STRONG L^p-SOLUTIONS TO THE NAVIER-STOKES FLOW PAST MOVING OBSTACLES: THE CASE OF SEVERAL OBSTACLES AND TIME DEPENDENT VELOCITY

EVA DINTELMANN, MATTHIAS GEISSERT, MATTHIAS HIEBER

Abstract. Consider the Navier-Stokes flow past several moving obstacles. It is shown that there exists a unique strong local solution in the L^p-setting, $1 < p < \infty$. Moreover, it is proved that the strong solution coincides with the known mild solution in the very weak sense.

1. Introduction

The mathematical description of the Navier-Stokes flow past rotating or moving obstacles gained quite some attention in the last years. The motion hereby is described by the equations of Navier-Stokes on an exterior domain depending on the time variable t. More precisely, consider the equation

$$(1.1)\qquad \left\{\begin{array}{rcllr} \partial_t v - \Delta v + (v\cdot\nabla)v + \nabla q &=& f & \text{in} \;\; \Omega(t), & t\in(0,T),\\ \operatorname{div} v &=& 0 & \text{in} \;\; \Omega(t), & t\in(0,T),\\ v(x,t) &=& M_i(t)x & \text{on} \;\; \Gamma_i(t),\; i=1,\dots,m, & t\in(0,T),\\ v(x,0) &=& v_0(x) & \text{in} \;\; \Omega(0). & \end{array}\right.$$

Here $v = v(x,t)$ and $q = q(x,t)$ denote the velocity and the pressure of the fluid, respectively. The boundary condition on $\Gamma_i(t)$ is the usual no-slip condition. In this paper we consider time-dependent domains $\Omega(t)$ of the following form: let $\mathcal{O}_1,\dots,\mathcal{O}_m \subset \mathbb{R}^n$, $n \ge 2$, be compact sets with boundaries $\Gamma_1,\dots,\Gamma_m$ of class $C^{1,1}$. We denote by $\Omega := \mathbb{R}^n \setminus \bigcup_{i=1}^m \mathcal{O}_i$ the exterior domain. For time-dependent matrices

$$M_i \in C^\infty([0,T];\mathbb{R}^{n\times n}),$$

with $\operatorname{tr} M_i(t) = 0$ for all $t\in[0,T]$, $i\in\{1,\dots,m\}$, we define the time dependent exterior domain

$$\Omega(t) := \mathbb{R}^n \setminus \bigcup_{i=1}^m \mathcal{O}_i(t)$$

with $\mathcal{O}_i(t) := \{y = G_{(i)}(t)x, x\in\mathcal{O}_i\}$, $\Gamma_i(t) := \{y = G_{(i)}(t)x, x\in\Gamma_i\}$ for $t\in[0,T]$ and a suitably defined isomorphism $G_{(i)}(t) : \overline{\mathcal{O}_i} \to \overline{\mathcal{O}_i(t)}$, for details see (2.5). As the obstacles shall not collide, we require

$$\operatorname{dist}\left(\overline{\mathcal{O}_i(t)}, \overline{\mathcal{O}_j(t)}\right) > 0, \qquad i\neq j,\; t\in[0,T].$$

Our aim is to construct a strong L^p-solution to (1.1).

It is interesting to compare our solution to problem (1.1) with the results which have been obtained recently by several different approaches. The situation of *one* obstacle rotating with *constant* angular velocity (i.e. M equals the rotation matrix) was considered first by Hishida [His99]. He proved the existence of a unique local *mild* solution to (1.1) in the context of L^2.

Strong solutions, again in the L^2-context and for one obstacle, were obtained by Galdi [Gal04], Galdi and Silvestre [GS05] by Galerkin methods as well as by Cumsille and Tucsnak [CT06].

1991 *Mathematics Subject Classification.* 76D03, 35Q30, 35B30.

Key words and phrases. Navier-Stokes equations, rotation, rigid bodies, maximal regularity.

In the case of two dimensions, these strong solutions are even global in time under appropriate assumptions on the data.

The situation where the data belong to L^p for $1 < p < \infty$, was considered first in [GHH06a], where the existence of a unique *mild* L^p*-solution* to (1.1) was established. For a different approach in this setting we refer to the recent work of Hishida and Shibata [HS06].

In this paper, we consider the situation of *strong* L^p*-solutions*. We prove that the existence of a local, strong solution to (1.1) in L^p even for *several* non colliding obstacles which may rotate or move with a *time-dependent* angular velocity. One of the main tools in the proof of our results will be the maximal $L^p - L^q$-regularity of the Stokes operator in exterior domains.

It is a natural question to ask whether the strong solution to (1.1) obtained in Theorem 3.1 below coincides with the mild solution constructed in [GHH06a]. We give an affirmative answer to this question in Theorem 3.3 below. Of course, we need to explain first the meaning of *coincides*, since mild and strong solutions are defined on different spaces. In this section we make use of the concept of *very weak solutions* which was introduced in [FJR72] for $\mathbb{R}^n$ and in [Ama00] for domains.

For more information about the Navier-Stokes equation in the rotating framework of all of $\mathbb{R}^n$ or $\mathbb{R}^n_+$, we refer to the papers [CM97], [BMN99], [HS05], [GIMM04] and [GIMMS05] dealing in particular with data non decaying at infinity.

2. Preliminaries

We start by transforming equation (1.1) on the time-dependent domain $\Omega(t)$ to an equation on a fixed cylindrical domain. More precisely, following the approach introduced by Inoue and Wakimoto [IW77], we introduce a change of coordinates which coincides in the special case of pure rotation, i.e. M equals the rotation matrix, with the rotation in a neighborhood of the rotating obstacle, but equals the identity far away from the rotating body; see also [CT06].

For the time being, assume there is only one moving obstacle. We then make the following assumption.

(A1) Let $\mathcal{O} \subset \mathbb{R}^n$, $n \geq 2$, be a compact set (the obstacle) with boundary $\Gamma := \partial\mathcal{O}$ of class $C^{1,1}$. Denote by $\Omega := \mathbb{R}^n \setminus \mathcal{O}$ the exterior domain corresponding to $\mathcal{O}$. For $M \in \mathbb{R}^{n\times n}$ with $\operatorname{tr} M = 0$ define $\Omega(t), \Gamma(t), \mathcal{O}(t)$ as $\Omega(t) := \{y = e^{tM}x, x \in \Omega\}$, $\Gamma(t) := \{y = e^{tM}x, x \in \Gamma\}$, $\mathcal{O}(t) := \{y = e^{tM}x, x \in \mathcal{O}\}$.

We then consider a ball $B_r(0)$ of radius $r > 0$ such that $B_r(0) \supset \mathcal{O}(t)$ for all $t \in [0, T]$ for some $T > 0$. Choose a cut-off function $\eta \in C_c^\infty(\mathbb{R}^n)$ satisfying $0 \leq \eta \leq 1$, $\eta = 1$ on $B_r(0)$ and $\eta = 0$ on $B_{2r}(0)^c$. Define $b : \mathbb{R}^n \to \mathbb{R}^n$ by

$$b(y) := \eta(y)My - B_K\left((\nabla\eta)M\cdot\right)(y), \tag{2.1}$$

where $K := \operatorname{supp}(\nabla\eta)$. Here B_K denotes the Bogovskiĭ operator; for details see [Bog79], [Gal94], [GHH06b]. Note that $b(y) = My$ for $y \in \overline{\mathcal{O}(t)}$. Moreover, since $\int_K(\nabla\eta)(y)My\, \mathrm{d}y = 0$ thanks to $\operatorname{tr} M = 0$, it follows from [Bog79] that $\operatorname{div} b = 0$ in $\mathbb{R}^n$ and $b \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$.

Consider then the initial value problem

$$\left\{\begin{array}{rcll} \partial_t X(y,t) & = & b(X(y,t)), & t > 0, \quad y \in \mathbb{R}^n, \\ X(y,0) & = & y, & \qquad\qquad y \in \mathbb{R}^n. \end{array}\right. \tag{2.2}$$

Then, by standard theory of ODE's, there exists a unique vector field $X \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ satisfying (2.2). Moreover, $X(\cdot, t)$ is a C^∞-diffeomorphism from Ω onto $\Omega(t)$. Its inverse $Y(\cdot, t)$ is of the same class of regularity and satisfies the initial value problem

$$\left\{\begin{array}{rcll} \partial_t Y(x,t) & = & -b(Y(x,t)), & t > 0, \quad x \in \mathbb{R}^n, \\ Y(x,0) & = & x, & \qquad\qquad x \in \mathbb{R}^n. \end{array}\right. \tag{2.3}$$

In fact, we only need the restrictions of X and Y on $[0,T]$, nevertheless (2.2) and (2.3) even can be solved on the whole of $\mathbb{R}^n \times \mathbb{R}_+$.

Denote by $J_X(\cdot,t)$ and $J_Y(\cdot,t)$ the Jacobian of $X(\cdot,t)$ and $Y(\cdot,t)$, respectively. Since div $b=0$, Liouville's theorem, see e. g. [Arn92], implies that

$$J_X(y,t)J_Y(X(y,t),t) = \mathrm{id} \text{ and } \det J_X(y,t) = \det J_Y(x,t) = 1 \tag{2.4}$$

for all $t \geq 0$ and $x, y \in \mathbb{R}^n$.

In the situation of several obstacles moving with time dependent velocity we make the following assumption:

(A2) Let $\mathcal{O}_1, \dots, \mathcal{O}_m \subset \mathbb{R}^n$, $n \geq 2$, be compact sets with boundaries $\Gamma_1, \dots, \Gamma_m$ of class $C^{1,1}$. We denote by $\Omega := \mathbb{R}^n \setminus \bigcup_{i=1}^m \mathcal{O}_i$ the exterior domain. For time-dependent matrices

$$M_i \in C^\infty([0,T];\mathbb{R}^{n\times n}),$$

with tr $M_i(t) = 0$ for all $t \in [0,T]$, $i \in \{1,\dots,m\}$, we define the time-dependent sets $\Omega(t), \Gamma_i(t)$ and $\mathcal{O}_i(t)$ for $t \in [0,T]$ by aid ot the unique solution $G_{(i)}(t) : \overline{\mathcal{O}_i} \to \overline{\mathcal{O}_i(t)}$ of the following ODE:

$$\left\{ \begin{array}{rcll} G'_{(i)}(t)x &=& M_i(t)G_{(i)}(t)x, & x \in \overline{\mathcal{O}_i}, \quad t \in (0,T), \\ G_{(i)}(0)x &=& x, & x \in \overline{\mathcal{O}_i}. \end{array} \right. \tag{2.5}$$

Note, that $G_{(i)}(t) : \overline{\mathcal{O}_i} \to \overline{\mathcal{O}_i(t)}$ is an isomorphism, as

$$\left\{ \begin{array}{rcll} [G'_{(i)}(t)x]_k &=& M_i(t)[G_{(i)}(t)x]_k, & x \in \overline{\mathcal{O}_i}, \quad t \in (0,T), \\ [G_{(i)}(0)x]_k &=& x_k, & x \in \overline{\mathcal{O}_i}, \end{array} \right.$$

with $1 \leq k \leq n$, $i \in \{1,\dots,m\}$, has a unique fundamental system of linear independent solutions $G_{(i)}(t) = \left(G^1_{(i)}(t), \dots, G^n_{(i)}(t)\right)$, due to the non-vanishing Wronski-determinant at zero. In particular, for the case of constant matrices $M_i \in \mathbb{R}^{n\times n}$, we are left with $G_{(i)}(t) = e^{tM_i}$ and inverse $G^{-1}_{(i)}(t) = e^{-tM_i}$.

As the obstacles shall not collide, we require

$$\mathrm{dist}\ \left(\overline{\mathcal{O}_i(t)}, \overline{\mathcal{O}_j(t)}\right) > 0, \qquad i \neq j,\ t \in [0,T].$$

We now choose open sets $B_{1_i}, B_{2_i} \subset \mathbb{R}^n$, such that $\overline{\mathcal{O}_i} \subset B_{1_i} \subset \overline{B_{1_i}} \subset B_{2_i}$ and set

$$B_{k_i}(t) := \left\{ y = G_{(i)}(t)x, x \in B_{k_i} \right\}, \quad t \in [0,T],\ k = 1,2,\ i \in \{1,\dots,m\}.$$

Then

$$\overline{\mathcal{O}_i(t)} \subset B_{1_i}(t) \subset \overline{B_{1_i}(t)} \subset B_{2_i}(t)$$

for $t \in [0,T]$. Moreover, we demand the sets B_{2_i} to be that small, that

$$\bigcap_{i=1}^m \overline{B_{2_i}(t)} = \emptyset, \qquad t \in [0,T].$$

Next, we introduce a time-dependent cut-off function $\eta \in C^\infty(\mathbb{R}^n \times [0,T])$, $0 \leq \eta \leq 1$, such that

$$\eta(y,t) := \left\{ \begin{array}{lll} 1 & \text{on} & \bigcup_{i=1}^m B_{1_i}(t), \qquad t \in [0,T], \\ 0 & \text{on} & \mathbb{R}^n \setminus \bigcup_{i=1}^m B_{2_i}(t), \quad t \in [0,T]. \end{array} \right.$$

We define the sets

$$K_i(t) := (\mathrm{supp} \nabla_y \eta(t,\cdot)) \cap \overline{B_{2_i}(t)}, \quad t \in [0,T], i \in \{1,\dots,m\},$$

and the time dependent vector field $b : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ by

$$b(y,t) := \eta(y,t) \sum_{i=1}^m M_i(t)y - \sum_{i=1}^m B_{K_i(t)} \left((\nabla_y \eta)(\cdot,t) M_i(t) \cdot \right)(y).$$

Note, that due to tr $M_i(t) = 0$, $t \in [0,T]$, $i = 1,\dots,m$, and properties of the Bogovskiĭ operator, the vector field b is solenoidal for $t \in [0,T]$ on all of $\mathbb{R}^n$. Further, $b(y,t) = M_i(t)y$ for $y \in \overline{\mathcal{O}_i(t)}$, $t \in [0,T]$, $i \in \{1,\dots,m\}$.

Note that for fixed $t^* \in [0,T]$

$$b(y,t^*) \quad = \quad \eta(y,t^*)\sum_{i=1}^m M_i(t^*)y - \sum_{i=1}^m B_{K_i^*}\left((\nabla_y \eta)(\cdot,t^*)M_i(t^*)\cdot\right)(y).$$

Since η is smooth in the first variable, the mapping properties of the Bogovskiĭ operator imply that $b(\cdot,t^*) \in C_c^\infty(\mathbb{R}^n)$. Now, freeze $y^* \in \mathbb{R}^n$ and consider

$$b(y^*,t) \quad = \quad \eta(y^*,t)\sum_{i=1}^m M_i(t)y^* - \sum_{i=1}^m B_{K_i^*}\left((\nabla_y \eta)(\cdot,t)M_i(t)\cdot\right)(y^*).$$

The smoothness of the cut-off function η and the matrices $M_i(\cdot)$ inherits to b and we thus see that

$$b \in C_{c,\sigma}^\infty\left(\mathbb{R}^n \times [0,T];\mathbb{R}^n\right).$$

In particular, b is uniformly Lipschitz continous with respect to the first variable and bounded on $[0,T] \times \mathbb{R}^n$.

Thus, the ordinary differential equation

$$\left\{\begin{array}{rcll} \partial_t X(y,t) & = & b(X(y,t),t), \quad t>0, & y \in \mathbb{R}^n, \\ X(y,0) & = & y, & y \in \mathbb{R}^n. \end{array}\right. \tag{2.6}$$

admits a unique solution by the Picard-Lindelöf theorem. Moreover, $X \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$.

As above, let $Y(\cdot,t)$ be the inverse of $X(\cdot,t)$. Then Y satisfies

$$\left\{\begin{array}{rcll} \partial_t Y(x,t) & = & -b(Y(x,t),t), \quad t>0, & x \in \mathbb{R}^n, \\ Y(x,0) & = & x, & x \in \mathbb{R}^n, \end{array}\right. \tag{2.7}$$

and (2.4). Again, only the restrictions of X, Y to $[0,T]$ will be relevant in the sequel.

We set

$$\begin{array}{rcll} U(y,t) & := & J_Y\left(X(y,t),t\right)v\left(X(y,t),t\right), & y \in \Omega, \quad t \in [0,T], \\ \pi(y,t) & := & q(X(y,t),t), & y \in \Omega, \quad t \in [0,T]. \end{array}$$

Then, (similarly to [IW77], [CT06, Prop.3.5]), a function

$$v \in L^p(0,T;W^{2,q}(\Omega(\cdot))) \cap W^{1,p}(0,T;L^q(\Omega(\cdot))), \quad q \in L^p(0,T;\hat{W}^{1,q}(\Omega(\cdot)))$$

is a solution to (1.1) if and only if

$$U \in L^p(0,T;W^{2,q}(\Omega)) \cap W^{1,p}(0,T;L^q(\Omega)), \quad \pi \in L^p(0,T;\hat{W}^{1,q}(\Omega))$$

and (U,π) satisfies the following set of equations

$$\left\{\begin{array}{rcll} \partial_t U + (\mathcal{M} - \mathcal{L})\,U & = & f - \mathcal{N}U - \mathcal{G}\pi, & \text{in} \quad \Omega \times (0,T), \\ \operatorname{div} U & = & 0, & \text{in} \quad \Omega \times (0,T), \\ U & = & M_i(t)x, & \text{on} \quad \Gamma_i \times (0,T),\ i = 1,\dots,m, \\ U(0) & = & v_0, & \text{in} \quad \Omega. \end{array}\right. \tag{2.8}$$

Here

$$
\begin{aligned}
(\mathcal{L}U)_i &:= \sum_{j,k=1}^n \partial_j(g^{jk}\partial_k U_i) + 2\sum_{j,k,l=1}^n g^{kl}\Gamma^i_{jk}\partial_l U_j + \sum_{j,k,l=1}^n \left(\partial_k(g^{kl}\Gamma^i_{jl}) + \sum_{m=1}^n g^{kl}\Gamma^m_{jl}\Gamma^i_{km}\right) U_j,\\
(\mathcal{N}U)_i &:= \sum_{j=1}^n U_j\partial_j U_i + \sum_{j,k=1}^n \Gamma^i_{jk}U_jU_k,\\
(\mathcal{M}U)_i &:= \sum_{j=1}^n \dot{Y}_j\partial_j U_i + \sum_{j,k=1}^n \left(\Gamma^i_{jk}\dot{Y}_k + (\partial_k Y_i)(\partial_j \dot{X}_k)\right) U_j,\\
(\mathcal{G}\pi)_i &:= \sum_{j=1}^n g^{ij}\partial_j\pi,
\end{aligned}
$$

with the metric contravariant tensor

$$g^{ij} = \sum_{k=1}^n (\partial_k Y_i)(\partial_k Y_j),$$

the metric covariant tensor

$$g_{ij} = \sum_{k=1}^n (\partial_i X_k)(\partial_j X_k)$$

and the Christoffel's symbol

$$\Gamma^k_{ij} = 1/2\sum_{l=1}^n g^{kl}\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right).$$

Note that $\mathcal{L}$ is the transformed Stokes operator, while $\mathcal{M}$ arises from transforming the time derivative. The nonlinearity $\mathcal{N}$ and modified gradient $\mathcal{G}$ correspond to $(v\cdot\nabla)v$ and ∇, respectively.

Setting $u(y,t) := U(y,t) - b(y,t)$, we see that

$$U \in L^p(0,T;W^{2,q}(\Omega)) \cap W^{1,p}(0,T;L^q(\Omega)), \quad \pi \in L^p(0,T;\hat{W}^{1,q}(\Omega))$$

is a solution of (2.8) if and only if

$$u \in L^p(0,T;W^{2,q}(\Omega)) \cap W^{1,p}(0,T;L^q(\Omega)), \quad \pi \in L^p(0,T;\hat{W}^{1,q}(\Omega))$$

and (u,π) solves

$$
(2.9)\quad \left\{
\begin{array}{rcll}
\partial_t u - \Delta u + \nabla\pi &=& F - \mathcal{N}u - \mathcal{B}u + (\mathcal{L}-\Delta)u & \\
&& \qquad - \mathcal{M}u + (\nabla - \mathcal{G})\pi & \text{in } \ \Omega\times(0,T),\\
\operatorname{div} u &=& 0 & \text{in } \ \Omega\times(0,T),\\
u &=& 0 & \text{on } \ \Gamma_i\times(0,T),\ i=1,\dots,m,\\
u(0) &=& v_0 - b(0) & \text{in } \ \Omega,
\end{array}
\right.
$$

with

$$(\mathcal{B}u)_i = \sum_{j=1}^n (u_j\partial_j b_i + b_j\partial_j u_i) + 2\sum_{j,k=1}^n \Gamma^i_{jk}b_k u_j, \quad F = f - \partial_t b - (\mathcal{M}-\mathcal{L})b - \mathcal{N}b.$$

In the sequel, maximal L^p-regularity of the Stokes operator in $L^q_\sigma(\Omega)$ plays an important role. More precisely, for $1<q<\infty$, we define the Stokes operator A_q in $L^q_\sigma(\Omega)$ by

$$
\left\{
\begin{array}{rcl}
A_q u &:=& P_q\Delta u,\\
D(A_q) &:=& W^{2,q}(\Omega)\cap W^{1,q}_0(\Omega)\cap L^q_\sigma(\Omega).
\end{array}
\right.
$$

As usual, P_q denotes the Helmholtz projection on $L^q(\Omega)$.

Let now $T_0 > 0$, $1 < p < \infty$, $T \in (0, T_0)$, $f \in L^p(0, T; L^q_\sigma(\Omega))$ and $u_0 \in (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}$. Then it follows from a classical result of Solonnikov [Sol77] (also see [Gig81] and [Frö02]) that there exists a unique solution

$$u \in X^T_{p,q} := W^{1,p}(0, T; L^q_\sigma(\Omega)) \cap L^p(0, T; W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega))$$

of the instationary Stokes problem

$$\left\{ \begin{array}{rcll} u'(t) - A_q u(t) &=& f(t), & t \in (0, T), \\ u(0) &=& u_0. & \end{array} \right.$$

Moreover, there exists $C > 0$, independent of T, f and u_0, such that

$$\|u\|_{X^T_{p,q}} \leq C(\|f\|_{L^p(0,T;L^q(\Omega))} + \|u_0\|_{(L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}}).$$

Setting $\nabla\pi := (\mathrm{Id} - P_q)\Delta u$, we see that (u, π) is a solution to

$$\left\{ \begin{array}{rcll} u'(t) - \Delta u(t) - \nabla\pi(t) &=& f(t), & t \in (0, T), \\ u(0) &=& u_0, & \end{array} \right.$$

satisfying

$$\|u\|_{X^T_{p,q}} + \|\pi\|_{Y^T_{p,q}} \leq C(\|f\|_{L^p(0,T;L^q(\Omega))} + \|u_0\|_{(L^q_\sigma,(\Omega)D(A_q))_{1-\frac{1}{p},p}}), \tag{2.10}$$

where $Y^T_{p,q} := L^p(0, T; \hat{W}^{1,q}(\Omega))$ and $C > 0$ is a constant independent of $T \in (0, T_0)$, f and u_0.

For the rest of this section, assume that $M = -M^t$, (A1) holds and consider

$$\left\{ \begin{array}{rclll} \partial_t v - \Delta v + (v \cdot \nabla)v + \nabla q &=& 0 & \text{in} \quad \Omega(t), & t \in (0, T), \\ \mathrm{div}\ v &=& 0 & \text{in} \quad \Omega(t), & t \in (0, T), \\ v(x, t) &=& Mx & \text{on} \quad \Gamma(t), & t \in (0, T), \\ v(x, 0) &=& v_0(x) & \text{in} \quad \Omega(0). & \end{array} \right. \tag{2.11}$$

It was shown in [GHH06a] that (2.11) admits a unique, local, mild solution w. In order to compare the solutions obtained in [GHH06a] and in Corollary 3.2 we introduce the notion of very weak solutions to equation (2.11). To this end, let $1 < p, q < \infty$. A function $u \in L^p(0, T_0; L^q_\sigma(\Omega(\cdot)))$ is called a *very weak solution* to (2.11) if

$$\begin{aligned} < v_0, \varphi >_\Omega \quad &- \int_0^{T_0} < u(t), \varphi'(t) + \Delta\varphi(t) + (u(t) \cdot \nabla)\varphi(t) >_{\Omega(t)} \,\mathrm{d}t \\ &+ \int_0^{T_0} < Mx, \nabla\varphi(t) >_{\Gamma(t)} \,\mathrm{d}t = 0, \end{aligned}$$

for $\varphi \in \mathrm{D}$, where

$$\mathrm{D} := \left\{ \varphi \in C^1\left([0, T_0]; C^\infty_c(\overline{\Omega(\cdot)})\right) : \varphi(T_0) = 0,\ \mathrm{div}\ \varphi(t) = 0,\ \varphi(t)|_{\Gamma(t)} = 0 \text{ for all } t \in [0, T_0] \right\}.$$

We say that two very weak solutions $u, v \in L^p(0, T_0; L^q_\sigma(\Omega(\cdot)))$ to (2.11) with initial value $v_0 \in L^q_\sigma(\Omega)$ coincide in the *very weak sense* if there exists $T \in (0, T_0)$ such that $u(t) = v(t)$ for a.a. $t \in (0, T)$.

In [GHH06a], problem (2.11) is transformed into the equation

$$\left\{ \begin{array}{rcll} \hat{u}' - A_{\Omega,b}\hat{u} + (\hat{u} \cdot \nabla\hat{u}) &=& F_2, & t \in (0, T), \\ \hat{u}(0) &=& v_0 - b, & \end{array} \right. \tag{2.12}$$

where $\hat{u}(x, t) := \mathrm{e}^{-tM} v(\mathrm{e}^{tM}x, t) - b(x)$. Here,

$$A_{\Omega,b}\hat{u} := P_q(\Delta\hat{u} + Mx \cdot \nabla\hat{u} - M\hat{u} - b \cdot \nabla\hat{u} - \hat{u} \cdot \nabla b)$$

with $D(A_{\Omega,b}) := \left\{\hat{u} \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^p_\sigma(\Omega) : Mx \cdot \nabla\hat{u} \in L^q(\Omega)\right\}$ and $F_2 := \Delta b + Mx \cdot \nabla b - Mb - b \cdot \nabla b$. It is proved that, for $v_0 - b \in L^r_\sigma(\Omega)$ with $r \geq n$, there exists a *mild solution* $\hat{u}$ to (2.12), i.e., $\hat{u} \in C([0,T]; L^r_\sigma(\Omega))$ satisfies the integral equation

$$\hat{u}(t) = T_{\Omega,b}(v_0 - b) - \int_0^t T_{\Omega,b}(t-s)P_q(\hat{u}\cdot\nabla\hat{u})(s)\,\mathrm{d}s + \int_0^t T_{\Omega,b}P_qF_2(s)\,\mathrm{d}s, \quad t \in (0,T). \tag{2.13}$$

Here, $(T_{\Omega,b}(t))_{t\geq 0} := \left(e^{tA_{\Omega,b}}\right)_{t\geq 0}$ is the semigroup generated by $(A_{\Omega,b}, D(A_{\Omega,b}))$ in $L^r_\sigma(\Omega)$. Moreover, this solution $\hat{u}$ satisfies:

$$t \mapsto t^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\hat{u}(\cdot) \in C([0,T]; L^q_\sigma(\Omega)), \tag{2.14}$$

$$t \mapsto t^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)+\frac{1}{2}}\nabla\hat{u}(\cdot) \in C([0,T]; L^q(\Omega)). \tag{2.15}$$

We show in Section 4 that w given by

$$w(x,t) := \mathrm{e}^{tM}\hat{u}(\mathrm{e}^{-tM}x, t) + \mathrm{e}^{tM}b(\mathrm{e}^{-tM}x), \tag{2.16}$$

with $\hat{u}$ given by the variation-of-constants formula (2.13) is a very weak solution to (2.11) for suitable choices of p, q and n. We are in the position to state our main results.

3. Main Results

Our existence and uniqueness result for equation (1.1) reads as follows.

Theorem 3.1. *Assume (A2) and let $p, q \in (1,\infty)$ such that $\frac{n}{2q} + \frac{1}{p} \leq \frac{3}{2}$. Assume that*

a) $f \in L^p(0,T; L^q_\sigma(\Omega(\cdot)))$,
b) $v_0 - b(\cdot,0) \in (L^q_\sigma(\Omega), W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega) \cap L^q_\sigma(\Omega))_{1-1/p,p}$.

Then there exists $T > 0$ such that the problem (1.1) admits a unique strong solution

$$v \in L^p(0,T; W^{2,q}(\Omega(\cdot))) \cap W^{1,p}(0,T; L^q_\sigma(\Omega(\cdot))), \quad q \in L^p(0,T; \hat{W}^{1,q}(\Omega(\cdot))).$$

Moreover, we can choose $T > 0$ sucht that either $T = +\infty$ or the function

$$t \mapsto \|v(t)\|_{\left(L^q_\sigma(\Omega(t)), W_0^{1,q}(\Omega(t)) \cap W^{2,q}(\Omega(t) \cap L^q_\sigma(\Omega(t)))\right)_{1-1/p,p}}$$

is unbounded on its maximal interval of existence $[0,T)$.

Corollary 3.2. *Assume (A1) and let $p, q \in (1,\infty)$ such that $\frac{n}{2q} + \frac{1}{p} \leq \frac{3}{2}$. Assume that*

a) $f \in L^p(0,T; L^q_\sigma(\Omega(\cdot)))$,
b) $v_0 - b \in (L^q_\sigma(\Omega), W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega) \cap L^q_\sigma(\Omega))_{1-1/p,p}$.

Then there exists $T > 0$ such that the problem (2.11) admits a unique strong solution

$$v \in L^p(0,T; W^{2,q}(\Omega(\cdot))) \cap W^{1,p}(0,T; L^q_\sigma(\Omega(\cdot))), \quad q \in L^p(0,T; \hat{W}^{1,q}(\Omega(\cdot))).$$

We have either $T = +\infty$ or

$$t \mapsto \|v(t)\|_{\left(L^q_\sigma(\Omega(t)), W_0^{1,q}(\Omega(t)) \cap W^{2,q}(\Omega(t) \cap L^q_\sigma(\Omega(t)))\right)_{1-1/p,p}}$$

is unbounded on its maximal interval of existence $[0,T)$.

The following theorem says that the two solutions v and w coincide in the very weak sense. More precisely, the following holds true.

Theorem 3.3. *Let $1 < p, q < \infty$ such that $\frac{n}{2q} + \frac{1}{p} \leq \frac{3}{2}$. Assume that*

a) $v_0 - b \in (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}$,
b) $v_0 - b \in L^r_\sigma(\Omega)$ *for some $r > n$ provided $\frac{n}{2q} + \frac{1}{p} = \frac{3}{2}$.*

Then w given by (2.16) and v given in Corollary 3.2 coincide in the very weak sense.

4. Proof of the first main result

Note that in order to prove our main result it suffices to construct a unique solution (u, π) to (2.9). The strategy of the proof is as follows: First, we derive estimates on the coefficients of the operators $\mathcal{N}, \mathcal{B}, \mathcal{L}, \mathcal{M}$ and $\mathcal{G}$ in the problem (2.9). Then, a fixed point argument in a suitable closed subspace of $X^T_{p,q} \times Y^T_{p,q}$ yields a unique solution $(u, \pi) \in X^T_{p,q} \times Y^T_{p,q}$ via maximal L^p-regularity of the Stokes operator.

Observe, that for a multi-index α and $k \in \mathbb{N}$ there is some constant $K_{|\alpha|,k,T} > 0$ such that

$$\left\|\partial^\alpha_y \partial^k_t b\right\|_{L^\infty(\mathbb{R}^n\times[0,T])} \le K_{|\alpha|,k,T}, \qquad |\alpha| + k > 0.$$

The following lemma yields estimates of the transformation mappings X and Y, respectively, that are defined by (2.6) and (2.7). Clearly, the assertions remain true in the case of b being independent from the time variable and X, Y defined as in (2.2) and (2.3).

Lemma 4.1. *Let $T_0 > 0$, $k \in \mathbb{N}$ and α a multi-index satisfying $|\alpha| + k > 0$. Then there exists $C_{|\alpha|,k,T_0} > 0$ such that*

$$\|\partial^\alpha_y \partial^k_t X\|_{L^\infty(\mathbb{R}^n\times[0,T_0])} \le C_{|\alpha|,k,T_0}.$$

The above estimates remain valid when $X(\cdot, t)$ is replaced by its inverse $Y(\cdot, t)$.

Proof. Let $T_0 > 0$, $k \in \mathbb{N}$ and α a multi-index satisfying $|\alpha| + k > 0$. By a direct calculation, we see that $X(t, y) = y$ for $y \notin \operatorname{supp} b$. Hence,

$$\|\partial^\alpha_y \partial^k_t X\|_{L^\infty((\operatorname{supp}\, b)^c\times[0,T_0])} \le 1.$$

Since $\operatorname{supp} b$ is compact and $X \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$, there exists $C_{|\alpha|,k,T_0}$, such that

$$\|\partial^\alpha_y \partial^k_t X\|_{L^\infty((\operatorname{supp}\, b)\times[0,T_0])} \le C_{|\alpha|,k,T_0}.$$

□

It follows from the definition of g^{ij}, g_{ij} and Γ^k_{ij} and the previous lemma that all coefficients of $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{B}$ and $\mathcal{G}$ are smooth and bounded on finite time intervals $[0, T_0]$ for any $T_0 > 0$. Moreover, by the mean value theorem, for $x \in \mathbb{R}^n$ we have $g^{ij}(x, t) - \delta_{ij} = t\partial_\tau g^{ij}(x, \tau)$ for some $\tau \in (0, t)$. Hence, it follows from Lemma 4.1 that

$$\|g^{ij} - \delta_{ij}\|_{L^\infty(\mathbb{R}^n\times[0,T_0])} \le t\|\partial_t g^{ij}\|_{L^\infty(\mathbb{R}^n\times[0,T_0])} \le Ct, \quad t \in (0, T_0). \tag{4.1}$$

The following embedding of $X^T_{p,q}$ in $L^k(0, T; W^{s,m}(\Omega))$ is needed to cope with the gradient terms. It mainly relies on the mixed derivatives theorem. Precisely, the following lemma holds:

Lemma 4.2. *Let $1 < p, q < \infty$, $T_0 > 0$ and $s = 0$ or $s = 1$. Assume that $k, m \in (1,\infty)$ obey $\frac{2-s}{2} + \frac{n}{2m} - \frac{n}{2q} \ge \frac{1}{p} - \frac{1}{k}$. Then $X^T_{p,q}$ is continously embedded in $L^k(0, T; W^{s,m}(\Omega))$ for any $T \in (0, T_0)$. Moreover, there exists $C_{T_0} > 0$ such that*

$$\|u\|_{L^k(0,T;W^{s,m}(\Omega))} \le C_{T_0}\|u\|_{X^T_{p,q}}, \qquad T \in (0, T_0),\ u \in X^T_{p,q},\ u(0) = 0.$$

Proof. By the mixed derivative theorem (see [Sob75], [DHP05]), for $\theta \in (0, 1)$ there exists $C > 0$ such that

$$\|u\|_{H^{\theta,p}(0,T_0;H^{2-2\theta,q}(\Omega))} \le C\|u\|_{X^{T_0}_{p,q}}, \quad u \in X^{T_0}_{p,q}.$$

It then follows from Sobolev embeddings that

$$\|u\|_{L^k(0,T;W^{s,m}(\Omega))} \le C_{T_0}\|u\|_{X^{T_0}_{p,q}}, \qquad u \in X^{T_0}_{p,q}.$$

Let $S : \left\{u \in X^T_{p,q} : u(0) = 0\right\} \to X^{T_0}_{p,q}$ be defined by

$$Su(t) := \begin{cases} u(t - (T_0 - T)), & t \ge T_0 - T, \\ 0, & 0 \le t < T_0 - T. \end{cases}$$

Then $\|Su\|_{X^{T_0}_{p,q}} = \|u\|_{X^T_{p,q}}$ and $\|Su\|_{L^k(0,T;W^{s,m}(\Omega))} = \|u\|_{L^k(0,T_0;W^{s,m}(\Omega))}$. We thus obtain

$$\|u\|_{L^k(0,T_0;W^{s,m}(\Omega))} = \|Su\|_{L^k(0,T;W^{s,m}(\Omega))} \leq C_{T_0}\|Su\|_{X^{T_0}_{p,q}} = C_{T_0}\|u\|_{X^T_{p,q}}$$

for all $u \in X^T_{p,q}$ satisfying $u(0) = 0$. □

Next we prove estimates for the terms on the right-hand side of (2.9).

Lemma 4.3. *Let $T_0 > 0$. Then there exists $C > 0$ such that for $T \in (0, T_0)$ and $(v_1, q_1), (v_2, q_2) \in X^T_{p,q} \times Y^T_{p,q}$ satisfying $v_1(0) = v_2(0) = 0$*

(a) $\|\mathcal{N}v_1 - \mathcal{N}v_2\|_{L^p(0,T;L^q(\Omega))} \leq C\left(\|v_1\|_{X^T_{p,q}} + \|v_2\|_{X^T_{p,q}}\right)\|v_1 - v_2\|_{X^T_{p,q}}$,

(b) $\|\mathcal{B}(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \leq CT^{\frac{1}{3p}}\|v_1 - v_2\|_{X^T_{p,q}}$,

(c) $\|\mathcal{M}(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \leq CT^{\frac{1}{3p}}\|v_1 - v_2\|_{X^T_{p,q}}$,

(d) $\|(\mathcal{L} - \Delta)(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \leq C(T + T^{\frac{1}{3p}})\|v_1 - v_2\|_{X^T_{p,q}}$,

(e) $\|(\nabla - \mathcal{G})(q_1 - q_2)\|_{L^p(0,T;L^q(\Omega))} \leq CT\|q_1 - q_2\|_{Y^T_{p,q}}$.

The estimates above are valid even if $v_1(0) \neq 0$ and $v_2(0) \neq 0$. However, in this case, C depends on T as well.

Proof. Set $k = 3p$, $k' = 3p/2$, $m = 3q$ and $m' = 3q/2$. By Hölder's inequality, we obtain for the first term of $\mathcal{N}v_1 - \mathcal{N}v_2$

$$\begin{aligned}\|v_1 \cdot \nabla v_1 - v_2 \cdot \nabla v_2\|_{L^p(0,T;L^q(\Omega))} &\leq \|(v_1 - v_2) \cdot \nabla v_1 + v_2 \cdot (\nabla v_1 - \nabla v_2)\|_{L^p(0,T;L^q(\Omega))} \\ &\leq \|v_1\|_{L^{k'}(0,T;W^{1,m'}(\Omega))}\|v_1 - v_2\|_{L^k(0,T;L^m(\Omega))} \\ &\quad + \|v_2\|_{L^k(0,T;L^m(\Omega))}\|v_1 - v_2\|_{L^{k'}(0,T;W^{1,m'}(\Omega))}.\end{aligned}$$

Hence, by Lemma 4.2, there exists $C > 0$, independent of $T \in (0, T_0)$ and v_1, v_2, such that

$$\|v_1 \cdot \nabla v_1 - v_2 \cdot \nabla v_2\|_{L^p(0,T;L^q(\Omega))} \leq C\left(\|v_1\|_{X^T_{p,q}} + \|v_2\|_{X^T_{p,q}}\right)\|v_1 - v_2\|_{X^T_{p,q}}.$$

Since the second term of $\mathcal{N}v_1 - \mathcal{N}v_2$ can be estimated similary, assertion (a) follows.

Similarly, (b) and (c) follow from the estimates

$$\|\mathcal{B}(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \leq CT^{\frac{1}{k}}\|v_1 - v_2\|_{L^{k'}(0,T;W^{1,q}(\Omega))} \leq CT^{\frac{1}{k}}\|v_1 - v_2\|_{X^T_{p,q}}$$

and

$$\|\mathcal{M}(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \leq CT^{\frac{1}{k}}\|v_1 - v_2\|_{L^{k'}(0,T;W^{1,q}(\Omega))} \leq CT^{\frac{1}{k}}\|v_1 - v_2\|_{X^T_{p,q}},$$

respectively, where $C > 0$ is independent of $T \in (0, T_0)$ and v_1, v_2.

As all coefficients of $\mathcal{L}$ are smooth we may rewrite it in non-divergence form and, by (4.1), we obtain

$$\begin{aligned}&\|(\mathcal{L} - \Delta)(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} \\ &\leq C\|g^{jk} - \delta_{jk}\|_{L^\infty(\mathbb{R}^n \times (0,T))}\|D^2(v_1 - v_2)\|_{L^p(0,T;L^q(\Omega))} + CT^{\frac{1}{k}}\|v_1 - v_2\|_{L^p(0,T;W^{1,q}(\Omega))} \\ &\leq C(T + T^{\frac{1}{k}})\|v_1 - v_2\|_{X^T_{p,q}}, \quad T \in (0, T_0),\ v_1, v_2 \in X^T_{p,q},\ v_1(0) = 0,\ v_2(0) = 0.\end{aligned}$$

Hence (d) follows. Assertion (e) similarly follows from (4.1). □

Let $T_0 > 0$, $T \in (0, T_0)$ and $p, q \in (1, \infty)$ satisfy $\frac{n}{2q} + \frac{1}{p} \leq \frac{3}{2}$. Next, we introduce the space for the fixed point argument. In order to do this, consider

$$\begin{cases} \partial_t u^* - \Delta u^* + \nabla \pi^* &= \hat{F}(0,0) \quad \text{in} \quad \Omega \times (0,T), \\ \text{div } u^* &= 0 \quad \text{in} \quad \Omega \times (0,T), \\ u^* &= 0 \quad \text{on} \quad \Gamma \times (0,T), \\ u^*(0) &= u_0 \quad \text{in} \quad \Omega, \end{cases}$$

where $\hat{F} : X_{p,q}^{T_0} \times Y_{p,q}^{T_0} \to L^p(0, T_0; L^q(\Omega))$ is defined by

$$\hat{F}(v,q) := F - \mathcal{N}v - \mathcal{B}v + (\mathcal{L} - \Delta)v - \mathcal{M}v + (\nabla - \mathcal{G})q.$$

Note that $\hat{F}$ is well-defined by Lemma 4.3 thanks to $\frac{n}{2q} + \frac{1}{p} \leq \frac{3}{2}$. Hence, by (2.10), there exists a unique solution $(u^*, \pi^*) \in X_{p,q}^T \times Y_{p,q}^T$.

Furthermore, let $(u^{**}, \pi^{**}) \in X_{p,q}^T \times Y_{p,q}^T$ denote the unique solution to

$$\begin{cases} \partial_t u^{**} - \Delta u^{**} + \nabla \pi^{**} &= \hat{F}(u^*, \pi^*) \quad \text{in} \quad \Omega \times (0,T), \\ \text{div } u^{**} &= 0 \quad \text{in} \quad \Omega \times (0,T), \\ u^{**} &= 0 \quad \text{on} \quad \Gamma \times (0,T), \\ u^{**}(0) &= u_0 \quad \text{in} \quad \Omega. \end{cases}$$

We rewrite (2.9) in terms of a fixed point problem equivalently by

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} + \nabla \tilde{\pi} &= \hat{F}(v,q) - \hat{F}(u^*, \pi^*) \quad \text{in} \quad \Omega \times (0,T), \\ \text{div } \tilde{u} &= 0 \quad \text{in} \quad \Omega \times (0,T), \\ \tilde{u} &= 0 \quad \text{on} \quad \Gamma \times (0,T), \\ \tilde{u}(0) &= 0 \quad \text{in} \quad \Omega, \end{cases} \tag{4.2}$$

where we used the notation

$$\tilde{u} := u - u^{**}, \qquad \tilde{\pi} := \pi - \pi^{**}.$$

In view of the Banach fixed point theorem we define for a given radius $R > 0$ and $T \in (0, T_0)$ the closed set

$$\mathcal{K}_{R,T} := \left\{ (v,q) \in X_{p,q}^T \times Y_{p,q}^T : v(0) = 0 \text{ and } \|v\|_{X_{p,q}^T} + \|q\|_{Y_{p,q}^T} \leq R \right\}$$

and a mapping

$$\Phi_{R,T} : \begin{cases} \mathcal{K}_{R,T} & \to \quad X_{p,q}^T \times Y_{p,q}^T, \\ (v,q) & \mapsto \quad (\tilde{u}, \tilde{\pi}) \qquad \text{such that (4.2) holds.} \end{cases}$$

In order to apply the Banach fixed point theorem to $\Phi_{R,T}$ we have to show that the mapping is well-defined, maps $\mathcal{K}_{R,T}$ into itself and is a contraction.

While $\phi_{R,T}$ is well-defined due to Lemma 4.3 and (2.10), the other two outstanding debits are shown in the two lemmata given below.

The next lemma shows that for suitable choices of $R > 0$ and $T > 0$ the closed set $\mathcal{K}_{R,T}$ is mapped by $\Phi_{R,T}$ into itself.

Lemma 4.4. *There exist $R > 0$ and $T_1 > 0$, such that $\Phi_{R,T} : \mathcal{K}_{R,T} \to \mathcal{K}_{R,T}$ for all $T \in (0, T_1)$.*

Proof. By (2.10) and Lemma 4.3, we obtain

$$\begin{aligned} \|\Phi_{R,T}(v,q)\|_{X_{p,q}^T \times Y_{p,q}^T} &\leq C \|\hat{F}(v,q)\|_{L^p(0,T;L^q(\Omega))} \\ &\leq C \big(\|F\|_{L^p(0,T;L^q(\Omega))} + \|\mathcal{N}v\|_{L^p(0,T;L^q(\Omega))} \\ &\qquad + \|\mathcal{B}v\|_{L^p(0,T;L^q(\Omega))} + \|(\mathcal{L} - \Delta)v\|_{L^p(0,T;L^q(\Omega))} + \|\mathcal{M}v\|_{L^p(0,T;L^q(\Omega))} \\ &\qquad + \|(\nabla - \mathcal{G})q\|_{L^p(0,T;L^q(\Omega))} \big) \\ &\leq C \left(\|F\|_{L^p(0,T;L^q(\Omega))} + R^2 + T^{\frac{1}{3p}} R + TR \right), \quad (v,q) \in \mathcal{K}_{R,T}. \end{aligned}$$

Since $\lim_{T\to 0} \|F\|_{L^p(0,T;L^q(\Omega))} = 0$, we obtain $\|\Phi_{R,T}(v,q)\|_{X^T_{p,q}\times Y^T_{p,q}} \le R$ provided R and T are small enough. $\square$

Lemma 4.5. *There exists $T_0 > 0$ such that $\Phi_{R,T} : \mathcal{K}_{R,T} \to \mathcal{K}_{R,T}$ is a contraction for all $T \in (0, T_0)$.*

Proof. Again, by (2.10) and Lemma 4.3, we obtain

$$\begin{aligned}
&\|\Phi_{R,T}(v_1,q_1) - \Phi_{R,T}(v_2,q_2)\|_{X^T_{p,q}\times Y^T_{p,q}} \\
&\le C\Big(\|\mathcal{N}v_1 - \mathcal{N}v_2\|_{L^p(0,T;L^q(\Omega))} + \|\mathcal{B}(v_1-v_2)\|_{L^p(0,T;L^q(\Omega))} \\
&\qquad + \|(\mathcal{L}-\Delta)(v_1-v_2)\|_{L^p(0,T;L^q(\Omega))} + \|\mathcal{M}(v_1-v_2)\|_{L^p(0,T;L^q(\Omega))} \\
&\qquad + \|(\nabla - \mathcal{G})(q_1-q_2)\|_{L^p(0,T;L^q(\Omega))} \Big) \\
&\le C(R + T + T^{\frac{1}{3p}})\,\|(v_1,q_1) - (v_2,q_2)\|_{X^T_{p,q}\times Y^T_{p,q}}, \quad (v_1,q_1),(v_2,q_2) \in \mathcal{K}_{R,T}.
\end{aligned}$$

Choosing T and R small enough, we obtain $C(R + T + T^{\frac{1}{3p}}) < 1$. $\square$

Proof of Theorem 3.1. The existence of a unique strong solution now follows from Lemma 4.4, Lemma 4.5 and the Banach fixed point theorem. Now, the theorem follows in a standard way from the fact, that $T > 0$ is uniform with respect to v_0, provided

$$\|v_0\|_{\left(L^q_\sigma(\Omega), W^{1,q}_0(\Omega)\cap W^{2,q}(\Omega)\cap L^q_\sigma(\Omega)\right)_{1-1/p,p}} < C_0,$$

and the continuous embedding

$$X^T_{p,q} \hookrightarrow C\left(0,T;\left(W^{1,q}_0(\Omega)\cap W^{2,q}(\Omega)\cap L^q_\sigma(\Omega), L^q_\sigma(\Omega)\right)_{1-1/p,p}\right).$$

$\square$

5. Comparison of strong and mild solutions

In this section we prove Theorem 3.3. For the notion of *very weak* solutions we refer back to Section 2. In a first step, we show that a mild solution is a very weak solution.

Lemma 5.1. *Let $(v_0 - b) \in L^q_\sigma(\Omega)$ for some $q \ge n$ and denote the mild solution to* (2.12) *on $[0,T]$ for some $T > 0$ by $\hat{u}$ (for a representation by the variation-of-constants formula see* (2.13)*). Then w, defined by* (2.16)*, is a very weak solution to* (2.11)*.*

Proof. By (2.14) and (2.15), we obtain $\hat{u} \in C([0,T]; L^q_\sigma(\Omega))$ and $\nabla\hat{u} \in L^1(0,T;L^q(\Omega))$. Choose $\hat{u}^1_n \in C^1([0,T]; C^\infty_{c,\sigma}(\Omega))$ and $\hat{u}^2_n \in C^1([0,T]; C^\infty_c(\overline{\Omega}))$ such that

$$\lim_{n\to\infty} \|\hat{u}^1_n - \hat{u}\|_{C([0,T];L^q(\Omega))} = 0 \text{ and } \lim_{n\to\infty} \|\nabla\hat{u}^2_n - \nabla\hat{u}\|_{L^1(0,T,L^q(\Omega))} = 0. \tag{5.1}$$

Then, by [Paz83, Chapter 4, Theorem 2.9], there exists a solution $\hat{v}_n \in L^1(0,T;D(A_{\Omega,b})) \cap W^{1,1}(0,T;L^q_\sigma(\Omega))$ satisfying

$$\left\{\begin{array}{rcll} \hat{v}_n'(t) - A_{\Omega,b}\hat{v}_n(t) + P_q\left((\hat{u}^1_n\cdot\nabla\hat{u}^2_n)(t)\right) &=& P_qF_2(t), & t \in (0,T), \\ \hat{v}_n(0) &=& \hat{u}^1_n(0). & \end{array}\right.$$

By the representation of $\hat{v}_n$ via variation-of-constants formula, we obtain

$$\lim_{n\to\infty} \|\hat{v}_n - \hat{u}\|_{L^1(0,T;L^{\frac{q}{2}}(K))} = 0 \tag{5.2}$$

for any compact $K \subset \overline{\Omega}$. Setting

$$\begin{aligned}\nabla\hat{\pi}_n &= (\mathrm{Id} - P_q)[F_2(t) + \hat{u}_n^1 \cdot \nabla \hat{u}_n^2(t) - \Delta \hat{v}_n(t) - Mx \cdot \nabla \hat{v}_n(t) \\ &\qquad + M\hat{v}_n(t) + b \cdot \nabla \hat{v}_n(t) + \hat{v}_n(t) \cdot \nabla b]\end{aligned}$$

for $t \in (0,T)$, we see that $(\hat{v}_n, \hat{\pi}_n)$ solves

$$\left\{\begin{array}{rcll} \hat{v}_n' - \Delta\hat{v}_n - Mx \cdot \nabla\hat{v}_n + M\hat{v}_n & & & \\ + b \cdot \nabla\hat{v}_n + \hat{v}_n \cdot \nabla b + \hat{u}_n^1 \cdot \nabla \hat{u}_n^2 + \nabla\hat{\pi}_n &=& F_2 & \text{in } \Omega, \quad t \in (0,T), \\ \operatorname{div} v_n &=& 0 & \text{in } \Omega, \quad t \in (0,T), \\ \hat{v}_n &=& 0 & \text{on } \Gamma, \quad t \in (0,T), \\ \hat{v}_n(0) &=& \hat{u}_n^1(0) & \text{in } \Omega. \end{array}\right.$$

Hence, $(v_n, \pi_n) := (\mathrm{e}^{tM}\hat{v}_n(\mathrm{e}^{-tM}x,t) + \mathrm{e}^{tM}b(\mathrm{e}^{-tM}x), \hat{\pi}(\mathrm{e}^{-tM}x,t))$ is a solution to

$$\left\{\begin{array}{rcll} \partial_t v_n - \Delta v_n + u_n^1 \cdot \nabla u_n^2 + (b\cdot\nabla)(v_n - u_n^2) & & & \\ + \left((v_n - u_n^1)\cdot\nabla\right) b + \nabla\pi_n &=& 0 & \text{in } \Omega(t), \quad t \in (0,T) \\ \operatorname{div} v_n &=& 0 & \text{in } \Omega(t), \quad t \in (0,T) \\ v_n &=& Mx & \text{on } \Gamma(t), \quad t \in (0,T) \\ v_n(0) &=& u_n^1(0) & \text{in } \Omega, \end{array}\right.$$

where $u_n^1 := \mathrm{e}^{tM}\hat{u}_n^1(\mathrm{e}^{-tM}x,t) + \mathrm{e}^{tM}b(\mathrm{e}^{-tM}x)$ and $u_n^2 := \mathrm{e}^{tM}\hat{u}_n^2(\mathrm{e}^{-tM}x,t) + \mathrm{e}^{tM}b(\mathrm{e}^{-tM}x)$. Moreover,

$$(v_n, \pi_n) \in \left(L^1(0,T; W^{2,q}(\Omega(\cdot))) \cap W^{1,1}(0,T; L^q(\Omega(\cdot)))\right) \times L^1(0,T; \hat{W}^{1,q}(\Omega(\cdot)))$$

thanks to $\hat{v}_n \in L^1(0,T; D(A_{\Omega,b})) \cap W^{1,1}(0,T; L^q(\Omega))$. Therefore, integration by parts yields

$$\begin{aligned} < v_n(0), \varphi >_\Omega - \int_0^T &< v_n(t), \varphi'(t) + \Delta\varphi(t) > + < u_n^2(t), (u_n^1(t)\cdot\nabla)\varphi(t) >_{\Omega(t)} \,\mathrm{d}t \\ &+ \int_0^T < (b\cdot\nabla)(v_n - u_n^2) + \left((v_n - u_n^1)\cdot\nabla\right) b, \varphi >_{\Omega(t)} \,\mathrm{d}t \\ &+ \int_0^T < Mx, \nabla\varphi(t) >_{\Gamma(t)} \,\mathrm{d}t = 0, \quad \varphi \in \mathrm{D}. \end{aligned}$$

Letting $n \to \infty$, it follows from (5.1) and (5.2) that u is a very weak solution. □

In a second step, the following lemma shows uniqueness of the very weak solution for suitable values of p and q. Especially, we will conclude coincidence of the mild and strong notion of solutions, as stated in Theorem 3.3.

Lemma 5.2. *Let $1 < p, q < \infty$ satisfy $\frac{n}{2q} + \frac{1}{p} \leq \frac{1}{2}$ and let $v_1, v_2 \in L^p(0,T_0; L^q(\Omega(\cdot)))$ be two very weak solutions to (2.11) for some $T_0 > 0$ with initial value $v_0 \in L^q(\Omega)$. Assume that $v_1 - v_2 \in L^p(0,T_0; L^q_\sigma(\Omega(\cdot)))$. Then there exists $T \in (0,T_0)$ such that $v_1(t) = v_2(t)$ for a.e. $t \in (0,T)$.*

Proof. For the time being let us assume that for some $T \in (0,T_0)$ and all $f \in L^{p'}(0,T; L^{q'}(\Omega(\cdot)))$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, there exists a solution

$$(\varphi, \pi) \in \mathrm{D}_{\mathrm{ext}} := \left(W^{1,p'}(0,T; L^{q'}_\sigma(\Omega(\cdot))) \cap L^{p'}(0,T; W^{2,q'}(\Omega(\cdot)))\right) \times L^{p'}(0,T; \hat{W}^{1,q'}(\Omega(\cdot)))$$

to the dual backward problem

$$
(5.3)\qquad \left\{\begin{array}{rcll}
-\partial_t\varphi - \Delta\varphi + \nabla\pi - (v_1+v_2)\cdot\nabla\varphi &=& f & \text{in } \Omega(t),\quad t\in(0,T),\\
\text{div } \varphi &=& 0 & \text{in } \Omega(t),\quad t\in(0,T),\\
\varphi &=& 0 & \text{on } \Gamma(t),\quad t\in(0,T),\\
\varphi(T) &=& 0 & \text{in } \Omega.
\end{array}\right.
$$

Then, we obtain for all $f \in L^{p'}(0,T;L^{q'}(\Omega(\cdot)))$

$$
\begin{aligned}
&\int_0^T < v_1(t) - v_2(t), f(t) >_{\Omega(t)}\ \mathrm{d}t\\
=\ &\int_0^T < v_1(t) - v_2(t), \left(-\partial_t\varphi - \Delta\varphi + \nabla\pi - (v_1+v_2)\cdot\nabla\varphi\right)(t) >_{\Omega(t)}\ \mathrm{d}t\\
=\ &\int_0^T < v_1(t) - v_2(t), \left(-\partial_t\varphi - \Delta\varphi - (v_1+v_2)\cdot\nabla\varphi\right)(t) >_{\Omega(t)}\ \mathrm{d}t = 0.
\end{aligned}
$$

Here, we have used that φ can be approximated by functions in D. This implies $v_1 = v_2$ in $(0,T)$.

It thus remains to show that for $f \in L^{p'}(0,T;L^{q'}(\Omega(\cdot)))$, there exists a solution $(\varphi,\pi) \in \mathrm{D}_{\mathrm{ext}}$ to (5.3). In order to do so, we first consider the forward problem

$$
(5.4)\qquad \left\{\begin{array}{rcll}
\partial_t\varphi - \Delta\varphi + \nabla\pi - (v_1^{\mathrm{T}}+v_2^{\mathrm{T}})\cdot\nabla\varphi &=& f^{\mathrm{T}} & \text{in } \Omega(t),\quad t\in(0,T),\\
\text{div } \varphi &=& 0 & \text{in } \Omega(t),\quad t\in(0,T),\\
\varphi &=& 0 & \text{on } \Gamma(t),\quad t\in(0,T),\\
\varphi(0) &=& 0 & \text{in } \Omega,
\end{array}\right.
$$

where $v_1^{\mathrm{T}}(t) = v_1(T-t)$, $v_2^{\mathrm{T}}(t) = v_2(T-t)$ and $f^{\mathrm{T}}(t) = f(T-t)$ for $t \in (0,T)$. Note first, that, by a scaling argument, we may assume that $\|f^T\|_{L^{p'}(0,T;L^{q'}(\Omega(\cdot)))}$ is arbitrary small. Then, similar to the proof of Theorem 3.1, it follows that there exists $T > 0$, independent of f^T and a solution $(\varphi,\pi) \in \mathrm{D}_{\mathrm{ext}}$ to (5.3). Indeed, we have an additional term coming from f^T, which is no problem since f^T is arbitrary small. Moreover, the nonlinear term has to be replaced by a term coming from $(v_1^{\mathrm{T}} + v_2^{\mathrm{T}})\cdot\nabla\varphi$. For convenience of the reader, we will give the estimates for $(\tilde{v}_1^{\mathrm{T}} + \tilde{v}_2^{\mathrm{T}})\cdot\nabla\tilde{\varphi}$. We choose $r, s \in (1,\infty)$ such that $\frac{1}{p'} = \frac{1}{p} + \frac{1}{r}$ and $\frac{1}{q'} = \frac{1}{q} + \frac{1}{s}$. Then, it follows from Hölder's inequality and Lemma 4.2

$$
\begin{aligned}
\|(\tilde{v}_1^{\mathrm{T}} - \tilde{v}_2^{\mathrm{T}})\cdot\nabla\tilde{\varphi}\|_{L^{p'}(0,T;L^{q'}(\Omega))} &\le \|\tilde{v}_1^{\mathrm{T}} - \tilde{v}_2^{\mathrm{T}}\|_{L^p(0,T;L^q(\Omega))}\|\nabla\tilde{\varphi}\|_{L^r(0,T;L^s(\Omega))}\\
&\le C\|\tilde{v}_1^{\mathrm{T}} - \tilde{v}_2^{\mathrm{T}}\|_{L^p(0,T;L^q(\Omega))}\|\tilde{\varphi}\|_{X^T_{p',q'}}.
\end{aligned}
$$

Since $\|\tilde{v}_1^{\mathrm{T}} - \tilde{v}_2^{\mathrm{T}}\|_{L^p(0,T;L^q(\Omega))} \to 0$ for $T \to 0$, we get a solution $(\varphi^{\mathrm{T}}, \pi^{\mathrm{T}})$ to (5.4) for some $T > 0$. Finally, (φ,π), where $\varphi(t) := \varphi^{\mathrm{T}}(T-t)$ and $\pi(t) := \pi^{\mathrm{T}}(T-t)$ for $t \in (0,T)$, is a solution to (5.3). □

We finally prove Theorem 3.3.

Proof of Theorem 3.3. Let us first assume that $\frac{n}{2q} + \frac{1}{p} < \frac{3}{2}$. Then, by Sobolev embeddings,

$$
(L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p} \hookrightarrow L^{\tilde{r}}_\sigma(\Omega)
$$

for some $\tilde{r} > n$. Hence, $v, w \in C([0,T_0]; L^{\tilde{r}}(\Omega)$ for some $T_0 > 0$. In particular, v, w satisfy the assumption of Lemma 5.2. Hence, the assertion follows from iteration in this case.

Let us now assume that $\frac{n}{2q} + \frac{1}{p} = \frac{3}{2}$ and $v_0 - b \in L^r_\sigma(\Omega)$ for some $r > n$. By Sobolev embeddings, we have

$$
v \in L^s(0,T;L^{\tilde{s}}(\Omega))
$$

for some $\tilde{s} \in (n, r)$ and $1 < s < \infty$ satisfying $\frac{n}{2\tilde{s}} + \frac{1}{s} \leq \frac{1}{2}$. Moreover, since $v_0 - b \in L^{\tilde{s}}_{\sigma}(\Omega)$, we have $v \in C([0,T]; L^{\tilde{s}}(\Omega))$. Hence, the assertion follows similar to above in this case. □

References

[Ama00] H. Amann, *On the strong solvability of the Navier-Stokes equations.* J. Math. Fluid Mech., **2** (2000), 16–98.

[Arn92] V. Arnol'd, *Ordinary differential equations*, Springer, Berlin, 1992.

[Bal77] J. M. Ball, *Strongly continuous semigroups, weak solutions, and the variation of constants formula*, Proc. Amer. Math. Soc. **63** (1977), (2), 370–373.

[BMN99] A. Banin, A. Mahalov and B. Nicolaenko, *Global regularity of 3D rotating Navier-Stokes equations for resonant domains.* Indiana Univ. Math. J. **48** (1999), 1133–1176.

[Bog79] M. E. Bogovskiĭ, *Solution of the first boundary value problem for an equation of continuity of an incompressible medium*, Dokl. Akad. Nauk SSSR **248** (1979), (5), 1037–1040.

[CT06] P. Cumsille and M. Tucsnak, *Wellposedness for the Navier-Stokes flow in the exterior of a rotating obstacle.* Math. Methods Appl. Sci. **29** (2006), (5), 595–623.

[CM97] Z. Chen and T. Miyakawa, *Decay properties of weak solutions to a perturbed Navier-Stokes system in* $\mathbb{R}^n$. Adv. Math. Sci. Appl. **7** (1997), 741–770.

[DHP03] R. Denk, M. Hieber and J. Prüss, *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc., **45**, (2003).

[DHP05] ———, *Optimal L^p-L^q-Estimates for parabolic boundary value problems with inhomogeneous data*, Konstanzer Schriften in Mathematik und Informatik, (2005), Preprint.

[FJR72] E.B. Fabes, B.F. Jones and N.M. Riviere, *The initial value problem for the Navier-Stokes equations with data in L^p.* Arch. Rational Mech. Anal., **45** (1972), 222–240.

[Frö02] A. Fröhlich, *Maximal regularity for the non-stationary Stokes system in an aperture domain.* J. Evol. Equ., **4** (2002), 471–493.

[Gal94] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, Springer, New York, 1994.

[Gal03] G.P. Galdi, *Steady flow of a Navier-Stokes fluid around a rotating obstacle*, J. Elasticity **71** (2003), no. 1-3, 1–31.

[Gal04] G. Galdi, *Stationary Navier-Stokes problem in a two-dimensional exterior domain.* Handb. Differ. Equ., 2004.

[GS05] P. Galdi, A. Silvestre, *Strong solutions to the Navier-Stokes equations around a rotating obstacle.* Arch. Rational Mech. Anal. **176** (2005), 331-350.

[GHH06a] M. Geissert, H. Heck, M. Hieber, *L^p-theory of the Navier-Stokes flow past rotating or moving obstacles.* J. Reine Angew. Math., (2006), 1-18.

[GHH06b] ———, *On the equation* div $u = f$ *and the Bogovskiĭ Operator*, Functional Analysis and PDE, Birkhäuser, G. Sweers, (ed.), to appear.

[Gig81] Y. Giga, *Analyticity of the semigroup generated by the Stokes operator in L_r spaces*, Math. Z., (1981), 297–329.

[GIMM04] Y. Giga, K. Inui, A. Mahalov, S. Matsui, *Navier-Stokes equations in a rotating frame in $\mathbb{R}^3$ with initial data nondecreasing at infinity.* Hokkaido Math. J., to appear.

[GIMMS05] Y. Giga, K. Inui, A. Mahalov, S. Matsui, J. Saal, *Rotating Navier-Stokes equations in $\mathbb{R}^3_+$ with initial data nondecreasing at infinity: the Ekman boundary layer problem.* Arch. Rat. Mech. Anal., to appear 2006.

[HS05] M. Hieber and O. Sawada, *The Navier-Stokes equations in R^n with linearly growing initial data.* Arch. Rational Mech. Anal., 2005.

[His99] T. Hishida, *An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle*, Arch. Ration. Mech. Anal. **150** (1999), (4), 307–348.

[HS06] T. Hishida, Y. Shibata, *Stability of the Navier-Stokes flow past a rotating obstacle*, Preprint, (2006).

[IW77] A. Inoue and M. Wakimoto, *On existence of solutions of the Navier-Stokes equation in a time dependent domain*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), (2), 303–319.

[Paz83] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[Sob75] P. E. Sobolevskiĭ, *Fractional powers of coercively positive sums of operators*, Dokl. Akad. Nauk SSSR **225** (1975), (6), 1271–1274.

[Sol77] V. Solonnikov, *The solvability of the second initial-boundary value problem for a linear nonstationary system of Navier-Stokes equations*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.(LOMI)**69** (1977), 200–218, 277.

Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgartenstr. 7, D-64289 Darmstadt, Germany

E-mail address: dintelmann@mathematik.tu-darmstadt.de, geissert@mathematik.tu-darmstadt.de, hieber@mathematik.tu-darmstadt.de