AN INTEGRATION FORMULA FOR POLAR ACTIONS

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ABSTRACT. We prove an analogue of Weyl's Integration Formula for compact Lie groups in the context of polar actions. We also show how certain classical examples from the literature can be viewed as special cases of our result.

1. THE INTEGRATION FORMULA

Let $\varphi : G \times M \to M$ be an smooth and proper isometric action action of a Lie group G on a Riemannian manifold M. φ is called *polar*, if there exists an embedded submanifold Σ of M which intersects with every G-orbit and is orthogonal to each orbit in the intersection points. Our main result is:

Theorem. *Let φ be a polar action and $\Sigma \subset M$ a section. We put $H := Z_G(\Sigma)$ and denote by $W = W(\Sigma)$ the generalized Weyl group of Σ. Furthermore, let $\omega_s : G/H \to M$, $gH \mapsto g \cdot s$ denote the 'orbit map' in the point $s \in M$ and assume W to be finite. Then the following holds:*

(i) *The function $\delta(s) := |\det(d\omega_s(eH))|$ on Σ is continuous and W-invariant. If G is compact, then δ is a volume scaling function in the following way:*

$$\delta(s) = \begin{cases} 0 & \text{if } s \text{ is singular,} \\ |G_s/H| \cdot \frac{\mathrm{vol}_l(G\cdot s)}{\mathrm{vol}_l(G/H)}. & \text{if } s \text{ is regular or exceptional,} \end{cases}$$

where $l := \dim(G/H)$ is the dimension of a principal orbit.

(ii) *The assignment $\Psi : f \in \mathcal{C}_c(M) \mapsto F \in \mathcal{C}_c(G/H \times \Sigma)^W$, given by*

$$F(gH, s) := f(g \cdot s)\,\delta(s),$$

extends to a continuous isomorphism from $L^1(M)$ onto $L^1(G/H \times \Sigma)^W$.

(iii) *For any $f \in L^1(M)$ we have the formula*

$$\int_M f(x)\,dx = \frac{1}{|W|}\int_\Sigma \left(\int_{G/H} f(g\cdot s)\,d(gH) \right) \delta(s)\,ds,$$

where $d(gH) = dg/dh$ is induced by the Haar measures dg on G, resp. dh on H. It coincides with the Riemannian measure of G/H with respect to the quotient metric.

(iv) *Let G be compact and $c := \mathrm{vol}_l(G/H)$. Then $f \mapsto \sqrt[p]{\frac{c\delta}{|W|}} f|_\Sigma$ is an isometry from $L^p(M)^G$ to $L^p(\Sigma)^W$ and for any $f \in L^1(M)^G$ we have the formula*

$$\int_M f(x)\,dx = \frac{c}{|W|}\int_\Sigma f(s)\delta(s)\,ds.$$

In particular, (iii) is a generalization of Weyl's celebrated integration formula. Now some remarks concerning our notation are in order:

Remark.

(i) Unless otherwise stated, we always consider the Riemannian measure associated to the Riemannian metric on a Riemannian manifold. In particular, we do not necessarily assume our manifolds to be oriented.

(ii) We equip Lie groups with a left invariant Riemannian metric and the corresponding Haar measure. The latter one then coincides with the Riemannian measure. For a compact Lie group, we further assume its invariant metric normalized such that its total volume becomes one.

(iii) For function spaces on manifolds with some G-action, a superscript denotes the subset of G-invariant functions. E.g. $\mathcal{C}(M)^G$ denotes the set of all G-invariant continuous functions on M.

(iv) The presentation of the theorem follows Theorem (3.14.1) and Corollary (3.14.2) of [DuiKol 2000], whose proofs include most of the necessary ideas for the generalized version we present in this paper. In [Hel 1984](Ch. I, §5), several important special cases of the integral formulæ are proved and explicit formulas for the Jacobians δ are given there. We have included some of them in section 2 below.

(v) For the proof of the theorem, we expect the reader to be familiar with certain basic properties of polar actions and sections which can be found in [PalTer 1988], or in the more recent literature [BerConOlm 2003]. Here are some of them:
  (a) The set of G-regular points lies open and dense in every section.
  (b) Sections are totally geodesic
  (c) The slice representation in each point is polar.
  (d) The generalized Weyl group $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$ parameterizes the intersection of every G-orbit with the section Σ.
  (e) The generalized Weyl group is a discrete group. The assumption on the finiteness of W in the theorem is met in the cases that G is a compact group, or if Σ is compact.
  (f) The G-regular set M^{reg} of M is covered $|W|$-fold by $G/H \times \Sigma^{\mathrm{reg}}$ where $H = Z_G(\Sigma)$. In case of a compact Lie group acting on itself via conjugation, this fact is sometimes called 'Weyl's covering theorem'.

(vi) The assumption that G acts properly, smoothly and in an isometric fashion on M means that we have a smooth homomorphism $G \to \mathrm{Iso}(M)$ whose image is a closed subgroup of the isometry group of M. In particular, the action need not be effective as it is the case with an arbitrary compact Lie group acting on itself via conjugation (see the section 'Examples' below).

Proof. To (i): We first deal with the invariance properties of δ. For this purpose, and for the integration formula in (iii), we need the following calculations. By abuse of notation, we denote by φ also the induced map $G/H \times \Sigma \to M$. For general $g \in G$, the defining properties of a group action imply

$$\varphi \circ (l_g \times \mathrm{id}_\Sigma) = \phi_g \circ \varphi, \tag{1}$$

where $l_g : G/H \to G/H$ is the left translation by g and similarly $\phi_g : M \to M$ is the isometry of M induced by g via the group action φ. Forming the differential in (eH, s), we obtain

$$d\varphi(gH, s) \circ (dl_g(eH) \times \mathrm{id}_{T_s\Sigma}) = d\phi_g(s) \circ d\varphi(eH, s) \tag{2}$$

which we rewrite in the form

$$d\varphi(gH, s) = d\phi_g(s) \circ d\varphi(eH, s) \circ (dl_{g^{-1}}(gH) \times \mathrm{id}_{T_s\Sigma}).$$

If in (1) we have that $g \cdot s$ lies in Σ again, we may further assume that $g \in N_G(\Sigma)$, because the intersection $\Sigma \cap (G \cdot s)$ is parameterized by the normalizer $N_G(\Sigma)$ of Σ. For such $g \in N_G(\Sigma)$, we get another equation similar to (2) in the following way. We first observe that:

$$\varphi \circ (r_g \times \mathrm{id}_\Sigma) = \varphi \circ (\mathrm{id}_{G/H} \times \phi_g),$$

where $r_g : G/H \to G/H,\ kH \mapsto kgH$, the right translation, is well defined since $g \in N_G(\Sigma)$. Forming the derivative in (eH, s) again, yields

$$d\varphi(gH, s) \circ (dr_g(eH) \times \mathrm{id}_{T_s\Sigma}) = d\varphi(eH, g\cdot s) \circ (\mathrm{id}_{\mathfrak{g}/\mathfrak{h}} \times d\phi_g(s)). \tag{3}$$

Combining (2) and (3), we obtain the following formula for $d\varphi(eH, g\cdot s)$:

$$d\varphi(eH, g\cdot s) = d\phi_g(s) \circ d\varphi(eH, s) \circ (\mathrm{Ad}_{g^{-1}} \times d\phi_g^{-1}(s)) \tag{4}$$

In other words, the differentials of φ in (eH, s) and $(eH, g\cdot s)$ for $g \in N_G(\Sigma)$ differ only by the isometries $d\phi_g(s)$ of M and $(\mathrm{Ad}_{g^{-1}} \times d\phi_g^{-1}(s))$ of $G/H \times \Sigma$. Note that the latter one is true, because $g \in N_G(\Sigma)$ and left translation by g is an isometry of G as well as G/H as a consequence of the remark (ii) above.

Recall that $\det(d\varphi(gH, s))$ is computed by choosing some orthonormal bases on $T_{gH}G/H \times T_s\Sigma$ and $T_{g\cdot s}M$ and then computing the usual determinant of the matrix representation of $d\varphi(gH, s)$ with respect to the given bases. The determinant obtained in this way is, up to a sign, independent of the choice of the bases. Now,

$$d\varphi(eH, s)(X + \mathfrak{h}, v) = d\omega_s(eH)(X) + d\phi_e(s)(v) = d\omega_s(eH)(X) + v.$$

By choosing orthonormal bases $x_1, \dots, x_l$ of $\mathfrak{g}/\mathfrak{h}$ and $y_{l+1}, \dots, y_{l+m}$ of $T_s\Sigma$ and completing the latter to an on-basis $y_1, \dots, y_l, y_{l+1}, \dots y_{l+m}$ on T_sM, we obtain the following matrix representation of $d\varphi(eH, s)$ with respect to these bases:

$$\begin{pmatrix} d\omega_s(eH) & 0 \\ 0 & \mathrm{id}_{T_s\Sigma} \end{pmatrix}.$$

We therefore have $\delta(s) = |\det(d\omega_s(eH))| = |\det(d\varphi(eH, s))|$, which we will need in the proof of the integral formula (iii); and by (4), we have that $\delta(s) = \delta(g\cdot s)$ for any $g \in N_G(\Sigma)$; proving the W-invariance of δ.

In order to show the continuity of δ, we express δ as a composition of continuous mappings by means of local frames. More precisely, consider an on-basis $x_1, \dots, x_l$ on $\mathfrak{g}/\mathfrak{h}$ and a local frame $Y_1, \dots, Y_l, Y_{l+1}, \dots, Y_{m+l}$ of M on $U \subset \Sigma$ about s which is adapted to Σ; i.e. $Y_{l+1}, \dots, Y_{l+m}$ is tangent to Σ. Mapping x_i to $Y_i(p)$, we obtain for every $p \in U$ a linear isometry $L_p : \mathfrak{g}/\mathfrak{h} \to \nu_p\Sigma$. This yields a smooth mapping $L : U \times \mathfrak{g}/\mathfrak{h} \to \nu(U)$ which is nothing but a local trivialization of the normal bundle $\nu(\Sigma)$ of Σ. Moreover, the pullback via L_p of the measure on $\nu_p\Sigma$ induced by dx_p gives $d(gH)_{eH}$. By means of L, we may write $\delta(p) = |\det(L_p^{-1} \circ d\omega_p(eH))|$ for all $p \in U$ where $\det(\cdot)$ now denotes the usual determinant for endomorphisms of $\mathfrak{g}/\mathfrak{h}$. This proves the continuity of δ. In fact, L_p and $d\omega_p(eH)$ depend smoothly on p on U; the latter because φ is a smooth action and $d\omega_p(eH) = \frac{\partial}{\partial p}\varphi(eH, p)|_{\mathfrak{g}/\mathfrak{h}\times\{0\}}$.

Now if s is G-regular, then ω_s is an embedding, if s is exceptional, ω_s is a $|G_p/H|$-fold covering and if s is singular, ω_s is a bundle over $G\cdot s$ with fibre G_s/H. If s is singular, then $l > \dim G\cdot s = \dim \mathrm{im}(d\omega_s(eH))$ and hence the rank of $d\omega_s(eH)$ is strictly less than l. Thus $\delta(s) = 0$ in that case.

If s is G-regular, then by definition $\mathrm{vol}_l(G\cdot s) = \int_{G\cdot s} 1\, dy$ where dy denotes the Riemannian measure on $G\cdot s$ related to the metric on $G\cdot s$ induced from M. Applying the transformation theorem to the embedding $\omega_s : G/H \to G\cdot s$, we have:

$$\mathrm{vol}_l(G\cdot s) = \int_{G\cdot s} 1\, dy = \int_{G/H} |\det(d\omega_s(gH))|\, d(gH) = \int_{G/H} \delta(s)\, d(gH) = \mathrm{vol}_l(G/H)\delta(s).$$

A similar consideration for exceptional $s \in \Sigma$ yields:

$$\mathrm{vol}_l(G\cdot s) = \frac{1}{|G_s/H|} \int_{G/H} |\det(d\omega_s(gH))|\, d(gH) = \frac{\mathrm{vol}_l(G/H)\delta(s)}{|G_s/H|}.$$

this concludes the proof of (i). Note that we did not use the finiteness of W here.

To (ii): First note that Ψ is well defined. In fact, F is certainly well defined, continuous with compact support and for every $(gH, s) \in G/H \times \Sigma$ and $n \in N_G(\Sigma)$ we have

$$F(gn^{-1}H, n \cdot s) = f(g \cdot s)\delta(n \cdot s) = f(g \cdot s)\delta(s).$$

That is, we used the Weyl group invariance of δ from (i). Thus F is Weyl-invariant too. It is also obvious that Ψ is a linear mapping. Concerning the continuity of Ψ, we should first note that we consider $\mathcal{C}_c(M)$ and $\mathcal{C}_c(G/H \times \Sigma)$ equipped with their particular L^1-norm. Now let f_n be a sequence of functions in $\mathcal{C}_c(M)$ converging to zero. Then for $F_n = \Psi(f_n)$ we have:

$$\begin{aligned} \|F_n\|_1 &= \int_{G/H \times \Sigma} |F_n(gH, s)| \, (dgH, s) = \int_\Sigma \int_{G/H} |f_n(g \cdot s)| \delta(s) \, dgH \, ds \\ &= |W| \cdot \int_M |f_n(x)| \, dx = |W| \cdot \|f_n\|_1 \to 0, \text{ as } n \to \infty. \end{aligned}$$

That is, we read formula (iii) backwards. Since $\mathcal{C}_c(\cdot)$ is a dense subspace of $L^1(\cdot)$, Ψ induces a continuous map $L^1(M) \to L^1(G/H \times \Sigma)^W$ which again we denote by Ψ. Concerning the surjectivity of Ψ, we suppose that $f_1, f_2 \in L^1(M)$ with $\Psi(f_1) = \Psi(f_2)$. Since $G/H \times \Sigma^{\text{reg}}$ is open and dense in $G/H \times \Sigma$, we have for almost every $(gH, s) \in G/H \times \Sigma$:

$$f_1(g \cdot s)\delta(s) = f_2(g \cdot s)\delta(s). \tag{5}$$

By (i), we have even in the case that G is not compact that $\delta(s) \neq 0$ for $s \in \Sigma^{\text{reg}}$. Therefore, we may divide by $\delta(s)$ in (5) and since $\varphi(G/H \times \Sigma^{\text{reg}}) = M^{\text{reg}}$ we obtain that $f_1 = f_2$ except for a set with measure zero.

Concerning the surjectivity of Ψ, we first define for every $F \in \mathcal{C}_c(G/H \times \Sigma^{\text{reg}})^W$ a function $f \in \mathcal{C}_c(M^{\text{reg}})$ by

$$f := \frac{F \circ \varphi^{-1}}{\delta \circ \text{pr}_2 \circ \varphi^{-1}}$$

where $\text{pr}_2 : G/H \times \Sigma^{\text{reg}} \to \Sigma^{\text{reg}}$ is the projection onto the second factor. Note that the functions $F \circ \varphi^{-1}$ and $\delta \circ \text{pr}_2 \circ \varphi^{-1}$ are well defined since F and $\delta \circ \text{pr}_2$ are W-invariant and the fibres of φ are precisely the W-orbits on $G/H \times \Sigma$. Continuity can easily be verified by considering local trivialization for the covering φ. Finally, $F \circ \varphi^{-1}$ has compact support as the fibres of φ are compact. Furthermore, δ has no zeros on Σ^{reg} so that $\delta \circ \text{pr}_2 \circ \varphi^{-1}$ has no zeros on M^{reg}. In conclusion, f is well defined.

We obviously have $\Psi(f) = F$ and by (iii) we obtain:

$$\begin{aligned} \int_{M^{\text{reg}}} f(x) \, dx &= \frac{1}{|W|} \int_{\Sigma^{\text{reg}}} \left(\int_{G/H} \frac{F(gH, s)}{\delta(s)} \, d(gH) \right) \delta(s) \, ds \\ &= \frac{1}{|W|} \int_{\Sigma^{\text{reg}}} \left(\int_{G/H} F(gH, s) \, d(gH) \right) ds. \end{aligned}$$

Now consider an arbitrary $F \in L^1(G/H \times \Sigma^{\text{reg}})^W$. We can approximate F by a sequence $(F_n)_{n \in \mathbf{N}} \subset \mathcal{C}_c(G/H \times \Sigma^{\text{reg}})^W$ and, as indicated above, form a sequence $(f_n)_{n \in \mathbf{N}} \subset \mathcal{C}_c(M^{\text{reg}})$ with $\Psi(f_n) = F_n$. Now the F_n form a Cauchy sequence and applying the integral identity above to $|f_n - f_m|$, we see that the f_n form a Cauchy sequence too. Since $L^1(M^{\text{reg}})$ is complete, there is a function $f \in L^1(M^{\text{reg}})$ with $\lim_{n \to \infty} f_n = f$ and by continuity of Ψ, we have $\Psi(f) = F$. Since M^{reg} (resp. $G/H \times \Sigma^{\text{reg}}$) is open and dense in M (resp. $G/H \times \Sigma$), we may identify $L^1(M^{\text{reg}})$ with $L^1(M)$ (resp. $L^1(G/H \times \Sigma^{\text{reg}})^W$ with $L^1(G/H \times \Sigma)^W$). This proves the surjectivity of Ψ. Using the open mapping theorem, we have thus established that Ψ is a continuous isomorphism between Banach spaces.

To (iii): Let us first assume that $f \in L^1(M) = L^1(M^{\mathrm{reg}})$ has support in some open subset $U \subset M^{\mathrm{reg}}$ such that $\varphi^{-1}(U)$ is the union of $|W|$-many disjoint open subsets $V_1, \dots, V_{|W|}$ of $G/H \times \Sigma^{\mathrm{reg}}$ each diffeomorphic via φ to U. The function $\Psi(f)$ has support in the union of these V_i. By the classical transformation formula, we then have

$$\begin{aligned}\int_M f(x)\,dx &= \frac{1}{|W|}\int_{G/H\times\Sigma} f(g\cdot s)|\det d\varphi(gH,s)|\,d(gH,s)\\ &\overset{(i)}{=} \frac{1}{|W|}\int_\Sigma\left(\int_{G/H} f(g\cdot s)\,d(gH)\right)\delta(s)\,ds.\end{aligned}$$

In the proof of (i), we have seen that $|\det d\varphi(gH,s)| = |\det d\omega(eH)| = \delta(s)$, whence the last equation above is true.

In the general case, with $f \in L^1(M^{\mathrm{reg}})$ arbitrary, we cover M^{reg} by a system $\mathcal{U}$ of local trivial charts like U above. Take a partition of unity subordinate to $\mathcal{U}$, say $(U_i, p_i)_{i\in I}$ with $\sum_{i\in I} p_i = 1$ and without loss of generality $U_i \in \mathcal{U}$. Then $f = \sum_{i\in I} f\cdot p_i$ and

$$\begin{aligned}\int_M f(x)\,dx &= \sum_{i\in I}\int_M f(x)p_i(x)\,dx\\ &= \frac{1}{|W|}\sum_{i\in I}\int_\Sigma\left(\int_{G/H} f(g\cdot s)p_i(g\cdot s)\,d(gH)\right)\delta(s)\,ds\\ &= \frac{1}{|W|}\int_\Sigma\left(\int_{G/H} f(g\cdot s)\,d(gH)\right)\delta(s)\,ds.\end{aligned}$$

To (iv): Restricting Ψ to the closed subspace $L^1(M)^G \subset L^1(M)$, we obtain the designated mapping up to a factor for the case $p = 1$, because $\Psi(f)$ then does not depend on the first variable anymore and the image of Ψ is clearly seen to be $L^1(\Sigma)^W$. The stated integral formula is an immediate consequence of (iii), as well as the statement concerning the isometries for the various L^p-spaces. □

2. Examples

The following two examples of polar actions and their corresponding volume scaling functions δ are well known in the literature (cf. [DuiKol 2000, Hel 1984]). In this context, the theorem above is called Weyl's Integration formula.

Example 1. Let G be a compact connected Lie group G and consider the action of G on itself via *conjugation*; that is $\varphi : G \times G \to G, (g,x) \mapsto gxg^{-1}$. Note that unless G has trivial center, this action is not effective. However, in view to the remark (vi) above, this is not a problem.

A section is given by any maximal torus T of G. It is pretty straightforward to show that $\delta(t) = |\det(\mathrm{id} - \mathrm{Ad}_t)|_{\mathfrak{g}/\mathfrak{t}}|$ for any $t \in T$. In the presence of a *root space decomposition* $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha\in P} L_\alpha$, we can compute δ more explicitly. For this purpose we recall some basic facts we need:

Fix a maximal torus T of G and let $\mathfrak{t}$ denote its Lie algebra. A (infinitesimal) *root* α is an element of $(\mathfrak{t}^{\mathbf{C}})^*$, the dual space of $\mathfrak{t}^{\mathbf{C}}$, such that $\mathfrak{g}_\alpha := \{Y \in \mathfrak{g}^{\mathbf{C}} \mid \mathrm{ad}_X(Y) = \alpha(X)\cdot Y \text{ for all } X \in \mathfrak{t}\} \neq 0$. If α is a root then so is $-\alpha$. Let P be a choice of *positive roots*; that is $0 \notin P$ and for each $\alpha \in P$ we have $-\alpha \notin P$. Now each L_α is defined by $L_\alpha := (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}$. Note that since ad_X is a skew endomorphism, with respect to some Ad-invariant inner product, we see that each α takes values in the imaginary numbers only. It is a fact that each nonzero $\mathfrak{g}_\alpha$ has complex dimension one, whereas the corresponding L_α has real dimension two.

We then have that $\mathrm{Ad}_{\exp(X)}$ with $X \in \mathfrak{t}$ acts on each nonzero $\mathfrak{g}_\alpha$ as the multiplication operator $e^{\alpha(X)} \in U(1)$. Hence, the action of $\mathrm{Ad}_{\exp(X)}$ on L_α is the rotation through the angle $\alpha(X)/i$. The root space decomposition above is invariant under $(\mathrm{id} - \mathrm{Ad}_t)$. If we write an arbitrary $t \in T$ as $t = \exp(X)$ for some $X \in \mathfrak{t}$, we obtain

$$\delta(t) = \prod_{\alpha \in P} \left[(1 - \cos \frac{\alpha(X)}{i})^2 + \sin \frac{\alpha(X)}{i} \right] = 4^{|P|} \prod_{\alpha \in P} \sin^2 \frac{\alpha(X)}{2i}.$$

Example 2. We may also consider the corresponding *adjoint representation* Ad of G on $\mathfrak{g}$. In this case, a section is given by any maximal Abelian subspace $\mathfrak{t} \subset \mathfrak{g}$ of $\mathfrak{g}$ which then corresponds to a maximal torus of G. Here, $\delta(X) = |\det(\mathrm{ad}_X)|$. Now ad_X acts on $\mathfrak{g}_\alpha$ as the multiplication operator $\alpha(X)$. Since $\alpha(X)$ assumes purely imaginary values only, the action of ad_X on L_α is a rotation through ninety degrees scaled by the factor $\frac{\alpha(X)}{i}$. Hence,

$$\delta(X) = (-1)^{|P|} \prod_{\alpha \in P} (\alpha(X))^2 = \prod_{\alpha \in P} |\alpha(X)|^2.$$

The following two examples are generalizations of the previous ones and they can, for instance, be found in [Hel 1984](Ch. I, §5). However, the notion of a polar action is not used there although the concept was already known at the time. Surprisingly, the bibliography of the book even lists the article of Dadok, where the polar representations on $\mathbf{R}^n$ are classified[Dad 1985].

Example 3. Let $M = G/K$ be a Riemannian symmetric space with $G = \mathrm{Iso}(M)^\circ$, the identity component of the isometry group of M, and $K = G_p$, the isotropy subgroup of G of some point $p \in M$. For simplicity reasons we will assume that M is either of compact type or noncompact type. It is then known that the action of K on $M = G/K$ by left translation is (hyper-)polar and a section is given by any maximal flat A of M through p. The situation here indeed generalizes example 1, because every connected compact Lie group is a compact Riemannian symmetric space. In order to compute $\delta(a)$, we start with some preliminaries. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to K and assume the usual Ad_K invariant inner product on $\mathfrak{g}$ derived from the Killing form; i.e. if B denotes the Killing form of $\mathfrak{g}$ then in the compact case, the inner product is $-B$ and in the noncompact case it is $-B(\cdot, \theta \cdot)$, where θ denotes the Cartan involution. We identify T_pM with $\mathfrak{p}$ and, under this identification, let $\mathfrak{a} := T_pA$. Then $\mathfrak{a} \subset \mathfrak{p}$ is a maximal Abelian subspace of $\mathfrak{p}$. If $\pi : G \to G/K$ denotes the canonical projection, then $d\pi(e) : \mathfrak{g} \to \mathfrak{p}$ is the corresponding projection with respect to the Cartan decomposition and the above identification. Let L denote the centralizer of A in K. Then $\omega_{aK} : K/L \to M,\ kL \mapsto kaK$ and we have

$$\begin{aligned} d\omega_{aK}(eL)(X + \mathfrak{l}) &= \frac{d}{dt}\Big|_{t=0} \omega_{aK}((\exp tX)L) = \frac{d}{dt}\Big|_{t=0} (\exp tX)aK \\ &= \frac{d}{dt}\Big|_{t=0} aa^{-1}(\exp tX)aK \\ &= \frac{d}{dt}\Big|_{t=0} a(\exp t\mathrm{Ad}_{a^{-1}}X)K \\ &= dl_a(p) \circ d\pi(e)(\mathrm{Ad}_{a^{-1}}X). \end{aligned}$$

Since $dl_a(p)$ is an isometry, we have $\delta(aK) = |\det(d\pi(e) \circ \mathrm{Ad}_{a^{-1}})|$. For further computations, we decompose $\mathfrak{p} = \mathfrak{a} \oplus \sum_{\alpha \in \Sigma^+} \mathfrak{p}_\alpha$ and $\mathfrak{k} = \mathfrak{l} \oplus \sum_{\alpha \in \Sigma^+} \mathfrak{k}_\alpha$. Here Σ denotes the set of restricted roots and Σ^+ a choice of positive roots. Furthermore, $\mathfrak{p}_\alpha = \{Y \in \mathfrak{p} \mid (\mathrm{ad}_X)^2 Y = \alpha(X)^2 Y \text{ for all } X \in \mathfrak{a}\}$ and $\mathfrak{k}_\alpha = \{Y \in \mathfrak{k} \mid (\mathrm{ad}_X)^2 Y = \alpha(X)^2 Y \text{ for all } X \in \mathfrak{a}\}$ and we put $m_\alpha := \dim \mathfrak{p}_\alpha = \dim \mathfrak{k}_\alpha$; see e.g. [Hel 1978] Ch.

VII, §11 for the details. Now $d\pi(e)\circ \mathrm{Ad}_{a^{-1}}$ leaves the decomposition of $\mathfrak{p}$ above invariant and acts on each direct summand $\mathfrak{p}_\alpha$ as the operator $-\sinh\alpha(H)\cdot\frac{\mathrm{ad}_H}{\alpha(H)}$ in case that M is of noncompact type and it acts as the operator $\sin\frac{\alpha(H)}{i}\cdot\frac{\mathrm{ad}_H}{\alpha(H)/i}$ in case that M is of compact type, where $a=\exp(H)$. In fact, if M is of noncompact type, then ad_H is a symmetric operator and thus it has only real eigenvalues, whereas if M is of compact type, ad_H is skew-symmetric and has only purely imaginary eigenvalues. In the noncompact case, we have for any $X\in\mathfrak{p}_\alpha$:

$$\begin{aligned}
\mathrm{Ad}_{a^{-1}}X &= e^{-\mathrm{ad}_H}(X)=\sum_{k=0}^{\infty}\frac{1}{k!}(-\mathrm{ad}_H)^k(X)\\
&= -\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}(\mathrm{ad}_H)^{2k+1}(X)+\sum_{k=0}^{\infty}\frac{1}{(2k)!}(\mathrm{ad}_H)^{2k}(X)\\
&= -\frac{1}{\alpha(H)}\sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\alpha(H)^{2k+1}\mathrm{ad}_H(X)+\sum_{k=0}^{\infty}\frac{1}{(2k)!}\alpha(H)^{2k}\cdot X\\
&= -\sinh\alpha(H)\cdot\frac{\mathrm{ad}_H(X)}{\alpha(H)}+\cosh\alpha(H)\cdot X.
\end{aligned}$$

Clearly, after applying the projection $d\pi(e)$, only the first term survives. Now $\frac{\mathrm{ad}_H}{\alpha(H)}$ is an isometry between $\mathfrak{k}_\alpha$ and $\mathfrak{p}_\alpha$ and hence

$$\delta(aK)=\prod_{\alpha\in\Sigma^+}|\sinh\alpha(H)|^{m_\alpha}.$$

An analogue computation in the compact case reveals:

$$\delta(aK)=\prod_{\alpha\in\Sigma^+}\left|\sin\frac{\alpha(H)}{i}\right|^{m_\alpha}.$$

Example 4. As before, there is a Lie algebra version of the previous class of examples, the so called *s-representations*: The action of K on $\mathfrak{p}$ (see example 3 for the notation) given by $k\cdot X:=\mathrm{Ad}_k(X)$ is (hyper-)polar. A section is any Cartan subspace $\mathfrak{a}$ of $\mathfrak{p}$. Quite straightforward, we then have $d\omega_H(eL)=\mathrm{ad}_H$ for any $H\in\mathfrak{a}$. In view to the root space decomposition in example 3, we obtain

$$\delta(H)=\prod_{\alpha\in\Sigma^+}|\alpha(H)|^{m_\alpha}.$$

Example 5. The last family of examples we give here are the so called Hermann actions (see [Her 1960]) which again generalize example 3. Assuming the notation of example 3 with G compact, let $H\subset G$ be another symmetric subgroup, i.e. G/H is a Riemannian symmetric space too. Then H acts on $M=G/K$ by left translation and this action is hyperpolar. A section is given in the following way: In addition to the Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ with respect to K, we also have the Cartan decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ with respect to H. Let $\mathfrak{a}\subset\mathfrak{m}\cap\mathfrak{p}$ be a maximal Abelian subspace. Then $A:=\exp(\mathfrak{a})$ is a flat section for the action of H on M (details can be found in [Conl 1964]). In opposition to the other examples, the author does not know yet wether it is possible to express the volume scaling function δ in terms of roots in general, or not. A computation, as in example 3, however reveals that

$$\delta(aK)=|\det(d\pi(e)\circ\mathrm{Ad}_{a^{-1}})|,$$

where $a\in A$. The difficulty which arises is that $d\pi(e)\circ\mathrm{Ad}_{a^{-1}}:\mathfrak{h}/\mathfrak{z}(A)\to\mathfrak{p}$ and it is not clear how to relate $\mathfrak{h}/\mathfrak{z}(A)$ with the normal space of $\mathfrak{a}$ in $\mathfrak{p}$ in a more or less canonical way like we did before, in the case when $\mathfrak{h}=\mathfrak{k}$.

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