Open Mapping Theorem for Topological Groups

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Abstract. We survey sufficient conditions that force a surjective continuous homomorphism between topological groups to be open. We present the shortest proof yet of an open mapping theorem between projective limits of finite dimensional Lie groups.
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Introduction

A proposition that lists conditions on topological groups G and H such that a surjective morphism $f: G \to H$ is an open map is called an *Open Mapping Theorem.* The morphism f factors in the form $f = f' \circ q$ with a quotient morphism $q: G \to G/\ker f$ and a *bijective* morphism $f': G/\ker f \to H$, and f is open if and only if f' has a continuous inverse and thus is an isomorphism of topological groups. If among the conditions formulated for the domain group G are those that are preserved by passage to quotients, then this simple remark reduces the search for Open Mapping Theorems to the search for conditions on a bijective morphism $f: G \to H$ to be an isomorphism of topological groups.

Example 0.1. If G is the additive group of real numbers with the discrete topology, and $H = \mathbb{R}$, this same group with its natural topology, then the identity map gives us an example of a bijective morphism between real abelian Lie groups which fails to be open. □

There is a considerable body of literature, notably in functional analysis on Open Mapping Theorems and their corollaries. Let us survey what is known on Open Mapping Therems for topological groups. In the classical situation we can often cite references for the proofs. The following is elementary:

0.2. Lemma. *A surjective morphism $f: G \to H$ of topological groups is open if and only if G has arbitrarily small identity neighborhoods U such that the interior of $f(U)$ is not empty.*

Proof. See for instance [5]. □

We shall agree that all topological groups considered here are Hausdorff spaces.

1. Countability at Work

In all classical Open Mapping Theorems some sort of countability is present. We shall observe this throughout the first part of our discourse.

Topological Vector Spaces

Standard texts on functional analysis have the following

Proposition 1.1. (The Classical Open Mapping Theorem) *Let $f\colon G \to H$ be a continuous linear map between complete first countable topological vector spaces, then f is open.*

Lie Groups

By a *Lie group* we understand a real Lie group as modelled on a Banach manifold as defined by Bourbaki [2], Ch.III, §1, n$^{\rm o}$ 1, Définition 1, or in textbooks or monographs like for instance [6].

1.2. Theorem. (The Open Mapping Theorem for Lie Groups) *A surjective morphism $f\colon G \to H$ of topological groups between connected Lie groups is open if G is separable.*

Proof. There is a commuting diagram of continuous functions

$$\begin{array}{ccc} \mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(f)} & \mathfrak{L}(H) \\ {\scriptstyle \exp_G}\downarrow & & \downarrow{\scriptstyle \exp_H} \\ G & \xrightarrow[f]{} & H. \end{array}$$

The Lie algebras are Banach spaces with respect to suitable norms and $\mathfrak{L}(f)$ is an operator between Banach spaces. The exponential functions implement local homeomorphisms at zero. Therefore, in view of Lemma 0.2, f is open if and only if $\mathfrak{L}(f)$ is open. By the Classical Open Mapping Theorem 1.1, this is the case iff $\mathfrak{L}(f)$ is surjective. This in turn is true if every one-parameter subgroup of H lifts, that is, if for every morphism $Y\colon \mathbb{R} \to H$ there is a morphism $X\colon \mathbb{R} \to G$ such that $Y = f \circ X$. Suppose that this fails for some Y, then an elementary argument shows that one finds an open identity neighborhood U of G and a $\delta > 0$ such that for $0 \le r < s < \delta$ one has $Y(r)f(U) \cap Y(s)f(U) = \emptyset$ (see e.g. [6], 5.52). Then $\{f^{-1}(Y(r))U : 0 \le r < \delta\}$ is an uncountable set of nonempty disjoint open subsets of G, contradicting the separability of G. □

Example 1.3. Let $G = \ell^1(\mathbb{R}_d)$ the Banach space of all real tuples $(x_r)_{r\in\mathbb{R}}$ such that the family $(|x_r|)_{r\in\mathbb{R}}$ is summable and let this sum designate the norm. The tuples

$$e_r = (\delta_{rs})_{s\in\mathbb{R}}, \quad \delta_{rs} = \begin{cases} 1 & \text{if } r = s, \\ 0 & \text{otherwise,} \end{cases}$$

generate a free discrete subgroup. In the product Banach space $G\times\mathbb{R}$ the subgroup D generated by the family $\{(e_r, -r) \mid r \in \mathbb{R}\}$ is discrete and free. Set $H = \frac{E\times\mathbb{R}}{D}$

and define $f: G \to H$, $f(x) = (x, 0) + D$. Then f is a continuous surjective homomorphism of Lie groups. We may identify $\mathfrak{L}(G)$ with G and $\exp_G$ with the identity functions; similarly we may take $\mathfrak{L}(H) = G \times \mathbb{R}$ and for $\exp_H$ the quotient map. Then $\mathfrak{L}(f): G \to G \times \mathbb{R}$ is given by $\mathfrak{L}(f)(x) = (x, 0)$ and is not surjective. The one-parameter group $t \mapsto (0, t) + D : \mathbb{R} \to H$ does not lift. The morphism f fails to be open. □

Polish Groups

In the presence of completeness of G and a countability condition on G, Lemma 0.2 can be strengthened in a significant way:

1.4. Lemma. *If G is a complete first countable topological group then a surjective morphism $f: G \to H$ is open if and only if G has arbitrarily small identity neighborhoods U such that the interior of $\overline{f(U)}$ is not empty.*

Proof. One adjusts a portion of the proof of the Classical Open Mapping Theorem to this situation. This is done for instance in the lecture notes [5], Theorem 3.2.□

Let us call a space *inexhaustible* if it cannot be a countable union of nowhere dense subsets. Certainly any Baire space is inexhaustible and by the Baire category theory any locally compact space and any locally completely metrizable space is inexhaustible. (See for instance [1], Chap. IX, §5, no 3, Theorem 1) A topological space is called *Polish* if it is completely metrizable and has a countable basis for its topology. With the aid of Lemma 1.4 one then obtains

1.5. Theorem. *A surjective morphism $f: G \to H$ of topological groups is open if G is Polish and H is inexhaustible. In particular, a surjective morphism of Polish groups is open.* □

Notice that the Lie groups G and H of Example 1.3 are completely metrizable but fail to be Polish. Also observe, that the Classical Open Mapping Theorem 1.1 is not implied by Theorem 1.5.

Compactness and the Open Mapping Theorem

A topological space is called *σ-compact* if it is a countable union of compact subsets. An inexhaustible σ-compact group is locally compact.

1.6. Theorem. *A surjective morphism $f: G \to H$ is open if G is σ-compact and H is inexhaustible.*

Proof. (H. Glöckner) Since a surjective continuous image of a σ-compact is σ-compact is suffices to consider bijective f. Let $G = \bigcup_{n=1}^{\infty} C_n$ for compact subsets C_n. Then $H = \bigcup_{n=1}^{\infty} f(C_n)$ and the $f(C_n)$ are compact, hence closed. Since H is inexhaustible there is an N such that $f(C_N)$ has an inner point h. The restriction and corestriction $f|C_N : C_N \to f(C_N)$ is a homeomorphism as C_N is compact. Let $j\colon C_N \to G$ be the inclustion. Now $f^{-1}|f(C_N) = j \circ f(C_N)^{-1} : f(C_N) \to G$ is continuous, and $f(C_N)$ is a neighborhood of $h \in H$. Thus f^{-1} is a homomorphism which is continuous at a point of its domain and is therefore continuous. □

Usually the texts require that G is locally compact such as [3], p. 42, Theorem 5.29, but Glöckner observed that this is not necessary. Of course from hindsight it follows from the hypotheses of Theorem 1.6 that $G/\ker f \cong H$ is necessarily locally compact.

1.7. Corollary. (The Open Mapping Theorem for Locally Compact Groups) *A surjective morphism $f\colon G \to H$ of locally compact groups is open if G is σ-compact.* □

Example 0.1 shows that this fails without the countability condition of σ-compactness. The identity morphism of the discrete additive group of rationals onto $\mathbb{Q}$ with its natural topology shows that a continuous bijective morphism from a σ-compact locally compact to a metrizable topological group need not be open unless H is inexhaustible. The dual of a Banach space is σ-compact in the weak star topology by the Theorem of Alaoglu–Banach–Bourbaki; by Theorem 1.6 therefore it is inexhaustible iff it is finite dimensional.

2. Projective Limits of Lie Groups

All of the above theorems rest on countability conditions. This is different for the Open Mapping Theorem for pro-Lie groups which we need to describe. They form a wide class of mostly infinite dimensional topological groups with an effective Lie theory.

Recall that a filter on a topological group G is called a *Cauchy filter* if for every identity neighborhood U there is a set F in the filter such that $F^{-1}F \subseteq U$. The group G is called *complete* if every Cauchy filter converges.

Proposition 2.1. *For a topological group G the following statements are equivalent:*

(i) *There is a projective system $\{f_{jk}\colon G_k \to G_j | j \le k, (j,k) \in J \times J\}$ of finite dimensional Lie groups such that $G = \lim_{j\in J} G_j$.*

(ii) *G is isomorphic as a topological group to a closed subgroup of a product $\prod_{j\in J} G_j$ of finite dimensional Lie groups.*

(iii) *G is complete and each identity neighborhood of G contains a normal subgroup N such that G/N is a finite dimensional Lie group.*

(See [8], or [7], Chapter 3. The proof of the implication "anything else $\Longrightarrow$(iii)" is nontrivial.)

Definition 2.2. A topological group satisfying the equivalent conditions of Proposition 2.1 is called a *pro-Lie group.* If every identity neighborhood of G contains a normal subgroup N such that G/N is a finite dimensional Lie group, then G is called a *proto-Lie group.* □

Accordingly, every pro-Lie group is a proto-Lie group, and a proto-Lie group is a pro-Lie group if and only if it is complete. To see this note that, for every proto-Lie group G, the set $\mathcal{N}(G)$ of closed normal subgroups of G such that G/N is a finite dimensional Lie group is a filter basis such that there is a natural embedding morphism $\gamma_G: G \to G_{\mathcal{N}(G)} \stackrel{\rm def}{=} \lim_{N\in\mathcal{N}(G)} G/N$ with dense image, and $G_{\mathcal{N}(G)}$ is a pro-Lie group and the completion of G ([7], Theorems 4.1). We shall also write the completion of G as $\overline{G}$, notably when we consider G as a dense subgroup of its completion. If G is a topological group and N a complete normal subgroup such that G/N is complete as well then G is complete (see for instance [15], p. 225, 12.3); therefore if a proto-Lie group G fails to be a pro-Lie group then none of the normal subgroups $N \in \mathcal{N}(G)$ is complete. Every closed subgroup of a pro-Lie group is a pro-Lie group as follows easily from Proposition 2.1. A Lie group is a proto-Lie group if and only if it is finite dimensional. If J is any set, the power $\mathbb{R}^J$ is a pro-Lie group. If J is infinite, then this group fails to be locally compact, and if J is uncountable, $\mathbb{R}^J$ fails to be metric. All locally compact abelian groups are pro-Lie groups. A topological group G is called *almost connected*, if the factor group G/G_0 modulo its identity component is compact. All almost connected locally compact groups are pro-Lie groups [16], [17], see also [14], p. 175.

Example 2.3. There is a pro-Lie group topology on the free abelian group $\mathbb{Z}^{(\mathbb{N})}$ of countably many generators making it into a nondiscrete pro-Lie group H ([12], [9], and [7], Chapter 5). If G denotes $\mathbb{Z}^{(\mathbb{N})}$ with the discrete topology, then the identity morphism $f: G \to H$ is a bijective morphism of pro-Lie groups that is not open where G is σ-compact and H is countable. □

After Example 2.3 it is hopeless to expect any Open Mapping Theorem for Pro-Lie Groups to arise from Baire category arguments directly, because a countable homogeneous Baire space is necessarily discrete. One therefore expects the following result to be nontrivial: This will be borne out by the proof.

Theorem 2.4. (The Open Mapping Theorem for Pro-Lie Groups) *A surjective morphism $f: G \to H$ of pro-Lie groups is open if G is almost connected.*

Example 2.5. Let C be the compact character group of the discrete abelian group $\mathbb{Z}^{\mathbb{N}}$, that is if $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, then $C = \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{T})$. Let $G = \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R}) \cong \mathbb{R}^{2^{\aleph_0}}$ and define $\mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, p): G \to C$, $p: \mathbb{R} \to \mathbb{T}$ being the quotient morphism, to be the induced map. Then the image of this morphism is the arc component H of zero in C. So the corestriction $f: G \to H$ is a surjective open morphism from a pro-Lie group onto an incomplete proto-Lie group. (See [12], or [8], Chapter 4.) □

Thus, unfortunately, a quotient group of a pro-Lie group need not be a pro-Lie group. This makes a reduction of the problem of analyzing a surjective morphism

$f: G \to H$ via canonical decomposition $f = f' \circ q$ with a quotient morphism $q: G \to G/\ker f$ and a *bijective* morphism $f': G/\ker f \to H$ very tricky.

The remainder of this article is devoted to a proof of this theorem. An earlier version of a proof is about to appear in [11]. The first part of the present proof will show that $G/\ker f$ is a pro-Lie group and the second part will prove the Open Mapping Theorem for bijective morphisms between connected pro-Lie groups.

3. Divibility of Groups and Connected Pro-Lie Groups

For what is to follow soon we need a new line of algebraic properties of pro-Lie groups and groups in general. Recall that a *divisible group* is one in which the equation $x^n = g$ has a solution for every group element g and every natural number n. Clearly homomorphic images of divisible groups are divisible.

3.1. Proposition. (i) *Let g be an element of a divisible group G. Then there is a group homomorphism $f: \mathbb{Q} \to G$ such that $f(1) = g$. Accordingly, g is contained in the divisible abelian subgroup $f(\mathbb{Q})$ of G.*

(ii) *Homomorphic images of $\mathbb{Q}$ are either singleton or infinite.*

(iii) *A finite group has no divisible subgroups other than the singleton one.*

Proof. (i) Using divisibility, recursively define elements $g_1 = g, g_2, \ldots$ such that $g_n^n = g_{n-1}$, $n = 2, 3, \ldots$. Every rational number $q \in \mathbb{Q}$ can be written in the form $q = \pm m/n!$. The function $f: \mathbb{Q} \to G$ sending $q = \pm m/n!$ to $g_n^{\pm m}$ is well defined and satisfies the requirements.

(ii) Let S be a subgroup of $\mathbb{Q}$. If $S = \{0\}$ then $\mathbb{Q}/S \cong \mathbb{Q}$. Assume that S contains a member $s \neq 0$; then $x \mapsto s^{-1}x : \mathbb{Q} \to \mathbb{Q}$ is an automorphism and $\mathbb{Q}/S \cong \mathbb{Q}/s^{-1}S$. Moreover, $1 = s^{-1}s \in s^{-1}S$. We may and will assume that $1 \in S$. Then $\mathbb{Q}/S \cong (\mathbb{Q}/\mathbb{Z})/(S/\mathbb{Z})$ is a homomorphic image of $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)$ where $\mathbb{Z}(p^\infty) = \{m/p^n : m \in \mathbb{Z}, n \in \mathbb{N}\}/\mathbb{Z}$ is the *Prüfer group* for the prime p. Since this is the decomposition into p-primary components or Sylow groups, $S/\mathbb{Z} = \bigoplus_{p \text{ prime}} S_p$ for $S_p = (S/\mathbb{Z}) \cap \mathbb{Z}(p^\infty)$, and

$$\mathbb{Q}/S \cong \bigoplus_{p \text{ prime}} \mathbb{Z}(p^\infty)/S_p.$$

Show that a quotient group of $\mathbb{Z}(p^\infty)$ is either singleton or isomorphic to $\mathbb{Z}(p^\infty)$. Thus $\mathbb{Q}/S$ is either singleton or infinite as a direct sum of Prüfer subgroups.

(iii) is a consequence of (i) and (ii). □

The structure of abelian divisible groups is completely known: See for instance [6], Appendix, 1, Theorem A1.42.

In any compact connected group, every element is contained in a maximal pro-torus (see for instance [6], Theorem 9.32). Compact connected abelian groups are divisible (see e.g. [6], Corollary 8.5). Therefore if a compact group G is connected, that is $G = G_0$, then it is divisible. The finite quotients of G/G_0 separate the points; thus if $G \neq G_0$, by Proposition 3.1(iii), G fails to be divisible. Therefore we have the

Fact. *A compact group is connected iff it is divisible.*

(For compact abelian groups see [6], Theorem 8.4.) The additive group of the field $\mathbb{Q}_p$ of p-adic numbers is a nondiscrete locally compact noncompact divisible group. By [7], Lemma 5.12, and the Fact above, every connected abelian pro-Lie group is divisible.

For an arbitrary (not necessarily topological) group G, let us denote by $D(G)$ the subgroup that is algebraically generated by all divisible subgroups. Then clearly

for any group homomorphism $f\colon G \to H$ the containment $f\big(D(G)\big) \subseteq D(H)$ holds.

In particular, recalling that a subgroup of a group is said to be *fully characteristic* if every endomorphism maps it into itself we have the following observation:

for any group G the subgroup $D(G)$ is fully characteristic.

This is understood in the sense of the category of groups; that is, no continuity is involved. However, let us now consider a connected pro-Lie group G with Lie algebra $\mathfrak{g}$. The subgroup $A(G,\mathfrak{g}) \subseteq G$ algebraically generated by the set $\exp_G \mathfrak{g}$ of all one-parameter subgroups (called the *minimal analytic subgroup with Lie algebra* $\mathfrak{g} = \mathfrak{L}(G)$ in [7], Chaptr 9) satisfies $D\big(A(G,\mathfrak{g})\big) = A(\mathfrak{g}, G)$.

We need several pieces of information from the theory of pro-Lie groups [7].

A. Lemma. *Let G be a connected pro-Lie group. Then $G = D(G)$. That is, G is generated by its divisible subgroups.*

Proof. By Theorem [7], 12.65 there is a compact connected abelian, hence divisible, subgroup C of G such that $G = C{\cdot}A(G,\mathfrak{g})$. Since $A(G,\mathfrak{g})$ is generated by its one parameter subgroups, G is generated by divisible subgroups. □

3.2. Proposition. *Let G be a connected pro-Lie group. Then G has no subgroups of finite index.*

Proof. By Lemma A, we have $G = D(G)$ and so for any homomorphism $f\colon G \to H$ the image $f(G)$ is contained in $D(H)$. If N is any normal subgroup of G, closed or not, then $G/N = D(G/N)$. If G/N is finite, then $D(G/N) = \{N\}$ and so $N = G$. If H is any subgroup of G of finite index, say, $G = g_1H\dot\cup\cdots\dot\cup g_nH$, then $N = \bigcap_{m=1}^{n} g_mHg_m^{-1}$ is a normal subgroup of finite index, and thus must equal G by what we saw. So $H = G$. □

The proof shows that G cannot contain any proper normal subgroup N such that G/N does not contain any nontrivial divisible subgroups.

3.3. Proposition. *Let C be a closed central totally disconnected subgroup of a connected pro-Lie group L. If G is any subgroup of L such that $L = CG$ and $C \cap G = \{1\}$. Then $G = L$.*

Proof. Let $N \in \mathcal{N}(L)$. Then $\overline{CN}/N$ is a closed central subgroup of the finite dimensional Lie group L/N. By the Closed Subgroup Theorem for Projective Limits [7] 1.34(i), C is canonically isomorphic to $\lim_{N\in\mathcal{N}(L)} \overline{CN}/N$ where $\overline{CN}/N$ is discrete since C is prodiscrete. Also, if L is connected, then $\overline{CN}/N$ is a closed central subgroup of the connected finite dimensional Lie group L/N. Hence it is finitely generated abelian, and so CN/N is finitely generated abelian, that is, it is isomorphic to a direct product of finitely many cyclic groups. By Theorem 1.34(iv) of [7] we know that $C/(C\cap N) \cong CN/N$ as topological groups, and so there is a closed subgroup $B_N = C\cap N$ of C such that C/B_N is finitely generated discrete, and $\lim_{N\in\mathcal{N}(L)} B_N = \lim_{N\in\mathcal{N}(L)} N = 1$.

Now any direct product of nondegenerate cyclic groups is profinite, that is, the subgroups of finite index separate the points.

Finally we suppose that $C \neq \{1\}$ and derive a contradiction. From the preceding paragraph we get a subgroup B of C such that C/B is finite.

Since L is algebraically the direct product $C{\cdot}G$ and B is contained in C, the factor group L/B is algebraically the direct product $(C/B){\cdot}(GB/B)$ Thus the connected pro-Lie group L has a normal subgroup GB of index $|C/B|$. This is impossible by Proposition 3.2 above. □

We introduce a technical, but convenient terminology:

Definition 3.4. A proto-Lie group P *has a stable Lie algebra* if $\mathfrak{L}(\overline{P}) = \mathfrak{L}(P)$ for its completion $\overline{P}$.

This applies to pro-Lie groups as follows:

B. Lemma. *Let $f\colon G \to H$ be a quotient morphism of topological groups. If G is a pro-Lie group then H is a proto-Lie group and the canonical embedding*

$$\gamma_H\colon H \to \overline{H} = H_{\mathcal{N}(H)} = \lim_{N\in\mathcal{N}(H)} H/N$$

induces an isomorphism

$$\mathfrak{L}(\gamma_H)\colon \mathfrak{L}(H) \to \mathfrak{L}(\overline{H})$$

of pro-Lie algebras. In particular $\mathfrak{L}(H)$ is a pro-Lie algebra.

Proof. [7], 4.20(i_0). □

It should be clear that this will be a crucial point in the proof of the Open Mapping Theorem 2.4. The following is now an important technical step in that proof of the which we shall attack presently.

3.5. Corollary. *Assume that G is a proto-Lie group with a stable Lie algebra and assume that its completion $\overline{G}$ is connected. If there is a bijective morphism $f\colon G \to H$ onto a complete topological group H, then G is complete, that is, G is a pro-Lie group.*

Proof. Consider G as a subgroup of its completion $\overline{G}$. The continuous morphism $f: G \to H$ into the complete group has a unique extension to a morphism $F: \overline{G} \to H$ of complete topological groups by the universal property of the completion. Let $C = \ker F$. Since $\mathfrak{L}$ preserves kernels by [7] Theorem 2.20 or 4.20(ii), $\mathfrak{L}(C) = \ker \mathfrak{L}(F) = \{X \in \mathfrak{L}(\overline{G}) : (\forall t \in \mathbb{R}) F(X(t)) = 1\} = \{X \in \mathfrak{L}(G) : (\forall t \in \mathbb{R}) f(X(t)) = 1\} = \{0\}$ since $\mathfrak{L}(\overline{G}) = \mathfrak{L}(G)$, $F|G = f$, and f is injective. Thus C, being a pro-Lie group as a closed subgroup of $\overline{G}$ by the Closed Subgroup Theorem for Pro-Lie Groups [7] 3.35, is totally disconnected by [7] Proposition 3.30. Since $\overline{G}$ is connected, C is central. Now f is bijective, and thus if $j: G \to \overline{G}$ is the inclusion morphism, then $\varphi = j \circ f^{-1} \circ F: \overline{G} \to G$ defines an algebraic endomorphism of $\overline{G}$ satisfying $\varphi^2 = \varphi$ such that $C = \ker \varphi$ and $G = \operatorname{im} \varphi$. Then $G = CG$ and $C \cap G = \{1\}$. Thus Proposition 3.3 shows $G = \overline{G}$. □

Clearly, the completion $\overline{G}$ of a topological group G is connected if G is connected, but $\mathbb{Q}$ in the induced topology of $\mathbb{R}$ a totally disconnected topological group, whose completion $\mathbb{R}$ is connected.

It was for this result and its consequences that we had to find an algebraic property of connected pro-Lie groups, such as being generated by divisible subgroups and thus having no finite algebraic homomorphic images. But now we improve it by considering *almost connected* proto-Lie and pro-Lie groups in place of connected ones;

3.6. Corollary. *Assume that G is an almost connected proto-Lie group with a stable Lie algebra. If there is a bijective morphism $f: G \to H$ onto a pro-Lie group H, then G is a pro-Lie group and $f(G_0) = H_0$.*

Moreover, f is an isomorphism of pro-Lie groups if $f|G_0 : G_0 \to H_0$ is open.

Proof. Let $P = f^{-1}(H_0)$. Then $G_0 \subseteq P$. Thus P is an almost connected proto-Lie group and $f|P : P \to H_0$ is surjective since f is surjective. We claim that P is connected and therefore equals G_0. As P is almost connected, the factor group P/G_0 is profinite. So if $P \neq G_0$, then P has an open normal proper subgroup Q of finite index. Since f is surjective, $f(Q)$ is a normal subgroup of finite index of H_0. Now H_0 is a closed subgroup of H which is assumed to be a pro-Lie group. Thus H_0 is a connected pro-Lie group. By Proposition 3.2, this implies $f(Q) = H_0$. Since f is also injective, $Q = P$ follows, in contradiction to the assumption that Q is a proper subgroup of P. Thus $G_0 = f^{-1}(H_0)$ as asserted.

We note that $\overline{G_0} \subseteq \overline{G}$ and $\mathfrak{L}(G_0) = \mathfrak{L}(G) = \mathfrak{L}(\overline{G})$, whence $\mathfrak{L}(G_0) = \mathfrak{L}(\overline{G_0})$. Now $f|G_0 : G_0 \to H_0$ is a bijective morphism, H_0 is complete, and G_0 is a connected proto-Lie group with stable Lie algebra. Hence Corollary 3.5 applies to $f|G_0$ and shows that G_0 is a pro-Lie group. We noted earlier that a topological group T with a a complete normal subgroup N such that T/N is complete is itself complete (see [15], p. 225, 12.3). Therefore, since G/G_0 is compact and thus complete, G is a pro-Lie group. Hence $f: G \to H$ is a bijective morphism between almost connected groups such that $f|G_0 : G_0 \to H_0$ is a bijective morphism between connected pro-Lie groups.

Now we assume that $f|G_0 : G_0 \to H_0$ is open and show that f is open. For a proof let U be an open identity neighborhood of G; we must show that $f(U)$ is open in H. First we find a normal subgroup $N \in \mathcal{N}(G)$ contained in U, and, by making U smaller if needed, we do not restrict generality by assuming that $UN = U$. (See e.g. [7] 1.27(i).) Observe that G_0N/N is an analytic subgroup of the finite dimensional Lie group G/N whose Lie algebra agrees with $\mathfrak{L}((G/N)_0)$ and which therefore is $(G/N)_0$, whence $G_0/(G_0 \cap N) \cong G_0N/N$ is a finite dimensional Lie group. Thus $M \stackrel{\rm def}{=} N \cap G_0 \in \mathcal{N}(G_0)$. Hence G_0/M is a finite dimensional Lie group and G/G_0 is compact since G is almost connected. So the factor group G/M is a locally compact almost connected and therefore σ-compact group. Since $f|G_0 : G_0 \to H_0$ is an isomorphism, $H_0/f(M)$ is isomorphic to G_0/M and is therefore a finite dimensional Lie group, whence $H/f(M)$ is locally compact. Thus the induced map $f_M: G/M \to H/f(M)$ is open by the Open Mapping Theorem for Locally Compact Groups 1.7. Thus $f_M(U/M) = f(U)/f(M)$ is open in $H/f(M)$ and so $f(U)$ is open in H as asserted. So f is open and therefore an isomorphism.□

From our discussion of finite-dimensional pro-Lie groups we can now turn the Open Mapping Theorem for Pro-Lie Groups 2.4.

3.7. Lemma. *Assume that $f: G \to H$ is a surjective morphism of pro-Lie groups and that G is almost connected. Then the following conclusions hold:*
(i) *The quotient group $G/\ker f$ is a pro-Lie group, and the induced bijective morphism $f': G/\ker f \to H$ maps $(G/\ker f)_0$ bijectively onto H_0.*
(ii) *f is open if the induced bijective morphism f' induces an open morphism $(G/\ker f)_0 \to H_0$ between connected pro-Lie groups.*

Proof. By Lemma B, the quotient $P = G/\ker f$ is a proto-Lie group by and has a stable Lie algebra. Now Corollary 3.6 applies to $f': G/\ker f \to H$ and proves claims (i) and (ii). □

This Lemma reduces the Open Mapping Theorem for surjective morphisms between almost connected pro-Lie groups to the Open Mapping Theorem for bijective morphisms between connected pro-Lie groups.

The following observation rounds off our general orientation on almost connected groups.

3.8. Proposition. *Let $f: G \to H$ be a morphism of topological groups and assume, firstly, that G is almost connected and, secondly, that $f(G)$ is dense in H. Then H is almost connected.*

Proof. Since $f(G_0)$ is connected and contains the identity, $f(G_0) \subseteq H_0$ and therefore the morphism $\varphi: G/G_0 \to H/H_0$, $\varphi(gG_0) = f(g)H_0$ is well-defined. By assumption G/G_0 is compact. Thus the continuous image $\varphi(G/G_0)$ is compact and therefore, since H/H_0 is Hausdorff, is closed in H/H_0. Since $f(G)$ is dense in

H, it follows that $\varphi(G/G_0)$ is dense in H/H_0. Therefore, $H/H_0 = \varphi(G/G_0)$, and so H/H_0 is compact. □

4. The Open Mapping Theorem for Bijective Morphisms

In this section we shall prove 2.4: The Open Mapping Theorem for Almost Connected Pro-Lie Groups saying that a surjective morphism between pro-Lie groups is open if its domain group is almost connected.

For a proof let $f: G \to H$ be a surjective morphism between pro-Lie groups and assume that G is almost connected. We must show that f is open. By Lemma 3.7 it is no loss of generality to assume that both G and H are connected and that f is bijective. We will do this from now on and prove the theorem through a sequence of steps.

⟨1⟩ *Claim*: G and H have the same Lie algebra $\mathfrak{g}$.

The morphism $\mathfrak{L}(f): \mathfrak{L}(G) \to \mathfrak{L}(H)$ is injective as $\mathfrak{L}$ preserves kernels by [7], Theorem 2.20; it is surjective by Corollary 4.22(ii) and so is an isomorphism by [7] Theorem A2.12(b) in Appendix 2. We may therefore set $\mathfrak{g} = \mathfrak{L}(G) = \mathfrak{L}(H)$ and keep the commutative diagram

$$(*)\qquad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \mathrm{id}\ } & \mathfrak{g} \\ {\scriptstyle \exp_G}\downarrow & & \downarrow{\scriptstyle \exp_H} \\ G & \xrightarrow[\ f\ ]{} & H \end{array}$$

in mind.

⟨2⟩ *Claims:* (i) $(\forall N \in \mathcal{N}(H)) f^{-1}(N) \in \mathcal{N}(G)$, (ii) f induces an isomorphism $G/f^{-1}(N) \to H/N$, and (iii) $\mathfrak{L}(f^{-1}(N)) = \mathfrak{L}^{-1}(f^{-1}(N)_0) = \mathfrak{n} \stackrel{\rm def}{=} \mathfrak{L}(N)$.

The structural invariants of G and H are the filters $\mathcal{N}(G)$ and $\mathcal{N}(H)$, respectively. Consider an $N \in \mathcal{N}(H)$ and $\mathfrak{n} \stackrel{\rm def}{=} \mathfrak{L}(N) \in \mathcal{I}(\mathfrak{g})$, the filter basis of all cofinite-dimensional closed ideals of $\mathfrak{g}$. Let V be an open identity neighborhood of H containing N such that $VN = V$ and V/N contains no subgroups other than the singleton one. Since f is continuous, $U \stackrel{\rm def}{=} f^{-1}(V)$ is an identity neighborhood of G. Set $M \stackrel{\rm def}{=} f^{-1}(N)$. Then $UM = U$ and UM/M contains no subgroup other than the singleton one. Since $\lim \mathcal{N}(G) = 1$ in G there is a $P \in \mathcal{N}(G)$ such that $P \subseteq U$. Then PM/M is a subgroup of G/M contained in U/M and thus agrees with M/M, that is, $PM = M$, and so $P \subseteq M$. Therefore we have a surjective morphism

$$f_{NP}: G/P \to H/N, \quad f_{NP}(gP) = f(g)N$$

of topological groups. By the definition of $\mathcal{N}(G)$ we know that G/P is a finite dimensional Lie group; it is connected since G is connected. Thus G/P is a locally compact σ-compact group. Also, H/N is a finite dimensional Lie group since $N \in \mathcal{N}(H)$. Hence it is locally compact. Thus the Open Mapping Theorem for Locally Compact Groups 1.7 applies and shows that f_{NP} is open. Since $M/P = \ker f_{NP}$, we know that $G/M \cong (G/P)/(M/P) \cong H/N$ is a Lie group. Therefore

$M \in \mathcal{N}(G)$. Thus

$$f^{-1}(\mathcal{N}(H)) \subseteq \mathcal{N}(G)$$

and $f_N: G/f^{-1}N \to H/N$, $f_N(gf^{-1}(N)) \stackrel{\rm def}{=} f(g)N$ is an isomorphism. From Corollary 4.21(i) we know that $\mathfrak{g}/\mathfrak{L}(f^{-1}(N)) \cong \mathfrak{L}(G/f^{-1}(N)) = \mathfrak{L}(H/N) \cong \mathfrak{g}/\mathfrak{n}$ It now follows from $(*)$ that $\mathfrak{L}(f^{-1}(N)) = \mathfrak{n}$. We shall abbreviate $(f^{-1}(N))_0$ by $f^{-1}(N)_0$ and recall $\mathfrak{L}(f^{-1}(N)_0) = \mathfrak{n}$.

$\langle 3\rangle$ *Claim:* $\{f^{-1}(N)_0 : N \in \mathcal{N}(H)\}$ is cofinal in $\{M_0 : M \in \mathcal{N}(G)\}$.

From $(*)$ and [7] Corollary 4.21(ii) we know that in the filter basis $\mathcal{I}(\mathfrak{g})$ of cofinite dimensional closed ideals of $\mathfrak{g}$ both of the filterbases $\mathcal{I}_G = \{\mathfrak{L}(M) : M \in \mathcal{N}(G)\}$ and $\mathcal{I}_H = \{\mathfrak{L}(N) : N \in \mathcal{N}(H)\}$ are cofinal in $\mathrm{id}\,\mathfrak{g}$; we shall use the cofinality of the latter. If $M \in \mathcal{N}(G)$, then $\mathfrak{m} \stackrel{\rm def}{=} \mathfrak{L}(M) \in \mathcal{I}_G$ and $M_0 = \overline{\langle \exp_G \mathfrak{m}\rangle}$ by Corollary 4.22(i) of [7]. Since $\mathcal{I}_H$ is cofinal in $\mathcal{I}(\mathfrak{g})$ there is an $N_M \in \mathcal{N}(H)$ such that $\mathfrak{L}(f^{-1}(N_M)) \subseteq \mathfrak{m}$. Consequently, using [7], 4.22(i) again we get $f^{-1}(N_M)_0 = \overline{\langle \exp_G \mathfrak{n}\rangle} \subseteq \overline{\langle \exp_G \mathfrak{m}\rangle} = M_0$, more specifically,

$$(\forall M \in \mathcal{N}(G))(\exists N_M \in \mathcal{N}(H)) f^{-1}(N_M)_0 \subseteq M_0.$$

$\langle 4\rangle$ *Claim:* $G \cong \lim_{N\in\mathcal{N}(H)} G/f^{-1}(N)_0$, that is, we have a limit representation of G indexed by $\mathcal{N}(H)$.

By $\langle 3\rangle$, the filterbasis $\{f^{-1}(N)_0 : N \in \mathcal{N}(H)\}$ is cofinal in the filterbasis $\{M_0 : M \in \mathcal{N}(G)\}$. From [7], Corollary 9.45 we know that $G \cong \lim_{M\in\mathcal{N}(G)} G/M_0$. Thus by the Cofinality Lemma 1.21 of [7],

$$\gamma: G \to \lim_{N\in\mathcal{N}(H)} G/f^{-1}(N)_0, \quad \gamma(g) = (gf^{-1}(N)_0)_{N\in\mathcal{N}(H)}$$

is an isomorphism.

$\langle 5\rangle$ *Claim:* $(\forall N \in \mathcal{N}(H)) f^{-1}(N_0) = f^{-1}(N)_0$.

This is a subtle but important point. Abbreviate $f^{-1}(N)$ by M. The bijective morphism f induces a bijective morphism $f|M: M \to N$ between pro-Lie groups; it clearly maps M_0 into N_0, and we claim that it maps M_0 onto N_0. Let $P \stackrel{\rm def}{=} f^{-1}(N_0) \subseteq M$, then $M_0 = f^{-1}(N)_0 \subseteq P$ and we have to show equality. From [7], Corollary 9.45(iii) it follows that $P/M_0 \subseteq M/M_0$ is isomorphic to a direct product $C \times \mathbb{Z}^n$ for a compact totally disconnected abelian group C and a discrete free group of rank n. These groups are residually finite, that is, the finite homomorphic images separate the points. Now suppose that $P \neq M_0$, then P contains a normal subgroup Q of finite positive index (containing M_0). But then $f(Q)$ is a normal subgroup of $f(P) = N_0$ of finite positive index since f is bijective. But this contradicts Proposition 3.2. Thus $P = M_0$ and the claim is proved.

$\langle 6\rangle$ *Claim: The bijection f induces a natural isomorphism of topological groups* $\alpha: \lim_{N\in\mathcal{N}(H)} G/f^{-1}(N)_0 \to \lim_{N\in\mathcal{N}(H)} H/N_0$.

Let $N \in \mathcal{N}(H)$. Then $f^{-1}(N)_0 = f^{-1}(N_0)$ by $\langle 5\rangle$, and thus f induces a bijective morphism $f^{-1}(N)_0 \to N_0$ and then also a bijective morphism

$$\alpha_N: G/f^{-1}(N)_0 \to H/N_0, \quad \alpha_N(gf^{-1}(N)_0) = f(g)N_0.$$

Since $f^{-1}(N) \in \mathcal{N}(G)$ by $\langle 2\rangle$ above, the factor group $G/f^{-1}(N)_0$ is locally compact by Corollary 9.45 of [7], as is the factor group H/N_0. Since the group G is connected, $G/f^{-1}(N)_0$ is σ-compact. The Open Mapping Theorem for Locally Compact Groups applies and shows that α_N is an isomorphism for each $N \in \mathcal{N}(H)$. This gives us an isomorphism

$$\alpha\colon \lim_{N\in\mathcal{N}(H)} G/f^{-1}(N)_0 \to \lim_{N\in\mathcal{N}(H)} H/N_0,$$
$$\alpha\big((g_N f^{-1}(N)_0)_{N\in\mathcal{N}(H)}\big) = (f(g_N)N_0)_{N\in\mathcal{N}(H)}$$

which is represented in the following diagram:

$$\begin{array}{ccc} G/f^{-1}(N)_0 & \xleftarrow{\mu_N} & \lim\limits_{Q\in\mathcal{N}(H)} G/f^{-1}(Q)_0 \\ {\scriptstyle\alpha_N}\downarrow & & \downarrow{\scriptstyle\alpha} \\ H/N_0 & \xleftarrow[\nu_N]{} & \lim\limits_{Q\in\mathcal{N}(H)} H/Q_0 \end{array}$$

for the respective limit morphisms μ, ν.

$\langle 7\rangle$ *Claim*: f is an isomorphism of topological groups.

The function $\gamma\colon G \to \lim_{N\in\mathcal{N}(H)} G/f^{-1}(N)_0$, $\gamma(g) = (gf^{-1}(N)_0)_{N\in\mathcal{N}(H)}$ is an isomorphism by $\langle 4\rangle$, and $\gamma_{\mathcal{N}_0(H)}\colon H \to \lim_{N\in\mathcal{N}(H)} H/N_0$, $\mathcal{N}_0(H) = \{N_0 : N \in \mathcal{N}(H)\}$ is an isomorphism by Corollary 9.45(ii). By $\langle 6\rangle$, the map α is an isomorphism. We have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma_{\mathcal{N}_0(H)}} \\ \lim\limits_{N\in\mathcal{N}(H)} \frac{G}{f^{-1}(N)_0} & \xrightarrow[\alpha]{} & \lim\limits_{N\in\mathcal{N}(H)} \frac{H}{N_0}. \end{array}$$

It follows that f is an isomorphism.

And this last step completes the proof of Theorem 2.4. □

5. Immediate Conclusions

We record some immediate consequences; applications abound in [7].

5.1. Corollary. (Closed Graph Theorem for Pro-Lie Groups) *Assume that G and H are pro-Lie groups and that $f\colon G \to H$ is a morphism of groups (algebraically) with graph* graph$(f) \overset{\text{def}}{=} \{(x, f(x)) : x \in G\} \subseteq G \times H$. *Consider the following statements:*

(i) *f is continuous.*

(ii) *The graph* graph(f) *is closed in $G \times H$*

(iii) graph(f) *is closed in $G \times H$ and is almost connected.*

Then (iii)$\Longrightarrow$(i)$\Longrightarrow$(ii).

Proof. (i)$\Longrightarrow$(ii) is a consequence of the general fact that the graph of any continuous function into a Hausdorff space is closed. We must show that (iii) implies (i). We define $\gamma: G \to \mathrm{graph}(f)$ by $\gamma(x) = \big(x, f(x)\big)$ and decompose f as follows:

$$G \xrightarrow{\gamma} \mathrm{graph}(f) \xrightarrow{\mathrm{pr}_H \mid \mathrm{graph}(f)} H.$$

We see that f is continuous if γ is continuous. The continuity of γ is equivalent to the openness of $\gamma^{-1} = \mathrm{pr}_G \,|\, \mathrm{graph}(f)$. By (ii) graph($f$) is a closed subgroup of the pro-Lie group $G \times H$. By the Closed Subgroup Theorem of Pro-Lie Groups [7] 3.35, it is a pro-Lie group. Now the Open Mapping Theorem 2.4 applies to the continuous morphism $\mathrm{pr}_G \,|\, \mathrm{graph}(f) : \mathrm{graph}(f) \to G$ and shows that it is open. □

5.2. Corollary. (Second Isomorphism Theorem for Pro-Lie Groups) *Assume that a pro-Lie group G is a product of an almost connected closed normal subgroup N and an almost connected closed subgroup H. Then $\beta: H/(H \cap N) \to G/N$ is an isomorphism.*

Proof. By the Closed Subgroup Theorem [7], 3.35, the closed subgroup H of the pro-Lie group G is a pro-Lie group. By Theorem [7], 4.28(i), G/N is a pro-Lie group since both G and N are almost connected. Hence the function $f: H \to G$, $f(h) = hN$ is a surjective morphism of pro-Lie groups whose domain is almost connected by hypothesis. therefore, by the Open Mapping Theorem 2.4, the morphism f is open. We have $\ker f = H \cap N$, and if $q: H \to H/\ker f$ is the quotient morphism we have $f = \beta \circ q$ and β is open and therefore is an isomorphism. □

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