem (1.1)–(1.3) such that

$$|\nabla\boldsymbol{U}^*| \in L^{3/2}(\Omega) \cap L^3(\Omega), \tag{1.4}$$

$$\lim_{R\to+\infty} \operatorname*{ess\,sup}_{|x|>R} |\boldsymbol{U}^* - \boldsymbol{U}^*_\infty| \, = \, 0 \tag{1.5}$$

where $\boldsymbol{U}^*_\infty = (\gamma, 0, 0)$, $\gamma \in \mathbb{R}$. The function $\boldsymbol{U} := \boldsymbol{U}^* - \boldsymbol{U}^*_\infty$ equals $\omega\boldsymbol{e}_1 \times \boldsymbol{x} - (\gamma, 0, 0)$ on $\partial\Omega$. Combining this information with the Sobolev inequality, see e.g. [8], p. 31, we can deduce that $\boldsymbol{U}$ satisfies $\boldsymbol{U} \in L^s(\Omega)^3$ for all $3 \leq s < +\infty$. In order to study the behavior of solutions near the steady solution $\boldsymbol{U}^*$, we put $\boldsymbol{u} = \boldsymbol{U}^* + \boldsymbol{v} = (\gamma, 0, 0) + \boldsymbol{U} + \boldsymbol{v}$. Then the perturbation $\boldsymbol{v}$ is a solution of the problem given by the equations

$$\partial_t\boldsymbol{v} - \nu\Delta\boldsymbol{v} - \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{v} + \omega\boldsymbol{e}_1 \times \boldsymbol{v} + \gamma\partial_1\boldsymbol{v} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{U}$$

$$+ (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p \, = \, \boldsymbol{0},$$

$$\nabla\cdot\boldsymbol{v} \, = \, 0$$

in $\Omega \times \mathcal{I}$ (where $\mathcal{I}$ is a time interval) and by the boundary condition

$$\boldsymbol{v}(\boldsymbol{x}, t) \, = \, 0 \qquad \text{for } \boldsymbol{x} \in \partial\Omega.$$

This problem can be written in the form of the operator equation

$$\partial_t\boldsymbol{v} \, = \, L^\omega_\gamma\boldsymbol{v} + N\boldsymbol{v} \tag{1.6}$$

in $L^2_\sigma(\Omega)$ where

$$L^\omega_\gamma\boldsymbol{v} \;=\; A^\omega_\gamma\boldsymbol{v} + B\boldsymbol{v}, \tag{1.7}$$

$$A^\omega_\gamma\boldsymbol{v} \;=\; \Pi_\sigma\nu\Delta\boldsymbol{v} + \Pi_\sigma[\omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{v} - \omega\boldsymbol{e}_1 \times \boldsymbol{v} - \gamma\partial_1\boldsymbol{v}], \tag{1.8}$$

$$B\boldsymbol{v} \;=\; -\Pi_\sigma[(\boldsymbol{U}\cdot\nabla)\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{U}], \tag{1.9}$$

$$N\boldsymbol{v} \;=\; -\Pi_\sigma(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}. \tag{1.10}$$

The operators A^ω_γ and L^ω_γ are defined in the same domains

$$D(A^\omega_\gamma) \;=\; D(L^\omega_\gamma) \;=\; \Big\{\boldsymbol{v}\in W^{2,2}(\Omega)^3\cap W^{1,2}_0(\Omega)^3\cap L^2_\sigma(\Omega);\ \omega(\boldsymbol{e}_1\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}\in L^2(\Omega)^3\Big\}$$

which are dense subsets of $L^2_\sigma(\Omega)$. The information on spectra of the linear operators A^ω_γ and L^ω_γ plays a fundamental role in studies of the evolution equation (1.6). Whereas the case $\gamma=0$ was treated in detail in our paper [6], here we consider the **important case** $\gamma\neq 0$. Our main theorem now reads as follows:

Theorem 1.1 *(i) The essential spectrum $\sigma_{ess}(A^\omega_\gamma)$ has the form*

$$\sigma_{ess}(A^\omega_\gamma) \;=\; \Lambda^\omega_\gamma \;:=\; \{\lambda=\alpha+\mathrm{i}\beta+\mathrm{i}k\omega\in\mathbb{C};\ \alpha,\beta\in\mathbb{R},\ k\in\mathbb{Z},\ \alpha\le-\nu\beta^2/\gamma^2\}, \qquad (1.11)$$

i.e., it consists of an infinite union of equally shifted filled parabolas in the left half–plane of $\mathbb{C}$, see Fig. 1.

(ii) The operator A^ω_γ is not normal.

(iii) If λ is an eigenvalue of A^ω_γ then $\mathrm{Re}\,\lambda<0$.

(iv) If the body $\mathcal{B}$ (and therefore also the domain Ω) is axially symmetric about the x_1–axis, then $\sigma(A^\omega_\gamma)=\sigma_{ess}(A^\omega_\gamma)=\Lambda^\omega_\gamma$.

(v) The operator L^ω_γ has the same essential spectrum as A^ω_γ.

(vi) $\sigma(L^\omega_\gamma)=\sigma_{ess}(L^\omega_\gamma)\cup\Gamma$ where Γ consists of an at most countable set of isolated eigenvalues of L^ω_γ which can possibly cluster only at points of $\sigma_{ess}(L^\omega_\gamma)$; each of them has finite algebraic multiplicity.

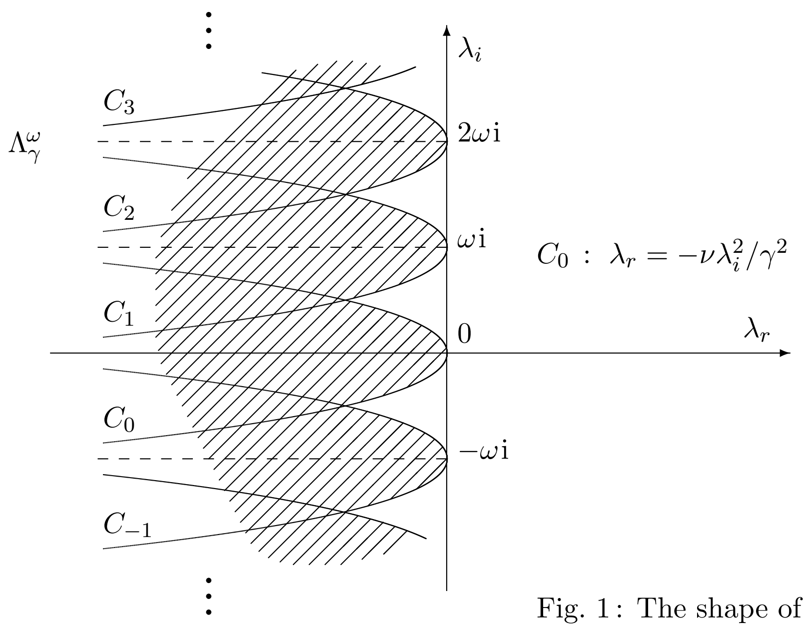

Fig. 1: The shape of set Λ^ω_γ

The proof of statements (i)–(iii) and (v), (vi) is given in Section 6. Statement (iv) is proved in Section 5.

2 Preliminaries

All function spaces needed in the following are considered to be spaces of complex–valued functions.

Lemma 2.1 *There exists $c_1 > 0$ such that if $\boldsymbol{v} \in D(A^\omega_\gamma)$ and $A^\omega_\gamma \boldsymbol{v} = \boldsymbol{f}$, then*

$$\|\boldsymbol{v}\|_{2,2} + \|(\omega \boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v}\|_{0,2} \le c_1(\gamma) \left(\|\boldsymbol{f}\|_{0,2} + \|\boldsymbol{v}\|_{0,2}\right). \tag{2.1}$$

Proof. The equation $A^\omega_\gamma \boldsymbol{v} = \boldsymbol{f}$ means that $A^\omega_0 \boldsymbol{v} = \boldsymbol{f} + \Pi_\sigma \gamma\, \partial_1 \boldsymbol{v}$. Applying the results from [5] ($\Omega = \mathbb{R}^3$) or from [16] (Ω being an exterior domain in $\mathbb{R}^3$) to the solution of the equation $A^\omega_0 \boldsymbol{v} = \boldsymbol{g}$ (with $\boldsymbol{g} = \boldsymbol{f} + \Pi_\sigma \gamma\, \partial_1 \boldsymbol{v}$), we obtain

$$\begin{aligned} \|\boldsymbol{v}\|_{2,2} + \|(\omega \boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v}\|_{0,2} &\le c_1(0) \left(\|A^\omega_0 \boldsymbol{v}\|_{0,2} + \|\boldsymbol{v}\|_{0,2}\right) \\ &\le c_1(0) \left(\|\boldsymbol{f}\|_{0,2} + |\gamma|\, \|\Pi_\sigma \partial_1 \boldsymbol{v}\|_{0,2} + \|\boldsymbol{v}\|_{0,2}\right). \end{aligned}$$

Interpolating suitably the norm $\|\Pi_\sigma \partial_1 \boldsymbol{v}\|_{0,2}$ between the norms $\|\boldsymbol{v}\|_{0,2}$ and $\|\boldsymbol{v}\|_{2,2}$, we arrive at (2.1). $\square$

Lemma 2.2 *A^ω_γ is a closed operator in $L^2_\sigma(\Omega)$ and its adjoint has the form*

$$(A^\omega_\gamma)^* \boldsymbol{v} = \Pi_\sigma \nu \Delta \boldsymbol{v} - \Pi_\sigma[\omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v} - \omega \boldsymbol{e}_1 \times \boldsymbol{v} - \gamma \partial_1 \boldsymbol{v}] = A^{-\omega}_{-\gamma} \boldsymbol{v} \tag{2.2}$$

with $D\big((A^\omega_\gamma)^*\big) = D(A^\omega_\gamma)$.

Proof. The operator A^ω_γ generates a C_0–semigroup, see [11]. Hence it is closed and there exists $\xi_0 \in \mathbb{R}$ such that $\xi \in \rho(A^\omega_\gamma)$ and $R(\zeta I - A^\omega_\gamma)$, the range of $\zeta I - A^\omega_\gamma$, equals $L^2_\sigma(\Omega)$ for $\xi > \xi_0$.

Let us denote by T^ω_γ the operator on the right hand side of (2.2) with $D(T^\omega_\gamma) = D(A^\omega_\gamma)$, i.e. $T^\omega_\gamma = A^{-\omega}_{-\gamma}$. Then T^ω_γ is closed and $R(\zeta I - T^\omega_\gamma) = L^2_\sigma(\Omega)$ if $\zeta > 0$ is sufficiently large. Using integration by parts, we can verify that

$$\big(\boldsymbol{u},\, A^\omega_\gamma \boldsymbol{v}\big)_{0,2} = \big(T^\omega_\gamma \boldsymbol{u},\, \boldsymbol{v}\big)_{0,2}$$

for all $\boldsymbol{u} \in D(T^\omega_\gamma)$ and $\boldsymbol{v} \in D(A^\omega_\gamma)$. It means that the operators A^ω_γ and T^ω_γ are adjoint to each other and $T^\omega_\gamma \subset (A^\omega_\gamma)^*$, see T. Kato [18], p. 167.

Suppose that $\boldsymbol{u} \in D((A^\omega_\gamma)^*)$. Then there exists $\boldsymbol{w} \in D(T^\omega_\gamma)$ such that $[\zeta I - (A^\omega_\gamma)^*]\boldsymbol{u} = (\zeta I - T^\omega_\gamma)\boldsymbol{w}$. Multiplying both sides of this identity by $\boldsymbol{v} \in D(A^\omega_\gamma)$, we arrive at

$$\big(\boldsymbol{u},\, (\zeta I - A^\omega_\gamma)\boldsymbol{v}\big)_{0,2} = \big(\boldsymbol{w},\, (\zeta I - A^\omega_\gamma)\boldsymbol{v}\big)_{0,2}.$$

As this holds for all $\boldsymbol{v} \in D(A^\omega_\gamma)$, we get $\boldsymbol{u} = \boldsymbol{w} \in D(T^\omega_\gamma)$ and consequently, $D((A^\omega_\gamma)^*) \subset D(T^\omega_\gamma)$. Thus, $(A^\omega_\gamma)^* = T^\omega_\gamma$. $\square$

Lemma 2.3 *If $\boldsymbol{v} \in D(A^\omega_\gamma)$, then both the terms $\omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla \boldsymbol{v} - \omega \boldsymbol{e}_1 \times \boldsymbol{v}$ and $\gamma \partial_1 \boldsymbol{v}$ belong to $L^2_\sigma(\Omega)$.*

Proof. It was already shown in [6] that $\omega(\boldsymbol{e}_1\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}-\omega\boldsymbol{e}_1\times\boldsymbol{v}\in L^2_\sigma(\Omega)$.

The space $C^\infty_{0,\sigma}(\Omega)$ is dense in $D(A^\omega_\gamma)$ in the topology of $W^{1,2}(\Omega)^3$; hence given $\boldsymbol{v}\in D(A^\omega_\gamma)$, there exists a sequence $\boldsymbol{v}^n\in C^\infty_{0,\sigma}(\Omega)$ such that $\boldsymbol{v}^n\to\boldsymbol{v}$ in $W^{1,2}(\Omega)^3$. Let ψ be a function from $W^{1,2}_{loc}(\Omega)$ such that $\nabla\psi\in L^2(\Omega)^3$. Then we have

$$\int_\Omega \gamma\partial_1\boldsymbol{v}\cdot\nabla\psi\,\mathrm{d}\boldsymbol{x} \;=\; \lim_{n\to+\infty}\int_\Omega \gamma\partial_1\boldsymbol{v}^n\cdot\nabla\psi\,\mathrm{d}\boldsymbol{x} \;=\; -\lim_{n\to+\infty}\int_\Omega \operatorname{div}(\gamma\partial_1\boldsymbol{v}^n)\,\psi\,\mathrm{d}\boldsymbol{x} \;=\; 0.$$

Thus the function $\gamma\partial_1\boldsymbol{v}$ is orthogonal to the subspace of all gradients in $L^2(\Omega)^3$, which implies that it belongs to $L^2_\sigma(\Omega)$, see e.g. G. P. Galdi [8], p. 103. □

Lemma 2.3 enables us to omit the projection Π_σ in front of the terms in the brackets on the right hand side of (1.8) and (2.2). The operator A^ω_γ can therefore be simplified to

$$\begin{aligned} A^\omega_\gamma\boldsymbol{v} &= A^0_0\boldsymbol{v}+\omega(\boldsymbol{e}_1\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}-\omega\boldsymbol{e}_1\times\boldsymbol{v}-\gamma\partial_1\boldsymbol{v}\\ &= A^0_\gamma\boldsymbol{v}+\omega(\boldsymbol{e}_1\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}-\omega\boldsymbol{e}_1\times\boldsymbol{v} \end{aligned}\tag{2.3}$$

where $A^0_0\equiv\nu\Pi_\sigma\Delta$ is the Stokes operator in $L^2_\sigma(\Omega)$ with domain $D(A^0_0)=W^{2,2}(\Omega)^3\cap W^{1,2}_0(\Omega)^3\cap L^2_\sigma(\Omega)$. The Stokes operator A^0_0 is selfadjoint in $L^2_\sigma(\Omega)$, see e.g. Y. Giga, H. Sohr [12]. Moreover, A^0_γ is the usual Oseen operator with the same domain as A^0_0. By analogy, the adjoint operator to A^ω_γ can be simplified to

$$\begin{aligned} (A^\omega_\gamma)^*\boldsymbol{v} &= A^0_0\boldsymbol{v}-\omega(\boldsymbol{e}_1\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}+\omega\boldsymbol{e}_1\times\boldsymbol{v}+\gamma\partial_1\boldsymbol{v}\\ &= A^0_{-\gamma}\boldsymbol{v}-\omega(\boldsymbol{e}_1\times\boldsymbol{x})\cdot\nabla\boldsymbol{v}+\omega\boldsymbol{e}_1\times\boldsymbol{v} \end{aligned}\tag{2.4}$$

Lemma 2.4 *The operator B, defined by (1.9), is A^ω_γ–compact.*

Proof. We have proved in [6] that the operator B is A^ω_0–compact. The proof in the case of $\gamma\neq 0$ can be done in the same way. The crucial step is an appropriate application of Lemma 2.1, which enables us to deduce that the boundedness of two sequences $\{\boldsymbol{\phi}_n\}$ and $\{A^\omega_\gamma\boldsymbol{\phi}_n\}$ in $L^2_\sigma(\Omega)$ implies the boundedness of $\{\boldsymbol{\phi}_n\}$ in $W^{2,2}(\Omega)^3$. □

Lemmas 2.2 and 2.4 imply that the operator L^ω_γ is closed in $L^2_\sigma(\Omega)$, see [18], p. 194.

It will be further advantageous to work in cylindrical coordinates. We shall denote by x_1, r and φ the cylindrical coordinate system whose axis is the x_1–axis such that the angle φ is measured from the positive part of the x_2–axis towards the positive part of the x_3–axis. The corresponding cylindrical components of vector functions will be denoted by the indices 1, r and φ, e.g. u_1, u_r and u_φ. In order to distinguish between the Cartesian and the cylindrical components of vectors, we shall write the Cartesian components in parentheses and the cylindrical components in brackets. Thus, we have $(u_1,u_2,u_3)\triangleq[u_1,u_r,u_\varphi]$. Using the transformations

$$\begin{aligned} u_r &= u_2\cos\varphi+u_3\sin\varphi, & u_2 &= u_r\cos\varphi-u_\varphi\sin\varphi,\\ u_\varphi &= -u_2\sin\varphi+u_3\cos\varphi, & u_3 &= u_r\sin\varphi+u_\varphi\cos\varphi, \end{aligned}$$

we can calculate that

$$(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\nabla\boldsymbol{u}-\boldsymbol{\omega}\times\boldsymbol{u} \;=\; \omega\,\partial_\varphi\boldsymbol{u}-(\boldsymbol{\omega}\times\boldsymbol{u}) \;=\; \omega\,\partial_\varphi(u_1,u_2,u_3)-\omega\,(0,-u_3,u_2)$$

$$
\begin{aligned}
&= \omega\,\partial_\varphi \begin{pmatrix} u_1 \\ u_r\,\cos\varphi - u_\varphi\,\sin\varphi \\ u_r\,\sin\varphi + u_\varphi\,\cos\varphi \end{pmatrix}^T - \omega \begin{pmatrix} 0 \\ -u_r\,\sin\varphi - u_\varphi\,\cos\varphi \\ u_r\,\cos\varphi - u_\varphi\,\sin\varphi \end{pmatrix}^T \\
&= \omega \begin{pmatrix} \partial_\varphi u_1 \\ (\partial_\varphi u_r)\,\cos\varphi - (\partial_\varphi u_\varphi)\,\sin\varphi \\ (\partial_\varphi u_r)\,\sin\varphi + (\partial_\varphi u_\varphi)\,\cos\varphi \end{pmatrix}^T \triangleq \omega \begin{bmatrix} \partial_\varphi u_1 \\ \partial_\varphi u_r \\ \partial_\varphi u_\varphi \end{bmatrix}^T = \omega\,\partial_\varphi\,[u_1, u_r, u_\varphi].
\end{aligned}
$$

In the following, the vector function $\boldsymbol{u}$ will be identified with $[u_1, u_r, u_\varphi]$; the same holds for other vectors or vector functions. Thus, the relation (2.3) between the operator A^ω_γ and the Stokes operator A^0_0 can be written in the form

$$
A^\omega_\gamma \boldsymbol{u} \;=\; A^0_0 \boldsymbol{u} + \omega\,\partial_\varphi \boldsymbol{u} - \gamma\,\partial_1 \boldsymbol{u} \tag{2.5}
$$

where A^0_0 now stands for the Stokes operator in cylindrical coordinates.

If T is a closed linear operator in a Hilbert space H, then we shall use the following notions and notation:

- $N(T)$ is the null space of T, $R(T)$ is the range and T^* is the adjoint operator to T.
- $\mathrm{nul}(T)$ is the nullity and $\mathrm{def}(T)$ is the deficiency of T.
- $\mathrm{ind}(T) = \mathrm{nul}(T) - \mathrm{def}(T)$ denotes the index of T.
- $\mathrm{nul}'(T)$ is the approximate nullity and $\mathrm{def}'(T)$ is the approximate deficiency of T.
- $\rho(T)$ denotes the resolvent set of T.
- $\sigma_p(T)$ is the point spectrum of T, $\sigma_c(T)$ its continuous spectrum and $\sigma_r(T)$ its residual spectrum.
- $\sigma(T)$ is the whole spectrum of T $(= \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T))$.
- $\sigma_{ess}(T)$ denotes the essential spectrum of T, i.e. the set of $\lambda \in \mathbb{C}$ such that $\mathrm{nul}'(T - \lambda I) = \mathrm{def}'(T - \lambda I) = +\infty$.
- $\widetilde{\sigma}_c(T)$ denotes the set of those $\lambda \in \mathbb{C}$ for which there exists a non–compact sequence $\{\boldsymbol{u}_n\}$ in the unit sphere in H such that $(T - \lambda I)\boldsymbol{u}_n \to \boldsymbol{0}$ for $n \to +\infty$. It is equivalent to the equality $\mathrm{nul}'(T - \lambda I) = +\infty$.
- T is said to be normal if $T^*T = TT^*$.

The definitions of these notions can be found in [18] or in [13], see [6] for the survey of their main properties. For the purposes of this paper, let us recall that $\sigma_p(T)$, $\sigma_r(T)$ and $\sigma_c(T)$ are mutually disjoint, $\sigma(T)$, $\sigma_{ess}(T)$ and $\widetilde{\sigma}_c(T)$ are closed sets in $\mathbb{C}$ and $\sigma_c(T) \subset \sigma_{ess}(T) \subset \widetilde{\sigma}_c(T) \subset \sigma(T)$.

3 The Oseen operator A^0_γ

It is known that the spectra of the Stokes operator A^0_0 satisfy the identities

$$
\sigma_p(A^0_0) \;=\; \sigma_r(A^0_0) \;=\; \emptyset, \tag{3.1}
$$

$$
\sigma(A^0_0) \;=\; \sigma_{ess}(A^0_0) \;=\; \sigma_c(A^0_0) \;=\; (-\infty, 0]. \tag{3.2}
$$

(The residual spectrum of A_0^0 is empty because A_0^0 is selfadjoint. The reasons why the point spectrum is also empty are explained in [6]. The identities $\sigma(A_0^0) = \sigma_c(A_0^0) = (-\infty, 0]$ follow from I. M. Glazman [13] and O. A. Ladyzhenskaya [20].)

The spectrum of the Oseen operator A_γ^0 was studied by K. I. Babenko in [1]. Considering the case $\Omega = \mathbb{R}^3$ and assuming that $\lambda \in \Lambda_\gamma^0$ where

$$\Lambda_\gamma^0 \;=\; \bigl\{\lambda = \alpha + \mathrm{i}\beta \in \mathbb{C};\; \alpha, \beta \in \mathbb{R},\; \alpha \leq -\nu\beta^2/\gamma^2\bigr\}, \tag{3.3}$$

K. I. Babenko mentions a construction (based on the Fourier transform) of a non–compact sequence $\{\boldsymbol{v}^n\}$ in the unit sphere in $L^2_\sigma(\Omega)$ such that $\|(A_\gamma^0 - \lambda I)\boldsymbol{v}^n\|_{0,2} \to 0$ as $n \to +\infty$. Then $\lambda \in \sigma_{ess}(A_\gamma^0)$ and consequently $\Lambda_\gamma^0 \subset \sigma_{ess}(A_\gamma^0)$. On the other hand, the author states that the equation

$$(A_\gamma^0 - \lambda I)\boldsymbol{v} \;=\; \boldsymbol{f}, \tag{3.4}$$

for Re $\lambda > 0$ and $\boldsymbol{f} \in L^2_\sigma(\Omega)$, can be solved by means of a Green's function of the Dirichlet problem with a reference to F. Odqvist [24] for more details concerning the construction of Green's function and its estimates. Furthermore, K. I. Babenko emphasizes that it is not difficult to treat the other cases of $\lambda \in \mathbb{C} - \Lambda_\gamma^0$. Thus, he arrives at Propositions 4 and 5 which imply that $\mathbb{C} - \Lambda_\gamma^0 \subset \rho(A_\gamma^0)$.

Since the information on the spectrum of the operator A_γ^0 is of fundamental importance, in the following theorem we present a complete proof based on a totally different approach.

Theorem 3.1 $\sigma(A_\gamma^0) = \sigma_{ess}(A_\gamma^0) = \Lambda_\gamma^0$.

Proof. I. Let us begin with the inclusion $\mathbb{C} - \Lambda_\gamma^0 \subset \rho(A_\gamma^0)$ to be proved by contradiction. Suppose that $\lambda = \alpha + \mathrm{i}\beta \in (\mathbb{C} - \Lambda_\gamma^0) \cap \sigma(A_\gamma^0)$. Assume that $\lambda \in \sigma_p(A_\gamma^0) \cup \sigma_c(A_\gamma^0)$ at first. Then there exists a sequence $\boldsymbol{v}^n$ in the unit sphere in $L^2_\sigma(\Omega)$ such that

$$(A_\gamma^0 - \lambda I)\boldsymbol{v}^n \;=\; \boldsymbol{\epsilon}^n \;\longrightarrow\; \boldsymbol{0} \qquad \text{in } L^2_\sigma(\Omega) \quad \text{as } n \to +\infty. \tag{3.5}$$

This sequence $\{\boldsymbol{v}^n\}$ can be constant if $\lambda \in \sigma_p(A_\gamma^0)$. We test (3.5) with $\boldsymbol{v}^n$ (in the L^2–sense for complex–valued functions) and get that

$$-\nu\,\|\nabla\boldsymbol{v}^n\|_{0,2}^2 - \lambda\,\|\boldsymbol{v}^n\|_{0,2}^2 - \gamma\,(\partial_1\boldsymbol{v}^n, \boldsymbol{v}^n)_{0,2} \;=\; (\boldsymbol{\epsilon}^n, \boldsymbol{v}^n)_{0,2}\,. \tag{3.6}$$

Note that Re $(\partial_1\boldsymbol{v}^n, \boldsymbol{v}^n)_{0,2} = 0$. Next we consider the real and imaginary part of (3.6) and see that

$$\nu\,\|\nabla\boldsymbol{v}^n\|_{0,2}^2 \;=\; -\alpha\,\|\boldsymbol{v}^n\|_{0,2}^2 - \mathrm{Re}\,(\boldsymbol{\epsilon}^n, \boldsymbol{v}^n)_{0,2} \;\leq\; -\alpha + \|\boldsymbol{\epsilon}^n\|_{0,2}\,. \tag{3.7}$$

Using (3.5), we observe that $\alpha \leq 0$. From (3.6) and (3.7) we obtain

$$\|\nabla\boldsymbol{v}^n\|_{0,2} \;\leq\; \Bigl(-\frac{\alpha}{\nu} + \frac{1}{\nu}\,\|\boldsymbol{\epsilon}^n\|_{0,2}\Bigr)^{1/2}, \tag{3.8}$$

as well as

$$\beta \;=\; \beta\,\|\boldsymbol{v}^n\|_{0,2}^2 \;=\; -\gamma\,\mathrm{Im}\,(\partial_1\boldsymbol{v}^n, \boldsymbol{v}^n)_{0,2} - \mathrm{Im}\,(\boldsymbol{\epsilon}^n, \boldsymbol{v}^n)_{0,2}$$

so that

$$|\beta| \;\leq\; \gamma\, \|\nabla \boldsymbol{v}^n\|_{0,2} + \|\boldsymbol{\epsilon}^n\|_{0,2}\,. \tag{3.9}$$

Inserting the estimate (3.8) into (3.9) we are led to the inequality

$$|\beta| \;\leq\; \gamma \left(-\frac{\alpha}{\nu} + \frac{1}{\nu}\, \|\boldsymbol{\epsilon}^n\|_{0,2} \right)^{1/2} + \|\boldsymbol{\epsilon}^n\|_{0,2}\,.$$

As $n \to +\infty$, (3.5) implies that $|\beta| \leq \gamma \sqrt{-\alpha/\nu}$, i.e. $\alpha \leq \nu\beta^2/\gamma^2$. Obviously, this inequality is in contradiction with the assumption $\lambda \in \mathbb{C} - \Lambda^0_\gamma$. Hence $\lambda \in \mathbb{C} - \Lambda^0_\gamma$ cannot belong to $\sigma_p(A^0_\gamma) \cup \sigma_c(A^0_\gamma)$. Now assume that $\lambda \in \sigma_r(A^0_\gamma)$. Then $\overline{\lambda}$ belongs to the point spectrum of the adjoint operator $(A^0_\gamma)^*$; this leads to the same contradiction as if $\lambda \in \sigma_p(A^0_\gamma)$. Thus, $\lambda \in \rho(A^0_\gamma)$ which implies that $\mathbb{C} - \Lambda^0_\gamma \subset \rho(A^0_\gamma)$.

II. Now we will prove that $\Lambda^0_\gamma \subset \widetilde{\sigma}_c(A^0_\gamma)$. Let $\lambda = \alpha + \mathrm{i}\beta \in (\Lambda^0_\gamma)^\circ$ be given; here $(\Lambda^0_\gamma)^\circ$ denotes the interior of Λ^0_γ, i.e. the set of $\alpha + \mathrm{i}\beta \in \mathbb{C}$ such that $\alpha < -\nu\beta^2/\gamma^2$. The number α can be written in the form $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 = -\nu\beta^2/\gamma^2$ and $\alpha_2 < 0$.

We shall explicitly define functions $\boldsymbol{v}^n \in L^2_\sigma(\Omega)$ such that $\|\boldsymbol{v}^n\|_{0,2} = 1$, $(A^0_\gamma - \lambda I)\boldsymbol{v}^n \to \boldsymbol{0}$ in $L^2_\sigma(\Omega)$ as $n \to +\infty$ and such that the sequence $\{\boldsymbol{v}^n\}$ does not contain any subsequence, convergent in $L^2_\sigma(\Omega)$. Let us denote by v^n_1, v^n_r and v^n_φ the cylindrical components of $\boldsymbol{v}^n$. Put

$$\begin{aligned}
v^n_1(x_1, r, \varphi) \;&:=\; 0, \\
v^n_r(x_1, r, \varphi) \;&:=\; \kappa_n\, U^n(x_1)\, V^n(r)\, \mathrm{e}^{\mathrm{i}k\varphi}, \\
v^n_\varphi(x_1, r, \varphi) \;&:=\; -\frac{1}{\mathrm{i}k}\, \partial_r \left[r\, v^n_r(x_1, r, \varphi) \right] \;=\; -\frac{1}{\mathrm{i}k}\, \kappa_n\, U^n(x_1) \left[V^n(r) + r\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r} \right] \mathrm{e}^{\mathrm{i}k\varphi}
\end{aligned}$$

where k is an arbitrary, but fixed chosen non–zero integer. Then, obviously, $\boldsymbol{v}^n$ satisfies the condition

$$\nabla \cdot \boldsymbol{v}^n \;\equiv\; \partial_1 v^n_1 + \frac{1}{r}\, \partial_r (r v^n_r) + \frac{1}{r}\, \partial_\varphi v^n_\varphi \;=\; 0.$$

Here the function U^n has the form

$$U^n(x_1) \;:=\; \eta^n_1(x_1)\, Y(x_1) \tag{3.10}$$

where η^n_1 is an infinitely differentiable function on $(-\infty, +\infty)$ such that $0 \leq \eta^n_1 \leq 1$,

$$\eta^n_1(x_1) \;=\; \begin{cases} 0 & \text{for } \; x_1 \leq -n - n^2 \; \text{ and } \; n + n^2 \leq x_1, \\ 1 & \text{for } \; -n^2 \leq x_1 \leq n^2, \end{cases}$$

and $Y(x_1) = \mathrm{e}^{\mathrm{i}ax_1}$. The identity $\alpha_1 = -\nu\beta^2/\gamma^2$ guarantees that the characteristic equation $\nu\zeta^2 - \gamma\zeta - (\alpha_1 + \mathrm{i}\beta) = 0$, corresponding to the equation (3.11) below, has the root $\zeta_1 = \mathrm{i}a$ where $a = -\beta/\gamma$. Thus, the function Y is a bounded non–trivial solution of the ordinary differential equation

$$\nu\, Y''(x_1) - \gamma\, Y'(x_1) - (\alpha_1 + \mathrm{i}\beta)\; Y(x_1) \;=\; 0 \tag{3.11}$$

in the interval $(-\infty, +\infty)$. The function V^n has the form

$$V^n(r) \,:=\, \eta_2^n(r)\, \mathrm{e}^{\mathrm{i}br}; \quad b = \sqrt{-\frac{\alpha_2}{\nu}} \tag{3.12}$$

where η_2^n is an infinitely differentiable function on $[0, +\infty)$ such that $0 \le \eta_2^n \le 1$ and

$$\eta_2^n(r) \;=\; \begin{cases} 0 & \text{for} \quad 0 \le r \le n \quad \text{and} \quad 3n + n^2 \le r, \\ 1 & \text{for} \quad 2n \le r \le 2n + n^2. \end{cases}$$

Both the functions η_1 and η_2 can be chosen so that their derivatives are of the order $1/n$. The definition of V^n guarantees that it satisfies

$$\nu\, \frac{\mathrm{d}^2}{\mathrm{d}r^2}\, V^n(r) - \alpha_2\, V^n(r) \;=\; 0 \tag{3.13}$$

for $2n < r < 2n + n^2$. Finally, the constant κ_n is chosen so that $\|\boldsymbol{v}^n\|_{0,2} = 1$. Thus, the support of $\boldsymbol{v}^n$ is a subset of

$$S^n \,:=\, \bigl\{\boldsymbol{x} = [x_1, r, \varphi] \in \mathbb{R}^3;\; -n - n^2 \le x_1 \le n + n^2,\; n \le r \le 3n + n^2,\; 0 \le \varphi < 2\pi\bigr\}. \tag{3.14}$$

Considering the norm of $\boldsymbol{v}^n$, we can observe that for large n the decisive contribution comes from the integral of $|v_\varphi^n|^2$, namely of its part $\bigl|(-1/\mathrm{i}k)\, \kappa_n\, U^n\, r\, (\mathrm{d}V^n/\mathrm{d}r)\, \mathrm{e}^{\mathrm{i}k\varphi}\bigr|^2$, on the region

$$D^n \,:=\, \bigl\{\boldsymbol{x} = [x_1, r, \varphi] \in \mathbb{R}^3;\; -n^2 < x_1 < n^2,\; 2n < r < 2n + n^2,\; 0 < \varphi < 2\pi\bigr\}. \tag{3.15}$$

The integrals of all other parts on other regions are of a lower order in n. Calculating the integral of $\bigl|(-1/\mathrm{i}k)\, \kappa_n\, U^n\, r\, (\mathrm{d}V^n/\mathrm{d}r)\, \mathrm{e}^{\mathrm{i}k\varphi}\bigr|^2$ on the domain D^n, we obtain

$$\begin{aligned} &\int_{-n^2}^{n^2} \int_{2n}^{2n+n^2} \int_0^{2\pi} \left| \frac{\kappa_n}{\mathrm{i}k}\, U^n(x_1)\, r\, \frac{\mathrm{d}V^n(r)}{\mathrm{d}r} \right|^2 r\, \mathrm{d}\varphi\, \mathrm{d}r\, \mathrm{d}x_1 \\ &\quad=\; 2\pi\, \frac{\kappa_n^2}{k^2} \int_{-n^2}^{n^2} |U^n(x_1)|^2\, \mathrm{d}x_1 \int_{2n}^{2n+n^2} r^3 \left| \frac{\mathrm{d}V^n(r)}{\mathrm{d}r} \right|^2 \mathrm{d}r \\ &\quad=\; 2\pi\, \frac{\kappa_n^2}{k^2}\, 2n^2\, \frac{b^2}{4} \Bigl((2n + n^2)^4 - (2n)^4 \Bigr). \end{aligned}$$

Here we have used the equalities $\eta_1^n(x_1) = \eta_2^n(r) = 1$, hence $|U^n(x_1)| = |V^n(r)| = 1$ for $(x_1, r, \varphi) \in D^n$. Thus, there exist $n_0 \in \mathbb{N}$ and positive constants c_2 and c_3 (independent of n) such that

$$\forall\, n \in \mathbb{N},\; n \ge n_0 \;:\quad \frac{c_2}{n^5} \;\le\; \kappa_n \;\le\; \frac{c_3}{n^5}\,. \tag{3.16}$$

Now looking at $(A_\gamma^0 - \lambda I)\boldsymbol{v}^n$, we can omit the projection Π_σ in front of the Laplace operator in $A_\gamma^0 \boldsymbol{v}^n$ because $\Delta \boldsymbol{v}^n$ is divergence–free and has a compact support in Ω. Thus, $(A_\gamma^0 - \lambda I)\boldsymbol{v}^n = \nu \Delta \boldsymbol{v}^n - \gamma \partial_1 \boldsymbol{v}^n - \lambda \boldsymbol{v}^n$. Calculating the norm of this expression in $L_\sigma^2(\Omega)$, we observe that the contributions coming from $\Omega - D^n$ tend to zero as $n \to +\infty$ because they represent square roots of integrals of functions bounded by $C\kappa_n^2 r^2$ on $S^n - D^n$. Due to

(3.16), this contribution is of the order $n^{-1/2}$. Concerning the integral on D^n, the decisive part again comes from $(\nu\Delta - \gamma\partial_1 - \lambda I)v^n_\varphi$, namely from $(\nu\Delta - \gamma\partial_1 - \lambda I)$ applied to the term $(-1/\mathrm{i}k)\,\kappa_n\,U^n\,r\,(\mathrm{d}V^n/\mathrm{d}r)\,\mathrm{e}^{\mathrm{i}k\varphi}$ because of the factor r inside this term. Note that due to (3.11) and (3.13)

$$
\begin{aligned}
&(\nu\Delta - \gamma\partial_1 - \lambda I)\Big(\frac{\kappa_n}{\mathrm{i}k}\,U^n(x_1)\,r\,\frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\,\mathrm{e}^{\mathrm{i}k\varphi}\Big)\\
&= \Big(\nu\partial_1^2 + \nu\partial_r^2 + \frac{\nu}{r}\,\partial_r + \frac{\nu}{r^2}\,\partial_\varphi^2 - \gamma\,\partial_1 - \lambda I\Big)\Big(\frac{\kappa_n}{\mathrm{i}k}\,U^n(x_1)\,r\,\frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\,\mathrm{e}^{\mathrm{i}k\varphi}\Big)\\
&= \frac{\kappa_n}{\mathrm{i}k}\,U^n(x_1)\,\Big(\nu\,\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\nu}{r}\,\frac{\mathrm{d}}{\mathrm{d}r} - \alpha_2 I\Big)\Big[r\,\frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\Big]\mathrm{e}^{\mathrm{i}k\varphi}\\
&\quad + \frac{\kappa_n}{\mathrm{i}k}\,\Big(\nu Y''(x_1) - \gamma Y'(x_1) - [\alpha_1 + \mathrm{i}\beta]\,Y(x_1)\Big)\Big[r\,\frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\Big]\mathrm{e}^{\mathrm{i}k\varphi}\\
&\quad - \frac{\kappa_n}{\mathrm{i}k}\,\frac{\nu k^2}{r^2}\,U^n(x_1)\;r\;\frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\;\mathrm{e}^{\mathrm{i}k\varphi}\\
&= \Big\{\frac{\kappa_n}{\mathrm{i}k}\;U^n(x_1)\;r\;\frac{\mathrm{d}}{\mathrm{d}r}\Big[\nu\,\frac{\mathrm{d}^2V^n(r)}{\mathrm{d}r^2} - \alpha_2\,V^n(r)\Big] + \frac{\kappa_n}{\mathrm{i}k}\;U^n(x_1)\;2\nu\;\frac{\mathrm{d}^2V^n(r)}{\mathrm{d}r^2}\\
&\quad + \frac{\kappa_n}{\mathrm{i}k}\;U^n(x_1)\;\frac{\nu}{r}\;\frac{\mathrm{d}}{\mathrm{d}r}\Big[r\,\frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\Big] - \frac{\kappa_n}{\mathrm{i}k}\;U^n(x_1)\;\frac{\nu k^2}{r}\;\frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\Big\}\;\mathrm{e}^{\mathrm{i}k\varphi}\\
&= \frac{\nu\kappa_n}{\mathrm{i}k}\,\Big(-3b^2 + \frac{\mathrm{i}b}{r} - \frac{k^2\,\mathrm{i}b}{r}\Big)\,\mathrm{e}^{\mathrm{i}(ax_1+br)}\,\mathrm{e}^{\mathrm{i}k\varphi}
\end{aligned}
$$

where in the last step we used the simple forms of the functions U^n and V^n on D^n, i.e. $U^n(x_1) = \mathrm{e}^{\mathrm{i}ax_1}$ and $V^n(r) = \mathrm{e}^{\mathrm{i}br}$. Hence

$$
\begin{aligned}
&\Big[\int_{-n^2}^{n^2}\int_{2n}^{2n+n^2}\int_0^{2\pi}\Big|\Big(\nu\Delta - \gamma\partial_1 - \lambda I\Big)\,\Big(\frac{\kappa_n}{\mathrm{i}k}\,U^n(x_1)\,r\,\frac{\mathrm{d}V^n(r)}{\mathrm{d}r}\,\mathrm{e}^{\mathrm{i}k\varphi}\Big)\Big|^2\,r\,\mathrm{d}\varphi\,\mathrm{d}r\,\mathrm{d}x_1\Big]^{1/2}\\
&\leq\; C(\nu,k,b)\,\kappa_n\,\Big[\int_{-n^2}^{n^2}\int_{2n}^{2n+n^2} r\;\mathrm{d}r\,\mathrm{d}x_1\Big]^{1/2} \;=\; C(\nu,k,b)\,\kappa_n\,n\,\Big[(2n+n^2)^2 - (2n)^2\Big]^{1/2}.
\end{aligned}
$$

The last term tends to zero as $n \to +\infty$ due to (3.16). In this way, we prove that $\|(A^0_\gamma - \lambda I)\boldsymbol{v}^n\|_{0,2} \to 0$ as $n \to +\infty$.

The sequence $\{\boldsymbol{v}^n\}$ does not contain any convergent subsequence because the intersection of supports of any infinite family of functions, chosen from $\{\boldsymbol{v}^n\}$, is empty.

Since λ was an arbitrarily chosen number from $(\Lambda^0_\gamma)^\circ$, we have obtained the inclusion $(\Lambda^0_\gamma)^\circ \subset \widetilde{\sigma}_c(A^0_\gamma)$. It means that $\mathrm{nul}'(A^0_\gamma - \lambda I) = +\infty$. Since the operators A^0_γ and $(A^0_\gamma)^*$ differ only in the sign in front of $\gamma\partial_1$, we can prove in the same way that $\mathrm{nul}'((A^0_\gamma)^* - \overline{\lambda}I) = +\infty$. It means that $\mathrm{def}'(A^0_\gamma - \lambda I) = +\infty$ and consequently, $\lambda \in \sigma_{ess}(A^0_\gamma)$. The essential spectrum is a closed set, hence $\Lambda^0_\gamma \subset \sigma_{ess}(A^0_\gamma)$. □

Theorem 3.1 provides an information on the shape of the whole spectrum $\sigma(A^0_\gamma)$, but it does not specify which numbers λ from $\sigma(A^0_\gamma)$ belong to $\sigma_p(A^0_\gamma)$, $\sigma_c(A^0_\gamma)$ or to $\sigma_r(A^0_\gamma)$. We do answer this question in this paper neither for the operator A^0_γ nor for the more general operator A^ω_γ.

The Oseen operator A^0_γ generates an analytic semigroup, see T. Miyakawa [22]. Therefore the operator $(-A^0_\gamma)$ is sectorial, see D. Henry [14], p. 20–21. The next theorem states the non–normality of the Oseen operator, which stresses the difference between the Stokes and Oseen operators.

Theorem 3.2 *The Oseen operator A^0_γ is not normal.*

Proof. Constructing a function $\boldsymbol{z} \in D((A^0_\gamma)^* A^0_\gamma)$ which is not in $D(A^0_\gamma (A^0_\gamma)^*)$, we show that the domains $D((A^0_\gamma)^* A^0_\gamma)$ and $D(A^0_\gamma (A^0_\gamma)^*)$ do not coincide.

Let $R > 0$ be so large that the body $\mathcal{B}$ is contained in the interior of the cube $[-R, R]^3$. Recall that $\Omega = \mathbb{R}^3 - \mathcal{B}$. Define the set

$$\begin{aligned} \Omega_{\rm per} \; := \; & \big\{ \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3;\ \exists\, k, l, m \in \mathbb{Z} : \\ & \big(x_1 - kR,\ x_2 - lR,\ x_3 - mR\big) \in [-R, R]^3 - \mathcal{B} \big\}. \end{aligned}$$

Note that $\Omega_{\rm per}$ is a domain in $\mathbb{R}^3$ which consists of infinitely many copies of the set $[-R, R]^3 - \mathcal{B}$, periodically repeated in directions parallel with the x_1–, x_2– and x_3–axis. We shall use the following function spaces:

- $(C_0^\infty)^3_{\rm per}$ is the space of infinitely differentiable vector functions $\boldsymbol{\phi}$ in $\Omega_{\rm per}$, R–periodic in the directions of all Cartesian axes and such that $\operatorname{dist}(\operatorname{supp} \boldsymbol{\phi};\, \mathcal{B}) > 0$.
- $(L^2)^3_{\rm per}$ is the completion of $(C_0^\infty)^3_{\rm per}$ in the norm identical with the L^2–norm on $(-R, R)^3 - \mathcal{B}$. The spaces $(W_0^{1,2})^3_{\rm per}$ and $(W^{2,2})_{\rm per}$ are defined analogously.
- $(L^2_\sigma)_{\rm per}$ is the closure of the space of divergence–free functions from $(C_0^\infty)^3_{\rm per}$ in $(L^2)^3_{\rm per}$.

Let $(A^0_0)_{\rm per}$ denote the Stokes operator in $(L^2_\sigma)_{\rm per}$ with the dense domain $D((A^0_0)_{\rm per}) = (W^{2,2})^3_{\rm per} \cap (W_0^{1,2})^3_{\rm per} \cap (L^2_\sigma)_{\rm per}$. Then $(A^0_0)_{\rm per}$ has a compact resolvent and its spectrum, as well as the spectrum of $(A^0_0)_{\rm per} - \gamma\partial_1$ (with the same domain), consists of a countable number of isolated eigenvalues with finite multiplicities and negative real parts. Choose an eigenvalue ζ of $(A^0_0)_{\rm per} - \gamma\partial_1$ and denote by $\boldsymbol{u}$ an associated eigenfunction so that the equation

$$(A^0_0)_{\rm per}\boldsymbol{u} - \gamma\partial_1\boldsymbol{u} - \zeta\boldsymbol{u} \; = \; \boldsymbol{0} \tag{3.17}$$

is satisfied in $\Omega_{\rm per}$.

Let us show, by contradiction, that the eigenfunction $\boldsymbol{u}$ can be chosen so that $\partial_1\boldsymbol{u} \not\equiv \boldsymbol{0}$ on $\partial\Omega_{\rm per}$. Assume the opposite, i.e. that all eigenfunctions $\boldsymbol{v}$ of the operator $(A^0_0)_{\rm per} - \gamma\partial_1$, corresponding to the eigenvalue ζ, satisfy $\partial_1\boldsymbol{v} \equiv \boldsymbol{0}$ on $\partial\Omega_{\rm per}$. Then, for each of them, there are two possibilities: either $\partial_1\boldsymbol{v} \equiv \boldsymbol{0}$ in $\Omega_{\rm per}$ (which can be easily excluded) or $\partial_1\boldsymbol{v}$ is also an eigenfunction of $(A^0_0)_{\rm per} - \gamma\partial_1$ corresponding to the same eigenvalue ζ. Since the eigenspace of $(A^0_0)_{\rm per} - \gamma\partial_1$, generated by all such eigenfunctions, is finite–dimensional, we can choose an eigenfunction $\boldsymbol{u}$ so that

$$\gamma\,\partial_1\boldsymbol{u} \; = \; \mu\,\boldsymbol{u} \tag{3.18}$$

with an appropriate constant μ. Since $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega_{\rm per}$ and equation (3.18) is satisfied in $\Omega_{\rm per}$, the integration of (3.18) on line segments parallel with the x_1–axis and starting from the boundary of Ω_{per} yields that $\boldsymbol{u} = \boldsymbol{0}$ on all such line segments. Thus, $\boldsymbol{u}$ vanishes identically in an open subset of Ω_{per}. Now the unique continuation principle, see e.g. R. Leiss [21], applied to $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{u}$, shows that $\boldsymbol{w} \equiv \boldsymbol{0}$ in Ω_{per}. Consequently, $\boldsymbol{u} \equiv \boldsymbol{0}$ in Ω_{per} which is impossible because $\boldsymbol{u}$ is an eigenfunction. Since the assumption that $\partial_1 \boldsymbol{u} \equiv \boldsymbol{0}$ on $\partial\Omega_{\text{per}}$ leads to contradiction, we have $\partial_1 \boldsymbol{u} \not\equiv \boldsymbol{0}$ on $\partial\Omega_{\text{per}}$.

Note that $\partial\Omega \subset \partial\Omega_{\text{per}}$ and $\partial\Omega_{\text{per}}$ consists of infinitely many copies of $\partial\Omega$ repeated periodically with the period R in the direction of each Cartesian coordinate. Now we multiply function $\boldsymbol{u}$ by an infinitely–differentiable cut–off function η_R which equals one in the neighborhood of $\partial\Omega$ and whose support is contained in $(-R, R)^3 - \mathcal{B}$, and correct the product $\eta_R \boldsymbol{u}$ by an appropriate function $\boldsymbol{U}_R$ which guarantees that $\operatorname{div}(\eta_R \boldsymbol{u} - \boldsymbol{U}_R) = 0$. By these means we can obtain a function $\boldsymbol{z}$ in $D((A^0_\gamma)^* A^0_\gamma)$ which coincides with the function $\boldsymbol{u}$ constructed above in the neighborhood of Ω and equals zero outside $(-R, R)^3$. The function $\boldsymbol{z}$ satisfies $\partial_1 \boldsymbol{z} \not\equiv \boldsymbol{0}$ on $\partial\Omega$. Then $\boldsymbol{z}$ cannot belong to $D(A^0_\gamma (A^0_\gamma)^*)$ because all functions from $D((A^0_\gamma)^* A^0_\gamma) \cap D(A^0_\gamma (A^0_\gamma)^*)$ satisfy on $\partial\Omega$ the conditions $\boldsymbol{z} = A^0_0 \boldsymbol{z} + \gamma \partial_1 \boldsymbol{z} = A^0_0 \boldsymbol{z} - \gamma \partial_1 \boldsymbol{z} = \boldsymbol{0}$, which implies that $\partial_1 \boldsymbol{z} = \boldsymbol{0}$ on $\partial\Omega$. □

4 Axially symmetric domains – decomposition of $L^2_\sigma(\Omega)$ and of A^0_γ

We shall assume that the domain $\Omega \subset \mathbb{R}^3$ is axially symmetric with respect to the x_1–axis in this section.

Let k be an integer. We introduce the following spaces and notation:

- $L^2(\Omega)^3_k = \{\boldsymbol{v} \in L^2(\Omega)^3;\ \boldsymbol{v} = \boldsymbol{V}(x_1, r)\, \mathrm{e}^{\mathrm{i}k\varphi}\}$
- $C^\infty_0(\Omega)^3_k = C^\infty_0(\Omega)^3 \cap L^2(\Omega)^3_k$
- $C^\infty_{0,\sigma}(\Omega)_k = C^\infty_0(\Omega)^3_k \cap C^\infty_{0,\sigma}(\Omega)$
- $L^2_\sigma(\Omega)_k$ = the closure of $C^\infty_{0,\sigma}(\Omega)_k$ in $L^2(\Omega)^3_k$
- P_k – the orthogonal projection of $L^2(\Omega)^3$ onto $L^2(\Omega)^3_k$
- $(A^0_\gamma)_k$ – the restriction of the operator A^0_γ to the space $L^2_\sigma(\Omega)_k$

Obviously, $L^2(\Omega)^3_k$, $k \in \mathbb{Z}$, is a closed subspace of $L^2(\Omega)^3$, and $L^2_\sigma(\Omega)_k$ is a closed subspace of $L^2_\sigma(\Omega)$. The domain of $(A^0_\gamma)_k$ equals $D(A^0_\gamma) \cap L^2_\sigma(\Omega)_k$.

Each function $\boldsymbol{v}$ from $L^2(\Omega)^3$ can uniquely be written in the form of a convergent Fourier series – with respect to the variable φ – of terms from $L^2(\Omega)^3_k$, $k \in \mathbb{Z}$:

$$\boldsymbol{v}(x_1, r, \varphi) = \sum_{k=-\infty}^{+\infty} \boldsymbol{V}^k(x_1, r)\, \mathrm{e}^{\mathrm{i}k\varphi}; \quad \boldsymbol{V}^k(x_1, r) = \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{v}(x_1, r, \varphi)\, \mathrm{e}^{-\mathrm{i}k\varphi}\, \mathrm{d}\varphi. \tag{4.1}$$

Thus, we have $L^2(\Omega)^3 = \ldots \oplus L^2(\Omega)^3_{-2} \oplus L^2(\Omega)^3_{-1} \oplus L^2(\Omega)^3_0 \oplus L^2(\Omega)^3_1 \oplus L^2(\Omega)^3_2 \oplus \ldots$. We have proved in [6] that

$$\Pi_\sigma\, L^2(\Omega)^3_k \;=\; L^2_\sigma(\Omega) \cap L^2(\Omega)^3_k \;=\; L^2_\sigma(\Omega)_k \;=\; P_k L^2_\sigma(\Omega). \tag{4.2}$$

The next lemma generalizes some results from [6].

Lemma 4.1 *Let $k \in \mathbb{Z}$. Then $(A^0_\gamma)_k$ is a closed operator in $L^2_\sigma(\Omega)_k$ with the dense domain $D((A^0_\gamma)_k)$; moreover $D((A^0_\gamma)_k) = P_k[D(A^0_\gamma)]$, $R((A^0_\gamma)_k) \subset L^2_\sigma(\Omega)_k$.*

Proof. The operator $(A^0_\gamma)_k$ is closed because it is the restriction of the closed operator A^0_γ onto a closed subspace of $L^2_\sigma(\Omega)$. The domain of $(A^0_\gamma)_k$ is the set of functions from $L^2_\sigma(\Omega)_k$, that belong to $W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3$. This set contains $C^\infty_{0,\sigma}(\Omega)$, hence it is dense in $L^2_\sigma(\Omega)_k$.

Let $\boldsymbol{v} \in D(A^0_\gamma) \equiv W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap L^2_\sigma(\Omega)$ and let (4.1) be its Fourier expansion in the variable φ. Then $\boldsymbol{V}^k(x_1,r)\,\mathrm{e}^{\mathrm{i}k\varphi} \equiv P_k\boldsymbol{v} \in W^{2,2}(\Omega)^3$, and, due to the axial symmetry of Ω and the boundary condition satisfied by $\boldsymbol{v}$ on $\partial\Omega$, $\boldsymbol{V}^k(x_1,r)\,\mathrm{e}^{\mathrm{i}k\varphi}$ also belongs to $W^{1,2}_0(\Omega)^3$. Using the equation $\operatorname{div}\boldsymbol{v} = 0$ and the orthogonality of the functions $\operatorname{div}\big[\boldsymbol{V}^k(x_1,r)\,\mathrm{e}^{\mathrm{i}k\varphi}\big]$ in $L^2(\Omega)^3$ (for different k), we can prove that $\operatorname{div}\big[\boldsymbol{V}^k(x_1,r)\,\mathrm{e}^{\mathrm{i}k\varphi}\big] = 0$. Hence $\boldsymbol{V}^k(x_1,r)\,\mathrm{e}^{\mathrm{i}k\varphi} \in L^2_\sigma(\Omega)_k$ and consequently, $P_k[D(A^0_\gamma)] \subset D((A^0_\gamma)_k)$.

On the other hand, if $\boldsymbol{v} \in D((A^0_\gamma)_k)$, then it belongs to $D(A^0_\gamma)$, and since $P_k\boldsymbol{v} = \boldsymbol{v}$, it also belongs to $L^2(\Omega)^3_k$. Hence $\boldsymbol{v} \in D(A^0_\gamma) \cap L^2(\Omega)^3_k = D(A^0_\gamma) \cap L^2_\sigma(\Omega)_k = P_k[D(A^0_\gamma)]$.

If $\boldsymbol{v} \in D((A^0_\gamma)_k)$, then $\Delta\boldsymbol{v}$, $\partial_1\boldsymbol{v} \in L^2(\Omega)^3_k$, and due to (4.2), $A^0_\gamma\boldsymbol{v} = \nu\Pi_\sigma\Delta\boldsymbol{v} - \gamma\partial_1\boldsymbol{v} \in L^2_\sigma(\Omega)_k$. Hence A^0_γ is reduced onto $L^2_\sigma(\Omega)_k$. □

Lemma 4.2 *Let $k \in \mathbb{Z}$. Then $\sigma((A^0_\gamma)_k) = \sigma_{ess}((A^0_\gamma)_k) = \Lambda^0_\gamma$ where Λ^0_γ is the parabolic region in $\mathbb{C}$ defined by (3.3): $\Lambda^0_\gamma = \big\{\lambda = \alpha + \mathrm{i}\beta \in \mathbb{C};\ \alpha,\, \beta \in \mathbb{R},\ \alpha \leq -\nu\beta^2/\gamma^2\big\}$.*

Proof. The operator $(A^0_\gamma)_k$ is a part of A^0_γ, hence $\sigma((A^0_\gamma)_k) \subset \sigma(A^0_\gamma) = \Lambda^0_\gamma$.

On the other hand, for $\lambda \in (\Lambda^0_\gamma)^\circ$, we have shown the existence of a non–compact sequence $\boldsymbol{v}^n \in L^2_\sigma(\Omega)$ such that $\|\boldsymbol{v}^n\|_{0,2} = 1$ and $(A^0_\gamma - \lambda I)\boldsymbol{v}^n \to \boldsymbol{0}$ in $L^2_\sigma(\Omega)$ as $n \to +\infty$ in the proof of Theorem 3.1, part II. The construction of $\boldsymbol{v}^n$ involved the choice of an arbitrary non–zero integer k. An easy examination shows that the functions $\boldsymbol{v}^n$ actually belong not only to $L^2_\sigma(\Omega)$, but to $L^2_\sigma(\Omega)_k$. Thus, we obtain that $(\Lambda^0_\gamma)^\circ \subset \widetilde{\sigma}_c((A^0_\gamma)_k)$ for $k \neq 0$. Using the same arguments as at the end of the proof of Theorem 3.1, we deduce that $\Lambda^0_\gamma \subset \sigma_{ess}((A^0_\gamma)_k)$ for $k \neq 0$. It completes the proof in the case when $k \neq 0$.

The case $k = 0$ must be treated separately. Suppose that $\lambda = \alpha + \mathrm{i}\beta \in (\Lambda^0_\gamma)^\circ$. Let us construct a non–compact sequence $\{\boldsymbol{v}^n\}$ in the unit sphere in $L^2_\sigma(\Omega)_0$ such that $(A^0_\gamma - \lambda I)\boldsymbol{v}^n \to \boldsymbol{0}$ as $n \to +\infty$. The requirement that $\boldsymbol{v}^n \in L^2_\sigma(\Omega)_0$ means that $\boldsymbol{v}^n \equiv [v^n_1, v^n_r, v^n_\varphi]$ does not depend on φ. Then the condition $\operatorname{div}\boldsymbol{v}^n = 0$ says that $\partial_1(rv^n_1) + \partial_r(rv^n_r) = 0$. This equation is automatically satisfied if $\boldsymbol{v}^n$ has the cylindrical components

$$\boldsymbol{v}^n(x_1,r) \;=\; \Big[\,\frac{1}{r}\,\partial_r\psi^n(x_1,r),\ -\frac{1}{r}\,\partial_1\psi^n(x_1,r),\ 0\Big]. \tag{4.3}$$

Put $\psi^n(x_1,r) = \delta_n\,U^n(x_1)\,V^n(r)$ where U^n and V^n are the same functions as in the proof of Theorem 3.1, i.e. the functions given by (3.10) and (3.12), and where the factor δ_n must be chosen so that $\|\boldsymbol{v}^n\|_{0,2} = 1$. Calculating the norm of $\|\boldsymbol{v}^n\|_{0,2}$, we can observe that for large n the decisive contribution comes from the integral on D^n, see (3.15). The contribution coming from $\Omega - D^n$ is of a lower order in powers of n. The cut–off functions η^n_1 and η^n_2 are both equal to 1 on D^n. Hence $U^n(x_1) = \mathrm{e}^{\mathrm{i}ax_1}$, where $a = -\beta/\gamma$, and $V^n(r) = \mathrm{e}^{\mathrm{i}br}$, where $b = \sqrt{-\alpha_2/\nu}$, see the proof of Theorem 3.1. Thus, $\psi(x_1,r) = \delta_n\,\mathrm{e}^{\mathrm{i}(ax_1+br)}$ on D^n

and

$$\begin{aligned}\int_{D^n} |\boldsymbol{v}^n|^2 \,\mathrm{d}\boldsymbol{x} &= \int_{-n^2}^{n^2} \int_{2n}^{2n+n^2} \int_0^{2\pi} \big(|v_1^n|^2 + |v_r^n|^2\big)\,\mathrm{d}\varphi\, r\,\mathrm{d}r\,\mathrm{d}x_1 \\ &= 2\pi \int_{-n^2}^{n^2} \int_{2n}^{2n+n^2} \frac{1}{r}\big(|\partial_r\psi^n|^2 + |\partial_1\psi^n|^2\big)\,\mathrm{d}r\,\mathrm{d}x_1 \\ &= 2\pi\,\delta_n^2 \int_{-n^2}^{n^2} \int_{2n}^{2n+n^2} \frac{1}{r}\,(a^2+b^2)\,\mathrm{d}r\,\mathrm{d}x_1 \\ &= 2\pi\,\delta_n^2\,(a^2+b^2)\,2n^2\Big(\ln(2n+n^2) - \ln(2n)\Big)\end{aligned}$$

The condition that this tends to 1 as $n \to +\infty$ leads to the existence of constants c_4, $c_5 > 0$ and $n_0 \in \mathbb{N}$ such that

$$\forall\, n \in \mathbb{N},\ n \geq n_0\ :\quad \frac{c_4}{n\sqrt{\ln n}} \leq \delta_n \leq \frac{c_5}{n\sqrt{\ln n}}\,. \tag{4.4}$$

Suppose that n is so large that the support of $\boldsymbol{v}^n$ is a subset of Ω. Then $(A_\gamma^0 - \lambda I)\boldsymbol{v}^n = \nu\Delta\boldsymbol{v}^n - \gamma\partial_1\boldsymbol{v}^n - \lambda\boldsymbol{v}^n$ and so $\|(A_\gamma^0 - \lambda I)\boldsymbol{v}^n\|_{0,2}^2$ equals

$$\delta_n^2 \int_{S^n - D^n} \big|\nu\Delta\boldsymbol{v}^n - \gamma\partial_1\boldsymbol{v}^n - \lambda\boldsymbol{v}^n\big|^2\,\mathrm{d}\boldsymbol{x} + \delta_n^2 \int_{D^n} \big|\nu\Delta\boldsymbol{v}^n - \gamma\partial_1\boldsymbol{v}^n - \lambda\boldsymbol{v}^n\big|^2\,\mathrm{d}\boldsymbol{x} \tag{4.5}$$

where S^n is defined by (3.14). The integrands are less than or equal to C/r^2. Thus, the first term in (4.5) can be estimated from above by

$$\begin{aligned}& C\delta_n^2 \int_{-n-n^2}^{n+n^2} \mathrm{d}x_1 \left(\int_n^{2n} + \int_{2n+n^2}^{3n+n^2}\right) \frac{r\,\mathrm{d}r}{r^2} + C\delta_n^2 \left(\int_{-n-n^2}^{-n^2} + \int_{n^2}^{n+n^2}\right)\mathrm{d}x_1 \int_n^{3n+n^2} \frac{r\,\mathrm{d}r}{r^2} \\ \leq\ & C\delta_n^2\,(2n+2n^2)\left(\ln\frac{2n}{n} + \ln\frac{3n+n^2}{2n+n^2}\right) + C\delta_n^2\left(2n\,\ln\frac{3n+n^2}{n}\right).\end{aligned}$$

Using (4.4), we verify that the right hand side tends to zero as $n \to +\infty$. In the second term in (4.5), we use the identity $\psi^n(x_1, r) = \delta_n\,\mathrm{e}^{\mathrm{i}(ax_1+br)}$ which holds on D^n. Note that the function $Y(x_1) = \mathrm{e}^{\mathrm{i}ax_1}$ satisfies the differential equation (3.11) and that the function e^{ibr} satisfies the differential equation (3.13). Then calculating the expression $\nu\Delta\boldsymbol{v}^n - \gamma\partial_1\boldsymbol{v}^n - \lambda\boldsymbol{v}^n$ we find that $|\nu\Delta\boldsymbol{v}^n - \gamma\partial_1\boldsymbol{v}^n - \lambda\boldsymbol{v}^n| \leq C\delta_n/r^2$. Consequently the second term in (4.5) can be estimated by

$$C\delta_n^2 \int_{-n^2}^{n^2} \int_{2n}^{2n+n^2} \int_0^{2\pi} \frac{1}{r^4}\,\mathrm{d}\varphi\, r\,\mathrm{d}r\,\mathrm{d}x_1 = C\delta_n^2\, n^2 \int_{2n}^{2n+n^2} \frac{\mathrm{d}r}{r^3} \leq C\delta_n^2\,.$$

Due to (4.4), the right hand side tends to zero as $n \to +\infty$. Hence we have shown that $\|(A_\gamma^0 - \lambda I)\boldsymbol{v}^n\|_{0,2} \to 0$ as $n \to +\infty$. The sequence $\{\boldsymbol{v}^n\}$ is non–compact because the intersection of the supports of any infinite family of functions chosen from $\{\boldsymbol{v}^n\}$ is empty. Since all the functions $\boldsymbol{v}^n$ belong to $L^2_\sigma(\Omega)_0$, we have proved that $\lambda \in \widetilde{\sigma}_c((A_\gamma^0)_0)$. Applying once again the same arguments as at the end of the proof of Theorem 3.1, we observe that $\lambda \in \sigma_{ess}((A_\gamma^0)_0)$ and due to the closedness of $\sigma_{ess}((A_\gamma^0)_0)$, we obtain the inclusion $\Lambda_\gamma^0 \subset \sigma_{ess}((A_\gamma^0)_0)$. This completes the proof in the case $k = 0$. □

Since $-(A_\gamma^0)_k$ is, by definition, the reduction of the sectorial operator $-A_\gamma^0$ on the space $L^2_\sigma(\Omega)_k$, $-(A_\gamma^0)_k$ is a sectorial operator in $L^2_\sigma(\Omega)_k$.

5 Axially symmetric domains Ω – the operator A^ω_γ and its decomposition

Let $k \in \mathbb{Z}$. We shall denote by $(A^\omega_\gamma)_k$ the restriction of A^ω_γ to $L^2_\sigma(\Omega)_k$. The domain of $(A^\omega_\gamma)_k$ is the same as the domain of $(A^0_\gamma)_k$, i.e.,

$$D((A^\omega_\gamma)_k) = D((A^0_\gamma)_k) \equiv W^{2,2}(\Omega)^3 \cap W^{1,2}_0(\Omega)^3 \cap L^2_\sigma(\Omega)_k.$$

If $\boldsymbol{u} \in L^2_\sigma(\Omega)_k$, then it has the form $\boldsymbol{u}(x_1, r, \varphi) = \boldsymbol{U}(x_1, r)\, \mathrm{e}^{\mathrm{i}k\varphi}$ and $\partial_\varphi \boldsymbol{u} = \mathrm{i}\, k\, \boldsymbol{U}\, \mathrm{e}^{\mathrm{i}k\varphi} = \mathrm{i}\, k\, \boldsymbol{u}$. Therefore, $(A^\omega_\gamma)_k$ can be rewritten as

$$(A^\omega_\gamma)_k \boldsymbol{u} \;=\; (A^0_\gamma)_k \boldsymbol{u} + \omega\, \partial_\varphi \boldsymbol{u} \;=\; (A^0_\gamma)_k \boldsymbol{u} + \mathrm{i}k\omega\, \boldsymbol{u}. \tag{5.1}$$

Thus, $(A^\omega_\gamma)_k$ is a closed and densely defined operator in $L^2_\sigma(\Omega)_k$. The representation (5.1) of the operator $(A^\omega_\gamma)_k$ and Lemma 4.2 imply that

$$\sigma((A^\omega_\gamma)_k) \;=\; \sigma_{ess}((A^\omega_\gamma)_k) \;=\; \{\lambda \in \mathbb{C};\; \lambda - \mathrm{i}k\omega \in \Lambda^0_\gamma\}. \tag{5.2}$$

The next lemma provides the information on the spectrum of the full operator A^ω_γ. It confirms that statement (iv) of Theorem 1.1 is true.

Lemma 5.1 *$\sigma(A^\omega_\gamma) = \sigma_{ess}(A^\omega_\gamma) = \Lambda^\omega_\gamma$ where set Λ^ω_γ is defined by (1.11): $\Lambda^\omega_\gamma = \{\lambda = \alpha + \mathrm{i}\beta + \mathrm{i}k\omega \in \mathbb{C};\; \alpha, \beta \in \mathbb{R},\; k \in \mathbb{Z},\; \alpha \le -\nu\beta^2/\gamma^2\}$.*

Proof. Each operator $(A^\omega_\gamma)_k$, $k \in \mathbb{Z}$, is a part of the operator A^ω_γ, hence $\sigma_{ess}((A^\omega_\gamma)_k) \subset \sigma_{ess}(A^\omega_\gamma)$. Thus, $\cup_{k\in\mathbb{Z}}\, \sigma_{ess}((A^\omega_\gamma)_k) = \Lambda^\omega_\gamma \subset \sigma_{ess}(A^\omega_\gamma) \subset \sigma(A^\omega_\gamma)$.

It remains to prove the opposite inclusion, i.e. that $\sigma(A^\omega_\gamma) \subset \Lambda^\omega_\gamma$ or equivalently that $(\mathbb{C} - \Lambda^\omega_\gamma) \subset \rho(A^\omega_\gamma)$. Suppose that $\lambda \equiv \alpha + \mathrm{i}\beta \in \mathbb{C} - \Lambda^\omega_\gamma$. We will show that the operator $A^\omega_\gamma - \lambda I$ has a bounded inverse in $L^2_\sigma(\Omega)$. Let $\boldsymbol{f} \in L^2_\sigma(\Omega)$ with Fourier expansion

$$\boldsymbol{f}(x_1, r, \varphi) \;=\; \sum_{k=-\infty}^{+\infty} \boldsymbol{f}_k(x_1, r)\; \mathrm{e}^{\mathrm{i}k\varphi},$$

where $\boldsymbol{f}_k\, \mathrm{e}^{\mathrm{i}k\varphi} \in L^2_\sigma(\Omega)_k$, be given. Let us at first solve the equation $((A^\omega_\gamma)_k - \lambda I)\boldsymbol{w}_k = \boldsymbol{f}_k\, \mathrm{e}^{\mathrm{i}k\varphi}$ in $L^2_\sigma(\Omega)_k$. Putting $\boldsymbol{w}_k = \boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi}$ and using (5.1), we observe that this equation is equivalent with

$$(A^0_\gamma)_k(\boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi}) - (\alpha + \mathrm{i}\beta - \mathrm{i}k\omega)(\boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi}) \;=\; \boldsymbol{f}_k\, \mathrm{e}^{\mathrm{i}k\varphi}. \tag{5.3}$$

Due to Lemma 4.2 we have $\alpha + \mathrm{i}\beta - \mathrm{i}k\omega \in \rho((A^0_\gamma)_k)$ for all $k \in \mathbb{Z}$. Since the operator $-(A^0_\gamma)_k$ is sectorial, we deduce from resolvent estimates for sectorial operators, see e.g. D. Henry [14], p. 23, that there exists a constant $M > 0$, independent of k, such that

$$\|\boldsymbol{u}_k\|_{0,2} \;=\; \|\boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi}\|_{0,2} \;\le\; \frac{M}{1+|k|}\, \|\boldsymbol{f}_k\, \mathrm{e}^{\mathrm{i}k\varphi}\|_{0,2} \;=\; \frac{M}{1+|k|}\, \|\boldsymbol{f}_k\|_{0,2}\,. \tag{5.4}$$

Then the series $\sum_{k=-\infty}^{+\infty} \boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi}$ converges in $L^2_\sigma(\Omega)$ and $\boldsymbol{u} = \sum_{k=-\infty}^{+\infty} \boldsymbol{u}_k\, \mathrm{e}^{\mathrm{i}k\varphi}$ satisfies the estimate

$$\|\boldsymbol{u}\|^2_{0,2} \;=\; \sum_{k=-\infty}^{+\infty} \|\boldsymbol{u}_k\|^2_{0,2} \;\le\; \sum_{k=-\infty}^{+\infty} \frac{M^2}{(1+|k|)^2}\, \|\boldsymbol{f}_k\|^2_{0,2} \;\le\; M^2\, \|\boldsymbol{f}\|^2_{0,2}\,. \tag{5.5}$$

From the equation (5.3) and the estimate (5.4), we have

$$\big\|(A^{\omega}_{\gamma})_k(\boldsymbol{u}_k\,\mathrm{e}^{\mathrm{i}k\varphi})\big\|_{0,2} \;\leq\; \|\boldsymbol{f}_k\,\mathrm{e}^{\mathrm{i}k\varphi}\|_{0,2} + |\alpha+\mathrm{i}\beta|\,\|\boldsymbol{u}_k\,\mathrm{e}^{\mathrm{i}k\varphi}\|_{0,2} \;\leq\; C\,\|\boldsymbol{f}_k\,\mathrm{e}^{\mathrm{i}k\varphi}\|_{0,2}$$

where C is independent of k. Using these inequalities and the closedness of the operator A^{ω}_{γ}, we deduce that $\boldsymbol{u} \in D(A^{\omega}_{\gamma})$ and $(A^{\omega}_{\gamma} - \lambda I)\boldsymbol{u} = \boldsymbol{f}$. This information, together with (5.5), completes the proof. □

6 General exterior domains – the operators A^{ω}_{γ} and L^{ω}_{γ}

Using the same procedure as in the proof of Theorem 3.2, we can show that the operator A^{ω}_{γ} is not normal, i.e., that the statement (ii) of Theorem 1.1 is true.

If $\lambda = \alpha + \mathrm{i}\beta$ is an eigenvalue of A^{ω}_{γ} and $\boldsymbol{v}$ is a corresponding eigenfunction, then, multiplying the equation $A^{\omega}_{\gamma}\boldsymbol{v} = \lambda\,\boldsymbol{v}$ by $\overline{\boldsymbol{v}}$ and integrating on Ω, we obtain the identity $-\nu\,\|\nabla\boldsymbol{v}\|^2_{0,2} \;=\; \alpha\,\|\boldsymbol{v}\|^2_{0,2}$; compare with that part of the proof of Theorem 3.1 which lead to (3.8). This verifies Theorem 1.1 (iii).

Let $R_0 = \max\{|\boldsymbol{x}|;\ \boldsymbol{x} \in \mathcal{B}\}$ and $\Omega_R = \Omega \cap B_R(\boldsymbol{0})$.

Lemma 6.1 *Let $\lambda \in \widetilde{\sigma}_c(A^{\omega}_{\gamma})$. Then there exists $R > R_0$ and a non–compact sequence $\{\boldsymbol{u}^n\}$ in $D(A^{\omega}_{\gamma})$ such that $\|\boldsymbol{u}^n\|_{0,2} = 1$, $\boldsymbol{u}^n = \boldsymbol{0}$ in Ω_R and*

$$(A^{\omega}_{\gamma} - \lambda I)\,\boldsymbol{u}^n \;\longrightarrow\; 0 \quad \textit{in } L^2_{\sigma}(\Omega) \quad \textit{for } n \to +\infty. \tag{6.1}$$

Proof. The condition $\lambda \in \widetilde{\sigma}_c(A^{\omega}_{\gamma})$ means that $\mathrm{nul}'(A^{\omega}_{\gamma} - \lambda I) = +\infty$. Then there exists an orthonormal sequence $\{\boldsymbol{v}^n\}$ in $L^2_{\sigma}(\Omega)$ such that

$$(A^{\omega}_{\gamma} - \lambda I)\,\boldsymbol{v}^n \;=\; \boldsymbol{\epsilon}^n \;\longrightarrow\; 0 \quad \text{in } L^2_{\sigma}(\Omega) \quad \text{for } n \to +\infty; \tag{6.2}$$

the construction of the sequence $\{\boldsymbol{v}^n\}$ is based on Lemma IV.2.3 in [18] and is explained in [6]. Obviously $\{\boldsymbol{v}^n\}$ converges to the zero function weakly in $L^2_{\sigma}(\Omega)$. Using (6.2) and the estimate (2.1), we get that the sequence $\{\boldsymbol{v}^n\}$ is bounded in $W^{1,2}_0(\Omega)^3 \cap W^{2,2}(\Omega)^3$. Then there exists a subsequence, again denoted by $\{\boldsymbol{v}^n\}$, which is weakly convergent to $\boldsymbol{0}$ in $W^{1,2}_0(\Omega)^3 \cap W^{2,2}(\Omega)^3$. Moreover, $\{\omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{v}^n\}$ converges weakly to $\boldsymbol{0}$ in $L^2_{\sigma}(\Omega)$. Suppose that $R \geq R_0 + 3$ is a fixed number. The compact imbedding $W^{2,2}(\Omega_R)^3 \hookrightarrow \hookrightarrow W^{1,2}(\Omega_R)^3$ yields

$$\boldsymbol{v}^n \;\longrightarrow\; \boldsymbol{0} \qquad \text{strongly in } W^{1,2}(\Omega_R)^3. \tag{6.3}$$

The first part of (6.2) can be written in the form

$$\nu\Delta\boldsymbol{v}^n + \omega(\boldsymbol{e}_1 \times \boldsymbol{x}) \cdot \nabla\boldsymbol{v}^n - \omega\boldsymbol{e}_1 \times \boldsymbol{v}^n - \gamma\partial_1\boldsymbol{v}^n - \lambda\boldsymbol{v}^n + \nabla q^n \;=\; \boldsymbol{\epsilon}^n \tag{6.4}$$

where q^n is an appropriate scalar function. It follows from (6.4) and (2.1) that $\nabla q^n \to \boldsymbol{0}$ weakly in $L^2(\Omega)^3$. Thus, the functions q^n, which are given uniquely up to an additive constant by (6.4), can be chosen so that $q^n \to q \equiv \mathit{const.}$ strongly in $L^2(\Omega_R)$. We may even assume that $q = 0$.

Denote by η an infinitely differentiable cut–off function in Ω such that

$$\eta(\boldsymbol{x}) \;=\; \begin{cases} 0 & \text{if} \quad |\boldsymbol{x}| < R-2, \\ 1 & \text{if} \quad |\boldsymbol{x}| > R-1, \end{cases}$$

and $0 \leq \eta(\boldsymbol{x}) \leq 1$ if $R-2 \leq |\boldsymbol{x}| \leq R-1$. Put $\boldsymbol{u}^n = \eta \boldsymbol{v}^n - \boldsymbol{V}^n$ where $\operatorname{div} \boldsymbol{V}^n = \nabla\eta \cdot \boldsymbol{v}^n$. Although $\boldsymbol{V}^n$ is not given uniquely, the results on solutions of the equation $\operatorname{div} \boldsymbol{V} = \boldsymbol{f}$, see e.g. [2], show that the function $\boldsymbol{V}^n$ can be chosen such that $\operatorname{supp} \boldsymbol{V}^n \subset \{\boldsymbol{x} \in \Omega;\ R-3 < |\boldsymbol{x}| < R\}$ and there exist $c_6 > 0$ such that

$$\|\boldsymbol{V}^n\|_{2,2} \;=\; \|\boldsymbol{V}^n\|_{2,2;\,\Omega_R} \;\leq\; c_6\, \|\nabla\eta \cdot \boldsymbol{v}^n\|_{1,2;\,\Omega_R} \;\longrightarrow\; 0 \tag{6.5}$$

as $n \to +\infty$. (Here $\|\,.\,\|_{2,2;\,\Omega_R}$ and $\|\,.\,\|_{1,2;\,\Omega_R}$ denote the norm in $W^{2,2}(\Omega_R)^3$ and in $W^{1,2}(\Omega_R)^3$, respectively.) The function $\boldsymbol{u}^n$ is divergence–free, equals $\boldsymbol{0}$ in Ω_{R-3}, equals $\boldsymbol{v}^n$ in $\Omega - \Omega_R$ and belongs to $L^2(\Omega)^3$. Due to the properties of the functions η and $\boldsymbol{V}^n$ we get $\boldsymbol{u}^n \in D(A^\omega_\gamma)$. Obviously $\boldsymbol{u}^n$ satisfies

$$\begin{aligned}
&\nu\Delta\boldsymbol{u}^n + \omega(\boldsymbol{e}_1 \times \boldsymbol{x})\cdot\nabla\boldsymbol{u}^n - \omega\boldsymbol{e}_1 \times \boldsymbol{u}^n - \gamma\partial_1\boldsymbol{u}^n - \lambda\boldsymbol{u}^n + \nabla(\eta q^n) \\
&\quad = \;\; \eta\left[\nu\Delta\boldsymbol{v}^n + \omega(\boldsymbol{e}_1 \times \boldsymbol{x})\cdot\nabla\boldsymbol{v}^n - \omega\boldsymbol{e}_1 \times \boldsymbol{v}^n - \gamma\partial_1\boldsymbol{u}_v - \lambda\boldsymbol{v}^n\right] + 2\nu\nabla\eta\cdot\nabla\boldsymbol{v}^n \\
&\quad\qquad + \nu(\Delta\eta)\boldsymbol{v}^n - \nu\Delta\boldsymbol{V}^n + \left[\omega(\boldsymbol{e}_1 \times \boldsymbol{x})\cdot\nabla\eta\right]\boldsymbol{v}^n - \gamma\,(\partial_1\eta)\,\boldsymbol{v}^n - \omega(\boldsymbol{e}_1 \times \boldsymbol{x})\cdot\nabla\boldsymbol{V}^n \\
&\quad\qquad + \omega\boldsymbol{e}_1 \times \boldsymbol{V}^n + \gamma\partial_1\boldsymbol{V}^n + \lambda\boldsymbol{V}^n + \nabla(\eta q^n) \\
&\quad = \;\; \eta\boldsymbol{\epsilon}^n + 2\nu\nabla\eta\cdot\nabla\boldsymbol{v}^n + \nu(\Delta\eta)\boldsymbol{v}^n - \nu\Delta\boldsymbol{V}^n + \left[\omega(\boldsymbol{e}_1 \times \boldsymbol{x})\cdot\nabla\eta\right]\boldsymbol{v}^n - \gamma\,(\partial_1\eta)\,\boldsymbol{v}^n \\
&\quad\qquad - \omega(\boldsymbol{e}_1 \times \boldsymbol{x})\cdot\nabla\boldsymbol{V}^n + \omega\boldsymbol{e}_1 \times \boldsymbol{V}^n + \gamma\partial_1\boldsymbol{V}^n + \lambda\boldsymbol{V}^n + (\nabla\eta)q^n
\end{aligned} \tag{6.6}$$

where $\eta\boldsymbol{\epsilon}^n \to \boldsymbol{0}$ in $L^2(\Omega)^3$ due to (6.2), and $\nu[2\nabla\eta\cdot\nabla\boldsymbol{v}^n + (\Delta\eta)\boldsymbol{v}^n] \to \boldsymbol{0}$ in $L^2(\Omega)^3$ because $\nabla\eta$ and $\Delta\eta$ are supported in Ω_R and due to (6.3). The terms $\left[\omega(\boldsymbol{e}_1 \times \boldsymbol{x})\cdot\nabla\eta\right]\boldsymbol{v}^n$ and $\gamma\,(\partial_1\eta)\,\boldsymbol{v}^n$ also tend to zero in in $L^2(\Omega)^3$ for the same reasons. Furthermore, all terms involving $\boldsymbol{V}^n$ tend to $\boldsymbol{0}$ in $L^2(\Omega)^3$ due to (6.5). Finally, $(\nabla\eta)q^n \to \boldsymbol{0}$ in $L^2(\Omega)^3$ because $q^n \to 0$ in $L^2(\Omega_R)$ and $\nabla\eta$ is supported in Ω_R. Thus,

$$\nu\Delta\boldsymbol{u}^n + \omega(\boldsymbol{e}_1 \times \boldsymbol{x})\cdot\nabla\boldsymbol{u}^n - \omega\boldsymbol{e}_1 \times \boldsymbol{u}^n - \gamma\partial_1\boldsymbol{u}^n - \lambda\boldsymbol{u}^n + \nabla(\eta q^n) \;\longrightarrow\; \boldsymbol{0} \quad \text{in } L^2(\Omega)^3$$

for $n \to +\infty$, and therefore $\{\boldsymbol{u}^n\}$ satisfies (6.1). Moreover, we have

$$\|\boldsymbol{u}^n\|_{0,2}^2 \;\geq\; \int_{|x|>R} |\boldsymbol{u}^n(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} \;=\; \int_{|x|>R} |\boldsymbol{v}^n(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} \;\longrightarrow\; 1 \quad \text{for } n \to +\infty$$

because $\|\boldsymbol{v}^n\|_{0,2} = 1$ and due to (6.3). If we divide each of the functions $\boldsymbol{u}^n$ by its norm in $L^2_\sigma(\Omega)$ and denote the new function again by $\boldsymbol{u}^n$, we obtain the sequence $\{\boldsymbol{u}^n\}$ with all the properties stated in Lemma 6.1. Finally, the orthonormality of $\{\boldsymbol{v}^n\}$ and (6.3) imply the non–compactness of the sequence $\{\boldsymbol{u}^n\}$. $\square$

We denote by $\widehat{A}^\omega_\gamma$ the operator which is defined in the same way as A^ω_γ, however on the whole space $\mathbb{R}^3$ rather than on the exterior domain $\Omega \subset \mathbb{R}^3$. Obviously, the operator $\widehat{A}^\omega_\gamma$ has all the properties derived in Sections 4 and 5.

Lemma 6.2 $\widetilde{\sigma}_c(A^\omega_\gamma) = \widetilde{\sigma}_c(\widehat{A}^\omega_\gamma) = \Lambda^\omega_\gamma$.

Proof. Suppose that $\lambda \in \widetilde{\sigma}_c(A^\omega)$. Let $R > 0$ and $\{\boldsymbol{u}^n\}$ be a number and a sequence, respectively, with the properties named in Lemma 6.1. All functions $\boldsymbol{u}^n$, extended by zero from Ω to the whole space $\mathbb{R}^3$, belong to the domain of operator $\widehat{A}^\omega_\gamma$. Thus, (6.1) shows that $\lambda \in \widetilde{\sigma}_c(\widehat{A}^\omega_\gamma)$.

On the other hand, if $\lambda \in \widetilde{\sigma}_c(\widehat{A}^\omega_\gamma)$ then we can use analogous arguments and prove that λ also belongs to $\widetilde{\sigma}_c(A^\omega_\gamma)$. □

Let us show that $\widetilde{\sigma}_c(A^\omega_\gamma) \subset \sigma_{ess}(A^\omega_\gamma)$; the opposite inclusion is trivial. For $\lambda \in \widetilde{\sigma}_c(A^\omega_\gamma)$ we have that $\mathrm{nul}'(A^\omega_\gamma - \lambda I) = +\infty$. Moreover, $\overline{\lambda} \in \widetilde{\sigma}_c((A^\omega_\gamma)^*) = \widetilde{\sigma}_c(A^{-\omega}_{-\gamma})$, so that $\mathrm{nul}'((A^\omega_\gamma)^* - \overline{\lambda} I) = +\infty$. Hence $\mathrm{def}'(A^\omega_\gamma - \lambda I) = +\infty$ which shows that $\lambda \in \sigma_{ess}(A^\omega_\gamma)$, see [18], p. 234. We have thus proved Theorem 1.1 (i). Theorem IV.5.35 in [18] and Lemma 2.4 imply that the essential spectrum of the operator L^ω_γ is the same as $\sigma_{ess}(A^\omega_\gamma)$; therefore it is also given by (1.11). Moreover, since $\mathrm{ind}(L^\omega_\gamma - \lambda I) = 0$ in $\mathbb{C} - \sigma_{ess}(L^\omega_\gamma)$ and due to Theorem IV.5.31 in [18], $\mathbb{C} - \sigma_{ess}(L^\omega_\gamma)$ can contain at most countably many eigenvalues λ of L^ω_ω, which can cluster only on the boundary of $\mathbb{C} - \sigma_{ess}(L^\omega_\gamma)$ and $0 < \mathrm{nul}(L^\omega_\gamma - \lambda I) = \mathrm{def}(L^\omega_\gamma - \lambda I) < +\infty$ at each of them. This implies Theorem 1.1 (vi).

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