Approximation of Prandtl-Reuss Plasticity through Cosserat-Plasticity

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Abstract

In this article we investigate the regularizing properties of Cosserat elasto-plastic models in a geometrically linear setting. The models feature an independent microrotation field which allow the Cauchy-stress to become non-symmetric while the contribution of the microrotations itself remains linear elastic. Extending previous work we show that for the large class of all quasistatic models of monotone type, solutions to the problem with microrotations are $\mathbb{H}^1$ well-posed. A similar result does not hold for the classical case without microrotations. For vanishing Cosserat effects we show also that the model with microrotations approximates the classical Prandtl-Reuss solution in an appropriate measure valued sense.

Key words: plasticity, polar-materials, coercive models, monotone flow rules, Cosserat continua.

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1 Introduction

Despite the claim to the contrary in most textbooks of mechanics, the symmetry of the Cauchy-stresses in classical elasticity is a postulate or a constitutive assumption but not

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a theorem. In the case of hyperelasticity for simple bodies the requirement of invariance of the free energy under superposed rigid-body motions, however, implies already the symmetry of the Cauchy stresses. If one wants to allow the stresses to become possibly non-symmetric in a variational context it is therefore necessary to introduce additional degrees of freedom, the simplest way consists of introducing additional proper rotations. Such a generalized concept of continua involving independent rotations has been introduced by the Cosserat brothers [CC09].

There is a vast and thriving literature on Cosserat materials. The mathematical analysis of Cosserat micropolar models for elastic continua in the infinitesimal case can be found in several articles, see e.g. [Duv70, HH69, Ghe74a, Ghe74b]. Existence results for a geometrically exact elastic Cosserat model are obtained in [Nef04, Nef05]. For an incomplete overview of the existing literature let us refer to the introduction in [NF04].

Allowing for non-symmetric stresses one should, however, first answer the question for what reason this undoubted complication of the theory should be endeavoured. Because we do not want to enter here in a discussion of the physical relevance of the Cosserat approach the reason for us for using the more complicated model consists in its ability to provide regularizations to non-well posed classical models, notably classical Prandtl-Reuss plasticity without hardening. It seems to be an interesting question to know exactly how much of the problems with the missing global regularity for Prandtl-Reusss is due in essence to the constitutive assumption of a symmetric Cauchy stress. For investigations of the regularity for classical plasticity we refer the reader to [BF96, FM99, FS00].

In line with this view, the Cosserat models have been advocated as a means to regularize the pathological mesh size dependence of localization computations where shear failure mechanisms [CH85, MV87, Müh89, BP91, Bar94] play a dominant role, for applications in plasticity see the non-exhaustive list [IW98, DSW93, RV96, dB91, dBS91, dB92]. The occurring mathematical difficulties reflect the physical fact that upon localization the validity limit of the classical models is reached. In models without any internal length the deformation should be homogeneous on the scale of a representative volume element of the material [MA91].

The model which we will investigate is based upon a model which has been introduced in [Nef03, Nef06b] in a finite strain framework. A geometrical linearization of this model has been investigated in [NC05, NC06] and is shown to be well-posed also in the rate-independent limit for both quasistatic and dynamic processes. In both articles the authors consider the simple case where the so called vector of internal variables z consists of the inelastic deformation tensor only. Here we study the class of all models of monotone type (for the definition see [Alb98]).

The proposed infinitesimal elasto-plastic model has also been implemented into a Finite-Element code [NCMW06] together with an investigation of the convergence of the numerical algorithm based on a time-incremental variational formulation. The first numerical results are promising and will be reported elsewhere. The finite elastic case has been considered in [NMW05, NM06].

This contribution presents the mathematical analysis of systems of equations modelling inelastic deformations of continua of Cosserat micropolar type.

In classical rate-independent elasto-plasticity without microrotations it is proved that global existence for the displacement u can be shown only in a measure-valued sense, while the symmetric Cauchy-stresses σ can be shown to remain in $L^2(\Omega)$. For results of this kind we refer for example to [AL87, Che02, Tem86]. If the Prandtl-Reuss model is considered with additional effects like hardening or viscosity, then global solutions $u \in \mathbb{H}^1$ exist (see [Alb98, Che01b, Che01a]). For the general models of monotone type the existence theory is not completely done (see the remarks in [Che02]). If additional Cosserat effects are taken into account (the stresses σ may get non-symmetric) then the existence of global in time $\mathbb{H}^1$-solutions for simple models (rate-independent and no hardening) was proved in the quasistatic case in [NC05] and in the dynamical case in [NC06].

In this work we study the general inelastic material of monotone type with independent microrotations in the infinitesimal setting of the problem. Similar to [NC05] we prove that in the quasistatic case the system of equations describing inelastic deformations of such a generalized material possesses global in time $\mathbb{H}^1$-solutions to general initial data and to boundary data satisfying fairly mild regularity assumptions. Moreover, we prove that the solution to the model of Melan-Prager with microrotations converges to a solution for the Cosserat-Prandtl-Reuss model without loosing the $\mathbb{H}^1$-regularity (compare with a similar result in the case without independent microrotations [Che01a]). In the last section we study the limit procedure $\mu_c \to 0^+$ of vanishing Cosserat effects in the Cosserat-Prandtl-Reuss model and obtain another approximation result for perfect elasto-plasticity, complementing the similar investigation in [Nef06a] for the elastic case.

2 Formulation of the problem and the main result

Let us start with the formulation of the initial boundary-value problem appearing in the infinitesimal elasto-plasticity model with Cosserat effects. Let us denote by $\Omega \subset \mathbb{R}^3$ a bounded domain with smooth boundary $\partial\Omega$. To determine a quasistatic deformation process of an inelastic body with microrotations in the infinitesimal setting we have to find the displacement vector $u : \Omega \times \mathbb{R} \to \mathbb{R}^3$, the microrotation matrix $A : \Omega \times \mathbb{R} \to \mathfrak{so}(3))$[1] and the vector of internal variables $z : \Omega \times \mathbb{R} \to \mathbb{R}^N$ such that

$$\begin{aligned} \operatorname{Div} \sigma &= -f\,, \\ \sigma &= 2\mu\,(\varepsilon - \varepsilon_p) + 2\,\mu_c\,(\operatorname{skew}(\nabla u) - A) + \lambda\operatorname{tr}[\varepsilon]\cdot 1\!\!1\,, \\ -l_c\,\Delta\operatorname{axl}(A) &= \mu_c\operatorname{axl}(\operatorname{skew}(\nabla u) - A)\,, \\ \dot z &\in F(\varepsilon, z)\,, \\ u_{|\partial\Omega} &= u_d\,,\quad A_{|\partial\Omega} = A_d\,,\quad z(0) = z^0\,. \end{aligned} \tag{2.1}$$

Here, $\varepsilon = \operatorname{sym}(\nabla u)$ denotes the classical infinitesimal elastic strain tensor and ε_p denotes the (still symmetric) inelastic strain tensor which belongs to the set of internal variables. Hence the vector z consists of ε_p and other components needed to describe the deformation process. Let us denote by B the projector $Bz = \varepsilon_p$. μ, λ are positive Lame constants, $\mu_c > 0$ is the Cosserat couple modulus and $l_c := \mu\, L_c^2 > 0$ is a material parameter where L_c with units of length defines an internal length scale. The operator skew denotes the skew-symmetric part of a 3×3 tensor and axl is the standard isomorphism between

[1] $\mathfrak{so}(3)$ denotes the Lie-algebra of skew-symmetric 3×3 matrices.

the set $\mathfrak{so}(3)$ and $\mathbb{R}^3$. This means that if $A = ((0,\alpha,\beta),(-\alpha,0,\gamma),(-\beta,-\gamma,0))$ then $\text{axl}(A) = (\alpha,\beta,\gamma)$. Moreover, F is the inelastic constitutive multifunction, u_d, A_d are given Dirichlet boundary data, z^0 are given initial data and f describes external body forces acting on the material. We refer to $\text{Div}\,\sigma = -f$ as balance of linear momentum with a possibly non-symmetric Cauchy stress tensor σ and the equation for the microrotations is the statement of balance of angular momentum.

Thermodynamical considerations yield (for details see [Alb98, Appendix A, p.143]) that there exists a free energy function $\psi : D(F) \times \mathfrak{so}(3) \to \mathbb{R}_+$ such that for all $(\varepsilon, z) \in D(F)$ and for all $A \in \mathfrak{so}(3)$

$$\rho\frac{\partial\psi(\varepsilon,z,A)}{\partial\varepsilon} = \sigma \qquad \text{(hyperelasticity)}\,, \tag{2.2}$$

$$\langle w^*, \rho\frac{\partial\psi(\varepsilon,z,A)}{\partial z}\rangle \leq 0 \qquad \text{for all} \quad w^* \in F(\varepsilon,z)\,, \tag{2.3}$$

where ρ is the mass density which we assume to be constant. By the second equation in system (2.1) we conclude that the free energy function has to be of the form

$$\rho\psi(\varepsilon,z,A) = \mu\|\varepsilon-\varepsilon_p\|^2 + \frac{\lambda}{2}(\text{tr}\,[\varepsilon])^2 + \mu_c\|\,\text{skew}(\nabla u) - A\|^2 + \psi_1(z,A)\,,$$

where the function ψ_1 is chosen such that the dissipation inequality (2.3) holds. It is not easy to describe all functions ψ_1 for which (2.3) holds. Therefore in the inelastic deformation theory it is usually assumed that ψ_1 is a quadratic form. In this article we additionally assume that

$$\psi_1(z,A) = 2l_c\|\nabla\,\text{axl}(A)\|^2 + \langle Lz, z\rangle$$

and $L \in \mathbb{R}^{N\times N}_{\text{sym}}$ is a positive semi-definite operator such that the operator $Mz = 2\mu B^T Bz + Lz$ is positive definite. Moreover we assume (compare with the monograph [Alb98]) that the constitutive multifunction F is given in the form

$$F(\varepsilon,z) = g\Big(-\rho\nabla_z\psi(\varepsilon,z,A)\Big)$$

with a multifunction $g : D(g) \subset \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ satisfying the monotonicity inequality

$$\forall\, z_1, z_2 \in D(g)\,, \quad \forall\, z_1^* \in g(z_1)\,,\ z_2^* \in g(z_2) \qquad \langle(z_1^*-z_2^*),(z_1-z_2)\rangle \geq 0 \tag{2.4}$$

and additionally $0 \in g(0)$. All models with this structure of inelastic constitutive function are called of **monotone type**.[2] The models of monotone type include e.g. the Prandtl-Ruess model, the Melan-Prager model, the Norton-Hoff model, the Ramberg-Osgood model, special cases of the Bodner-Partom model and many others. It is easy to see that all monotone models are thermodynamical admissible which means that the dissipation inequality is automatically satisfied. Hence, according to all these assumptions system

[2]of pre-monotone type if g is monotone at the point zero only, i.e. $\langle g(\xi),\xi\rangle \geq 0\,\forall\,\xi \in \mathbb{R}^N$.

(2.1) has the form

$$
\begin{aligned}
\operatorname{Div}\sigma &= -f\,,\\
\sigma &= 2\mu\,(\varepsilon-\varepsilon_p)+2\,\mu_c\,(\operatorname{skew}(\nabla u)-A)+\lambda\operatorname{tr}[\varepsilon]\cdot 1\!\!1\,,\\
-l_c\,\Delta\operatorname{axl}(A) &= \mu_c\operatorname{axl}(\operatorname{skew}(\nabla u)-A)\,,\\
\dot z &\in g(-\rho\nabla_z\psi(\varepsilon,z,A))\,,\\
\rho\psi(\varepsilon,z,A) &= \mu\|\varepsilon-\varepsilon_p\|^2+\frac{\lambda}{2}(\operatorname{tr}[\varepsilon])^2+\mu_c\|\operatorname{skew}(\nabla u)-A\|^2+2l_c\|\nabla\operatorname{axl}(A)\|^2+\langle Lz,z\rangle,\\
u_{|\partial\Omega} &= u_d\,,\quad A_{|\partial\Omega}=A_d\,,\quad z(0)=z^0\,.
\end{aligned}
\tag{2.5}
$$

The goal of this article is to prove that system (2.5) is $\mathbb{H}^1$ well-posed if the given data g, f, u_d, A_d, z^0 satisfy some natural restrictions. In this article we use the following standard notation: for an open set $U\subset\mathbb{R}^n$ the symbol $\mathbb{W}^{k,p}(U,\mathbb{R}^N)$ denotes the usual Sobolev space of vector-valued L^p-functions possessing L^p-weak derivatives up to the order k. For $p=2$ we write $\mathbb{W}^{k,2}(U,\mathbb{R}^N)=\mathbb{H}^k(U,\mathbb{R}^N)$. The main result of this article is the following existence and uniqueness theorem for system (2.5):

Theorem 2.1 (Main Theorem)
Suppose that the constitutive multifunction g is a maximal monotone mapping and the given data f, u_d, A_d satisfy: for all times $T>0$

$$
\begin{aligned}
&f\in C^1([0,T],L^2(\Omega,\mathbb{R}^3))\,, && \dddot f\in L^2((0,T)\times\Omega,\mathbb{R}^3)\,,\\
&u_d\in C^2([0,T],\mathbb{H}^{\frac12}(\partial\Omega,\mathbb{R}^3))\,, && \dddot u_d\in L^2((0,T),\mathbb{H}^{\frac12}(\partial\Omega,\mathbb{R}^3))\,,\\
&A_d\in C^2([0,T],\mathbb{H}^{\frac32}(\partial\Omega,\mathfrak{so}(3)))\,, && \dddot A_d\in L^2((0,T),\mathbb{H}^{\frac12}(\partial\Omega,\mathfrak{so}(3)))\,.
\end{aligned}
$$

Moreover, assume that the initial data $z^0\in L^2(\Omega,\mathbb{R}^N)$ are chosen such that the initial value of $\nabla_z\psi$ belongs to the domain of the maximal monotone operator g. Then the system (2.5) possesses a global in time, unique solution (u,z,A) with the regularity: for all $T>0$

$$
\begin{aligned}
&u\in\mathbb{W}^{1,\infty}((0,T),\mathbb{H}^1(\Omega,\mathbb{R}^3))\,, \qquad z\in\mathbb{W}^{1,\infty}((0,T),L^2(\Omega,\mathbb{R}^N))\,,\\
&A\in\mathbb{W}^{1,\infty}((0,T),\mathbb{H}^2(\Omega,\mathfrak{so}(3)))\,.
\end{aligned}
$$

Our Main Theorem implies that for all monotone models in the inelastic deformation theory the independent microrotations have a regularizing effect: the strains remain in L^2 and the solution is found in $\mathbb{H}^1$. This is at variance with the case without Cosserat effects where we observe in the noncoercive case (the operator L is only positive semi-definite as e.g. in plasticity without hardening) a lack of regularity of the strain and the inelastic strain tensors (see for example [Che97, Che98, Che99, Che01a]).

3 Yosida approximation and energy estimates

In this section we present an approximation process for system (2.5) and prove the main estimates for the approximation sequence. The idea of the approximation is very simple. We use the fact that maximal monotone mappings can be approximated by global Lipschitz single-valued functions, in the literature called the Yosida approximation (see for example [AC84, Theorem 2, page 144]). Hence, we rewrite system (2.5) with the Yosida approximation g_η instead of g and try to pass to the limit $\eta \to 0^+$. Thus, for $\eta > 0$ we study the following system of equations

$$
\begin{aligned}
\operatorname{Div} \sigma^\eta &= -f\,, \\
\sigma^\eta &= 2\mu\,(\varepsilon^\eta - \varepsilon_p^\eta) + 2\,\mu_c\,(\operatorname{skew}(\nabla u^\eta) - A) + \lambda\,\operatorname{tr}[\varepsilon^\eta]\cdot \mathbb{1}\,, \\
-l_c\,\Delta\,\operatorname{axl}(A^\eta) &= \mu_c\,\operatorname{axl}(\operatorname{skew}(\nabla u^\eta) - A^\eta)\,, \\
\dot{z^\eta} &= g_\eta(-\rho\nabla_z\psi(\varepsilon^\eta, z^\eta, A^\eta))\,, \\
\rho\psi(\varepsilon^\eta, z^\eta, A^\eta) &= \mu\|\varepsilon^\eta - \varepsilon_p^\eta\|^2 + \frac{\lambda}{2}(\operatorname{tr}[\varepsilon]^\eta)^2 + \mu_c\|\operatorname{skew}(\nabla u^\eta) - A^\eta\|^2, \\
&\quad + 2l_c\|\nabla\,\operatorname{axl}(A^\eta)\|^2 + \langle L z^\eta, z^\eta\rangle\,, \\
u^\eta_{|\partial\Omega} &= u_d\,, \quad A^\eta_{|\partial\Omega} = A_d\,, \quad z^\eta(0) = z^0
\end{aligned}
\tag{3.1}
$$

with the same data f, u_d, A_d, z^0 as for the system (2.5). The next theorem presents existence and uniqueness result for system (3.1).

Theorem 3.1 (Existence and uniqueness for approximated problem) *Let us assume that the given data possess the following regularity: for all $T > 0$*

$$f \in C([0,T], L^2(\Omega,\mathbb{R}^3))\,,\ u_d \in C([0,T], \mathbb{H}^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3))\,,\ A_d \in C([0,T], \mathbb{H}^{\frac{3}{2}}(\partial\Omega, \mathfrak{so}(3)))$$

and the initial data z^0 belong to $L^2(\Omega,\mathbb{R}^N)$. Then the approximated problem has a global in time, unique solution (u^η, z^η, A^η) with the regularity

$$
\begin{aligned}
&u^\eta \in C([0,T], \mathbb{H}^1(\Omega,\mathbb{R}^3))\,, \qquad z^\eta \in C^1([0,T], L^2(\Omega,\mathbb{R}^N))\,, \\
&A^\eta \in C([0,T], \mathbb{H}^2(\Omega, \mathfrak{so}(3)))\,.
\end{aligned}
$$

If the given data are more regular, specifically

$$
\begin{aligned}
&\dot{f} \in C([0,T], L^2(\Omega,\mathbb{R}^3))\,,\ \dot{u}_d \in C([0,T], \mathbb{H}^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3))\,, \\
&\dot{A}_d \in C([0,T], \mathbb{H}^{\frac{3}{2}}(\partial\Omega, \mathfrak{so}(3)))\,,
\end{aligned}
\tag{3.2}
$$

then the solution is C^1 in time.

Proof. The approximated system contains only global Lipschitz nonlinearities. Hence, the proof is standard. For more information we refer to [NC05]. ■

Next, we are going to obtain some estimates for the approximate sequence (u^η, z^η, A^η). The first one is the energy estimate, this means that the energy associated with the system (3.1) is controlled in time by the given data. Let us start with the definition of the energy function. The energy is defined as

$$\mathcal{E}(u,z,A)(t) = \int_\Omega \Big(\mu\|\varepsilon - Bz\|^2 + \frac{\lambda}{2}(\operatorname{tr}[\varepsilon])^2 + \mu_c\|\operatorname{skew}(\nabla u) - A\|^2 + 2l_c\,\|\nabla\,\operatorname{axl}(A)\|^2 + \langle Lz, z\rangle\Big)\,dx\,.$$

The crucial property of the energy is the coerciveness with respect to the displacement-gradient ∇u. The next theorem recalls this crucial result.

Theorem 3.2 (Coerciveness of the energy)
The energy function is elastically coercive with respect to ∇u. This means that
$\exists\, C_E > 0\,,\ \forall\, u \in \mathbb{H}^1_0(\Omega,\mathbb{R}^3)\,,\ \forall\, A \in \mathbb{H}^1(\Omega,\mathfrak{so}(3))\,,\ \forall\, z \in L^2(\Omega,\mathbb{R}^N)$

$$\mathcal{E}(u,z,A) \geq C_E(\|u\|^2_{H^1(\Omega)} + \|A\|^2_{H^1(\Omega)})\,.$$

Moreover,
$\exists\, C_E > 0,\ \forall\, u_d \in \mathbb{H}^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)\,,\quad \forall\, A_d \in \mathbb{H}^{\frac{1}{2}}(\partial\Omega,\mathfrak{so}(3)),\ \exists\, C_d > 0,\ \forall\, z \in L^2(\Omega,\mathbb{R}^N),$
$\forall\, u \in \mathbb{H}^1(\Omega,\mathbb{R}^3),\ \forall\, A \in \mathbb{H}^1(\Omega,\mathfrak{so}(3))$ *with* $u_{|\partial\Omega} = u_d$ *and* $A_{|\partial\Omega} = A_d$ *it holds that*

$$\mathcal{E}(u,z,A) + C_d \geq C_E(\|u\|^2_{H^1(\Omega)} + \|A\|^2_{H^1(\Omega)})\,.$$

A proof of this statement can be found in [NC05, Theorem 3.2]. ■

To estimate the energy function we need initial values of the displacement $u^\eta(0)$ and of the microrotations $A^\eta(0)$. These values are determined by the initial value z^0, more precisely by Bz^0 only. Using the continuity with respect to time we conclude that the initial values $u^\eta(0)$, $A^\eta(0)$ are solutions of the following linear elliptic boundary-value problem

$$\begin{aligned}
&\operatorname{Div}\sigma^\eta(0) = -f(0)\,,\\
&\sigma^\eta(0) = 2\mu\,(\varepsilon^\eta(0) - Bz^\eta(0)) + 2\,\mu_c\,(\operatorname{skew}(\nabla u^\eta(0)) - A^\eta(0)) + \lambda\operatorname{tr}[\varepsilon^\eta(0)]\cdot 1\!\!1\,,\\
&-l_c\Delta\operatorname{axl}(A^\eta(0)) = -\mu_c\ \operatorname{axl}(A^\eta(0)) + \mu_c\ \operatorname{axl}(\operatorname{skew}(\nabla u^\eta(0)))\,,\\
&u^\eta(0)_{|\partial\Omega} = u_d\,,\quad A^\eta(0)_{|\partial\Omega} = A_d\,,
\end{aligned} \tag{3.3}$$

where $\varepsilon^\eta(0) = 1/2(\nabla u^\eta(0) + \nabla^T u^\eta(0))$. The elliptic system (3.3) possesses a unique solution with the regularity $u^\eta(0) \in H^1(\Omega,\mathbb{R}^3)$, $A^\eta(0) \in H^2(\Omega,\mathfrak{so}(3)))$ which is independent of η. This implies that the initial energy value $\mathcal{E}(u^\eta,z^\eta,A^\eta)(0)$ is a constant.

Theorem 3.3 (Energy estimate for the approximate sequence)
Let us assume that the given data have the regularity (3.2) and $\{(u^\eta,z^\eta,A^\eta)\}$ is the solution of the approximate problem (3.1). Then for all $T > 0$ there exists a positive constant $C(T)$, independent of η, such that

$$\mathcal{E}(u^\eta,z^\eta,A^\eta)(t) \leq C(T) \quad \text{for all}\ \ t \in [0,T)\,. \tag{3.4}$$

Proof. Calculating the time derivative of the energy and using the symmetry of the matrix L we obtain

$$\begin{aligned}
\dot{\mathcal{E}}(u^\eta,z^\eta,A^\eta)(t) = \int_\Omega \Big(&2\mu\langle\varepsilon^\eta - Bz^\eta, \dot\varepsilon^\eta - B\dot z^\eta\rangle + \lambda\operatorname{tr}[\varepsilon^\eta]\operatorname{tr}[\dot\varepsilon^\eta] + \langle Lz^\eta,\dot z^\eta\rangle\\
&+2\mu_c\langle\operatorname{skew}(\nabla u^\eta) - A^\eta, \operatorname{skew}(\nabla\dot u^\eta) - \dot A^\eta\rangle + 4l_c\langle\nabla\operatorname{axl}(A^\eta),\nabla\operatorname{axl}(\dot A^\eta)\rangle\Big)\,dx\,.
\end{aligned}$$

Let us observe that $-\langle\varepsilon^\eta - Bz^\eta, B\dot z^\eta\rangle + \langle Lz^\eta,\dot z^\eta\rangle = -\langle B^T(\varepsilon^\eta - Bz^\eta) + Lz^\eta, \dot z^\eta\rangle$. Moreover, from the definition of the free energy we have that $\rho\nabla\psi(\varepsilon^\eta,z^\eta,A^\eta) = B^T(\varepsilon^\eta - Bz^\eta) + Lz^\eta$.

These equalities allow us to write that

$$\dot{\mathcal{E}}(u^\eta, z^\eta, A^\eta)(t) = -\int\limits_\Omega \langle \rho\nabla_z\psi(\varepsilon^\eta, z^\eta, A^\eta), \dot{z}^\eta\rangle dx$$
$$+\int\limits_\Omega \langle \sigma^\eta, \nabla\dot{u}^\eta\rangle dx - 2\mu_c\int\limits_\Omega \langle \mathrm{skew}(\nabla u^\eta) - A^\eta, \dot{A}^\eta\rangle dx$$
$$+4l_c\int\limits_\Omega \langle \nabla\,\mathrm{axl}(A^\eta), \nabla\,\mathrm{axl}(\dot{A}^\eta)\rangle dx\,.$$

By the monotonicity of the Yosida approximation g_η and the property $0 = g_\eta(0)$ we see that the first term on the right hand side of the last inequality is nonpositive. Next, we integrate partially in the second and in the fourth integral. Finally, using the balance of linear and angular momentum and the boundary conditions we arrive at the inequality

$$\dot{\mathcal{E}}(u^\eta, z^\eta, A^\eta)(t) \le \int\limits_\Omega \langle f, \dot{u}^\eta\rangle dx + \int\limits_{\partial\Omega} \langle \sigma^\eta.n, \dot{u}_d\rangle ds$$
$$+4l_c\int\limits_{\partial\Omega} \langle \nabla\,\mathrm{axl}(A^\eta).n, \mathrm{axl}(\dot{A}_d)\rangle ds\,. \tag{3.5}$$

Integrating (3.5) in time we have

$$\mathcal{E}(u^\eta, z^\eta, A^\eta)(t) \le \mathcal{E}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta)(0) + \int\limits_0^t\int\limits_\Omega \langle f, \dot{u}^\eta\rangle dx$$
$$+\int\limits_0^t\int\limits_{\partial\Omega} \langle \sigma^\eta.n, \dot{u}_d\rangle ds + 4l_c\int\limits_0^t\int\limits_{\partial\Omega} \langle \nabla\,\mathrm{axl}(A^\eta).n, \mathrm{axl}(\dot{A}_d)\rangle ds\,. \tag{3.6}$$

This inequality is the same as in the proof of Theorem 3.3 given in [NC05]. Hence, for the remaining part of the proof we refer to this article. ■

The result obtained in the last theorem yields some boundedness of the sequence (u^η, z^η, A^η). To pass to the limit in system (3.1) we need estimates for the time derivatives of this sequence. In all initial boundary-value problems in the inelastic deformation theory such an estimate is the crucial ingredient.

Theorem 3.4 (Energy estimate for time derivatives)
Suppose that the given data possess the following regularity : for all times $T > 0$

$$\begin{array}{ll} f \in C^1([0,T], L^2(\Omega,\mathbb{R}^3))\,, & \ddot{f} \in L^2((0,T)\times\Omega,\mathbb{R}^3)\,,\\ u_d \in C^2([0,T], \mathbb{H}^{\frac12}(\partial\Omega,\mathbb{R}^3))\,, & \dddot{u}_d \in L^2((0,T), \mathbb{H}^{\frac12}(\partial\Omega,\mathbb{R}^3))\,,\\ A_d \in C^2([0,T], \mathbb{H}^{\frac32}(\partial\Omega,\mathfrak{so}(3)))\,, & \dddot{A}_d \in L^2((0,T), \mathbb{H}^{\frac12}(\partial\Omega,\mathfrak{so}(3)))\,. \end{array} \tag{3.7}$$

Moreover, assume that the initial data z^0 are chosen such that the initial value of the argument of the constitutive multifunction $-\rho\nabla_z\psi(\varepsilon^\eta(0), z^0, A^\eta(0))$ defined by the system

(3.3) belongs to the domain of the maximal monotone operator g. Then there exists a positive constant $C(T)$, independent of the parameter η, such that

$$\mathcal{E}(\dot{u}^\eta, \dot{z}^\eta, \dot{A}^\eta)(t) \leq C(T) \quad \text{for all } \; t \in [0, T)\,.$$

Proof. For $h > 0$ let us denote by $(u^\eta_h(t), z^\eta_h(t), A^\eta_h(t))$ the shifted functions $(u^\eta(t+h), z^\eta(t+h), A^\eta(t+h))$ and calculate the time derivative of the energy evaluated on the differences $(u^\eta_h - u^\eta, z^\eta_h - z^\eta, A^\eta_h - A^\eta)$. Then we arrive at the equality

$$\begin{aligned}
\dot{\mathcal{E}}(u^\eta_h - u^\eta, z^\eta_h - z^\eta, A^\eta_h - A^\eta)(t) = &\int_\Omega 2\mu\, \langle \varepsilon^\eta_h - \varepsilon^\eta - Bz^\eta_h + Bz^\eta, \dot{\varepsilon}^\eta_h - \dot{\varepsilon}^\eta - B\dot{z}^\eta_h + B\dot{z}^\eta\rangle dx \\
&+2\mu_c \int_\Omega \langle \mathrm{skew}(\nabla u^\eta_h - \nabla u^\eta) - A^\eta_h + A^\eta, \mathrm{skew}(\nabla \dot{u}^\eta_h - \nabla \dot{u}^\eta) - \dot{A}^\eta_h + \dot{A}^\eta\rangle dx \\
&+\lambda \int_\Omega \mathrm{tr}\,[\varepsilon^\eta_h - \varepsilon^\eta]\mathrm{tr}\,[\dot{\varepsilon}^\eta_h - \dot{\varepsilon}^\eta]\, dx + 4l_c \int_\Omega \langle \nabla\,\mathrm{axl}(A^\eta_h - A^\eta), \nabla\,\mathrm{axl}(\dot{A}^\eta_h - \dot{A}^\eta)\rangle dx \\
&+ \int_\Omega \langle Lz^\eta_h - Lz^\eta, \dot{z}^\eta_h - \dot{z}^\eta\rangle dx
\end{aligned} \tag{3.8}$$

where $\varepsilon^\eta_h(t) = \varepsilon^\eta(t+h)$. Next, using the definition of the free energy we see that

$$\rho\nabla_z\psi(\varepsilon^\eta_h - \varepsilon^\eta, z^\eta_h - z^\eta, A^\eta_h - A^\eta) = B^T(\varepsilon^\eta_h - \varepsilon^\eta - Bz^\eta_h + Bz^\eta) + L(z^\eta_h - z^\eta)\,.$$

Let us define the shifted value of the stress tensor $\sigma^\eta_h(t) = \sigma^\eta(t+h)$. Then using the elastic constitutive relation and the linearity of the derivative of the free energy we obtain

$$\begin{aligned}
&\dot{\mathcal{E}}(u^\eta_h - u^\eta, z^\eta_h - z^\eta, A^\eta_h - A^\eta)(t) \\
&= -\int_\Omega \langle -\rho\nabla_z\psi(\varepsilon^\eta_h, z^\eta_h, A^\eta_h) + \rho\nabla_z\psi(\varepsilon^\eta, z^\eta, A^\eta), \dot{z}^\eta_h - \dot{z}^\eta\rangle dx \\
&+ \int_\Omega \langle \sigma^\eta_h - \sigma^\eta, \nabla \dot{u}^\eta_h - \nabla\dot{u}^\eta\rangle dx \\
&-4\mu_c \int_\Omega \langle \mathrm{axl}\,\mathrm{skew}(\nabla u^\eta_h - \nabla u^\eta) - \mathrm{axl}(A^\eta_h - A^\eta), \mathrm{axl}(\dot{A}^\eta_h - \dot{A}^\eta)\rangle dx \\
&+4l_c \int_\Omega \langle \nabla\,\mathrm{axl}(A^\eta_h - A^\eta), \nabla\,\mathrm{axl}(\dot{A}^\eta_h - \dot{A}^\eta)\rangle dx\,,
\end{aligned} \tag{3.9}$$

By the monotonicity of the Yosida approximation the first term on the right hand side of (3.9) is nonpositive. In the same manner as in the energy estimate in Theorem 3.3 we integrate by parts in the second and in the fourth integral and use the balance of linear momentum and the equation for the microrotations. Hence, we may conclude that

$$\begin{aligned}
\dot{\mathcal{E}}(u^\eta_h - u^\eta, z^\eta_h - z^\eta, A^\eta_h - A^\eta)(t) \leq &\int_\Omega \langle f_h - f, \dot{u}^\eta_h - \dot{u}^\eta\rangle dx \\
+ \int_{\partial\Omega} \langle (\sigma^\eta_h - \sigma^\eta).n, \dot{u}_{d,h} - \dot{u}_d\rangle ds &+ 4l_c \int_{\partial\Omega} \langle \nabla\,\mathrm{axl}(A^\eta_h - A^\eta).n, \mathrm{axl}(\dot{A}_{d,h} - \dot{A}_d\rangle ds\,,
\end{aligned} \tag{3.10}$$

where $f_h(t) = f(t+h)$, $u_{d,h}(t) = u_d(t+h)$ and $A_{d,h}(t) = A_d(t+h)$. Integrating (3.10) in time we arrive at the inequality

$$\mathcal{E}(u^\eta_h - u^\eta, z^\eta_h - z^\eta, A^\eta_h - A^\eta)(t) \le \mathcal{E}(u^\eta_h - u^\eta, z^\eta_h - z^\eta, A^\eta_h - A^\eta)(0)$$
$$+ \int_0^t \int_\Omega \langle f_h - f, \dot{u}^\eta_h - \dot{u}^\eta\rangle dx\, d\tau + \int_0^t \int_{\partial\Omega} \langle(\sigma^\eta_h - \sigma^\eta).n, \dot{u}_{d,h} - \dot{u}_d\rangle ds\, d\tau \qquad (3.11)$$
$$+4l_c \int_0^t \int_{\partial\Omega} \langle \nabla \operatorname{axl}(A^\eta_h - A^\eta).n, \operatorname{axl}(\dot{A}_{d,h} - \dot{A}_d\rangle ds\, d\tau\,.$$

The last inequality is the same as inequality [NC05, 3.25]. Hence, the remaining part of the proof is the same as in [NC05]. For the convenience of the reader we add a sketch of this part. In (3.11) in the integral term on the right hand side we shift the shift operator onto given data. Next, we divide the result by h^2 and pass to the limit $h \to 0^+$. By assumption the initial value $-\rho\nabla_z\psi(\varepsilon^\eta(0), z^0, A^\eta(0))$ belongs to the domain of the maximal monotone operator g. This implies that the sequence $g_\eta\Big(-\rho\nabla_z\psi(\varepsilon^\eta(0), z^0, A^\eta(0))\Big)$ is bounded in $L^2(\Omega, \mathbb{R}^N)$. Thus, the sequence $B\dot{z}^\eta(0)$ is bounded in $L^2(\Omega, \mathrm{Sym}(3))$. Calculating the initial values $\dot{u}^\eta, \dot{A}^\eta$ from (3.3) with $B\dot{z}^\eta$ instead of Bz^0 we conclude that the initial value of the energy evaluated for the time derivatives is bounded. Using the regularity assumptions of the data and coerciveness of the energy we complete the proof. ∎

4 Estimate for differences of two approximation steps and proof of the main theorem

The energy estimates proved in the last section yield that the sequence of stresses $\{\sigma^\eta\}$ and the sequence of their time derivatives $\{\dot{\sigma}^\eta\}$ are bounded in $L^\infty((0,T), L^2(\Omega, \mathbb{R}^{3\times3}))$. By the coerciveness of the energy the strains $\{\varepsilon^\eta\}$ and the strain rates $\{\dot{\varepsilon}^\eta\}$ are bounded in $L^\infty((0,T), L^2(\Omega, \mathbb{R}^{3\times3}))$. Moreover, using that the operator $Mz = 2\mu B^T Bz + Lz$ is positive definite we have that the sequences $\{z^\eta\}$, $\{\dot{z}^\eta\}$ are bounded in $L^\infty((0,T), L^2(\Omega, \mathbb{R}^N))$. Finally, the microrotations $\{A^\eta\}$ and their time derivatives $\{\dot{A}^\eta\}$ are bounded in the space $L^\infty((0,T), H^1(\Omega, \mathfrak{so}(3)))$. Hence, for a subsequence (again denoted using the superscript η) we have

$$\begin{aligned}
\sigma^\eta &\overset{*}{\rightharpoonup} \sigma && \text{in} \quad L^\infty((0,T), L^2(\Omega, \mathbb{R}^{3\times3}))\,,\\
\dot{\sigma}^\eta &\overset{*}{\rightharpoonup} \dot{\sigma} && \text{in} \quad L^\infty((0,T), L^2(\Omega, \mathbb{R}^{3\times3}))\,,\\
A^\eta &\overset{*}{\rightharpoonup} A && \text{in} \quad L^\infty((0,T), \mathbb{H}^1(\Omega, \mathfrak{so}(3)))\,,\\
\dot{A}^\eta &\overset{*}{\rightharpoonup} \dot{A} && \text{in} \quad L^\infty((0,T), \mathbb{H}^1(\Omega, \mathfrak{so}(3)))\,,\\
u^\eta &\overset{*}{\rightharpoonup} u && \text{in} \quad L^\infty((0,T), \mathbb{H}^1(\Omega, \mathbb{R}^3))\,,\\
\dot{u}^\eta &\overset{*}{\rightharpoonup} \dot{u} && \text{in} \quad L^\infty((0,T), \mathbb{H}^1(\Omega, \mathbb{R}^3))\,,\\
z^\eta &\overset{*}{\rightharpoonup} z && \text{in} \quad L^\infty((0,T), L^2(\Omega, \mathbb{R}^N))\,,\\
\dot{z}^\eta &\overset{*}{\rightharpoonup} \dot{z} && \text{in} \quad L^\infty((0,T), L^2(\Omega, \mathbb{R}^N))\,,
\end{aligned}$$

for all $T > 0$. Hence, the sequence of strains $\{\varepsilon^\eta\}$ converges weakly to $\varepsilon = \frac{1}{2}(\nabla u + \nabla^T u)$. Consequently, the limit functions satisfy

$$\begin{aligned} \operatorname{Div}\sigma &= -f\,, \\ \sigma &= 2\mu\,(\varepsilon - Bz) + 2\,\mu_c\,(\operatorname{skew}(\nabla u) - A) + \lambda\operatorname{tr}[\varepsilon]\cdot 1\!\!1\,, \\ -l_c\,\Delta\operatorname{axl}(A) &= \mu_c\ \operatorname{axl}(\operatorname{skew}(\nabla u) - A)\,, \\ \dot z &= \mathrm{w-}\lim_{\eta\to 0^+} g_\eta\Big(-\rho\nabla_z\psi(\varepsilon^\eta, z^\eta, A^\eta)\Big)\,, \\ u_{|\partial\Omega} &= u_d\,,\quad A_{|\partial\Omega} = A_d\,,\quad z(0) = z^0\,. \end{aligned} \tag{4.1}$$

To end the existence theory we need to prove that

$$\mathrm{w-}\lim_{\eta\to 0^+} g_\eta\Big(-\rho\nabla_z\psi(\varepsilon^\eta, z^\eta, A^\eta)\Big) \in g\Big(-\rho\nabla_z\psi(\varepsilon, z, A)\Big)\,. \tag{4.2}$$

where $\mathrm{w-}\lim$ denotes the weak limit in the space $L^\infty((0,T), L^2(\Omega,\mathbb{R}^N))$. This can be done if we improve the convergence of the sequence $\{\nabla_z\psi(\varepsilon^\eta, z^\eta, A^\eta)\}$.

Theorem 4.1 (Estimate for difference of two approximation steps)
Let us assume that the given data satisfy all requirements of Theorem 3.4. Then $\mathcal{E}(u^\eta - u^\nu, z^\eta - z^\nu, A^\eta - A^\nu)(t) \to 0$ for $\eta,\nu\to 0^+$ uniformly on bounded time intervals.

Proof. Calculating the time derivative of the energy evaluated on the differences of two approximation steps we obtain

$$\begin{aligned} \dot{\mathcal{E}}(u^\eta - u^\nu, z_p^\eta - z^\nu, A^\eta - A^\nu)(t) &= 2\mu\int_\Omega \langle \varepsilon^\eta - \varepsilon^\nu - Bz^\eta + Bz^\nu, \dot\varepsilon^\eta - \dot\varepsilon^\nu - B\dot z^\eta + B\dot z^\nu\rangle dx \\ &+\lambda\int_\Omega \operatorname{tr}[\varepsilon^\eta - \varepsilon^\nu]\operatorname{tr}[\dot\varepsilon^\eta - \dot\varepsilon^\nu]\,dx + 4l_c\int_\Omega \langle\nabla\operatorname{axl}(A^\eta - A^\nu), \nabla\operatorname{axl}(\dot A^\eta - \dot A^\nu)\rangle dx \\ &+2\mu_c\int_\Omega \langle\operatorname{skew}(\nabla u^\eta - \nabla u^\nu) - A^\eta + A^\nu, \operatorname{skew}(\nabla\dot u^\eta - \nabla\dot u^\nu) - \dot A^\eta + \dot A^\nu\rangle dx\,. \\ = -\int_\Omega &\langle -\rho\nabla_z\psi(\varepsilon^\eta, z^\eta, A^\eta) + \rho\nabla_z\psi(\varepsilon^\nu, z^\nu, A^\nu), \\ &\qquad g_\eta\Big(-\rho\nabla_z\psi(\varepsilon^\eta, z^\eta, A^\eta)\Big) - g_\nu\Big(-\rho\nabla_z\psi(\varepsilon^\nu, z^\nu, A^\nu)\Big)\rangle dx\,. \end{aligned}$$

(to obtain the last equality we have used that the given data for both approximation steps are the same). Next, to estimate the integral on the right hand side we use the method from [AC84, Theorem 1 p. 147]) and conclude that

$$\begin{aligned} -\int_\Omega &\langle -\rho\nabla_z\psi(\varepsilon^\eta, z^\eta, A^\eta) + \rho\nabla_z\psi(\varepsilon^\nu, z^\nu, A^\nu), \\ &\qquad g_\eta\Big(-\rho\nabla_z\psi(\varepsilon^\eta, z^\eta, A^\eta)\Big) - g_\nu\Big(-\rho\nabla_z\psi(\varepsilon^\nu, z^\nu, A^\nu)\Big)\rangle dx \\ &\le \frac{\eta}{4}\|\dot z^\nu\|^2_{L^2} + \frac{\nu}{4}\|\dot z^\eta\|^2_{L^2}\,. \end{aligned}$$

The energy estimate for the time derivatives implies that the norms $\|\dot{z}^\nu\|^2_{L^2}$, $\|\dot{z}^\eta\|^2_{L^2}$ are bounded independent of η, ν. Hence, integrating in time the last inequality completes the proof. ■

Theorem 4.1 yields that the sequence $\{\nabla_z\psi(\varepsilon^\eta, z^\eta, A^\eta)\}$ converges strongly to $\nabla_z\psi(\varepsilon, z, A)$ in the space $L^\infty((0,T), L^2(\Omega, \mathbb{R}^N))$. Thus, using the standard properties of maximal monotone operators and the Yosida approximation we deduce that the inclusion (4.2) holds. To end the proof of the Main Theorem we have to prove uniqueness of solutions to system (2.5).

Theorem 4.2 (Uniqueness of solutions)
Let us assume that the given data f, u_d, A_d, z^0 satisfy all requirements of Theorem 3.4 Then system (2.5) possesses a unique, global in time solution (u, z, A).

Proof. Assume that (u^1, z^1, A^1) and (u^2, z^2, A^2) are two solutions of (2.5) for the same given data. Calculating the time derivative of the energy evaluated on the differences of two solutions we arrive at the inequality

$$
\begin{aligned}
\dot{\mathcal{E}}(u^1 - u^2, z^1 - z^2, A^1 - A^2)(t) &= 2\mu \int_\Omega \langle \varepsilon^1 - \varepsilon^2 - Bz^1 + Bz^2, \dot{\varepsilon}^1 - \dot{\varepsilon}^2 - B\dot{z}^1 + B\dot{z}^2\rangle dx \\
&\quad + \lambda \int_\Omega \mathrm{tr}\,[\varepsilon^\eta - \varepsilon^\nu]\mathrm{tr}\,[\dot{\varepsilon}^\eta - \dot{\varepsilon}^\nu]\, dx + 4l_c \int_\Omega \langle \nabla\,\mathrm{axl}(A^1 - A^2), \nabla\,\mathrm{axl}(\dot{A}^1 - \dot{A}^2)\rangle dx \\
&\quad + 2\mu_c \int_\Omega \langle \mathrm{skew}(\nabla u^1 - \nabla u^2) - A^1 + A^2, \mathrm{skew}(\nabla \dot{u}^1 - \nabla \dot{u}^2) - \dot{A}^1 + \dot{A}^2\rangle dx \\
&\quad + \int_\Omega \langle Lz^1 - Lz^2, \dot{z}^1 - \dot{z}^2\rangle dx \\
&= -\int_\Omega \langle -\rho\nabla_z\psi(\varepsilon^1, z^1, A^1) + \rho\nabla_z\psi(\varepsilon^2, z^2, A^2), \dot{z}^1 - \dot{z}^2\rangle dx \le 0\,.
\end{aligned}
$$

Consequently

$$
\mathcal{E}(u^1 - u^2, z^1 - z^2, A^1 - A^2)(t) \le \mathcal{E}(u^1 - u^2, z^1 - z^2, A^1 - A^2)(0) = 0
$$

and get the result using the coerciveness of the energy function. ■

5 Cosserat perfect elasto-plasticity as a limit of Melan-Prager models

In [Che01a] it is proved that in the case without independent microrotations solutions of the Melan-Prager problem converge to a solution of perfect elasto-plasticity if the evolution of the additional kinematic hardening "tends to zero". Now we are able to prove a similar result for inelastic continua with Cosserat effects. Let us recall the structure of the Melan-Prager model and of the perfect elasto-plasticity model. Following [Che01a], the system of equations modeling perfect elasto-plasticity belongs to the class of simple models of monotone type. This means that the vector of internal variables z consists of ε_p only and the inelastic constitutive equation is of monotone type:

$$\begin{aligned} \operatorname{Div}\sigma &= -f\,, \\ \sigma &= 2\mu\,(\varepsilon-\varepsilon_p) + 2\,\mu_c\,(\operatorname{skew}(\nabla u) - A) + \lambda\,\operatorname{tr}[\varepsilon]\cdot 1\!\!1\,, \\ -l_c\,\Delta\,\operatorname{axl}(A) &= \mu_c\,\operatorname{axl}(\operatorname{skew}(\nabla u) - A)\,, \\ \dot\varepsilon_p &\in \partial I_K\Big(2\mu\,(\varepsilon-\varepsilon_p)\Big)\,. \end{aligned} \tag{5.1}$$

Here K denotes the set of admissible stresses, which we assume to be of the form $K^d\times\mathbb{R}$ and $K^d \subset \operatorname{dev}\operatorname{Sym}(3)$ is closed, convex, bounded set with $0\in\operatorname{int}(K^d)$. $\operatorname{dev}\operatorname{Sym}(3)$ denotes the subspace of $\operatorname{Sym}(3)$ consisting of deviatoric (trace-free) parts of symmetric matrices. The function $I_K : \operatorname{Sym}(3)\to\mathbb{R}_+$ is the indicator function of the set K, this means that

$$I_K(\tau) = \begin{cases} 0 & \text{for } \tau\in K\,, \\ \infty & \text{for } \tau\notin K\,. \end{cases}$$

Finally, ∂I_K denotes the subgradient of the convex function I_K. It is easy to see that the constitutive multifunction ∂I_K is maximal monotone and the existence theory presented in this article (for simple models without Cosserat effects see also [NC05]) yields $\mathbb{H}^1$ well-posedness of this model.

The Melan-Prager model is a modification of the perfect elasto-plasticity. In this model the vector z contains ε_p and additionally the backstress $b\in\operatorname{Sym}(3)$. The system of equations has now the form

$$\begin{aligned} \operatorname{Div}\sigma &= -f\,, \\ \sigma &= 2\mu\,(\varepsilon-\varepsilon_p) + 2\,\mu_c\,(\operatorname{skew}(\nabla u) - A) + \lambda\,\operatorname{tr}[\varepsilon]\cdot 1\!\!1\,, \\ -l_c\,\Delta\,\operatorname{axl}(A) &= \mu_c\,\operatorname{axl}(\operatorname{skew}(\nabla u) - A)\,, \\ \dot\varepsilon_p &\in \partial I_K\Big(2\mu\,(\varepsilon-\varepsilon_p) - b\Big)\,, \\ \dot b &= \gamma\dot\varepsilon_p\,, \end{aligned} \tag{5.2}$$

where $\gamma>0$ is a material parameter and the set of admissible stresses K is defined in the same manner as in system (5.1). Similar to perfect elasto-plasticity the Melan-Prager model is also $\mathbb{H}^1$ well-posed:

Theorem 5.1
Let us assume that $\{(u^\gamma,\varepsilon_p^\gamma,b^\gamma,A^\gamma)\}$ is a sequence of solutions to the Melan-Prager model for $\gamma>0$ with boundary data independent of γ. Suppose that the initial value $b^{0,\gamma}$ for the backstress is equal to $\gamma\varepsilon_p^0$ where ε_p^0 is the initial value of the inelastic strain tensor. Moreover, assume that the initial data is chosen such that the initial value of the argument of the constitutive multifunction satisfies $2\mu(\varepsilon^\gamma(0)-\varepsilon_p^0)-b^{0,\gamma}\in D(\partial I_K)$ for each γ. Then the sequence $\{(u^\gamma,\varepsilon_p^\gamma,A^\gamma)\}$ converges to the solution of the perfect elasto-plasticity model considered with the same boundary data as for the Melan-Prager model and with the initial inelastic strain equal to ε_p^0.

Proof. The proof is a simple consequence of the Main Theorem. The free energy function associated with the Melan-Prager model has the form

$$\mathcal{E}(u^\gamma, \varepsilon_p^\gamma, b^\gamma, A^\gamma)(t) = \int_\Omega \Big(\mu\|\varepsilon^\gamma - \varepsilon_p^\gamma\|^2 + \frac{\lambda}{2}(\operatorname{tr}[\varepsilon^\gamma])^2 + \mu_c\|\operatorname{skew}(\nabla u^\gamma) - A^\gamma\|^2 + 2l_c\,\|\nabla\operatorname{axl}(A^\gamma)\|^2 + \frac{\gamma}{2}|b^\gamma|^2\Big)dx\,.$$

Hence, the operator L is defined by $Lz^\gamma = L(\varepsilon_p^\gamma, b^\gamma) = (0, \frac{\gamma}{2}b^\gamma)$ and the operator $M(\varepsilon_p^\gamma, b^\gamma) = (2\mu\varepsilon_p^\gamma, \frac{\gamma}{2}b^\gamma)$ is positive definite. By Theorem 3.2 the energy function is coercive with respect to variable ε^γ and the coerciveness constant does not depend on γ. Consequently, the energy is coercive with respect to variable ε_p^γ independent of γ. This observation implies that the constants in Theorems 3.3 and 3.4 do not depend on γ. Hence, going to a subsequence if necessary we conclude that the sequence $\{(u^\gamma, \varepsilon_p^\gamma, b^\gamma, A^\gamma)\}$ converges weakly to (u, ε, b, A) in the space $\mathbb{W}^{1,\infty}(L^2)$. By assumption the sequence of the initial data $\{b^{0,\gamma}\}$ converges strongly to zero and consequently the sequence b^γ converges strongly to zero. To prove that functions (u, ε_p, A) satisfy (5.1) we use Theorem 4.1. This theorem gives that $\{2\mu(\varepsilon^\gamma - \varepsilon_p^\gamma) - b^\gamma\}$ converges strongly. Hence, the limit function is equal to $2\mu(\varepsilon - \varepsilon_p)$ where ε is the symmetric part of the gradient of u. Using standard properties of maximal monotone operators we conclude that (u, ε_p, A) is a solution to (5.1). Theorem 4.2 immediately yields that the whole approximating sequence $\{(u^\gamma, \varepsilon_p^\gamma, b^\gamma, A^\gamma)\}$ converges to $(u, \varepsilon_p, 0, A)$. ■

Note, that our result yields that the inelastic deformation theory with independent microrotations is stable under limit procedures involving some parameters appearing in the components Bz of the vector z. This result differs from similar limit procedures in the classical inelastic deformation theory (compare with [Che97, Che98, Che99, Che00, Che01a, CN02]).

6 Approximation of perfect elasto-plasticity

In this section we are going to study the limit process $\mu_c \to 0^+$ of vanishing Cosserat effects in the Cosserat-Prandtl-Reuss model (5.1). The same investigation for pure elasticity has been done in [Nef06a]. Let us denote by $(u^{\mu_c}, \varepsilon^{\mu_c}, A^{\mu_c})$ the global in time L^2-solution of the model. This means that these functions satisfy

$$\begin{aligned}
\operatorname{Div}\sigma^{\mu_c} &= -f\,,\\
\sigma^{\mu_c} &= 2\mu\,(\varepsilon^{\mu_c} - \varepsilon_p^{\mu_c}) + 2\,\mu_c\,(\operatorname{skew}(\nabla u^{\mu_c}) - A^{\mu_c}) + \lambda\operatorname{tr}[\varepsilon^{\mu_c}]\cdot 1\!\!1\,,\\
-l_c\,\Delta\operatorname{axl}(A^{\mu_c}) &= \mu_c\operatorname{axl}(\operatorname{skew}(\nabla u^{\mu_c}) - A^{\mu_c})\,,\\
\dot\varepsilon_p^{\mu_c} &\in \partial I_K\Big(2\mu\,(\varepsilon^{\mu_c} - \varepsilon_p^{\mu_c})\Big)\,,\\
u^{\mu_c}_{|\partial\Omega} &= u_d\,,\quad A^{\mu_c}_{|\partial\Omega} = A_d\,,\quad \varepsilon_p^{\mu_c}(0) = \varepsilon_p^0\,.
\end{aligned} \tag{6.3}$$

The goal of this section is to prove that under the so called **safe load condition** for the forces acting on the material the sequence of solutions converges to a solution in the measure-valued sense of the Prandtl-Reuss model for $\mu_c \to 0^+$. Moreover, we show

that the field of microrotations A^{μ_c} converges to the unique harmonic function A which satisfies the boundary condition $A_{|\partial\Omega} = A_d$. In other words, in the limit of vanishing Cosserat effects the microrotations are decoupled from the equations of perfect elasto-plasticity.
Let us denote by $\mathcal{M}^{3\times 3}_{\rm sym}(\Omega)$ the Banach space containing all bounded Radon measures in Ω with values in $\mathrm{Sym}(3)$, by $\operatorname{dev}\mathcal{M}^{3\times 3}_{\rm sym}(\Omega)$ the subspace of $\mathcal{M}^{3\times 3}_{\rm sym}(\Omega)$ consisting of measures with values in $\operatorname{dev}\mathrm{Sym}(3)$ and by $\mathcal{BD}(\Omega)$ the space of bounded deformations $\{u \in L^1(\Omega,\mathbb{R}^3) : 1/2(\nabla u + \nabla^T u) \in \mathcal{M}^{3\times 3}_{\rm sym}(\Omega)\}$ (for more details see [EG92, Tem83]). Moreover, let us denote with $L^\infty_w((0,T),X)$ the space of bounded and weakly measurable functions defined on the interval $(0,T)$ with values in the Banach space X. Now we can define solutions in the measure-valued sense of the perfect elasto-plasticity model.

Definition 6.1 (measure-valued solutions)
Let f be a given external force and u_d be a given Dirichlet boundary data. We say that a pair (u,ε_p) satisfies the equations of the Prandtl-Reuss model in the sense of measures if $u \in \mathbb{W}^{1,\infty}_w((0,T),\mathcal{BD}(\Omega))$, $\varepsilon_p \in \mathbb{W}^{1,\infty}_w((0,T),\operatorname{dev}\mathcal{M}^{3\times 3}_{\rm sym}(\Omega))$,

$$\frac{1}{2}(\nabla u + \nabla^T u) - \varepsilon_p \in \mathbb{W}^{1,\infty}((0,T),L^2(\Omega,\mathrm{Sym}(3)))\,,$$

the balance of linear momentum

$$-\operatorname{Div}\sigma = -\operatorname{Div}(2\mu(\varepsilon-\varepsilon_p) + \lambda\mathrm{tr}\,[\varepsilon]\cdot 1\!\!1) = f$$

is satisfied in the L^2-sense (here $\varepsilon = 1/2(\nabla u + \nabla^T u)$) and the inelastic constitutive equation

$$\dot\varepsilon_p \in \partial I_K(\sigma)$$

is satisfied in the measure sense. This means that for all $\tau \in L^2(\Omega,\mathrm{Sym}(3))$, such that $\operatorname{Div}\tau \in L^2(\Omega,\mathbb{R})$ and $\tau(x)\in K$ for a.e. $x\in\Omega$ the expression

$$\langle\dot\varepsilon_p,(\sigma-\tau)\rangle \quad \textit{is a nonnegative measure}\,.$$

Moreover, the Dirichlet boundary condition is satisfied in the normal direction, this means that $u_{|\partial\Omega}\cdot n = u_d\cdot n$, where n is the normal unit vector to the boundary $\partial\Omega$.

Next we define a condition for the given force f. This is the so called safe-load condition.

Definition 6.2 (safe-load condition)
We say that the given force f satisfies the safe-load condition if there exists a function $u^*_d \in \mathbb{W}^{1,\infty}((0,T),\mathbb{H}^1(\Omega,\mathbb{R}^3))$ such that the following linear elastic problem

$$\begin{aligned} -\operatorname{Div}\sigma^*(x,t) &= f(x,t)\,,\\ \sigma^*(x,t) &= 2\mu\,\varepsilon(u^*(x,t)) + \lambda\mathrm{tr}\,[\varepsilon(u^*(x,t))]\cdot 1\!\!1\,,\\ u^*(x,t)_{|\partial\Omega} &= u^*_d(x,t)_{|\partial\Omega} \end{aligned}$$

possesses a solution $u^* \in \mathbb{W}^{1,\infty}((0,T),\mathbb{H}^1(\Omega,\mathbb{R}^3))$ such that $\sigma^*(x,t)\in K$ for a.e. $(x,t)\in\Omega\times(0,T)$ and

$$\exists\, c^*>0 \quad dist(\sigma^*(x,t),\partial K)\geq c^* \quad for\ a.e.(x,t)\in\Omega\times(0,T)\,,$$

where $\varepsilon(u^*) = 1/2(\nabla u^* + \nabla^T u^*)$.

Theorem 6.3 (Cosserat-plasticity and $\mu_c \to 0^+$)
Let us assume that $f \in C^1([0,T], L^2(\Omega,\mathbb{R}^3))$, $\dddot{f} \in L^2((0,T)\times\Omega,\mathbb{R}^3)$ and f satisfies the safe load condition. Moreover, assume that the Dirichlet data possess the regularity

$$u_d \in C^2([0,T],\mathbb{H}^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3))\,, \qquad \dddot{u}_d \in L^2((0,T),\mathbb{H}^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3))\,,$$
$$A_d \in C^2([0,T],\mathbb{H}^{\frac{3}{2}}(\partial\Omega,\mathfrak{so}(3)))\,, \quad \dddot{A}_d \in L^2((0,T),\mathbb{H}^{\frac{1}{2}}(\partial\Omega,\mathfrak{so}(3)))\,.$$

Further, suppose that the initial data $\varepsilon_p^0 \in L^2(\Omega, \operatorname{dev}\operatorname{Sym}(3))$ are chosen such that the initial value of ε^{μ_c} belongs to $L^2(\Omega,\operatorname{Sym}(3))$ and $2\mu\,(\varepsilon^{\mu_c}(x,0) - \varepsilon_p^0(x)) \in K$ for a.e $(x,t) \in \Omega\times(0,T)$.
Then for $\mu_c \to 0^+$ the sequence $\{(u^{\mu_c}, \varepsilon_p^{\mu_c})\}$ of solutions to the problem (6.3) possesses a subsequence which converges weakly to a solution in the measure sense of the perfect elasto-plasticity and the sequence $\{A^{\mu_c}\}$ converges strongly in the space $C([0,T],\mathbb{H}^1(\Omega,\mathfrak{so}(3)))$ to a harmonic (with respect to x) function, which satisfies the Dirichlet boundary condition $A_{|\partial\Omega} = A_d$.

Proof. The energy associated with the problem (6.3) has the form

$$\mathcal{E}(u^{\mu_c}, \varepsilon_p^{\mu_c}, A^{\mu_c})(t) = \int_\Omega \Big(\mu\|\varepsilon^{\mu_c} - \varepsilon_p^{\mu_c}\|^2 + \frac{\lambda}{2}(\operatorname{tr}[\varepsilon^{\mu_c}])^2 + \mu_c\|\operatorname{skew}(\nabla u^{\mu_c}) - A^{\mu_c}\|^2 + 2l_c\,\|\nabla\operatorname{axl}(A^{\mu_c})\|^2\Big)dx\,.$$

We see that in this case the operator $L \equiv 0$ and the operator $Mz^{\mu_c} = M\varepsilon_p^{\mu_c} = 2\mu\varepsilon_p^{\mu_c}$ is positive definite. Hence, we can use the Main Theorem and obtain that for all $\mu_c > 0$ the energy $\mathcal{E}(u^{\mu_c}, \varepsilon_p^{\mu_c}, A^{\mu_c})$ and the energy evaluated for the time derivatives $\mathcal{E}(\dot{u}^{\mu_c}, \dot{\varepsilon}_p^{\mu_c}, \dot{A}^{\mu_c})$ are both bounded on bounded time intervals. If $\mu_c \to 0^+$ we loose the coerciveness of the energy and can conclude only that the sequence

$$\begin{aligned}
\{\varepsilon^{\mu_c} - \varepsilon_p^{\mu_c}\} &\quad \text{is bounded in the space} \quad \mathbb{W}^{1,\infty}((0,T),L^2(\Omega,\operatorname{Sym}(3)))\,,\\
\{\operatorname{tr}[\varepsilon^{\mu_c}]\} &\quad \text{is bounded in the space} \quad \mathbb{W}^{1,\infty}((0,T),L^2(\Omega,\mathbb{R}))\,,\\
\{A^{\mu_c}\} &\quad \text{is bounded in the space} \quad \mathbb{W}^{1,\infty}((0,T),\mathbb{H}^1(\Omega,\mathfrak{so}(3)))\,,\\
\{\sqrt{\mu_c}(\operatorname{skew}(\nabla u^{\mu_c}) - A^{\mu_c})\} &\quad \text{is bounded in the space} \quad \mathbb{W}^{1,\infty}((0,T),L^2(\Omega,\mathfrak{so}(3)))\,.
\end{aligned}$$

The last observation yields that the sequence $\{\mu_c(\operatorname{skew}(\nabla u^{\mu_c}) - A^{\mu_c})\}$ converges strongly to zero in the space $\mathbb{W}^{1,\infty}((0,T),L^2(\Omega,\mathfrak{so}(3)))$. Using the standard elliptic estimate

$$\begin{aligned}
\|A^{\mu_c}\|_{\mathbb{H}^2(\Omega)} + \|\dot{A}^{\mu_c}\|_{\mathbb{H}^2(\Omega)} \le C\Big(&\|A_d\|_{\mathbb{H}^{\frac{3}{2}}(\partial\Omega)} + \|\dot{A}_d\|_{\mathbb{H}^{\frac{3}{2}}(\partial\Omega)}\\
&+\mu_c\|\operatorname{skew}(\nabla u^{\mu_c}) - A^{\mu_c}\|_{L^2(\Omega)} + \mu_c\|\operatorname{skew}(\nabla\dot{u}^{\mu_c}) - \dot{A}^{\mu_c}\|_{L^2(\Omega)}\Big)
\end{aligned}$$

we obtain that from the sequence $\{A^{\mu_c}\}$ we can select a subsequence (further on denoted with the same symbol) which converges weakly in the space $\mathbb{W}^{1,\infty}((0,T),\mathbb{H}^2(\Omega,\mathfrak{so}(3)))$ and consequently strongly in the space $C([0,T],\mathbb{H}^1(\Omega,\mathfrak{so}(3)))$. The limit function A satisfies the boundary value problem

$$-\Delta A(x,t) = 0\,, \quad A_{|\partial\Omega} = A_d\,.$$

Uniqueness of such harmonic functions immediately yields the last statement of the theorem. Next, we want to prove that the sequence $\{(u^{\mu_c}, \varepsilon_p^{\mu_c})\}$ possesses a weak accumulation point, which solves, in the measure-valued sense, the (symmetric) Prandtl-Reuss system. According to the safe load condition we conclude that

$$|\dot{\varepsilon}_p^{\mu_c}| \le \frac{1}{c^*}\langle \dot{\varepsilon}_p^{\mu_c}, (S^{\mu_c} - \sigma^*)\rangle$$

where $S^{\mu_c} = 2\mu(\varepsilon^{\mu_c} - \varepsilon_p^{\mu_c}) + \lambda \mathrm{tr}\,[\varepsilon^{\mu_c}] \cdot 1\!\!1$, the stress tensor σ^* and the constant c^* are from the Definition 6.2. Hence, using that $(\mathrm{skew}(\nabla u^{\mu_c}) - A^{\mu_c}) \in \mathfrak{so}(3)$ we obtain

$$\begin{aligned}
\int_\Omega |\dot{\varepsilon}_p^{\mu_c}|\, dx &\le \frac{1}{c^*}\int_\Omega \langle \dot{\varepsilon}_p^{\mu_c}, (S^{\mu_c} - \sigma^*)\rangle\, dx \\
&= \frac{1}{c^*}\int_\Omega \langle (\dot{\varepsilon}_p^{\mu_c} - \dot{\varepsilon}^{\mu_c}), (S^{\mu_c} - \sigma^*)\rangle\, dx + \frac{1}{c^*}\int_\Omega \langle \dot{\varepsilon}^{\mu_c}, (\sigma^{\mu_c} - \sigma^*)\rangle\, dx \\
&= \frac{1}{c^*}\int_\Omega \langle (\dot{\varepsilon}_p^{\mu_c} - \dot{\varepsilon}^{\mu_c}), (S^{\mu_c} - \sigma^*)\rangle\, dx + \frac{1}{c^*}\int_\Omega \langle \nabla \dot{u}^{\mu_c}, (\sigma^{\mu_c} - \sigma^*)\rangle\, dx \\
&-\frac{\mu_c}{c^*}\int_\Omega \langle \mathrm{skew}(\nabla \dot{u}^{\mu_c}) - \dot{A}^{\mu_c}, \mathrm{skew}(\nabla u^{\mu_c}) - A^{\mu_c}\rangle\, dx - \frac{\mu_c}{c^*}\int_\Omega \langle \dot{A}^{\mu_c}, \mathrm{skew}(\nabla u^{\mu_c}) - A^{\mu_c}\rangle\, dx \\
&\le C(\mathcal{E}(\dot{u}^{\mu_c}, \dot{\varepsilon}_p^{\mu_c}, \dot{A}^{\mu_c}) + \mathcal{E}(u^{\mu_c} - u^*, \varepsilon_p^{\mu_c}, A^{\mu_c})) + \frac{1}{c^*}\int_{\partial\Omega} \langle \dot{u}_d, (\sigma^{\mu_c} - \sigma^*)\cdot n\rangle\, ds \\
&\qquad + \|\dot{A}^{\mu_c}\|_{L^2(\Omega)} \|\mu_c/c^*(\mathrm{skew}(\nabla u^{\mu_c}) - A^{\mu_c})\|_{L^2(\Omega)} \\
&\le C(\mathcal{E}(\dot{u}^{\mu_c}, \dot{\varepsilon}_p^{\mu_c}, \dot{A}^{\mu_c}) + \mathcal{E}(u^{\mu_c} - u^*, \varepsilon_p^{\mu_c}, A^{\mu_c})) + \frac{1}{c^*}\|\dot{u}_d\|_{\mathbb{H}^{\frac{1}{2}}(\partial\Omega)} \|\sigma^{\mu_c} - \sigma^*\|_{L^2(\Omega)} \\
&\qquad + \|\dot{A}^{\mu_c}\|_{L^2(\Omega)} \|\mu_c/c^*(\mathrm{skew}(\nabla u^{\mu_c}) - A^{\mu_c})\|_{L^2(\Omega)}
\end{aligned}$$

(note that $\mathrm{Div}(\sigma^{\mu_c} - \sigma^*) = 0$) where the constant $C > 0$ do not depend on μ_c. Using the boundedness of the energy and regularity of the solution u^* we conclude that the sequence $\{\dot{\varepsilon}_p^{\mu_c}\}$ is bounded in the space $L^\infty((0,T), L^1(\Omega, \mathrm{Sym}(3)))$. The energy estimate yields also that the sequence $\{\dot{\varepsilon}^{\mu_c} - \dot{\varepsilon}_p^{\mu_c}\}$ is bounded in the space $L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))$ and consequently the sequence $\{\dot{\varepsilon}^{\mu_c}\}$ is bounded in the space $L^\infty((0,T), L^1(\Omega, \mathrm{Sym}(3)))$. These results allow us to select a subsequence (further on denoted with the same symbol) such that

$$\begin{aligned}
\varepsilon^{\mu_c} &\rightharpoonup \varepsilon \quad \text{in} \quad \mathbb{W}_w^{1,\infty}((0,T), \mathcal{M}^{3\times 3}_{\mathrm{sym}}(\Omega))\,, \\
\varepsilon_p^{\mu_c} &\rightharpoonup \varepsilon_p \quad \text{in} \quad \mathbb{W}_w^{1,\infty}((0,T), \mathrm{dev}\,\mathcal{M}^{3\times 3}_{\mathrm{sym}}(\Omega))\,.
\end{aligned}$$

The energy estimate implies that the function $\sigma = 2\mu(\varepsilon - \varepsilon_p) + \lambda \mathrm{tr}\,[\varepsilon] \cdot 1\!\!1$ belongs to the space $\mathbb{W}^{1,\infty}((0,T), L^2(\Omega, \mathrm{Sym}(3)))$. The boundedness of the sequence $\{\mathrm{tr}\,[\varepsilon^{\mu_c}]\}$ in the space $\mathbb{W}^{1,\infty}((0,T), L^2(\Omega, \mathbb{R}))$ implies that the Dirichlet boundary condition is satisfied in the normal direction to the boundary. The proof that $\langle \dot\varepsilon_p, (\sigma - \tau)\rangle$ is a nonnegative measure for all $\tau \in L^2(\Omega, \mathrm{Sym}(3))$, such that $\mathrm{Div}\, \tau \in L^2(\Omega, \mathbb{R})$ and $\tau(x) \in K$ for a.e. $x \in \Omega$ is the same as in the article [Tem86] or [Che01a]. ■

Remark 6.4
(a) In the same manner as in the proof of Theorem 6.3 we can prove that the sequence $\{(u^{\mu_c}, \varepsilon_p^{\mu_c}, b^{\mu_c}, A^{\mu_c})\}$ *of solutions to the Melan-Prager model with microrotations converges for* $\mu_c \to 0^+$ *to a global in time* $\mathbb{H}^1$*-solution of the Melan-Prager model without Cosserat effects. Existence of such solutions in the quasistatic case can be found in [AC04, AC07].*

(b) For simplicity we study in this article the Dirichlet boundary value problem only. The presented theory works well if the mixed boundary value problem for the displacement and the Dirichlet boundary value problem for microrotations is required. We use in the existence theory the $\mathbb{H}^2$*-regularity for microrotations only.*

7 Remarks and open problems

In this contribution we have shown that all models of monotone type from the inelastic deformation theory are $\mathbb{H}^1$ well-posed if Cosserat effects are taken into account. Thus, the following question appears: is the regularizing effect of Cosserat media so strong that the Main Theorem can be improved to hold for some non-monotone models as well? For example, inelastic deformations described by Armstrong-Frederick inelastic constitutive relation with independent microrotations are solutions of the following system of equations

$$\begin{aligned}
\mathrm{Div}\, \sigma &= -f\,, \\
\sigma &= 2\mu\,(\varepsilon - \varepsilon_p) + 2\,\mu_c\,(\mathrm{skew}(\nabla u) - A) + \lambda\,\mathrm{tr}\,[\varepsilon]\cdot \mathbb{1}\,, \\
-l_c\,\Delta\,\mathrm{axl}(A) &= \mu_c\ \mathrm{axl}(\mathrm{skew}(\nabla u) - A)\,, \\
\dot\varepsilon_p &\in \partial I_K\Big(2\mu\,(\varepsilon - \varepsilon_p) - b\Big)\,. \\
\dot b &= c\dot\varepsilon_p - d|\dot\varepsilon_p| b
\end{aligned} \tag{7.1}$$

where $c, d > 0$ are material parameters and the set of admissible stresses K has the same properties as in (5.1). System (7.1) without Cosserat effects was studied in [Che03]. There it is shown that the Armstrong-Frederick flow rule is thermodynamically admissible, belongs to the class of pre-monotone models, but does not belong to the class of monotone models and the initial boundary-value problem possesses "weak-type admissible solutions" only. Hence, the following open problem is stated:

Open problem: For which given boundary and initial data does system (7.1) possess global in time $\mathbb{H}^1$ solutions?

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