# Approximative Computation
# and Generalizations of Metric Spaces

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**Abstract**. To find a general framework that is suitable for applications of computation theory, we introduce and investigate *'computation spaces'*, which correspond to certain neighbourhood spaces or to certain 'generalized' metric spaces. Due to Ceitin, G.S., [3] (see also [1], [2] or [7]), each effective operator on a metric space (which satisy weak assumptions) is *effectively continuous*, i.e. has a *modulus of continuity*. Spreen, D., and Young, P., [10] have generalized that and similar results uniformly to proper *topological spaces*. (See also [5] and [6].) Those results and their proofs, however, do not yet yield a *convenient* method to find moduli of continuity that are useful for individual applications. Therefore, we shall present calculus-like methods to obtain both programs to compute functions on computation spaces (§1) and, for such functions, computable moduli of continuity, which are suitable for individual applications (§2).

In §3 we generalize the notion of H-quasi-metric spaces (i.e. quasi-metric spaces which satisfy the Hausdorff axiom) and show the following: Every 'generalized' H-quasi-metric space 'induces' a neighbourhood space of a particular sort, which is similar to that of computation spaces. Every neighbourhood space of that sort can be induced by a generalized H-quasi-metric space. All generalized H-quasi-metric spaces that induce the same neighbourhood space differ from each other only insignificantly. - Certain results of §1 and §2 concerning H-quasi-metric spaces can be generalized.

**Contents**: 

## §0 Introduction

**Definition**: A pair $(X, c)$ is said to be an **H-quasi-metric space** (here "H" abbreviates "Hausdorff") iff $X$ is a set, and $c: X^2 \to [0, \infty]$ satisfies the following conditions for all $x, y, z \in X$:

(c1) $c(x, x) = 0$

(c2) $c(x, y) > 0$ if $x \neq y$

(c3) $c(x, z) \leq c(x, y) + c(y, z)$

(c4) $x \neq y \quad \Rightarrow \quad \exists \varepsilon > 0.\ \forall z \in X.\ \Big(c(z, x) > \varepsilon \ \vee\ c(z, y) > \varepsilon\Big).$

**Notes**: (c2) is a consequence of (c1) and (c4). Moreover, (c4) is satisfied, if there exists a metric $d$ on $X$ such that $d(x, y) \leq c(x, y)$ for all $x, y \in X$.

Examples for quasi-metric spaces are given in [4] and [9]. We give some further examples of H-quasi-metric spaces which enable concrete numerical applications.

(E1) $X$ is an interval of $\mathbb{R}$, $c(x, y) = a \cdot (x - y)$ if $x \geq y$, $c(x, y) = b \cdot (y - x)$ if $x < y$ (where $a, b \in \mathbb{R}^+ \cup \{\infty\}$). - A 'concrete interpretation' of this example is the following:

(E2) $X$ is a river, and $c(x, y)$ is the time needed to sail with a particular boat from $x$ to $y$. (This example can also be transferred to aviation as well as to space-flight.)

(E3) $X$ is a rising ground or a mountainous terrain, and $c(x, y)$ is the energy needed to go from $x$ to $y$. (Here we assume that even for walking downhill one needs some energy.)

(E4) $X$ is the set of all spots on the streets of a town, where some streets are one-way. $c(x, y)$ is the length of the shortest *permitted* way on $X$ from $x$ to $y$. - This can more precisely be modeled in the following form:

(E5) Let $(X, d)$ be a metric space, and let $C \subseteq X^2$ satisfy $\forall x \in X.\ (x, x) \in C$.
$x$ is said to be *connected* with $y$ by $x_0, x_1, \ldots, x_n$ $(n \in \mathbb{N}^+)$ iff

$$x = x_0, \;\; y = x_n, \;\; \text{and} \;\; (x_i, x_{i+1}) \in C \;\; \text{for all} \;\; i = 0, \ldots, n-1.$$

Let $S(x, y)$ be the set of all sums $\sum_{i=0}^{n-1} d(x_i, x_{i+1})$ such that $x$ is connected with $y$ by $x_0, \ldots, x_n$. Let $c : X^2 \to [0, \infty]$ be defined by

$$c(x, y) = \inf S(x, y)$$

(where $\inf \emptyset = \infty > \xi$ for all $\xi \in \mathbb{R}$).

(E6) Let $(X_i, c_i)$ be H-quasi-metric spaces such that $c_i : X_i^2 \to \mathbb{R}_0^+$ for $i = 1, \ldots, \mu$, $\mathbf{X} = X_1 \times \ldots \times X_\mu$, and let $\mathbf{c}, \mathbf{c}' : \mathbf{X^2} \to \mathbb{R}_0^+$ be defined by

$$\mathbf{c}(x, y) = \max\{c_1(x_1, y_1), \ldots, c_\mu(x_\mu, y_\mu)\},$$

$$\mathbf{c}'(x, y) = \sqrt{c_1(x_1, y_1)^2 + \ldots + c_\mu(x_\mu, y_\mu)^2}$$

for $\mu$-tuples $x = (x_1, \ldots, x_\mu)$ and $y = (y_1, \ldots, y_\mu)$. Then $(\mathbf{X}, \mathbf{c})$ and $(\mathbf{X}, \mathbf{c}')$ are H-quasi-metric spaces, too. (This is well-known.)

The following proposition is concerned with *systems of neighbourhoods* in H-quasi-metric spaces. (This proposition will be generalized in §3.)

**0.1 Proposition**: Let $(X, c)$ be an H-quasi-metric space, $\alpha \in ]0,1[ \cap \mathbb{Q}$, $\alpha^{-\infty} := \infty$, $\alpha^{\infty} := 0$, $\mathbb{Z}^{\mathrm{C}} := \mathbb{Z} \cup \{-\infty, \infty\}$,

$$B(y,k) := \{x \in X\colon c(x,y) \leq \alpha^k\} \text{ for } (y,k) \in X \times \mathbb{Z}^{\mathrm{C}},$$

especially $B(y, -\infty) = X$, $B(y, \infty) = \{y\}$, and let

$$k * m := \max\{j \in \mathbb{Z}^{\mathrm{C}} : \alpha^j \geq \alpha^k + \alpha^m\} \quad \text{for} \quad k, m \in \mathbb{Z}^{\mathrm{C}}.$$

Then we have, for all $x, y, z \in X$ and all $k, m \in \mathbb{Z}^{\mathrm{C}}$:

(B1) $x \in B(x,k)$
(B2) $x \in B(y,k) \ \wedge \ y \in B(z,m) \ \Rightarrow \ x \in B(z, k*m)$
(B3) $B(x,m) \subseteq B(x,k)$ for $m \geq k$.
(B4) $x \neq y \quad \Rightarrow \quad \exists k \in \mathbb{Z}.\ B(x,k) \cap B(y,k) = \emptyset$

($*$) $\lim_{m\to\infty}(m * m) = \infty$.

Therefore, the sets $O \subseteq X$ satisfying $\forall x \in O.\ \exists k \in \mathbb{Z}.\ B(x,k) \subseteq O$ form a Hausdorff topology on $X$. (The proof is left to the reader.)

**Definition:** $(X, B, *, U, \nu)$ is said to be a **computation space**, iff $X$ is a set, $B\colon X \times \mathbb{Z}^{\mathrm{C}} \to \mathcal{P}(X)$ (the power set of $X$), $*\colon \mathbb{Z}^{\mathrm{C}} \times \mathbb{Z}^{\mathrm{C}} \to \mathbb{Z}^{\mathrm{C}}$ is computable, the above conditions (B1) - (B4) and ($*$) are satisfied, moreover, there exists a finite alphabet $\Sigma$ such that $U \subseteq \Sigma^*$ (the set of all words over $\Sigma$), and $\nu : U \to X$ is a function (called a 'notation' of $\nu(U)$).

A further motivation for the choice of the axioms (B1) - (B4) and ($*$) (and thus also for (c4)) will be given below in **(A1)** and **(A3)**.

We identify $\nu(u)$ with $u$ for $u \in U$, assume that $U \subseteq X$, and we briefly write $(X, B, *, U)$ for $(X, B, *, U, \nu)$. (Notice, however, that then for all $u, v \in U$ the equation "$u = v$" does not generally mean the literal equality of $u$ and $v$ as words.) - If $U$ is dense in $X$, then $U$ plays in $X$ an analogous role as $\mathbb{Q}$ in $\mathbb{R}$.

For applications we are especially interested in computation spaces that can be defined in a (sufficiently extensive) language $\mathcal{L}$ the use of its sentences is introduced *'predicatively'*. We especially suppose that "$x \in B(y,k)$" is defined as a formula of $\mathcal{L}$.

The applicability of a simulation of the ordinary computation concept in set theory is rather problematic since in any nonstandard model of set theory there exists a nonstandard sequence of elements of $\{0,1\}$ (e.g.) which is nonstandard computable but the restriction of which to the standard part of $\mathbb{N}$ is *not* computable. Moreover, I do not know whether if set theory is consistent, then there exists a standard model of set theory, i.e. a model such that the set of all *finite* natural numbers is an element of that model. (Here we do not discuss the question how the concept of finite natural numbers can be characterized.

We do also not investigate the question on which conditions the value $\inf S(x,y)$ defined in (E5) exists in a predicative model.)

**Definition:** A computation space $(X,B,*,U)$ is said to be **induced** by $(X,c,\alpha,U)$, iff $(X,c)$ is an H-quasi-metric space, $\alpha \in \,]0,1[\, \cap \mathbb{Q}$, and both $B$ and $*$ are defined as in 0.1.

Then, any element of $B(x,k)$ can be regarded as an *approximation* of $x$ to the precision $\alpha^k$. We say that $k$ is the *degree* of this precision. Here, we also consider *negative* integers $k$ since for several practical purposes it will be useful to admit a precision $\alpha^k$ that is greater than 1. (For the pure theory of comput*ability*, however, it would be sufficient to consider positive degrees of precision only.) The choice of $\alpha \in \,]0,1[\, \cap \mathbb{Q}$ can be adapted to particular purposes and means. - If a computation space $(X,B,*,U)$ is given, we also say that $x$ is an approximation of $y$ to the *degree* $k$ of precision to mean that $x \in B(y,k)$.

In this paper, we shall provide means for applications as considered in **(A1) - (A3)**. For their formulations we assume that two computation spaces $(X,B,*,U)$ and $(X',B',*',U')$ are given.

**(A1)** For certain applications of geometry we have to compute rational numbers which approximate $\sqrt{2}$, e.g., to any desired degree of precicion. To provide means for similar applications we shall deal with programs to approximately compute given elements of $X$ by elements of $U$. Accordingly, an element $x$ of $X$ is said to be *computable* iff it can effectively be approximated to any desired degree $k \in \mathbb{Z}$ of precision by means of a *program* or a *computable* function $p : \mathbb{Z} \to U$, i.e. such that $p_k \in B(x,k)$ for all $k \in \mathbb{Z}$. (Then, due to (B4), $x$ is uniquely determined by $p$. - By (B1), all elements of $U$ are computable.)

**(A2)** Given a function $f : X \to X'$. If we have obtained programs to compute a great number of elements, $x_1,\ldots,x_n$, of $X$ and if we want to compute the corresponding values $f(x_1),\ldots,f(x_n)$, then it would in general be helpful to find a *'metaprogram'* that can be applied to any program for computing an element $x$ of $X$ and produces then a program to compute $f(x)$. Accordingly, we say that $f$ is computable iff there exists such a metaprogram.

**(A3)** Sometimes we need an approximation of a value $f(a)$ of a given computable function $f : X \to X'$ corresponding to an argument $a$ $(\in X)$ for which, however, we can obtain rough estimates only. To compute, nevertheless, $f(a)$ to a given degree $j$ of precision we could *try* to do the following: Find a computable '*modulus of continuity*' $\delta : \mathbb{Z} \times X \to \mathbb{Z}$ for $f$, i.e. satisfying

$$\forall m \in \mathbb{Z}.\ \forall x,y \in X.\ (x \in B(y,\delta(m,x)) \ \Rightarrow\ f(x) \in B'(f(y),m)).$$

Choose $k,m \in \mathbb{Z}$ with $k * m \geq j$ (cf. $(*)$ in 0.1), find then a computable $b \in B(a,\delta(m,b))$, and compute $f(b)$ to the degree $k$, i.e. determine a $v \in U' \cap B'(f(b),k)$. Then we obtain $f(b) \in B'(f(a),m)$ and hence $v \in B'(f(a),k*m) \subseteq B'(f(a),j)$ (by

(B2) and (B3)). - However, there is *no guarantee* that we can find a computable element $b$ for which we can prove that $b \in B(a, \delta(m, b))$. Therefore, particularly useful for applications are moduli of continuity, whose values are not unnecessarily large (cf. (B3)).

Since we shall particularly deal with 'metaprograms' for transforming programs, our approach is similar to that of the 'Russian school' (see [3], [7], or also [1], [2]). However, we shall in general argue classically. - Generalized metric spaces are investigated in [4], [8] and [9], e.g. - In [10] there is shown that all effective operators on topological spaces (which satisfy certain assumptions) are effectively continuous. This is a uniform generalization of former results concerning effective operators on metric spaces (see [7, p.297], e.g.) and on cpo's. Those results and their proofs, however, do not yet yield a *convenient* method to find moduli of continuity that are suitable for concrete applications. Our investigations in §2 are concerned with such methods.

Some of the following propositions have the form $\forall x\ \exists y\ A(x, y)$ or a similar one. We shall state such a proposition to announce that its direct proof will specify a method to find a 'solution' $y$ satisfying $A(x, y)$, if any admissible $x$ is given. In the following investigations, several programs and moduli of continuity will occur as such solutions.

## §1 Programs to compute functions on computation spaces

Given a finite alphabet $\Sigma$. Then, by a **program** we mean a program for register machines, e.g., which operates on words over $\Sigma$. Given sets $V_1, \ldots, V_\mu, W \subseteq \Sigma^*$ ($\mu \geq 1$), we denote the set of all programs that compute functions of type $V_1 \times \ldots \times V_\mu \to W$ by

$$P(V_1 \times \ldots \times V_\mu \to W).$$

For $p \in P(V_1 \times \ldots \times V_\mu \to W)$, $v_1 \in V_1, \ldots, v_\mu \in V_\mu$, and $w \in W$, we write

$$p(v_1, \ldots, v_\mu) = w$$

to mean that $p$ with the input $(v_1, \ldots, v_\mu)$ halts with the unique output $w$. - We define:

$$p_k := p(k) \quad \text{for} \quad p \in P(\mathbb{Z} \to W),\ k \in \mathbb{Z}.$$

Since programs operating on words over $\Sigma$ can be considered as words, too, there exists another finite alphabet $\Sigma_2 \supset \Sigma$ such that $P(V_1 \times \ldots \times V_\mu \to W) \subset \Sigma_2^*$ for all $V_1, \ldots, V_\mu, W \subseteq \Sigma^*$. Accordingly, if $V_1, V_2, W_1, W_2 \subseteq \Sigma^*$, the set $P(P(V_1 \to W_1) \to P(V_2 \to W_2))$, e.g., of 'metaprograms' is defined.

**Presupposition**: Let

$$\underline{X} := (X, B, *, U), \ \ \underline{X'} := (X', B', *', U'), \ \text{and} \ \underline{X''} := (X'', B'', *'', U'')$$

be computation spaces, and let

$$A \subseteq X \ \text{ and } \ A' \subseteq X'.$$

Motivated by §0 **(A1)**, we define:

**Definition**: Let $P(A)$ be the set of all programs $p \in P(\mathbb{Z} \to U)$ satisfying

$$\forall k \in \mathbb{Z}. \ p_k \in B(\sharp p, k),$$

for some $\sharp p \in A$. In the following, we use the corresponding abbreviation

$$\sharp p := \lim_{k \to \infty} p_k.$$

An element $a \in X$ is said to be ($\underline{X}$-) **computable** iff $P(\{a\}) \neq \emptyset$, i.e. $a = \sharp p$ for some $p \in P(X)$.

According to §0 **(A2)**, a function $f \colon A \to X'$ is (roughly) said to be computable iff there exists a ('meta-') program by which, for each computable $x \in A$, we can also compute the value $f(x)$. We define more precisely:

**Definition**: For functions $f : A \to X'$ let $P(f)$ be the set of all programs $F \in P(P(A) \to P(X'))$ with $\sharp F(p) = f(\sharp p)$ for all $p \in P(A)$ (i.e. $F(p)_k \in B'(f(\sharp p), k)$ for all $(k, p) \in \mathbb{Z} \times P(A)$). $f$ is said to be $\underline{X}, \underline{X'}$**-computable** iff $P(f) \neq \emptyset$. - Let $cp.(A \to A')$ be the set of all $\underline{X}, \underline{X'}$-computable functions of $A$ into $A'$.

For functions $f \colon A \to U'$ with $A \subseteq U$, the $\underline{X}, \underline{X'}$-computability must be distinguishes from the original computability of word functions. Nevertheless, for functions considered in the following we simply say "*computable*" for "$\underline{X}, \underline{X'}$-computable", e.g.

**1.1 Proposition**: For any computable element $b \in X'$, the constant function $x \mapsto b, \ x \in A$ is computable. $\square$

"$\square$" at the end of a proposition or proof is to mean that the proof is left to the reader or complete, respectively.

**1.2 Lemma**: A function $f : A \to X'$ is computable iff there exists a program $\Phi \in P(\mathbb{Z} \times P(A) \to U')$ such that

$$\Phi(k, p) \in B'(f(\sharp p), k) \quad \text{for all} \quad (k, p) \in \mathbb{Z} \times P(A).$$

Proof: 1. Let $U' \subseteq \Sigma^*$. Due to the Normal Form Theorem of Kleene there exists a 'universal' program $\Omega \in P(P(\mathbb{Z} \to \Sigma^*) \times \mathbb{Z} \to \Sigma^*)$ such that, for all $q \in P(\mathbb{Z} \to \Sigma^*)$ and all $k \in \mathbb{Z}$, we have

$$q_k = \Omega(q, k).$$

Thus, for each $F \in P(f) \subseteq P(P(A) \to P(\mathbb{Z} \to U'))$ there exists $\Phi \in P(\mathbb{Z} \times P(A) \to U')$ such that

$$\Phi(k,p) = \Omega(F(p),k) = F(p)_k \quad \text{for all} \quad (k,p) \in \mathbb{Z} \times P(A).$$

2. Due to the so called Parametrisation Theorem (see [2, p. 43], e.g.), if $\Phi \in P(\mathbb{Z} \times P(A) \to U')$, then there exists an $F \in P(P(A) \to P(\mathbb{Z} \to U'))$ such that

$$F(p)_k = \Phi(k,p) \quad \text{for all} \quad (k,p) \in \mathbb{Z} \times P(A). \;\square$$

**1.3 Lemma**: For any $L \in P(\mathbb{Z} \times P(A) \to \mathbb{Z})$ there exists $\Phi \in P(\mathbb{Z} \times P(A) \to U)$ such that $\Phi(k,p) = p_{L(k,p)}$ for all $(k,p) \in \mathbb{Z} \times P(A)$.

Proof: $p_{L(k,p)} = \Omega(p, L(k,p))$. $\square$

**Presupposition and Definition:** Let $\underline{X}_i := (X_i, B_i, *_i, U_i)$ $(i = 1, \ldots, \mu)$ be computation spaces, and let their *product space*

$$\underline{\mathbf{X}} := \underline{X}_1 \times \ldots \times \underline{X}_\mu := (\mathbf{X}, \mathbf{B}, *, \mathbf{U})$$

be defined by

$$\begin{aligned} &\mathbf{X} = X_1 \times \ldots \times X_\mu, \quad \mathbf{U} = U_1 \times \ldots \times U_\mu, \\ &\mathbf{B}((x_1, \ldots, x_\mu), k) = B_1(x_1, k) \times \ldots \times B_\mu(x_\mu, k), \\ &k{*}m = \min\{k *_1 m, \ldots, k *_\mu m\}. \end{aligned}$$

Let the *projections* $\mathrm{id}_i : \mathbf{X} \to X_i$ $(i = 1, \ldots, \mu)$ be defined by

$$\mathrm{id}_i(x_1, \ldots, x_\mu) = x_i.$$

Moreover, let $\underline{\mathbb{R}}$ be the computation space induced by $(\mathbb{R}, d_1, \alpha_{\mathbb{R}}, \mathbb{Q})$ where $d_1(\xi, \eta) = |\xi - \eta|$ for all $\xi, \eta \in \mathbb{R}$, and $\alpha_{\mathbb{R}} \in \,]0,1[\, \cap \mathbb{Q}$. Accordingly, let

$$P(\mathbb{R}) := \{r \in P(\mathbb{Z} \to \mathbb{Q}) \colon \forall k \in \mathbb{Z}.\ |r_k - \sharp r| \leq \alpha_{\mathbb{R}}^k\}.$$

**1.4 Proposition**: $\underline{\mathbf{X}}$ is a computation space. $\square$

**1.5 Proposition**: If $p \in P(\mathbf{X})$ and $i = 1, \ldots, \mu$, then $\mathrm{id}_i(p_k) \in B_i(\mathrm{id}_i(\sharp p), k)$. Hence, $\mathrm{id}_i$ is computable. $\square$

**1.6 Proposition**: Let $(X, c)$ be an H-quasi-metric space satisfying $c : X^2 \to \mathbb{R}_0^+$ and

$$c(\sharp p, u) \leq \Lambda(p) \cdot c(u, \sharp p) \quad \text{for all} \quad (p, u) \in P(X) \times U$$

where $\Lambda \in P(P(X) \to \mathbb{Q}^+)$, and let $\underline{X}$ be induced by $(X, c, \alpha, U)$. Assume that there exists a program $\Gamma \in P(\mathbb{Z} \times U \times U \to \mathbb{Q})$ such that

$$|c(u,v) - \Gamma(m,u,v)| \leq \alpha^m \quad \text{for all} \quad (m,u,v) \in \mathbb{Z} \times U \times U.$$

Then $c$ is $(\underline{X} \times \underline{X}), \underline{\mathbb{R}}$-computable.

Proof: Let $p, q \in P(X)$; $k, m \in \mathbb{Z}$; and $\lambda := \max\{\Lambda(p), \Lambda(q)\}$. Then we easily obtain

$$|c(\sharp p, \sharp q) - c(p_m, q_m)| \leq \alpha^m \cdot (1 + \lambda)$$

and hence

$$|c(\sharp p, \sharp q) - \Gamma(m, p_m, q_m)| \leq \alpha^m \cdot (2 + \lambda) \leq \alpha_{\mathbb{R}}^k,$$

if $m = \min\{n \in \mathbb{Z} : \alpha^n \cdot (2+\lambda) \leq \alpha_{\mathbb{R}}^k\}$. By 1.3 and 1.2 it follows that $c$ is computable. □

**1.7 Proposition**: If $f_i \in cp.(A \to X_i)$ for $i = 1, \ldots, \mu$, then $(f_1, \ldots, f_\mu) : A \to \mathbf{X}$ is $\underline{X}, \underline{\mathbf{X}}$-computable.

Proof: Let $F_i \in P(f_i)$ for $i = 1, \ldots, \mu$, and let $F \in P(P(A) \to P(\mathbb{Z} \to \mathbf{U}))$ satisfy $F(p)_k = (F_1(p)_k, \ldots, F_\mu(p)_k)$ for all $(p, k) \in P(A) \times \mathbb{Z}$. Because of $F_i(p)_k \in B_i(f_i(\sharp p), k)$ for $i = 1, \ldots, \mu$ we obtain $F(p)_k \in \mathbf{B}((f_1(\sharp p), \ldots, f_\mu(\sharp p)), k)$. Accordingly, $F \in P(f_1, \ldots, f_\mu)$. □

**1.8 Proposition**: If $f \in cp.(A \to A')$ and $g \in cp.(A' \to X'')$, then $g \circ f \in cp.(A \to X'')$.

Proof: Let $F \in P(f)$ and $G \in P(g)$. Then, for all $p \in P(A)$, we have $\sharp F(p) = f(\sharp p) \in A'$, hence $F(p) \in P(A')$, and hence $\sharp G(F(p)) = g(\sharp F(p)) = g(f(\sharp p))$. Therefore, $G \circ F \in P(g \circ f)$ (where $\circ$ also denotes the composition of programs.) □

**Definition**: Let us supply $\mathbb{N}$ with the discrete topology. Accordingly, let (for $k > -\infty$) $B_{\mathbb{N}}(n, k) := \{n\}$ and $\underline{\mathbb{N}} := (\mathbb{N}, B_{\mathbb{N}}, \min_{\mathbb{N}\times\mathbb{N}}, \mathbb{N})$. A function $f : \mathbb{N} \times A \to X'$ is simply said to be **computable** iff it is $(\underline{\mathbb{N}} \times \underline{X}), \underline{X'}$-computable, i.e. iff there exists a program $F \in P(\mathbb{N} \times P(A) \to P(X'))$ such that $f(n, \sharp p) = \sharp F(n, p)$ for all $(n, p) \in \mathbb{N} \times P(A)$. Accordingly, let then $P(f)$ be the set of all programs of this kind. - Moreover, let us identify a *sequence* of functions $f_n : A \to X'$ with the corresponding function $f \colon \mathbb{N} \times A \to X'$ satisfying $f(n, x) = f_n(x)$.

**1.9 Proposition**: If $f \in cp.(A \to A')$, $g \in cp.(\mathbb{N} \times A \times A' \to A')$, and if $h : \mathbb{N} \times A \to A'$ is recursively defined by

$$h(0, x) = f(x), \quad h(n + 1, x) = g(n, x, h(n, x))$$

for all $(n, x) \in \mathbb{N} \times A$, then $h$ is computable.

Proof: Let $F \in P(f)$ and $G \in P(g)$. Then there exists $H \in P(\mathbb{N} \times P(A) \to P(A'))$ such that

$$H(0, p) = F(p), \quad H(n + 1, p) = G(n, p, H(n, p)).$$

Thus we have $h(0, \sharp p) = \sharp H(0, p)$, and the inductive hypothesis $h(n, \sharp p) = \sharp H(n, p)$ implies $h(n + 1, \sharp p) = \sharp H(n + 1, p)$. Accordingly, $H \in P(h)$. □

**1.10 Proposition**: Let $f \in cp.(\mathbb{N} \times A \to X')$, $g : A \to X'$ and $M \in P(\mathbb{Z} \times P(A) \to \mathbb{N})$ such that

$$\forall p \in P(A).\ \forall k \in \mathbb{Z}.\ f(M(k,p), \sharp p) \in B'(g(\sharp p), k).$$

Then $g$ is computable.

Proof: There exist $\sigma', \tau' \in P(\mathbb{Z} \to \mathbb{Z})$ such that $\sigma'_k *' \tau'_k \geq k$ for all $k \in \mathbb{Z}$. (Example: $\sigma'_k = \tau'_k = \min\{m \in \mathbb{Z} : m \geq k \ \wedge \ m *' m \geq k\}$, cf. §0 $(*)$.) Since $f$ is computable and due to 1.2, there exists $\Phi \in P(\mathbb{Z} \times \mathbb{N} \times P(A) \to U')$ such that, for all $(k,n,p) \in \mathbb{Z} \times \mathbb{N} \times P(A)$, we especially have $\Phi(\sigma'_k, n, p) \in B'(f(n, \sharp p), \sigma'_k)$.
Setting $n := M(\tau'_k, p)$ we obtain $f(n, \sharp p) \in B'(g(\sharp p), \tau'_k)$, and hence, by (B2) and (B3), $\Phi(\sigma'_k, n, p) \in B'(g(\sharp p), \sigma'_k *' \tau'_k) \subseteq B'(g(\sharp p), k)$. So, by 1.2, $g$ is computable. □

**1.11 Corollary**: Let $f \in cp.(\mathbb{N} \times A \to X')$ and $K \in P(\mathbb{N} \times P(A) \to \mathbb{Z})$ such that $\lim_{n\to\infty} K(n,p) = \infty$ for all $p \in P(A)$. Assume that the function sequence $f$ converges pointwise to $g : A \to X'$ such that

$$\forall n \in \mathbb{N}.\ \forall p \in P(A).\ f(n, \sharp p) \in B'(g(\sharp p), K(n,p)).$$

Then $g$ is computable.

Proof: Let $m := \min\{n \in \mathbb{N} : K(n,p) \geq k\}$. Then we have $f(m, \sharp p) \in B'(g(\sharp p), K(m,p)) \subseteq B'(g(\sharp p), k)$ by (B3). Thus we can apply 1.10. □

**1.12 Corollary**: Let $\underline{X'}$ be induced by $(X', c', \beta, U')$ $(\beta \in \,]0,1[\, \cap \mathbb{Q})$.
Let $f \in cp.(\mathbb{N} \times A \to X')$, and let $\eta : \mathbb{N} \times A \to \mathbb{R}_0^+$ be $(\underline{\mathbb{N}} \times \underline{X}), \underline{\mathbb{R}}$-computable and satisfy

$$\forall p \in P(A).\ \lim_{n\to\infty} \eta(n, \sharp p) = 0.$$

Let the function sequence $f$ converge pointwise to $g : A \to X'$ such that

$$\forall n \in \mathbb{N}.\ \forall p \in P(A).\ c'\big(f(n, \sharp p), g(\sharp p)\big) \leq \eta(n, \sharp p).$$

Then $g$ is computable.

Proof: Since $\eta$ is computable, by 1.2 there exists $E \in P(\mathbb{Z} \times \mathbb{N} \times P(A) \to \mathbb{Q})$ such that

$$|\eta(n, \sharp p) - E(k,n,p)| \leq \alpha_{\mathbb{R}}^k.$$

Let $K(n,p) := \max\big\{k \in \mathbb{Z} : \beta^k \geq E(n,n,p) + \alpha_{\mathbb{R}}^n\big\}$. Then we easily obtain

$$c'\big(f(n, \sharp p), g(\sharp p)\big) \leq \beta^{K(n,p)} < \beta^{-1} \cdot \big(\eta(n, \sharp p) + 2\alpha_{\mathbb{R}}^n\big) \to 0 \ \text{ as } \ n \to \infty.$$

Thus, by 1.11, $g$ is computable. □

**1.13 Corollary**: Let $\underline{X}_i$ be induced by $(X_i, c_i, \alpha, U_i)$ where $c_i \in cp.(X_i^2 \to \mathbb{R}_0^+)$ $(i = 1, \ldots, \mu)$. Then the H-quasi-metrics $\mathbf{c}, \mathbf{c}' : \mathbf{X}^2 \to \mathbb{R}_0^+$ defined in §0 (E6) are computable, and $\underline{\mathbf{X}}$ is induced by $(\mathbf{X}, \mathbf{c}, \alpha, \mathbf{U})$.

Proof: The both functions of $(\mathbb{R}_0^+)^\mu$ into $\mathbb{R}_0^+$, $(\xi_1,\ldots,\xi_\mu) \mapsto \max\{\xi_1,\ldots,\xi_\mu\}$ and $(\xi_1,\ldots,\xi_\mu) \mapsto \sqrt{\xi_1^2+\ldots+\xi_\mu^2}$, are computable (see Appendix 1: Proposition A1.1 and the subsequent remark). Therefore, the computability of $\mathbf{c}$ and $\mathbf{c}'$ follows by 1.5, 1.7, and 1.8. - The proof of the remainder is left to the reader. $\square$

## §2 Programs to compute moduli of continuity

With respect to applications as described in §0 **(A3)** we define:

**Definition**: Given $f\colon A \to X'$, a program $\Delta \in P(\mathbb{Z} \times P(A) \to \mathbb{Z})$ is said to be a $B, B'$**-modulus of continuity** for $f$ iff

$$\forall k \in \mathbb{Z}.\ \forall p \in P(A).\ \forall y \in A.\ \Big(\sharp p \in B(y, \Delta(k,p)) \ \Rightarrow\ f(\sharp p) \in B'(f(y), k)\Big)$$

Let $\mathcal{M}(f, B, B')$ be the set of all $B, B'$-moduli of continuity for $f$.

In H-quasi-metric spaces, a similar concept in place of the $B, B'$-moduli of continuity is more convenient:

**Definition**: Let $(X, c, \alpha, U)$ and $(X', c', \beta, U')$ induce $\underline{X}$ and $\underline{X'}$, respectively. Given $f : A \to X'$, a function $\delta : \mathbb{R}^+ \times A \to \mathbb{R}^+$ is said to be a $c, c'$**-modulus of continuity** for $f$, iff $\delta$ is $(\underline{\mathbb{R}} \times \underline{X}), \underline{\mathbb{R}}$-computable (for $\underline{\mathbb{R}}$ see §1) and satisfies

$$\forall \varepsilon > 0.\ \forall x, y \in A.\ \Big(c(x,y) \le \delta(\varepsilon, x) \ \Rightarrow\ c'(f(x), f(y)) \le \varepsilon\Big).$$

Particularly suitable for concrete applications are $c, c'$-moduli of continuity whose values are not unnecessarily small. - By the following proposition, each $c, c'$-modulus of continuity can be replaced by a $B, B'$-modulus of continuity.

**2.1 Proposition**: If $\delta \in cp.(\mathbb{R}^+ \times A \to \mathbb{R}^+)$ and $\alpha, \beta, \gamma \in \,]0,1[\, \cap \mathbb{Q}$, then there exists $\Delta \in P(\mathbb{Z} \times P(A) \to \mathbb{Z})$ such that, for all $k \in \mathbb{Z}$ and all $p \in P(A)$,

$$\alpha^{1+\gamma} \cdot \delta(\beta^k, \sharp p) < \alpha^{\Delta(k,p)} \le \delta(\beta^k, \sharp p).$$

Proof: Let $n \in \mathbb{N}$ such that $\alpha_{\mathbb{R}}^n \le \frac{\gamma}{2} < \alpha_{\mathbb{R}}^{n-1}$. Then there exists a program $\Theta \in P(P(\mathbb{R}) \to \mathbb{Z})$ such that $\Theta(r) = \min\{j \in \mathbb{Z} : j \ge r_n + \alpha_{\mathbb{R}}^n\}$ for all $r \in P(\mathbb{R})$. We easily obtain

$$\sharp r \le \Theta(r) < \sharp r + 1 + \gamma \quad \text{for all} \quad r \in P(\mathbb{R}).$$

Now, let $\log_\alpha$ be the logarithm to the base $\alpha$ (which is computable due to Appendix 1: the remark following A1.1), let $D \in P(\log_\alpha \circ \delta)$, and define $\Delta \in P(\mathbb{Z} \times P(A) \to \mathbb{Z})$ by

$\Delta(k,p) = \Theta(D(\pi(\beta^k),p))$ with $\pi \in P(\mathbb{Q} \to P(\mathbb{Q}))$, $\pi(t)_m = t$ for all $(m,t) \in \mathbb{Z} \times \mathbb{Q}$. Then we have $\sharp D(\pi(\beta^k),p) = \log_\alpha(\delta(\beta^k, \sharp p))$, hence

$$\log_\alpha(\delta(\beta^k, \sharp p)) \leq \Delta(k,p) < \log_\alpha(\delta(\beta^k, \sharp p)) + 1 + \gamma$$

and hence $\delta(\beta^k, \sharp p) \geq \alpha^{\Delta(k,p)} > \alpha^{1+\gamma} \cdot \delta(\beta^k, \sharp p)$. $\square$

**Definition**: For any $c : X^2 \to [0,\infty]$ let $c^2 : X^2 \times X^2 \to [0,\infty]$ be defined by $c^2((x_1,x_2),(y_1,y_2)) = \max\{c(x_1,y_1), c(x_2,y_2)\}$.

**2.2 Proposition**: Let $c : X^2 \to \mathbb{R}_0^+$ be a quasi-metric which satisfies

$$\forall x,y \in X.\ c(y,x) \leq \lambda(x) \cdot c(x,y)$$

where $\lambda \in cp.(X \to \mathbb{R}^+)$. Let $\delta_c : \mathbb{R}^+ \times X^2 \to \mathbb{R}^+$ be defined by

$$\delta_c(\varepsilon,(x_1,x_2)) = \frac{\varepsilon}{1 + \max\{\lambda(x_1), \lambda(x_2)\}}.$$

Then $\delta_c$ is a $c^2, d_1$-modulus of continuity for $c$ (where $d_1(\xi,\eta) = |\xi - \eta|$). $\square$

By the following propositions we have calculus-like rules to produce moduli of continuity of type $(B, B')$ or of an analogous type. - If we say that a modulus of continuity (which is a program) is defined by a given equation, we mean that a modulus of continuity which satisfies that equation can be defined by means of a well-known procedure.

**2.3 Proposition**: A $\mathbf{B}, B_i$-modulus of uniform continuity for $\mathrm{id}_i : \mathbf{X} \to X_i$ $(i = 1,\ldots,\mu)$ is defined by $\Delta_{\mathrm{id}_i}(k,p) = k$ (for all $(k,p) \in \mathbb{Z} \times P(\mathbf{X})$). $\square$

**2.4 Proposition**: Let $f_i : A \to X_i$ and $\Delta_i \in \mathcal{M}(f_i, B, B_i)$ for $i = 1,\ldots,\mu$. Then a $B, \mathbf{B}$-modulus of continuity for $(f_1,\ldots,f_\mu)$ is defined by

$$\Delta(k,p) = \max\{\Delta_1(k,p),\ldots,\Delta_\mu(k,p)\}. \quad \square$$

**2.5 Proposition**: Let $f \in cp.(A \to A')$, $g : A' \to X''$, $\Delta_f \in \mathcal{M}(f, B, B')$, $\Delta_g \in \mathcal{M}(g, B', B'')$, and $F \in P(f)$. Then a $B, B''$-modulus of continuity for $g \circ f$ is defined by

$$\Delta_{g \circ f}(k,p) = \Delta_f(\Delta_g(k, F(p)), p). \quad \square$$

**2.6 Proposition**: Let $f \in cp.(A \to A')$, $g \in cp.(\mathbb{N} \times A \times A' \to A')$, and let $h : \mathbb{N} \times A \to A'$ be recursively defined by

$$h(0,x) = f(x), \quad h(n+1,x) = g(n,x,h(n,x)).$$

Let, moreover, $\Delta_f \in \mathcal{M}(f,B,B')$, $\Delta_g \in \mathcal{M}(g, B \times B', B')$ (where $(B \times B') : ((y,z),k) \mapsto B(y,k) \times B'(z,k)$ ), and let $\Delta_h \in P(\mathbb{Z} \times \mathbb{N} \times P(A) \to \mathbb{Z})$ satisfy

$$\begin{array}{rcl} \Delta_h(k,0,p) & = & \Delta_f(k,p) \\ \Delta_h(k,n+1,p) & = & \max\{\vartheta, \Delta_h(\vartheta,n,p)\} \end{array}$$

where $\vartheta := \Delta_g(k,n,p,H(n,p))$ with $H \in P(h)$. Then $\Delta_h \in \mathcal{M}(h,B,B')$.

Proof: We have to show that

(i) $\forall k \in \mathbb{Z}. \forall p \in P(A). \forall y \in A. \Big(\sharp p \in B(y, \Delta_h(k,n,p)) \;\Rightarrow\; h(n,\sharp p) \in B'(h(n,y),k)\Big)$

holds for all $n \in \mathbb{N}$. For $n = 0$, (i) holds obviously. Moreover, for every $n \in \mathbb{N}$, (i) implies by (B3) that the following holds for all $k \in \mathbb{Z}$, $p \in P(A)$, and all $y \in A$:

$$\sharp p \in B(y, \Delta_h(k,n+1,p)) \;\Rightarrow\; \left\{ \begin{array}{c} \sharp p \in B(y,\vartheta) \text{ (see above)} \\ h(n,\sharp p) \in B'(h(n,y),\vartheta) \end{array} \right\} \;\Rightarrow$$

$$\Rightarrow\; h(n+1,\sharp p) = g(n,\sharp p,h(n,\sharp p)) \in B'(g(n,y,h(n,y)),k) = B'(h(n+1,y),k). \;\Box$$

**Presupposition**: In the following let $\sigma', \tau' \in P(\mathbb{Z} \to \mathbb{Z})$ satisfy $\sigma'_k *' \tau'_k \geq k$ for all $k \in \mathbb{Z}$ (cf. proof of 1.10), and so

(B2') $\qquad x \in B(y,\sigma'_k) \;\wedge\; y \in B(z,\tau'_k) \;\Rightarrow\; x \in B(z,k).$

**2.7 Proposition**: Let $f : \mathbb{N} \times A \to X'$ converge pointwise to $g : A \to X'$ such that

$$\begin{array}{l} \forall x \in A.\, \forall k \in \mathbb{Z}.\, \exists m \in \mathbb{N}.\, \forall n > m. \\ \big[g(x) \in B'(f(n,x),k) \;\wedge\; f(n,x) \in B'(g(x),k)\big]. \end{array}$$

Let $\Delta_f \in \mathcal{M}(f_n,B,B')$ for all $n \in \mathbb{N}$, and let $\Delta_g : \mathbb{Z} \times P(A) \to \mathbb{Z}$ be defined by $\Delta_g(k,p) = \Delta_f(\tau'(\sigma'_k),p)$. Then $\Delta_g \in \mathcal{M}(g,B,B')$.

Proof: Let $y \in A$, $k \in \mathbb{Z}$, $p \in P(A)$, $x := \sharp p \in B(y, \Delta_f(\tau'(\sigma'_k),p))$. Then for some sufficiently large $n \in \mathbb{N}$ we have

$$\begin{array}{rcl} g(x) & \in & B'(f(n,x),\sigma'(\sigma'_k)) \\ f(n,x) & \in & B'(f(n,y),\tau'(\sigma'_k)) \\ f(n,y) & \in & B'(g(y),\tau'_k), \end{array}$$

hence $g(x) \in B'(f(n,y),\sigma'_k)$, and hence $g(x) \in B'(g(y),k)$. $\Box$

Now we show that for certain functions the concept of computability can be simplified.

**2.8 Proposition**: Let $A$ be open and $A \cap U$ be decidable on U. Let $f : A \to X'$. Assume that there exists a program $\Gamma \in P(\mathbb{Z} \times (A \cap U) \to \mathbb{Z})$ satisfying

(ii) $$\forall k \in \mathbb{Z}.\ \forall u \in A \cap U.\ \forall x \in A.\ \Big(u \in B(x, \Gamma(k,u)) \ \Rightarrow\ f(u) \in B'(f(x), k)\Big),$$

(iii) $$\forall k \in \mathbb{Z}.\ \forall r \in P(\mathbb{Z} \to A \cap U).\ \exists n \in \mathbb{Z}.\ n \geq \Gamma(k, r_n).$$

Then $f$ is computable if and only if there exists a program $F \in P(\mathbb{Z} \times (A \cap U) \to U')$ such that

(iv) $$F(k,u) \in B'(f(u), k) \quad \text{for all} \quad (k,u) \in \mathbb{Z} \times (A \cap U).$$

**Notes**: 1. $\Gamma$ and $F$ do not operate on programs.
2. Assume that $f$ has a modulus $\Delta \in P(\mathbb{Z} \to \mathbb{Z})$ of uniform continuity. If we define $\Gamma \in P(\mathbb{Z} \times (A \cap U) \to \mathbb{Z})$ by $\Gamma(k,u) = \Delta(k)$, then $\Gamma$ satisfies both (ii) and (iii).

For the proof of 2.8 we show at first:

**2.9 Lemma**: If $A$ is open and $A \cap U$ is decidable on $U$, then there exists a program $Q \in P(P(A) \to P(A))$ such that $\sharp Q(p) = \sharp p$ and $Q(p)_n \in A \cap U$ for all $(n,p) \in \mathbb{Z} \times P(A)$.

Proof: Let $p \in P(A)$. Since $A$ is open, there exists $j \in \mathbb{Z}$ such that $B(\sharp p, j) \subseteq A$. Then, for all $k \geq j$, we have $p_k \in B(\sharp p, k) \subseteq A$. Accordingly, there exists $K \in P(\mathbb{Z} \times P(A) \to \mathbb{Z})$ such that $K(n,p) = \min\{k \geq n : p_k \in A\}$. Due to 1.3 and the proof of 1.2 there exists $Q \in P(P(A) \to P(\mathbb{Z} \to U))$ such that

$$Q(p)_n = p_{K(n,p)} \in A \cap B(\sharp p, K(n,p)) \subseteq B(\sharp p, n). \ \Box$$

Proof of 2.8: Let $\Gamma$ satisfy (ii) and (iii), and define $N \in P(\mathbb{Z} \times P(\mathbb{Z} \to A \cap U) \to \mathbb{Z})$ by the condition that, for all $(k,r) \in \mathbb{Z} \times P(\mathbb{Z} \to A \cap U)$, $N(k,r)$ is the first member $m$ of the sequence $0, -1, 1, -2, 2, -3, 3, \ldots$ (e.g.) such that $m \geq \Gamma(k, r_m)$.
Assume now that $k \in \mathbb{Z}$, $p \in P(A)$, and $q := Q(p)$ (with $Q$ as above). For $n := N(\tau'_k, q)$ we successively obtain $n \geq \Gamma(\tau'_k, q_n)$, $q_n \in B(\sharp p, n) \subseteq B(\sharp p, \Gamma(\tau'_k, q_n))$, and hence $f(q_n) \in B'(f(\sharp p), \tau'_k)$ (by (ii)). From the additional assumptum (iv) we especially obtain $F(\sigma'_k, q_n) \in B'(f(q_n), \sigma'_k)$ and so, by (B2'), $F(\sigma'_k, q_n) \in B'(f(\sharp p), k)$ where $q_n = \Omega(Q(p), N(\tau'_k, Q(p)))$ (for $\Omega$ as in the proof of 1.2). So, by 1.2, $f$ is computable. - The converse holds obviously. $\Box$

# §3 Generalized H-quasi-metric spaces

**Definition**: $(X, c, \oplus)$ is said to be a **generalized H-quasi-metric space** iff $X$ is a set, $c : X^2 \to [0, \infty]$, $\oplus : [0, \infty]^2 \to [0, \infty]$, for all $x, y, z \in X$ we have

(c1) $c(x, x) = 0$

(c2) $c(x, y) > 0$ if $x \neq y$

(c3)* $c(x, z) \leq c(x, y) \oplus c(y, z)$

(c4) $x \neq y \;\Rightarrow\; \exists \varepsilon > 0.\ \forall z \in X.\ \Big(c(z, x) > \varepsilon \;\vee\; c(z, y) > \varepsilon\Big)$,

and $\oplus$ satisfies the following conditions:

(1) $\oplus$ is monotonic increasing in both arguments.

(2) $\lim_{\xi \to 0}(\xi \oplus \xi) = 0$.

**Example:** Let $\|.\|$ be a norm of $\mathbb{R}^\mu$. Define $c(x, y) = \frac{\|x-y\|}{\|y\|}$ for all $x, y \in \mathbb{R}^\mu$ with $y \neq \theta$ (where $\|\theta\| = 0$), $c(\theta, \theta) = 0$, and $c(x, \theta) = \infty$ if $x \neq \theta$. Define, moreover, $\xi \oplus \eta = \xi + \xi\eta + \eta$ for $\xi, \eta \in [0, \infty]$. Then $(\mathbb{R}^\mu, c, \oplus)$ is a generalized H-quasimetric space. (For the proof of (c4) consider $\varepsilon = \frac{\|x-y\|}{3 \cdot \max\{\|x\|, \|y\|\}}$.) □

For the following we fix a number $\alpha \in \,]0, 1[\, \cap \mathbb{Q}$.

**Definition**: $(X, B, *)$ is said to be **induced** by $(X, c, \oplus)$, iff the latter is a generalized H-quasi-metric space, and $B : X \times \mathbb{Z}^{\mathrm{C}} \to \mathcal{P}(X)$ as well as $* : (\mathbb{Z}^{\mathrm{C}})^2 \to \mathbb{Z}^{\mathrm{C}}$ are defined by

(3) $B(y, k) = \{x \in X : c(x, y) \leq \alpha^k\}$

(4) $k * m = \max\{n \in \mathbb{Z}^{\mathrm{C}} : \alpha^n \geq \alpha^k \oplus \alpha^m\}$.

**3.1 Proposition**: If $(X, B, *)$ is induced by $(X, c, \oplus)$, then $(X, B, *)$ satisfies (B1) - (B4) (see §0) and the following conditions:

(5) $*$ is monotonic increasing in both arguments.

(6) $\lim_{m \to \infty}(m * m) = \infty$. (This is the same as $(*)$ in §0.)

Proof: Obviously, $B$ satisfies (B1), (B3), and (B4).
Ad (B2): Due to (3), (c3)*, (1), and (4), we have

$$\begin{aligned} & x \in B(y, k) \;\wedge\; y \in B(z, m) \;\Rightarrow \\ \Rightarrow\; & c(x, y) \leq \alpha^k \;\wedge\; c(y, z) \leq \alpha^m \\ \Rightarrow\; & c(x, z) \leq \alpha^k \oplus \alpha^m \leq \alpha^{k*m} \\ \Rightarrow\; & x \in B(z, k * m). \end{aligned}$$

Ad (5): If $k \leq l$ we have $\alpha^k \geq \alpha^l$, $\alpha^{k*m} \geq \alpha^k \oplus \alpha^m \geq \alpha^l \oplus \alpha^m$ and hence $k * m \leq l * m$ by (4). In the same way, $*$ is monotonic increasing in the second argument.
Ad (6): By (2), for all $k \in \mathbb{Z}$ there exists $\xi > 0$ such that $\xi \oplus \xi < \alpha^k$. For all $m \in \mathbb{Z}$ with $\alpha^m \leq \xi$ we obtain $\alpha^{(m*m)+1} < \alpha^m \oplus \alpha^m \leq \xi \oplus \xi < \alpha^k$, and hence $m * m \geq k$. □

**Presupposition**: In the following, let $(X, B, *)$ satisfy (B1) - (B4), $B(y, -\infty) = X$ for all $y \in X$, and (5) - (6). - Then we can define a generalized H-quasi-metric space $(X, \hat{c}, \hat{\oplus})$ that induces $(X, B, *)$:

**Definition**: Let the functions $\lambda : X^2 \to \mathbb{Z}^{\mathrm{C}}$, $\hat{c} : X^2 \to [0, \infty]$, $\varphi : [0, \infty] \to [0, 1]$, and $\hat{\oplus} : [0, \infty]^2 \to [0, \infty]$ be defined by

(7) $\lambda(x, y) = \max\{k \in \mathbb{Z}^{\mathrm{C}} : x \in B(y, k)\}$

(8) $\hat{c}(x, y) = \alpha^{\lambda(x,y)}$

(9) $\alpha^k \hat{\oplus} \alpha^m = \alpha^{k*m}$

(10) $\varphi(\xi) = \frac{\xi}{1+\xi}$ if $0 \leq \xi < \infty$, $\quad \varphi(\infty) = 1$

(11) $\xi \mapsto \varphi(\xi \hat{\oplus} \alpha^m)$ is affinely linear in $[\alpha^k, \alpha^{k-1}]$ for all $k \in \mathbb{Z}$, $m \in \mathbb{Z}^{\mathrm{C}}$.

(12) $\eta \mapsto \varphi(\xi \hat{\oplus} \eta)$ is affinely linear in $[\alpha^m, \alpha^{m-1}]$ for all $\xi \in [0, \infty]$, $m \in \mathbb{Z}$.

**Notes:** Due to the Hausdorff axiom (B4), $\lambda(x, y)$ exists for all $x, y \in X$. -
In the definition of $\hat{\oplus} : [0, \infty]^2 \to \infty$ by (9), (11) and (12) we should use a function like $\varphi$ in case $\alpha^k \hat{\oplus} \alpha^m = \infty$ for some $k, m \in \mathbb{Z}$.

We easily obtain:

**3.2 Lemma**: For all $x, y \in X$ and all $k \in \mathbb{Z}^{\mathrm{C}}$,

$$x \in B(y, k) \;\Leftrightarrow\; \hat{c}(x, y) \leq \alpha^k. \quad \square$$

**3.3 Proposition**: $(X, \hat{c}, \hat{\oplus})$ is a generalized H-quasi-metric space.

Proof: First we show that $\hat{\oplus}$ satisfies (1) - (2):
Ad (1): Let $\alpha^k \leq \alpha^l$. Then $k \geq l$, $k * m \geq l * m$ by (5), and hence $\alpha^k \hat{\oplus} \alpha^m = \alpha^{k*m} \leq \alpha^{l*m} = \alpha^l \hat{\oplus} \alpha^m$. Similarly, if $\alpha^m \leq \alpha^n$, then we obtain $\alpha^k \hat{\oplus} \alpha^m \leq \alpha^k \hat{\oplus} \alpha^n$. By (10) - (12) it follows that $\hat{\oplus}$ is monotonic increasing in both arguments.
Ad (2): In case $k * k \geq n$ (cf. (6)) and $0 < \xi \leq \alpha^k$ we have $0 \leq \xi \hat{\oplus} \xi \leq \alpha^k \hat{\oplus} \alpha^k \leq \alpha^n$.

By definition, $\hat{c}$ satisfies (c1). By (B4) and 3.2 it is easily seen that $\hat{c}$ satisfies (c4). (c2) follows from (c1) and (c4).
Ad (c3)*: Let $k := \lambda(x, y)$ and $m := \lambda(y, z)$. Then we successively obtain:

$$\hat{c}(x, y) = \alpha^k, \;\; \hat{c}(y, z) = \alpha^m,$$
$$x \in B(y, k), \;\; y \in B(z, m), \;\; x \in B(z, k * m),$$
$$\hat{c}(x, z) \leq \alpha^{k*m} = \alpha^k \hat{\oplus} \alpha^m = \hat{c}(x, y) \hat{\oplus} \hat{c}(y, z). \quad \square$$

**3.4 Proposition**: $(X, \hat{c}, \hat{\oplus})$ induces $(X, B, *)$.

Proof: By 3.2, $B$ and $\hat{c}$ satisfy (3). - By (9), $*$ and $\hat{\oplus}$ satisfy (4). $\square$

**3.5 Proposition**: If $(X, B, *)$ is induced by $(X, c, \oplus)$, and if $\hat{c}, \hat{\oplus}$ are defined as above, then we have, for all $x, y \in X$ and all $m \in \mathbb{Z}$,

$$\begin{array}{ll} c(x, y) \leq \hat{c}(x, y) < \alpha^{-1} \cdot c(x, y), & \text{if } \; 0 < c(x, y) < \infty, \\ \hat{c}(x, y) = c(x, y), & \text{if } \; c(x, y) \in \{0, \infty\}, \\ \alpha^k \oplus \alpha^m \leq \alpha^k \hat{\oplus} \alpha^m < \alpha^{-1} \cdot (\alpha^k \oplus \alpha^m) & \text{if } \; 0 < \alpha^k \oplus \alpha^m < \infty, \\ \alpha^k \hat{\oplus} \alpha^m = \alpha^k \oplus \alpha^m, & \text{if } \; \alpha^k \oplus \alpha^m \in \{0, \infty\}. \end{array}$$

Proof: Ad $c, \hat{c}$: By (3) and the definitions of $\lambda$ and $\hat{c}$ we have $c(x,y) \leq \alpha^{\lambda(x,y)} = \hat{c}(x,y)$.
If $0 < c(x,y) < \infty$, then $\lambda(x,y) \in \mathbb{Z}$ and hence $\alpha \cdot \hat{c}(x,y) = \alpha^{\lambda(x,y)+1} < c(x,y)$.
If $c(x,y) = 0$, then $x = y,\ x \in B(y,\infty),\ \lambda(x,y) = \infty$, and hence $\hat{c}(x,y) = 0$.
If $c(x,y) = \infty$, then $\hat{c}(x,y) = \infty$.

Ad $\oplus, \hat{\oplus}$: By the definition of $\hat{\oplus}$ and (4) we have $\alpha^k \hat{\oplus} \alpha^m = \alpha^{k*m} \geq \alpha^k \oplus \alpha^m$.
If $0 < \alpha^k \oplus \alpha^m < \infty$, then $k * m \in \mathbb{Z}$ and hence $\alpha \cdot (\alpha^k \hat{\oplus} \alpha^m) = \alpha^{(k*m)+1} < \alpha^k \oplus \alpha^m$.
If $\alpha^k \oplus \alpha^m = 0$, then $k * m = \infty,\ \alpha^k \hat{\oplus} \alpha^m = 0$.
If $\alpha^k \oplus \alpha^m = \infty$, then $\alpha^k \hat{\oplus} \alpha^m = \infty$. $\square$

**Definition**: Let $\overline{\mathbb{Q}} := \mathbb{Q}_0^+ \cup \{\infty\}$. A generalized H-quasimetric space $(X, c, \oplus)$ and its function $\oplus$ are said to be **nice** iff the restriction of $\oplus$ to $\overline{\mathbb{Q}} \times \overline{\mathbb{Q}}$ maps this set into $\overline{\mathbb{Q}}$ and is computable in the original sense for word functions.

**Notes: 1.** If $(X, B, *, U)$ is a computation space, then (by the definition of this concept in §0) $*$ is computable. If, moreover, $*$ satisfies (5), and $\hat{\oplus}$ is defined as above, then $\hat{\oplus}$ is obviously nice.
**2.** 'Conversely', if $(X, B, *)$ is induced by a nice generalized H-quasi-metric space $(X, c, \oplus)$, then $*$ is computable so that, for any set $U \subseteq X$ of words over a common finite alphabet, $(X, B, *, U)$ is a computation space.
**3.** 1.12 also holds in case we have a nice generalized H-quasi-metric space $(X', c', \oplus')$.
**4.** The definition of $c, c'$-moduli of continuity (see §2) can also be applied to functions of type $A \to X'$ $(A \subseteq X)$ if generalized H-quasi-metric spaces $(X, c, \oplus)$ and $(X', c', \oplus')$ are given. For the corresponding 'generalized' $c, c'$-moduli of continuity there also hold propositions which are analogous to 2.3 - 2.6, and, for nice spaces $(X', c', \oplus')$, 2.7.

* * *

In §1 and §2 we have used the existence of a computable function $* : \mathbb{Z}^{\mathrm{C}} \times \mathbb{Z}^{\mathrm{C}} \to \mathbb{Z}^{\mathrm{C}}$ with the properties (B2) and $(*)$ (of §0, i.e. (6) of §3) for no other purpose than to prove the existence of computable functions $\sigma, \tau : \mathbb{Z} \to \mathbb{Z}$ satisfying

(B2') $$x \in B(y, \sigma_k) \ \wedge \ y \in B(z, \tau_k) \ \Rightarrow \ x \in B(z, k).$$

(see the proof of 1.10). So we obtain all corresponding results of §1 and §2 if we only presuppose the existence of such functions $\sigma$ and $\tau$ in place of $*$. On the other hand, a computation space $(X, B, *, U)$ can in general be applied more 'elastically' than a corresponding space $(X, B, \sigma, \tau, U)$ with *particular* functions $\sigma, \tau$. We shall show that every space of this kind that satisfies certain additional conditions can be considered as 'induced' by a computation space.

To this end we now suppose that we have computable functions $\sigma, \tau : \mathbb{Z} \to \mathbb{Z}$ satisfying (B2') and, of course, that we have $X, B$ satisfying (B1) and (B3). Does there exist a computable $* : \mathbb{Z}^{\mathrm{C}} \times \mathbb{Z}^{\mathrm{C}} \to \mathbb{Z}^{\mathrm{C}}$ satisfying (B2) such that $\sigma$ and $\tau$ can be characterized by means of $*$?

By (B2') and (B3) we have $B(y, \sigma_k) = B(y, \max\{k, \sigma_k\})$ and $B(z, \tau_k) = B(z, \max\{k, \tau_k\})$ for all $y, z \in X$ and $k \in \mathbb{Z}$. So we may assume that

$$\sigma_k, \tau_k \geq k \quad \text{for all} \quad k \in Z. \tag{13}$$

In the following we use the abbreviations

$$\sigma_{-\infty} := \inf_{k \in \mathbb{Z}} \sigma_k, \quad \tau_{-\infty} := \inf_{k \in \mathbb{Z}} \tau_k,$$

$$K_{mn} := \{k \in \mathbb{Z} : \sigma_k \leq m \ \wedge \ \tau_k \leq n\}.$$

**3.6 Proposition**: Given $X, B$ satisfying (B1) and (B3), and functions $\sigma, \tau \colon \mathbb{Z} \to \mathbb{Z}$, which are computable, monotonic increasing, and satisfy both (B2') and (13). Assume, moreover, that the values of $\sigma_{-\infty}$ and $\tau_{-\infty}$ have been computed. Define $* : \mathbb{Z}^{\mathrm{C}} \times \mathbb{Z}^{\mathrm{C}} \to \mathbb{Z}^{\mathrm{C}}$ by

$$m * n = \sup K_{mn}.$$

Then $*$ is computable and satisfies (5), (6), and (B2). The functions $\sigma$ and $\tau$ can be characterized by means of $*$:

$$\sigma_k = \min\{m \in \mathbb{Z} : \exists n \in \mathbb{Z}.\ m * n \geq k\},$$

$$\tau_k = \min\{n \in \mathbb{Z} : \exists m \in \mathbb{Z}.\ m * n \geq k\}.$$

Proof: At first we show that $\{(m, n) \in (\mathbb{Z}^{\mathrm{C}})^2 : K_{mn} = \emptyset\}$ is decidable. Given $m, n \in \mathbb{Z}^{\mathrm{C}}$. If $m = -\infty$ or $n = -\infty$ then $K_{mn} = \emptyset$. Now let $m, n > -\infty$. If $\sigma_{-\infty} \leq m$ and $\tau_{-\infty} \leq n$ then there exist $k_1, k_2 \in \mathbb{Z}$ with $\sigma_{k_1} \leq m$ and $\tau_{k_2} \leq n$. Setting $k := \min\{k_1, k_2\}$ we obtain $\sigma_k \leq \sigma_{k_1}$, $\tau_k \leq \tau_{k_2}$, so $k \in K_{mn}$, and so $K_{mn} \neq \emptyset$. Now let $\sigma_{-\infty} > m$ or $\tau_{-\infty} > n$. Then, for all $k \in \mathbb{Z}$, $\sigma_k > m$ or $\tau_k > n$, i.e. $k \notin K_{mn}$. So $K_{mn} = \emptyset$.

Computability of $*$: Given $m, n \in \mathbb{Z}^{\mathrm{C}}$. We consider the case $m \leq n$, e.g. If $m = -\infty$ then $K_{mn} = \emptyset$ so that $m * n = -\infty$. If $m = \infty$, then $n = \infty$, so $K_{mn} = \mathbb{Z}$, and so $m * n = \infty$. Now let $m \in \mathbb{Z}$. Because of (13), the set $K_{mn}$ has an upper bound (namely $m$). So in case $K_{mn} \neq \emptyset$, $K_{mn}$ has a greatest element, namely $m * n$, which we can compute. In case $K_{mn} = \emptyset$ we have $m * n = -\infty$. (Note that we can decide whether $K_{mn} = \emptyset$.
Ad (5): Let $m \leq m'$ and $n \leq n'$. Then $K_{mn} \subseteq K_{m'n'}$. So $m' * n'$ is an upper bound of $K_{mn}$ and so $m * n \leq m' * n'$.
Ad (6): For all $m \geq \max\{\sigma_k, \tau_k\}$ we have $k \in K_{mm}$ and hence $m * m \geq k$.
(B2) easily follows by means of (B3), (B2') and the fact that $m \geq \sigma_{m*n}$, $n \geq \tau_{m*n}$ in case $m * n \in \mathbb{Z}$.

Characterization of $\sigma$: Let $M_k := \{m \in \mathbb{Z} : \exists n \in \mathbb{Z}.\ m * n \geq k\}$. Because of $\sigma_k \leq \sigma_k$ and $\tau_k \leq \tau_k$ we have $k \in K_{\sigma_k \tau_k}$, so $k \leq \sigma_k * \tau_k$, and so $\sigma_k \in M_k$. Moreover, for all $m \in M_k$ there exists $n \in \mathbb{Z}$ such that $m * n \geq k$, and hence $m \geq \sigma_{m*n} \geq \sigma_k$. So we have $\sigma_k = \min M_k$. - $\tau$ can be characterized similarly by means of $*$. $\square$.

**Notes**: 1. By the first part of the latter proof, $\{(m,n) \in (\mathbb{Z}^{\mathrm{C}})^2 : K_{mn} = \emptyset\}$ is at any rate decidable in the classical sense of the word, but if we do not know the values of $\sigma_{-\infty}$ and $\tau_{-\infty}$, we can perhaps not *find* a corresponding decision procedure. 2. Now we only suppose that $\sigma, \tau$ are computable and satisfy (B2') and (13). Then we can define similar functions $\sigma^+, \tau^+ \colon \mathbb{Z} \to \mathbb{Z}$ by

$$\begin{array}{lll} \sigma_k^+ = \sigma_{k_0}, & \tau_k^+ = \tau_{k_0}, & \text{if} \quad k \le k_0 \\ \sigma_k^+ = \max\{\sigma_k, \sigma_{k-1}^+\}, & \tau_k^+ = \max\{\tau_k, \tau_{k-1}^+\}, & \text{if} \quad k > k_0. \end{array}$$

Of course, $\sigma^+, \tau^+$ are computable and monotonic increasing. As easily seen, they also satisfy (B2'), (13), $\sigma_{-\infty}^+ = \sigma_{k_0}$, $\tau_{-\infty}^+ = \tau_{k_0}$, and so all assumptions of 3.6. (Regard that for individual applications we need not consider sentences of the form $x \,\varepsilon\, B(z,k)$ with 'very small' degrees $k$ of approximation.)

## Appendix 1: Programs to compute 'real functions'

In the following, let $\mu \in \mathbb{N}^+$ and let $\underline{\mathbb{R}^\mu}$ be the computation space induced by $(\mathbb{R}^\mu, d_\mu, \alpha, \mathbb{Q}^\mu)$ where $\alpha \in ]0,1[$ is given and $d_\mu \colon (\mathbb{R}^\mu)^2 \to \mathbb{R}_0^+$ is defined by

$$d_\mu(x,y) := |x-y| := \max_{i=1,\dots,\mu} |x_i - y_i|.$$

The proofs of the Lemmata and of the Propositions of §1 (except 1.4) yield very general 'calculus-like' rules to construct programs for computing functions. To obtain more rules for this purpose we further deal with programs for computing functions of type $A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^\mu$. Such a function is simply said to be computable iff it is $\underline{\mathbb{R}^\mu}, \underline{\mathbb{R}}$-computable.

**A1.1 Proposition**: The following functions are computable: Addition, subtraction, multiplication, and division of real numbers, the functions max, min: $\mathbb{R}^2 \to \mathbb{R}$, hence the usual norm $|.| \colon \mathbb{R} \to \mathbb{R}$, and hence the distance $d_1$.

By the 1.2 and 1.3 it is sufficient to show that the following holds for all $r, s \in P(\mathbb{R})$:

$$|(\sharp r \pm \sharp s) - (r_k \pm s_k)| \le \alpha^n \quad \text{for} \quad k = n + \min\left\{j \in \mathbb{Z} : 2\alpha^j \le 1\right\}$$

$$|\sharp r \cdot \sharp s - r_k \cdot s_k| \le \alpha^n \quad \text{for} \quad k = \min\left\{j \in \mathbb{Z} : \alpha^j \le \frac{\alpha^n}{|r_j| + |s_j| + \alpha^j}\right\}$$

$$\left|\frac{1}{\sharp r} - \frac{1}{r_k}\right| \le \alpha^n \quad \text{for} \quad \sharp r \ne 0, \;\; k = \min\left\{j \in \mathbb{Z} : \alpha^j \le \alpha^n(|r_j| - \alpha^j)|r_j|\right\}$$

$$|\max(\sharp r, \sharp s) - \max(r_n, s_n)| \le \alpha^n$$

$$\Big||\sharp r| - |r_n|\Big| \le \alpha^n. \quad \square$$

,

**Remark**: By applying the proofs of the Propositions A1.1, 1.1, 1.5, 1.7 - 1.9, and 1.12 we can now construct programs to compute functions like exp, sin, tan, ln, $\arctan\big|]-1,1[$, $\arcsin\big|]-1,1[$, and the square-root, e.g. - To generalize this we define:

**Definition**: For $\lambda = (\lambda_1, \ldots, \lambda_\mu) \in \mathbb{N}^\mu$ and $x = (x_1, \ldots, x_\mu) \in \mathbb{R}^\mu$ let

$$|\lambda| := \lambda_1 + \ldots + \lambda_\mu, \text{ and } x^\lambda := x_1^{\lambda_1} \cdots x_\mu^{\lambda_\mu}.$$

We easily obtain:

**A1.2 Proposition** (Corollary): For each $\mu \in \mathbb{N}^+$ and each $(a_\lambda)_{\lambda\in\mathbb{N}^\mu} \in cp.(\mathbb{N}^\mu \to \mathbb{R})$, the following function sequence is computable:

$$(n,x) \mapsto \sum_{|\lambda|\leq n} a_\lambda x^\lambda := \sum_{\lambda_1=0}^{n} \sum_{\lambda_2=0}^{n-\lambda_1} \cdots \sum_{\lambda_\mu=0}^{n-\lambda_1-\ldots-\lambda_{\mu-1}} a_\lambda x^\lambda. \quad \Box$$

Concerning the computability of *power series* we have:

**A1.3 Proposition** (Corollary): Let $f : ]-r,r[^\mu \to \mathbb{R}$,

$$f(x) = \sum_{\lambda\in\mathbb{N}^\mu} a_\lambda x^\lambda := \lim_{n\to\infty} \sum_{|\lambda|\leq n} a_\lambda x^\lambda \text{ for all } x \in ]-r,r[^\mu,$$

where $(a_\lambda)_{\lambda\in\mathbb{N}^\mu} \in cp.(\mathbb{N}^\mu \to \mathbb{R})$; $r, c \in cp.\mathbb{R}^+$, and $\sum_{|\lambda|=n} |a_\lambda| \leq \frac{c}{r^n}$ for all $n$. Then $f$ is computable

Proof: For $x \in ]-r,r[^\mu$ we have

$$\left|\sum_{|\lambda|\geq k} a_\lambda x^\lambda\right| \leq \sum_{n\geq k} \left(\sum_{|\lambda|=n} |a_\lambda|\right) |x|^n \leq \sum_{n\geq k} \frac{c}{r^n}|x|^n = c \cdot \left(\frac{|x|}{r}\right)^k \cdot \frac{1}{1-\frac{|x|}{r}} \to 0$$

as $k \to \infty$. By 1.12 and A1.2 it follows that $f$ is computable. $\Box$

The following proposition is concerned with the computability of inverse functions.

**A1.4 Proposition**: If $a, b \in cp.\mathbb{R}$, $a < b$, and if $f \in cp.([a,b] \to \mathbb{R})$ is strictly monotonic, then the inverse function $f^{-1}$ is computable. (Cf. [5], Chap.5, §4.)

Proof: Assume that $f$ is strictly monotonic *increasing*, for instance. Let $F \in P(f)$, and $q \in P(f[a,b])$. We recursively define two sequences $p_*, r_* : \mathbb{N} \to P([a,b])$ such that $\sharp p_n \leq f^{-1}(\sharp q) \leq \sharp r_n$ and

$$f^{-1}(\sharp q) = \lim_{n\to\infty} \sharp p_n = \lim_{n\to\infty} \sharp r_n.$$

Let $p_0 \in P(\{a\})$, $r_0 \in P(\{b\})$, and if $p_n, r_n \in P([a,b])$ have been defined, let

$$p_n^+ := \frac{2}{3}p_n + \frac{1}{3}r_n, \;\; r_n^- := \frac{1}{3}p_n + \frac{2}{3}r_n$$

(i.e., for all $k \in \mathbb{Z}$, $p_{nk}^+ = \frac{2}{3}p_{nk} + \frac{1}{3}r_{nk}$, e.g.)

$$k_n := \min\left\{k \in \mathbb{Z} : F(p_n^+)_k + 2\alpha^k \leq q_k \;\;\vee\;\; q_k \leq F(r_n^-)_k - 2\alpha^k\right\}$$

(for the existence of $k_n$ see below),

$$A_n := F(p_n^+)_{k_n} + 2\alpha^{k_n}, \quad B_n := F(r_n^-)_{k_n} - 2\alpha^{k_n},$$

$$(p_{n+1}, r_{n+1}) := \begin{cases} (p_n^+, r_n^-), & \text{if} \;\; A_n \leq q_{k_n} \leq B_n, \\ (p_n^+, r_n), & \text{if} \;\; A_n \leq q_{k_n} > B_n, \\ (p_n, r_n^-), & \text{if} \;\; A_n > q_{k_n} \leq B_n. \end{cases}$$

(A distinction of 5 corresponding cases would be more suitable for actual computations of $f^{-1}(\sharp q)$.) We use, moreover, the abbreviations

$$a_n := \sharp p_n, \; b_n := \sharp r_n, \; a_n^+ := \sharp p_n^+, \; b_n^- := \sharp r_n^-.$$

Inductive hypothesis: Assume that

(IH) $\qquad p_n, r_n \in P([a,b]), \;\; a_n < b_n, \;\; f(a_n) \leq \sharp q \leq f(b_n), \;\; b_n - a_n \leq \left(\frac{2}{3}\right)^n (b-a).$

Then we have $f(a_n^+) < f(b_n^-)$, and hence

$$\sharp F(p_n^+) = f(a_n^+) < \sharp q \;\;\vee\;\; \sharp q < f(b_n^-) = \sharp F(r_n^-).$$

Accordingly, $k_n$ exists, and thus $p_{n+1}$ and $r_{n+1}$ are defined as elements of P([a,b]). It follows that

$$a_n \leq a_{n+1} < b_{n+1} \leq b_n, \;\; b_{n+1} - a_{n+1} \leq \left(\frac{2}{3}\right)^{n+1} (b-a).$$

If $A_n = F(p_n^+)_{k_n} + 2\alpha^{k_n} \leq q_{k_n}$, we have

$$f(a_{n+1}) = f(a_n^+) \leq F(p_n^+)_{k_n} + \alpha^{k_n} \leq q_{k_n} - \alpha^{k_n} \leq \sharp q,$$

for instance. Accordingly, in every case,

$$f(a_{n+1}) \leq \sharp q \leq f(b_{n+1}).$$

We have shown that (IH) holds *for all n*. Thus $a_n \leq f^{-1}(\sharp q) \leq b_n$, hence

$$|f^{-1}(\sharp q) - a_n| \leq \left(\frac{2}{3}\right)^n (b-a) \leq \left(\frac{2}{3}\right)^n (r_{0n} - p_{0n} + 2\alpha^n)$$

and hence (by the triangle inequality), for all $m \in \mathbb{Z}$,

$$|f^{-1}(\sharp q) - p_{nm}| \leq \left(\frac{2}{3}\right)^n (r_{0n} - p_{0n} + 2\alpha^n) + \alpha^m.$$

Setting $n_k := \min\left\{n \in \mathbb{N} : \left(\frac{2}{3}\right)^n (r_{0n} - p_{0n} + 2\alpha^n) \leq \frac{1}{2}\alpha^k\right\}$, and $m_k := \min\{m \in \mathbb{Z} : \alpha^m \leq \frac{1}{2}\alpha^k\}$, we obtain

$$|f^{-1}(\sharp q) - p_{n_k,m_k}| \leq \alpha^k.$$

However, $p_*$ and $r_*$ depend on $q$. Accordingly, we we can specify a program $G \in P(\mathbb{Z} \times P(f[a,b]) \to \mathbb{Q})$ such that

$$|f^{-1}(\sharp q) - G(k,q)| \leq \alpha^k.$$

It follows by 1.2 that $f^{-1}$ is computable. $\square$

In Appendix 2 we also show that certain integral functions are computable (see A2.5).

## Appendix 2: Moduli of continuity for 'real functions'

By 2.1 - 2.7 or their proofs we can specify 'calculus-like' rules to construct moduli of continuity. To obtain more rules for this purpose, we shall especially deal with $d_\mu, d_1$-moduli of continuity for functions of type $A \to \mathbb{R}$ ($A \subseteq \mathbb{R}^\mu, \mu \in \mathbb{N}^+$). (For $d_\mu$ see Appendix 1.)

**A2.1 Proposition**: The functions $\delta_h$ described by the following equations are moduli of continuity (or even of *uniform* continuity) for the cited funtions $h$ of type $\mathbb{R}^2 \to \mathbb{R}$ or $(\subseteq)\ \mathbb{R} \to \mathbb{R}$, respectively:

| | |
|---|---|
| $\delta_{\text{add}}(\varepsilon) = \frac{\varepsilon}{2}$ | $(\text{add}(x,y) = x + y)$ |
| $\delta_{\text{sub}}(\varepsilon) = \frac{\varepsilon}{2}$ | $(\text{sub}(x,y) = x - y)$ |
| $\delta_{\text{mul}}(\varepsilon, x, y) = \frac{1}{2}\left(\sqrt{(\lvert x\rvert + \lvert y\rvert)^2 + 4\varepsilon} - (\lvert x\rvert + \lvert y\rvert)\right)$ | $(\text{mul}(x,y) = xy)$ |
| $\delta_{\text{div}}(\varepsilon, x) = \frac{\varepsilon x^2}{1+\varepsilon\lvert x\rvert}$ | $(\text{div}(x) = 1/x,\ x \neq 0)$ |
| $\delta_{\text{max}}(\varepsilon) = \varepsilon$ | $(\max : \mathbb{R}^2 \to \mathbb{R})$ |
| $\delta_{\text{norm}}(\varepsilon) = \varepsilon$ | $(\text{norm}(x) = \lvert x\rvert)$. $\square$ |

**A2.2 Proposition**: If $a, b \in cp.\mathbb{R}$, $a < b$, and if $f \in cp.([a,b] \to \mathbb{R})$ is strictly monotonic increasing and continuous, then the function $\delta : \mathbb{R}^+ \times [a,b] \to \mathbb{R}^+$ defined as follows is a modulus of continuity for $f$:

$$\delta(\varepsilon, x) = \begin{cases} \min\{x - f^{-1}(f(x) - \hat{\varepsilon}),\ f^{-1}(f(x) + \hat{\varepsilon}) - x\}, \\ \qquad \text{if } f^{-1}(f(a) + \hat{\varepsilon}) < x < f^{-1}(f(b) - \hat{\varepsilon}), \\ \min\left\{f^{-1}(f(a) + \hat{\varepsilon}) - a,\ f^{-1}(f(x) + \hat{\varepsilon}) - x\right\} \\ \qquad \text{if } \ a \le x \le f^{-1}(f(a) + \hat{\varepsilon}), \\ \min\left\{x - f^{-1}(f(x) - \hat{\varepsilon}),\ b - f^{-1}(f(b) - \hat{\varepsilon})\right\}, \\ \qquad \text{if } \ f^{-1}(f(b) - \hat{\varepsilon}) \le x \le b \end{cases}$$

where $\hat{\varepsilon} := \min\{\varepsilon,\ \frac{1}{2}(f(b) - f(a))\}$.

To prove the computability of $\delta$, we show:

**Lemma**: If $f \in cp.(A_1 \to \mathbb{R})$, $g \in cp.(A_2 \to \mathbb{R})$, $A_1 \subseteq\ ]-\infty, a]$, $A_2 \subseteq [a, \infty[$, $a \in cp.\mathbb{R} \cap A_1 \cap A_2$, $f(a) = g(a)$, and if $h : A_1 \cup A_2 \to \mathbb{R}$ is defined by

$$h(x) = \begin{cases} f(x), & \text{if } x \in A_1, \\ g(x), & \text{if } x \in A_2, \end{cases}$$

then $h$ is computable. (Here, $A_i$ can also be replaced by $A_i \times B$ $(i = 1, 2)$.)

**Proof**: $h(x) = f(\min(x,a)) + g(\max(x,a)) - f(a)$. $\square$

In many cases we can find moduli of continuity by the following proposition in which we use the definition $\overline{B}(x,r) := \{y \in \mathbb{R}^\mu : |y - x| \le r\}$ (for $x \in \mathbb{R}^\mu, r \ge 0$).

**A2.3 Proposition**: Let $A \subseteq \mathbb{R}^\mu$ be open, and $f : A \to \mathbb{R}$ be partially differentiable. Let $\varrho \in cp.(A \to \mathbb{R}^+)$ satisfy $\overline{B}(x, \varrho(x)) \subseteq A$ for all $x \in A$. Let, moreover, $g \in cp.(A \to \mathbb{R})$ such that

$$g(x) \ge \sum_{i=1}^{\mu} |D_i f(y_1, \ldots, y_{i-1}, z_i, x_{i+1}, \ldots, x_\mu)|, \quad g(x) > 0$$

for all $x \in A, y, z \in \overline{B}(x, \varrho(x))$ with $x_i < z_i < y_i$ or $y_i < z_i < x_i$. Let $\delta : \mathbb{R}^+ \times A \to \mathbb{R}^+$ be defined by

$$\delta(\varepsilon, x) = \min\left\{\varrho(x), \frac{\varepsilon}{g(x)}\right\}.$$

Then $\delta$ is a modulus of continuity for $f$.

**Proof**: For all $x, y \in A$ such that $|x - y| \le \delta(\varepsilon, x)$, $\varepsilon > 0$ we have

$$\begin{aligned} & |f(x_1, \ldots, x_\mu) - f(y_1, \ldots, y_\mu)| \\ \le\ & |f(x_1, x_2, \ldots) - f(y_1, x_2, \ldots)| + |f(y_1, x_2, x_3, \ldots) - f(y_1, y_2, x_3, \ldots)| + \ldots \\ & + |f(y_1, \ldots, y_{\mu-1}, x_\mu) - f(y_1, \ldots, y_{\mu-1}, y_\mu)| \\ \le\ & |x_1 - y_1| \cdot |D_1 f(z_1, x_2, \ldots)| + |x_2 - y_2| \cdot |D_2 f(y_1, z_2, x_3, \ldots)| + \ldots \\ & + |x_\mu - y_\mu| \cdot |D_\mu f(y_1, \ldots, y_{\mu-1}, z_\mu)| \\ \le\ & \delta(\varepsilon, x) \cdot g(x) \le \varepsilon \end{aligned}$$

for proper $z_1, \ldots, z_\mu$ such that $z_i$ between $x_i$ and $y_i$. $\square$

Moduli of continuity for power series:

**A2.4 Proposition**: Let $f : ]-r, r[^\mu \to \mathrm{IR}$,

$$f(x) = \sum_{\lambda \in \mathrm{IN}^\mu} a_\lambda x^\lambda \quad \text{for} \quad x \in ]-r, r[^\mu$$

where $r \in cp.\mathrm{IR}^+$ and $a_\lambda \in \mathrm{IR}$ for all $\lambda \in \mathrm{IN}^\mu$. Let, moreover, $c \in cp.\mathrm{IR}^+$ and $\sum_{|\lambda|=n} |a_\lambda| \leq \frac{c}{r^n}$ for all $n \in \mathrm{IN}^+$. Assume that $\delta \in cp.(\mathrm{IR}^+ \times ]-r, r[^\mu \to \mathrm{IR}^+)$ satisfies

$$\delta(\varepsilon, x) = \min\left\{\frac{r-|x|}{2}, \frac{\varepsilon}{cr}\left(\frac{r-|x|}{2}\right)^2\right\}.$$

Then $\delta$ is a modulus of continuity for $f$. If even $\sum_{|\lambda|=n} |a_\lambda| \leq \frac{c}{nr^{n-1}}$ for all $n \in \mathrm{IN}^+$, then $\delta(\varepsilon, x)$ can be replaced by $\frac{r-|x|}{2} \cdot \min\left\{1, \frac{\varepsilon}{cr}\right\}$.

**Proof**: Let $x, y \in ]-r, r[^\mu$ and $|x-y| \leq \varrho(x) := \frac{r-|x|}{2}$. Then we have

$$|y| \leq |x| + |y-x| \leq |x| + \frac{r-|x|}{2} = \frac{|x|+r}{2} < r.$$

Let, moreover, $\xi := \frac{|x|+r}{2}$ and $q := \frac{\xi}{r}$ (hence $0 < q < 1$), and $\sum_{|\lambda|=n} |a_\lambda| \leq \frac{c}{r^n}$. It follows that

$$|D_i f(y)| \leq \sum_{n\geq 1} \sum_{|\lambda|=n} \lambda_i |a_\lambda y^\lambda y_i^{-1}| \leq \sum_{n\geq 1} \sum_{|\lambda|=n} \lambda_i \cdot |a_\lambda| \xi^{n-1},$$

$$\sum_{i=1}^{\mu} |D_i f(y)| \leq \sum_{n\geq 1} \sum_{|\lambda|=n} n |a_\lambda| \xi^{n-1} \leq \sum_{n\geq 1} n \cdot \frac{c}{r^n} \xi^{n-1} = \frac{c}{r} \cdot \sum_{n\geq 1} n q^{n-1} = \frac{c}{r(1-q)^2}.$$

It follows by A2.3 that a modulus of continuity $\delta$ for $f$ is defined by

$$\delta(\varepsilon, x) = \min\{\varrho(x), \frac{\varepsilon r}{c}(1-q)^2\} = \min\left\{\frac{r-|x|}{2}, \frac{\varepsilon}{cr}\left(\frac{r-|x|}{2}\right)^2\right\}.$$

The remainder can be proved similarly. $\square$

Now we investigate integrals:

**A2.5 Proposition**: Let $f \in cp.(I \times A \to \mathbb{R})$, where $I$ is a (finite or infinite) intervall, $I \subseteq [a, \infty[ \subseteq \mathbb{R}$, $a \in cp.I$, $A \subseteq \mathbb{R}^\mu$ (e.g.), and let $g : I \times A \to \mathbb{R}$ be defined by

$$g(x, y) = \int_a^x f(t, y)dt.$$

Assume that there exists a modulus of continuity, $\delta_f$, for $f$ such that, for all $\varepsilon \in \mathbb{R}^+$, $x_0 \in I$, $y, y_0 \in A$, and all $s, t \in I$, we have

$$a \le s, t \le x_0 \;\wedge\; |(t, y) - (s, y_0)| \le \delta_f(\varepsilon, x_0, y_0) \implies |f(t, y) - f(s, y_0)| \le \varepsilon.$$

Then $g$ is computable (cp. [5], p.226) and has a modulus of continuity of the same kind. (This proposition can also successively be applied to proper multiple integrals).

Proof: 1. Computability of $g$: By 2.1 there also exists a modulus $\Delta_f : \mathbb{Z} \times P(I) \times P(A) \to \mathbb{Z}$ such that, for all $k \in \mathbb{Z}$, $p \in P(I)$, $q \in P(A)$, $y \in A$ and all $s, t \in I$, we have

$$a \le s, t \le \sharp p \;\wedge\; |(t, y) - (s, \sharp q)| \le \alpha^{\Delta_f(k,p,q)} \implies |f(t, y) - f(s, \sharp q)| \le \alpha^k.$$

We may assume that $a = 0$. Let $p \in P(I)$, $q \in P(A)$, $k \in \mathbb{Z}$, $\phi(p) := p_m + \alpha^m$ (for some $m \in \mathbb{Z}$),

$$k' := \min\{i \in \mathbb{Z} : \alpha^i \le \frac{\alpha^k}{\phi(p)}\}, \quad n := \min\left\{j \in \mathbb{N} : \; \frac{\phi(p)}{j} \le \alpha^{\Delta_f(k',p,q)}\right\},$$

and $s_i := i \cdot \frac{\sharp p}{n}$ (for $i = 0, \ldots, n$). We successively obtain for $i = 0, \ldots, n-1$:

$$0 \le s_{i+1} - s_i = \frac{\sharp p}{n} \le \frac{\phi(p)}{n} \le \alpha^{\Delta_f(k',p,q)},$$

$$|f(t, \sharp q) - f(s_i, \sharp q)| \le \alpha^{k'} \le \frac{\alpha^k}{\phi(p)} \le \frac{\alpha^k}{\sharp p} \quad \text{for} \quad s_i \le t \le s_{i+1},$$

$$\left| \int_{s_i}^{s_{i+1}} f(t, \sharp q)dt - (s_{i+1} - s_i) \cdot f(s_i, \sharp q) \right| \le \frac{\alpha^k}{n},$$

$$\left| \int_0^{\sharp p} f(t, \sharp q)dt - \frac{\sharp p}{n} \cdot \sum_{i<n} f(s_i, \sharp q) \right| \le \alpha^k.$$

It follows by 1.10 that $g$ is computable.

2. Now we specify a modulus of continuity, $\delta_g$, for $g$. Let $M : \mathbb{R}^+ \times I \times A \to \mathbb{R}^+$ be defined by

$$M(\varepsilon, x_0, y_0) = |f(0, y_0)| + \left( \frac{x_0}{\delta_f(\frac{\varepsilon}{2x_0}, x_0, y_0)} + 1 \right) \cdot \frac{\varepsilon}{2x_0}.$$

$M$ is computable, and it can be shown that $M(\varepsilon, x_0, y_0) \geq |f(t, y)|$ for all $\varepsilon \in \mathbb{R}^+$, $x_0 \in I$, $y, y_0 \in A$, $0 \leq t \leq x_0$ and $|y - y_0| \leq \delta_f(\frac{\varepsilon}{2x_0}, x_0, y_0)$. - Now we assume, moreover, that $0 \leq s, x \leq x_0$ and

$$|(x, y) - (s, y_0)| \leq \delta_g(\varepsilon, x_0, y_0) := \min \left\{ \frac{\varepsilon}{2 \cdot M(\varepsilon, x_0, y_0)}, \delta_f(\frac{\varepsilon}{2x_0}, x_0, y_0) \right\}.$$

Then we obtain

$$\begin{aligned} |g(x, y) - g(s, y_0)| &\leq |g(x, y) - g(s, y)| + |g(s, y) - g(s, y_0)| \\ &\leq \left|\int_s^x f(t, y) dt\right| + \left|\int_0^s (f(t, y) - f(t, y_0)) dt\right| \\ &\leq |x - s| \cdot M(\varepsilon, x_0, y_0) + x_0 \cdot \frac{\varepsilon}{2x_0} \leq \varepsilon. \qquad \square \end{aligned}$$

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