The Dirichlet problem for graphs of prescribed anisotropic mean curvature

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Abstract

We consider the Dirichlet problem for two-dimensional graphs of prescribed mean curvature in $\mathbb{R}^3$ where the prescribed mean curvature function $H = H(X,N)$ may depend on the point X in space and on the normal N of the graph as well. In special situations this Dirichlet problem arises as the Euler equation of a generalised nonparametric area funcional. Under certain smallness conditions we will solve the Dirichlet problem and construct minimizers of the generalized area functional.

Introduction

In this paper we study the Dirichlet problem for two-dimensional graphs of prescribed anisotropic mean curvature in $\mathbb{R}^3$: Given a $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^2$ and Dirichlet boundary values $g \in C^{2+\alpha}(\partial\Omega)$ we want to find a solution $\zeta \in C^{2+\alpha}(\overline{\Omega})$ of

$$\begin{aligned}
&\operatorname{div}\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = 2H(x,y,\zeta,N) \quad \text{in } \Omega\,,\\
&N(x,y) = \frac{1}{\sqrt{1+|\nabla\zeta|^2}}(-\zeta_x,-\zeta_y,1) \quad \text{in } \Omega\,,\\
&\zeta = g \quad \text{on } \partial\Omega\,.
\end{aligned} \tag{1}$$

Here, $N(x,y)$ denotes the upper unit normal vector of the graph ζ. The function $H : \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$, which is called the prescribed mean curvature, depends on the point $(x,y,\zeta(x,y))$ in space and on normal $N(x,y)$ as well. At each point $(x,y) \in \Omega$ the geometric mean curvature of the graph ζ is equal to the value $H(x,y,\zeta(x,y),N(x,y))$, thus a solution ζ is also called a graph of prescribed mean curvature H.

For the special case $H = H(x,y)$ several existence results based on different methods have already been proven. Assuming the smallness condition $|H| \leq h$ for some constant $h > 0$, the Dirichlet problem was solved in [6] with the Leray-Schauder method for convex domains satifying an enclosing sphere condition of radius $\frac{1}{2h}$. Furthermore, since the Dirichlet problem is in that case given by the Euler equation of the functional

$$A(\zeta) := \int\limits_{\Omega} \left(\sqrt{1+|\nabla\zeta|^2} + 2H(x,y)\zeta\right)dxdy\;,$$

direct methods from the calculus of variations can be applied to obtain a solution (see e.g. [5]).

The case $H = H(x,y,z)$ was treated in [10] by Sauvigny where a stable parametric solution from the corresponding Plateau problem is taken and shown to be a graph over the x,y-plane. Here one needs the monotonocity assumption $H_z \geq 0$, which by the maximum principle also guarantees the uniqueness of solutions.

In this paper we study and solve the Dirichlet problem for graphs of prescribed mean curvature $H = H(X,N)$ depending on the point X in space and on the normal N as well. In general this problem does not necessarily arise as the Euler equation of some geometric functional. However, we want to point out that in special situations, namely when H depends linearly on N, such a problem appears when considering certain generalized nonparametric area functionals such as

$$A(\zeta) := \int\limits_{\Omega} \Big(a(x,y,\zeta)\sqrt{1+|\nabla\zeta|^2} + b(x,y,\zeta) \Big) dxdy$$

with functions $a : \mathbb{R}^3 \to (0,+\infty)$ and $b : \mathbb{R}^3 \to \mathbb{R}$. The Euler equation is then given by

$$\operatorname{div} \frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = \frac{\nabla a(x,y,\zeta)\cdot N + b_z(x,y,\zeta)}{a(x,y,\zeta)} =: 2H(x,y,\zeta,N) \quad \text{in } \Omega \, .$$

When $b \equiv 0$ minimizers of this nonparametric functional A were constructed in [13] and the Plateau problem for the parametric version of this functional was studied in [2]. The functional A then measures area of the graph ζ in the Riemannian space

$$(\mathbb{R}^3, ds^2) \ , \ ds^2 = a(x,y,z)(dx^2+dy^2+dz^2)$$

and solutions of the Euler equation are minimal graphs in this Riemannian space. The existence theorem we will prove in this paper will also apply to the case that $b \neq 0$. By making a special choice for the function b, namely if $b_z = 2ha^{\frac{3}{2}}$ with $h \in \mathbb{R}$, this enables us also to construct graphs of constant mean curvature h in the Riemannian space $(\mathbb{R}^3, ds^2)$.

This paper is organized as follows: In section 1 we prove a differential equation for the normal (Theorem 1) for a conformally parametrized surface. We use it to prove an interior gradient bound of the graph in terms of its boundary gradient (Corollary 1). In section 2 we then solve the Dirichlet problem (see Theorem 2) using the Leray-Schauder method from [6]. In section 3 we then apply the existence theorem to the functional A and construct critical points and minimizers of this functional (see Theorem 3).

1. A differential equation for the normal and its application

Let $B := \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 < 1\}$ denote the open unit disc in $\mathbb{R}^2$. Given a solution $\zeta \in C^{2+\alpha}(\overline{\Omega})$ of (1) on a simply connected $C^{2+\alpha}$-domain Ω, we can introduce conformal paramters on this graph (see [11]): There exists a positively oriented $C^{2+\alpha}$-diffeomorphism $f : \overline{B} \to \overline{\Omega}$ such that

$$X(u,v) = (x(u,v), y(u,v), z(u,v)) \in C^{2+\alpha}(\overline{B}, \mathbb{R}^3) \ , \ X(u,v) := (f(u,v), \zeta \circ f(u,v))$$

is given in conformal parameters. The vector valued function X satisfies the prescribed mean curvature system

$$\triangle X = 2H(X,N) X_u \wedge X_v \quad \text{in } B \tag{2}$$

where

$$N(u,v) := \frac{X_u \wedge X_v}{|X_u \wedge X_v|}(u,v) \quad \text{for } (u,v) \in B$$

denotes the normal of X. Furthermore, we have the conformality relations

$$|X_u|^2 - |X_v|^2 = 0 = X_u \cdot X_v \quad \text{in } B\ . \tag{3}$$

Note that, since the surface is a graph, the normal N satisfies $N \cdot e_3 = N_3 > 0$ with $e_3 = (0,0,1)$. This also means that the image of the normal mapping is included in the upper hemisphere

$$S^2_+ := \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\ ,\ z > 0\}\ .$$

For the case of constant mean curvature $H \equiv$ const, it is well known that the normal vector N satisfies the differential equation

$$\triangle N + 2W(2H^2 - K)N = 0 \quad \text{in } B$$

with $W := |X_u \wedge X_v|$ and the Gaussian curvature K. We now show a generalisation of this equation for the case of prescribed mean curvature.

Theorem 1: *Let $X \in C^{2+\alpha}(B,\mathbb{R}^3)$ be a solution of (2) and (3) for the prescribed mean curvature $H \in C^{1+\alpha}(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R})$. Then the normal N of the surface belongs to the space $C^{2+\alpha}(B,\mathbb{R}^3)$ and it satisfies the differential equation*

$$\triangle N + 2(\nabla_N H \cdot X_u)N_u + 2(\nabla_N H \cdot X_v)N_v + 2W(2H^2 - K - \nabla_X H \cdot N)N = -2W\nabla_X H\ . \tag{4}$$

Here, $K = K(u,v)$ denotes the Gauss curvature and W is defined by $W := |X_u \wedge X_v|$. Furthermore, we have set $\nabla H = (\nabla_X H, \nabla_N H) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Proof:
From $X \in C^{2+\alpha}(B,\mathbb{R}^3)$ we first conclude that the right hand side of the differential equation

$$\triangle X = 2H(X,N)X_u \wedge X_v \quad \text{in } B$$

belongs to $C^{1+\alpha}(B,\mathbb{R}^3)$. By potential theory it follows $X \in C^{3+\alpha}(B,\mathbb{R}^3)$ and thus $N \in C^{2+\alpha}(B,\mathbb{R}^3)$. Now at any point $(u,v) \in B$ the vectors X_u, X_v and N are linearly independent and form a orthogonal basis of $\mathbb{R}^3$. From the conformality relations for X we conclude

$$X_v \wedge N = X_u \quad \text{and} \quad N \wedge X_u = X_v\ . \tag{5}$$

Furthermore we have the following Weingarten equations

$$N_u = -\frac{b_{11}}{W}X_u - \frac{b_{12}}{W}X_v \quad \text{and} \quad N_v = -\frac{b_{12}}{W}X_u - \frac{b_{22}}{W}X_v\ , \tag{6}$$

where

$$b_{11} = X_{uu} \cdot N \quad , \quad b_{12} = X_{uv} \cdot N \quad \text{and} \quad b_{22} = X_{vv} \cdot N$$

are the coefficients of the second fundamental form. For any given vector $\gamma \in \mathbb{R}^3$ we now calculate

$$\begin{aligned}
&(\gamma \cdot N_u)X_u + (\gamma \cdot N_v)X_v \\
&= \Big(-\frac{b_{11}}{W}(X_u \cdot \gamma) - \frac{b_{12}}{W}(X_v \cdot \gamma)\Big)X_u + \Big(-\frac{b_{12}}{W}(X_u \cdot \gamma) - \frac{b_{22}}{W}(X_v \cdot \gamma)\Big)X_v \\
&= \Big(-\frac{b_{11}}{W}X_u - \frac{b_{12}}{W}X_v\Big)(X_u \cdot \gamma) + \Big(-\frac{b_{12}}{W}X_u - \frac{b_{22}}{W}X_v\Big)(X_v \cdot \gamma) \\
&= (\gamma \cdot X_u)N_u + (\gamma \cdot X_v)N_v\ .
\end{aligned} \tag{7}$$

For the Gauss curvature $K(u,v)$ of the surface X we have the relation

$$N_u \wedge N_v = K X_u \wedge X_v = KWN \; . \tag{8}$$

Using (5) and (6) we now calculate

$$\begin{aligned} N \wedge N_u &= -\frac{b_{11}}{W} X_v + \frac{b_{12}}{W} X_u = \frac{b_{12}}{W} X_u + \frac{b_{22}}{W} X_v - \frac{b_{11}+b_{22}}{W} X_v = -N_v - 2H(X,N) X_v \; , \\ N \wedge N_v &= -\frac{b_{12}}{W} X_v + \frac{b_{22}}{W} X_u = -\frac{b_{11}}{W} X_u - \frac{b_{12}}{W} X_v + \frac{b_{11}+b_{22}}{W} X_u = N_u + 2H(X,N) X_v \; . \end{aligned}$$

Differentiating the first equation w.r.t. v and the second one w.r.t. u we obtain

$$\begin{aligned} N_{vv} &= -N_v \wedge N_u - N \wedge N_{uv} - (2H(X,N)X_v)_v \\ N_{uu} &= N_u \wedge N_v + N \wedge N_{uv} - (2H(X,N)X_u)_u \; . \end{aligned}$$

The sum of the two equation yields

$$\begin{aligned} \frac{1}{2} \triangle N &= \frac{1}{2}(N_{uu} + N_{vv}) \\ &= N_u \wedge N_v - (H(X,N)X_u)_u - (H(X,N)X_v)_v \\ &= KWN - (\nabla_X H \cdot X_u) X_u - (\nabla_X H \cdot X_v) X_v \\ &\quad -(\nabla_N H \cdot N_u) X_u - (\nabla_N H \cdot N_v) X_v - H(X,N) \triangle X \\ &= KWN - W \nabla_X H + W(\nabla_X H \cdot N) N \\ &\quad -(\nabla_N H \cdot X_u) N_u - (\nabla_N H \cdot X_v) N_v - 2WH^2 N \; . \end{aligned}$$

Here we have used (7) for $\gamma = \nabla_N H$ and (8). After some regrouping we obtain the desired differential equation (4). $\square$

Remark: This differential equation is a generalisation of one proven by Sauvigny in [10, Satz 1] for the case $H = H(X)$. In [3, Theorem 1.1] a differential equation for the normal of surfaces of so-called prescribed F-mean curvature is proven.

We now want to derive a lower bound for the function $\xi(u,v) := N_3(u,v) = N(u,v) \cdot e_3 > 0$ in B in terms of the boundary values of ξ. We note that by Theorem 1 ξ satifies

$$\triangle \xi + 2(\nabla_N H \cdot X_u)\xi_u + 2(\nabla_N H \cdot X_v)\xi_v + 2W(2H^2 - K - \nabla_X H \cdot N)\xi = -2WH_z \; .$$

We first impose the following structural condition on the prescribed mean curvature:

Assumption (A):
The function H has to satisfy the structure condition

$$H(X,N) = H_1(X,N) + H_2(X,N)\, N_3 \quad \text{for } (X,N) \in \mathbb{R}^3 \times S^2_+$$

with two functions $H_1, H_2 \in C^1(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R})$ where H_1 is monotone in the z-variable, i.e. $(H_1)_z \geq 0$.
Remarks:

1.) If we assume $H_z \geq 0$ then assumption (A) is satisfied with $H_1 \equiv H$ and $H_2 \equiv 0$. From $H_z \geq 0$ one can also deduce the uniqeness of solutions to the Dirichlet problem using the maximum principle for quasilinear elliptic equations (see [6, Theorem 10.2]). For a presribed mean curvature $H = H(X)$ only depending on the point in space, assumption (A) is actually equivalent to $H_z \geq 0$.

2.) In general assumption (A) is weaker then $H_z \geq 0$. In fact, it does not imply uniqueness for the Dirichlet problem. To see this, consider the disc $\Omega := \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < \frac{3}{4}\}$ and the prescribed mean curvature

$$H(X,N) := -2(X \cdot N) + 1 = -2(xN_1 + yN_2 + zN_3) + 1$$

satifying assumption (A). It is easy to see that the two graphs

$$\zeta_1(x,y) := \frac{1}{2} \quad \text{and} \quad \zeta_2(x,y) := \sqrt{1 - x^2 - y^2} \quad \text{for } (x,y) \in \overline{\Omega}$$

are graphs of prescribed mean curvature H. They also have the same boundary values $\zeta_1 = \zeta_2 = \frac{1}{2}$ on $\partial\Omega$.

Using assumption (A) together with the relation $2H^2 - K \geq 0$ between mean and Gaussian curvature, we now obtain for $\xi(u,v) = N_3(u,v)$ the differential inequality

$$\triangle \xi + a\xi_u + b\xi_v + c\xi \leq 0 \quad \text{in } B \tag{9}$$

with the coefficients

$$a(u,v) := 2\nabla_N H \cdot X_u \ , \ b(u,v) := 2\nabla_N H \cdot X_v \text{ and } c(u,v) := 2W(-\nabla_X H \cdot N + (H_2)_z) \ . \tag{10}$$

We now want to give an interior lower bound for ξ in terms of its boundary values. Since the coefficient c may have positive or negative sign, we cannot directly apply the maximum principle to ξ. Instead, we use the following product ansatz

$$\tilde{\xi}(u,v) := \frac{\xi(u,v)}{\psi(u,v)} \quad \text{for } (u,v) \in \overline{B}$$

for some positive function $\psi \in C^2(\overline{B}, (0,+\infty))$ to be choosen later. Then putting $\xi(u,v) = \tilde{\xi}(u,v)\psi(u,v)$ into (9) we obtain for $\tilde{\xi}$ the differential inequality

$$\psi \triangle \tilde{\xi} + \tilde{a}\tilde{\xi}_u + \tilde{b}\tilde{\xi}_v + \tilde{c}\tilde{\xi} \leq 0 \tag{11}$$

for some coefficients $\tilde{a}, \tilde{b}, \tilde{c}$ where in particular

$$\tilde{c} = \triangle\psi + a\psi_u + b\psi_v + c\psi \ .$$

To have a minimum principle for $\tilde{\xi}$ we need to choose ψ such that $\tilde{c} \geq 0$. Therefore, we show

Proposition 1: *Let $X(u,v) = (x(u,v), y(u,v), z(u,v)) \in C^{2+\alpha}(\overline{B}, \mathbb{R}^3)$ be a solution of (2) and (3) for the prescribed mean curvature $H \in C^{1+\alpha}(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R})$ satifying assumption (A) and the estimates*

$$|H| \leq h \quad , \quad |\nabla_N H| \leq h_1 \quad \textit{and} \quad |\nabla_X H| \leq h_2$$

with constants $h, h_1, h_2 \geq 0$. Defining $Y(u,v) := (x(u,v), y(u,v), 0)$ we assume the smallness condition

$$|Y(u,v)| \leq r \quad \textit{in } \overline{B}$$

for a constant $r < \frac{1}{2h + 4h_1}$.
Then there exists a constant $\lambda = \lambda(h, h_1, h_2, r) > 0$ such that

$$\psi(u,v) := e^{\lambda|Y|^2} \quad \textit{for } (u,v) \in \overline{B}$$

satisfies the differential inequality

$$\triangle\psi + a\psi_u + b\psi_v + c\psi \geq 0 \tag{12}$$

with a, b and c defined by (10).

Proof:
Using the inequaltity $|Y_u|^2+|Y_v|^2 \geq |X_u|^2$, which is a consequence of the conformal parametrization of X, we first calculate

$$\begin{aligned}
\triangle\psi &= \Big(2\lambda e^{\lambda|Y|^2}\, Y\cdot Y_u\Big)_u + \Big(2\lambda e^{\lambda|Y|^2}\, Y\cdot Y_v\Big)_v \\
&= 2\lambda\psi\Big(\triangle Y\cdot Y + |Y_u|^2+|Y_v|^2+2\lambda(Y\cdot Y_u)^2+2\lambda(Y\cdot Y_v)^2\Big) \\
&\geq 2\lambda\psi\Big(\triangle X\cdot Y + |Y_u|^2+|Y_v|^2\Big) \\
&= 2\lambda\psi\Big(|Y_u|^2+|Y_v|^2+2H(X,N)X_u\wedge X_v\cdot Y\Big) \\
&\geq 2\lambda\psi\Big(|X_u|^2-2h|X_u|^2|Y|\Big) = 2\lambda\psi W(1-2h|Y|)\ .
\end{aligned}$$

Next we estimate

$$\begin{aligned}
a\psi_u+b\psi_v &= 2(\nabla_N H\cdot X_u)(2\lambda\psi\, Y\cdot Y_u)+2(\nabla_N H\cdot X_v)(2\lambda\psi\, Y\cdot Y_v) \\
&\leq 4\lambda\psi|\nabla_N H|\,|X_u|\,|Y|(|Y_u|+|Y_v|) \\
&\leq 8\lambda\psi h_1|Y|\,|X_u|^2 = 8\lambda\psi W h_1|Y|\ .
\end{aligned}$$

Combining these two estimates we reach

$$\begin{aligned}
\triangle\psi+a\psi_u+b\psi_v+c\psi &\geq 2\lambda\psi W(1-2h|Y|-4h_1|Y|)+2W(-\nabla_X H\cdot N+(H_2)_z)\psi \\
&\geq 2\lambda\psi W(1-2h|Y|-4h_1|Y|)-2W(3h_2)\psi \\
&= 2\psi W\Big(\lambda(1-2h|Y|-4h_1|Y|)-3h_2\Big) \\
&\geq 2\psi W\Big(\lambda(1-\{2h+4h_1\}r)-3h_2\Big)\ .
\end{aligned}$$

Now, using the smallness assumption on r of this lemma, we can choose a $\lambda=\lambda(h,h_1,h_2,r)>0$ large enough such that (12) holds. $\square$

As a consequence we obtain the following gradient estimate.

Corollary 1: *Given a simply connected $C^{2+\alpha}$-domain $\Omega\subset\mathbb{R}^2$ let $\zeta\in C^{2+\alpha}(\overline{\Omega},\mathbb{R})$ be a solution of (1) for the prescribed mean curvature $H\in C^{1+\alpha}(\mathbb{R}^3\times\mathbb{R}^3,\mathbb{R})$ satisfying assumption (A) and the estimates*

$$|H|\leq h \quad,\quad |\nabla_N H|\leq h_1 \quad\text{and}\quad |\nabla_X H|\leq h_2\ .$$

Assume that the domain Ω is included in a ball $B_r(0,0)\subset\mathbb{R}^2$ of some radius $r<\frac{1}{2h+4h_1}$. Then there exists a constant $C=C(h,h_1,h_2,r)$ such that

$$\sup_{(x,y)\in\Omega}|\nabla\zeta(x,y)|\leq C\Big(1+\sup_{(x,y)\in\partial\Omega}|\nabla\zeta(x,y)|\Big)\ .$$

Proof:
Given a solution ζ of (1) we introduce conformal parameters $X=X(u,v)=(f(u,v),\zeta\circ f(u,v))\in C^{2+\alpha}(\overline{B},\mathbb{R}^3)$ on the graph as mentioned in the beginning of this section. We consider the third component of the normal $\xi(u,v):=N_3(u,v)$ and remark $\xi(u,v)=(1+|\nabla\zeta\circ f(u,v)|^2)^{-\frac{1}{2}}$. Using (12) of Proposition 1 together with (11) we find a constant $\lambda>0$ such that for $\tilde{\xi}(u,v)=\xi(u,v)e^{-\lambda(x^2+y^2)}>0$ we have the differential inequality

$$\psi\triangle\tilde{\xi}+\tilde{a}\tilde{\xi}_u+\tilde{b}\tilde{\xi}_v\leq 0\ .$$

By the maximum principle $\tilde{\xi}$ must achieve its minimum on ∂B, so for any $(u_0, v_0) \in B$ we have

$$\xi(u_0, v_0) \geq \tilde{\xi}(u_0, v_0) \geq \inf_{\partial B} \tilde{\xi}(u, v) = \inf_{\partial B} \xi(u, v) e^{-\lambda(x^2+y^2)} \geq e^{-\lambda r^2} \inf_{\partial B} \xi(u, v) \ .$$

Setting $(x_0, y_0) = f^{-1}(u_0, v_0) \in \Omega$ we conclude

$$\sqrt{1 + |\nabla\zeta(x_0, y_0)|^2} \leq e^{\lambda r^2} \sup_{(x,y)\in\partial\Omega} \sqrt{1 + |\nabla\zeta(x, y)|^2}$$

which yields the desired estimate for $C := e^{\lambda r^2}$. □

Remarks: By a different choice of the function ψ in Lemma 1 it may be possible to improve the assumption on the radius r of the ball containing Ω. The smallness assumption on r can be removed completely provided a global a priori estimate of $W = |X_u \wedge X_v|$ in $\overline{B}$ is first shown where X is the conformal reparametrization of the graph ζ. Proposition 1 can then be proven for the function $\psi(u, v) := e^{\lambda u}$. Such a global estimate of W can be proven using interior estimates for elliptic systems with quadratic growth in gradient (see [8]) combined with a boundary regularity theorem for such systems (see [9]) and is worked out in the author's dissertation thesis [1].

2. Solution of the Dirichlet problem

In the first section we have provided an interior gradient bound in terms of the boundary gradient. We now need a boundary gradient estimate. Such an estimate holds for a large class of elliptic differential equations including our one (see [6, chapter 14]). However, the following convexity assumption on the domain must be imposed.

Definition 1*:* Let $\Omega \subset \mathbb{R}^2$ be a C^2-domain and $\kappa : \partial\Omega \to \mathbb{R}$ the curvature of $\partial\Omega$ w.r.t. the inner normal. Then Ω is called k-convex for a constant $k \geq 0$ if the inequality

$$\kappa(x, y) \geq k \quad \text{for } (x, y) \in \partial\Omega$$

holds.

Remarks:

1.) A k-convex domain for $k \geq 0$ is necessarily a convex domain.

2.) A k-convex domain satisfies a uniform enclosing sphere condition in the following sence: For each point $(x, y) \in \partial\Omega$ there exists a ball $B_{\frac{1}{k}}(x_1, y_1) \subset \mathbb{R}^2$ of radius $\frac{1}{k}$ centered at some point $(x_1, y_1) \in \mathbb{R}^2$ such that

$$\Omega \subset B_{\frac{1}{k}}(x_1, y_1) \quad \text{and} \quad (x, y) \in \partial B_{\frac{1}{k}}(x_1, y_1) \ .$$

We can now solve the Dirichlet problem.

Theorem 2*:* *Assumptions:*

1.) Let a $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^2$ be given.

2.) The prescribed mean curvature $H \in C^{1+\alpha}(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R})$ satifies assumption (A) and

$$|H| \leq h \quad , \quad |\nabla_N H| \leq h_1 \quad in\, \overline{\Omega} \times \mathbb{R} \times S^2_+$$

for some constants $h > 0$ and $h_1 \geq 0$.

3.) The domain Ω is $2h$-convex and contained in a ball $B_r(0,0)$ of some radius $r < \frac{1}{2h+4h_1}$.

Then for any Dirichlet boundary values $g \in C^{2+\alpha}(\partial\Omega, \mathbb{R})$ the Dirichlet problem (1) has a solution $\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R})$.

Proof:
Consider the family of Dirichlet problems

$$\zeta \in C^{2+\alpha}(\overline{\Omega}, \mathbb{R}) \quad , \quad \operatorname{div}\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = 2t\, H(x,y,\zeta,N) \quad \text{in } \Omega \quad \text{and} \quad \zeta = t\, g \quad \text{on } \partial\Omega \tag{13}$$

with a parameter $t \in [0,1]$ and let ζ be a solution for some $t \in [0,1]$. We define the spherical caps

$$\eta^{\pm}(x,y) := \pm\Big(||g||_{C^0(\partial\Omega)} + \sqrt{\frac{1}{h^2} - x^2 - y^2}\Big) \quad \text{for } (x,y) \in \overline{\Omega}$$

which are well defined because Ω is contained in the ball $B_r(0,0)$ of radius $r < \frac{1}{2h+4h_1} < \frac{1}{h}$. Noting $\eta^- \le \zeta \le \eta^+$ on $\partial\Omega$ the comparision principle for quasilinear eliptic equations (see [6, Theorem 10.1]) yields $\eta^- \le \zeta \le \eta^+$ in Ω and we obtain the C^0-estimate

$$||\zeta||_{C^0(\overline{\Omega})} \le ||g||_{C^0(\partial\Omega)} + \frac{1}{h} \,.$$

Now, using the $2h$-convexity of the domain together with this C^0-estimate, a boundary gradient estimate

$$||\zeta||_{C^1(\partial\Omega)} \le C_1$$

can be proven with a constant C_1 only depending on $||g||_{C^2(\partial\Omega)}$ and h (see [6, Corollary 14.5] and the remark following there). Using Corollary 1 we obtain the global C^1-estimate

$$||\zeta||_{C^1(\overline{\Omega})} \le C_2$$

with a constant C_2 independent of t. The Leray-Schauder method [6, Theorem 13.8] yields a solution of the Dirichlet problem (13) for each $t \in [0,1]$. For $t = 1$ we obtain the desired solution of (1). □

3. Ciritical points and minimizers of a generalized nonparametric area functional

We consider the generalized nonparametric area functional

$$A(\zeta) := \int_{\Omega} \Big(a(x,y,\zeta)\sqrt{1+|\nabla\zeta|^2} + b(x,y,\zeta)\Big)\, dx dy$$

for a domain $\Omega \subset \mathbb{R}^2$ and given functions $a : \mathbb{R}^3 \to (0,+\infty)$ and $b : \mathbb{R}^3 \to \mathbb{R}$.
Examples:

1.) In case $a \equiv 1$ and $b \equiv 0$ we obtain the standard Euklidean area of a graph in $\mathbb{R}^3$. The Euler equation is the nonparametric minimal surface equation.

2.) For $a \equiv 1$ and $b = b(x, y, z)$ we obtain the functional

$$A(\zeta) = \int\limits_{\Omega} \Big(\sqrt{1 + |\nabla\zeta|^2} + b(x, y, \zeta) \Big) dxdy \ .$$

The corresponding Euler equation leads to graphs of prescribed mean curature $H = H(x, y, z) = \frac{1}{2} b_z(x, y, z)$.

3.) Taking some function for $a = a(x, y, z)$ but $b \equiv 0$ one obtains

$$A(\zeta) = \int\limits_{\Omega} a(x, y, \zeta) \sqrt{1 + |\nabla\zeta|^2} dxdy \ .$$

This functional measures the area of the graph ζ in the Riemannian space

$$(\mathbb{R}^3, ds^2) \quad , \quad ds^2 = a(x, y, z)(dx^2 + dy^2 + dz^2)$$

which is a space conformally equivalent to Euklidian $\mathbb{R}^3$. The Euler equation leads to minimal graphs in this Riemannian space.

4.) Setting

$$U := \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \Omega \ , \ \min(0, \zeta(x, y)) < z < \max(0, \zeta(x, y))\}$$

the oriented volume $\mathrm{vol}(U)$ of U in the Riemannian space $(\mathbb{R}^3, ds^2)$ is computed by

$$\mathrm{vol}(U) = \int\limits_{\Omega} \int\limits_{0}^{\zeta(x,y)} \sqrt{a(x, y, z)^3} dzdxdy = \int\limits_{\Omega} \int\limits_{0}^{\zeta(x,y)} a(x, y, z)^{\frac{3}{2}} dzdxdy \ .$$

We now define

$$b(x, y, z) := \int\limits_{0}^{z} a(x, y, s)^{\frac{3}{2}} ds$$

and consider the functional

$$A(\zeta) = \int\limits_{\Omega} \Big(a(x, y, \zeta) \sqrt{1 + |\nabla\zeta|^2} + 2h \, b(x, y, \zeta) \Big) dxdy$$

with a parameter $h \in \mathbb{R}$. Here one looks for critical points of the area under a volume constraint in the Riemannian space $(\mathbb{R}^3, ds^2)$ if one considers h to be a Lagrange parameter. The corresponding Euler equation leads to graphs of constant mean curvature h in the Riemannian space.

We now derive the Euler equation of the functional A which in the case $b \equiv 0$ was already established in [2].

Lemma 1: *Given two functions $a, b \in C^1(\mathbb{R}^3, \mathbb{R})$ with $a > 0$ in $\mathbb{R}^3$, the Euler equation of the functional A is given by*

$$\mathrm{div} \, \frac{a \nabla\zeta}{\sqrt{1 + |\nabla\zeta|^2}} = a_z \sqrt{1 + |\nabla\zeta|^2} + b_z \quad in \ \Omega$$

or equivalenty

$$\operatorname{div} \frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = \frac{\nabla a \cdot N + b_z}{a} \quad \text{in } \Omega$$

with the upper unit normal of the graph

$$N = N(x,y) = (1+|\nabla\zeta|^2)^{-\frac{1}{2}}(-\zeta_x, -\zeta_y, 1) \ .$$

Proof:
Setting

$$F(x,y,z,p,q) := a(x,y,z)\sqrt{1+p^2+q^2} + b(x,y,z) \quad \text{for } x,y,z,p,q \in \mathbb{R} \ ,$$

the functional can written in the form

$$A(\zeta) = \int_\Omega F(x,y,\zeta,\zeta_x,\zeta_y)dxdy \ .$$

We first calculate

$$F_z = a_z\sqrt{1+p^2+q^2} + b_z \quad , \quad F_p = \frac{a\,p}{\sqrt{1+p^2+q^2}} \quad \text{and} \quad F_q = \frac{a\,q}{\sqrt{1+p^2+q^2}} \ .$$

Given a test function $\delta \in C_0^\infty(\Omega)$ and a parameter $t \in \mathbb{R}$ we set $\zeta^t := \zeta + t\delta$ and obtain for the first variation

$$\begin{aligned} 0 &= \frac{d}{dt}A(\zeta^t)\Big|_{t=0} = \int_\Omega (F_z\delta + F_p\delta_x + F_q\delta_y)dxdy \\ &= \int_\Omega \Big(F_z\delta - \operatorname{div}(F_p,F_q)\,\delta\Big)dxdy = \int_\Omega \Big(F_z - \operatorname{div}(F_p,F_q)\Big)\delta\,dxdy \ , \end{aligned}$$

In the last step we have integrated by parts and used the zero boundary values of δ. As this equation has to hold for all function $\delta \in C_0^\infty(\Omega)$ we obtain as the Euler equation

$$\begin{aligned} 0 &= F_z - \operatorname{div}(F_p,F_q) = a_z\sqrt{1+\zeta_x^2+\zeta_y^2} + b_z - \operatorname{div}\frac{a\nabla\zeta}{\sqrt{1+\zeta_x^2+\zeta_y^2}} \\ &= \frac{a_z(1+\zeta_x^2+\zeta_y^2)}{\sqrt{1+\zeta_x^2+\zeta_y^2}} + b_z - a\operatorname{div}\frac{\nabla\zeta}{\sqrt{1+\zeta_x^2+\zeta_y^2}} - \frac{(a_x+a_z\zeta_x)\zeta_x + (a_y+a_z\zeta_y)\zeta_y}{\sqrt{1+\zeta_x^2+\zeta_y^2}} \\ &= \frac{a_z - a_x\zeta_x - a_y\zeta_y}{\sqrt{1+\zeta_x^2+\zeta_y^2}} + b_z - a\operatorname{div}\frac{\nabla\zeta}{\sqrt{1+\zeta_x^2+\zeta_y^2}} \\ &= \nabla a \cdot N + b_z - a\operatorname{div}\frac{\nabla\zeta}{\sqrt{1+\zeta_x^2+\zeta_y^2}} \quad \text{in } \Omega \end{aligned}$$

with the normal $N(x,y)$ of the graph. Using the assumption $a > 0$ in $\mathbb{R}^3$, after some regrouping we obtain

$$\operatorname{div}\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = \frac{1}{a}(\nabla a \cdot N + b_z) \quad \text{in } \Omega \ ,$$

which is the Euler equation in the desired form. □

If we now define

$$H = H(X,N) := \frac{\nabla a(X) \cdot N + b_z(X)}{2a(X)} \tag{14}$$

then by Lemma 1 we see that any critical point the functional A is a solution of

$$\operatorname{div}\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = 2H(x,y,\zeta,N) \quad \text{in } \Omega \quad \text{in } \Omega\,. \tag{15}$$

We now want to apply Theorem 2, but we first have to check when the special function H here satisfies assumption (A) needed for Theorem 2. We therefore write

$$H = \frac{\nabla a\cdot N + b_z}{2a} = \frac{a_xN_1 + a_yN_2 + b_z}{2a} + \frac{a_z}{2a}N_3 =: H_1(X,N) + H_2(X,N)\,N_3\,.$$

Now, since assumption (A) is satisfies if $(H_1)_z \geq 0$, we calculate

$$(H_1)_z = \left(\frac{a_x}{2a}\right)_z N_1 + \left(\frac{a_y}{2a}\right)_z N_2 + \frac{b_{zz}a - b_za_z}{2a^2} \geq 0\,.$$

As this inequality must hold for all $N_1, N_2 \in (-1,1)$ we assume both $b_{zz}a - b_za_z \geq 0$ as well as

$$0 = \left(\frac{a_x}{a}\right)_z = (\log a)_{zx} \quad \text{and} \quad 0 = \left(\frac{a_y}{a}\right)_z = (\log a)_{zy}\,,$$

which is equivalent to the product representation $a(x,y,z) = a_1(x,y)a_2(z)$ with certain functions $a_1 : \mathbb{R}^2 \to (0,+\infty)$ and $a_2 : \mathbb{R} \to (0,+\infty)$. We can now prove

Theorem 3*:*
Assumptions:

a) Let $a, b \in C^{2+\alpha}(\mathbb{R}^3,\mathbb{R})$ satisfy $a(x,y,z) = a_1(x,y)a_2(z)$ with functions $a_1 : \mathbb{R}^2 \to (0,+\infty)$, $a_2 : \mathbb{R} \to (0,+\infty)$ and

$$b_{zz}a - b_za_z \geq 0 \quad \textit{in } \mathbb{R}^3\,.$$

b) For some constant $c, d \geq 0$ let

$$\frac{|\nabla a| + |b_z|}{a} \leq d \quad \textit{and} \quad \frac{|\nabla a|}{a} \leq c \quad \textit{in } \mathbb{R}^3\,.$$

c) Let a d-convex $C^{2+\alpha}$-domain $\Omega \subset \mathbb{R}^2$ be given which is included in a ball $B_r(0,0)$ of radius $r < \frac{1}{d+2c}$.

Then for any Dirichlet boundary values $g \in C^{2+\alpha}(\partial\Omega,\mathbb{R})$ there exists at least one solution $\zeta \in C^{2+\alpha}(\overline{\Omega},\mathbb{R})$ of the Euler equation (15) of the functional A.
In the case $a = a(x,y)$, i.e. $a_2(z) \equiv 1$, that solution ζ is unique and minimizes the functional A in the class of $C^1(\overline{\Omega},\mathbb{R})$-functions with boundary values g.

Proof:
We set

$$H(X,N) := \frac{1}{2a(X)}\Big(\nabla a(X)\cdot N + b_z(X)\Big)\,.$$

From $a, b \in C^{2+\alpha}(\mathbb{R}^3,\mathbb{R})$ it follows $H \in C^{1+\alpha}(\mathbb{R}^3\times\mathbb{R}^3,\mathbb{R})$. We have already shown that this H satisfies assumption (A) needed for Theorem 1. Now note that

$$|H| \leq \frac{1}{2a}\Big(|\nabla a| + |b_z|\Big) \leq \frac{1}{2}d \quad \text{and} \quad |\nabla_N H| = \frac{|\nabla a|}{2a} \leq \frac{1}{2}c\,.$$

Since by assumption Ω is d-convex and contained in a ball of radius $r < \frac{1}{d+2c}$, Theorem 2 yields a solution of (15) having prescribed boundary values $g \in C^{2+\alpha}(\partial\Omega,\mathbb{R})$.

We now consider the case $a = a(x,y)$. Assumption a) of this theorem then gives $b_{zz} \geq 0$. By Lemma 1 the Euler equation reads

$$\operatorname{div} \frac{a\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} = b_z(x,y,\zeta) \quad \text{in } \Omega \ .$$

For any function $\eta \in C^1(\overline{\Omega}, \mathbb{R})$ with $\eta = g$ on $\partial\Omega$ we will now show $A(\zeta) \leq A(\eta)$ and use arguments similar to those given in [7, chapter 13] for the standard nonparametric area functional. By the divergence theorem we first have

$$\begin{aligned} 0 &= \int_\Omega \operatorname{div} \frac{a(\eta-\zeta)\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} dxdy = \int_\Omega \Big((\eta-\zeta) \operatorname{div} \frac{a\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}} + \frac{a\nabla\eta\cdot\nabla\zeta - a|\nabla\zeta|^2}{\sqrt{1+|\nabla\zeta|^2}} \Big) dxdy \\ &= \int_\Omega \Big((\eta-\zeta) b_z + \frac{a\nabla\eta\cdot\nabla\zeta - a|\nabla\zeta|^2}{\sqrt{1+|\nabla\zeta|^2}} \Big) dxdy \ . \end{aligned}$$

We use this to obtain

$$\begin{aligned} A(\zeta) &= \int_\Omega \Big(a(x,y)\sqrt{1+|\nabla\zeta|^2} + b(x,y,\zeta) \Big) dxdy \\ &= \int_\Omega \Big(a(x,y) \frac{1+|\nabla\zeta|^2}{\sqrt{1+|\nabla\zeta|^2}} + b(x,y,\zeta) \Big) dxdy \\ &= \int_\Omega \Big(a(x,y) \frac{1+\nabla\zeta\cdot\nabla\eta}{\sqrt{1+|\nabla\zeta|^2}} + (\eta-\zeta) b_z(x,y,\zeta) + b(x,y,\zeta) \Big) dxdy \\ &\leq \int_\Omega \Big(a(x,y)\sqrt{1+|\nabla\eta|^2} + (\eta-\zeta) b_z(x,y,\zeta) + b(x,y,\zeta) \Big) dxdy \\ &\leq \int_\Omega \Big(a(x,y)\sqrt{1+|\nabla\eta|^2} + b(x,y,\eta) \Big) dxdy = A(\eta) \ . \end{aligned}$$

In the last step we have used the assumption $b_{zz} \geq 0$. Thus we have shown $A(\zeta) \leq A(\eta)$ and equality can only hold if $\nabla\zeta \equiv \nabla\eta$ which by the same boundary values of g is only possible for $\zeta \equiv \eta$. This also shows the uniqueness of the solution ζ. □

Remarks:

1.) In the paper [13] of Tausch a minimizer for functionals including

$$A(\zeta) = \int_\Omega a(x,y,\zeta)\sqrt{1+|\nabla\zeta|^2} dxdy$$

in the class $C^2(\Omega) \cap C^{0,1}(\overline{\Omega})$ was constructed using nonparametric variational methods. This corresponds to our functional for the case $b \equiv 0$. However, the existence result proven there is not applicable in the case $b \not\equiv 0$.

2.) Uniqueness of solutions still holds under the assumption $H_z \geq 0$ on the prescribed mean curvature function H of (14). This is, under the assumptions of Theorem 3, equivalent to $(\log a)_{zz} \geq 0$, i.e. the function $\log a$ is convex as a function of the z-variable. Under this assumption one can also show that the second variation of the functional A is positive (see [1]).

3.) However, if one does not assume the function $\log a$ to be convex in the z-variable, the solution may not be unique anymore. Also, critical points of the functional A may not be minimizers anymore.

4.) In [2] the Plateau problem for the parametric version of our functional

$$A(X) := \int_B a(X)|X_u \wedge X_v| dudv$$

was treated (see also [4]). For more general parametric functionals of the form

$$A(X) := \int_B F(X, X_u \wedge X_v) dudv$$

a projectability theorem can be found in [3], which says that under certain assumptions any stable parametric solution X of the Plauteau problem must be a graph over the x, y-plane.

Acknowledgements: This paper is part of the author's dissertation thesis. The author would like to express his sincere gratitute to his advisors Prof. K. Große-Brauckmann and Prof. F. Sauvigny for their support and many helpful discussions.

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