Local in time regularity properties of the Navier-Stokes equations beyond Serrin's condition

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Abstract

Let u be a weak solution of the Navier-Stokes equations in a domain $\Omega \subseteq \mathbb{R}^3$ and a time interval $[0,T)$, $0 < T \leq \infty$, with initial value u_0, and vanishing external force. As is well known, global regularity of u for general u_0 is an unsolved problem unless we pose additional assumptions on u_0 or on the solution u itself such as Serrin's condition $\|u\|_{L^s(0,T;L^q(\Omega))} < \infty$ where $\frac{2}{s} + \frac{3}{q} = 1$. In the present paper we prove several new local and global regularity properties by using assumptions beyond Serrin's condition e.g. as follows: If the norm $\|u\|_{L^r(0,T;L^q(\Omega))}$, with Serrin's number $\frac{2}{r} + \frac{3}{q} = 1 + \alpha$ ($\alpha > 0$) strictly larger than 1, is sufficiently small, or if u satisfies a *local leftward* $L^s(L^q(\Omega))$–condition for every $t \in (0,T)$, where $\frac{2}{s} + \frac{3}{q} = 1$, then u is regular in $(0,T)$. Further results deal with similar regularity conditions based on energy quantities such as $\|u\|_{L^\infty(T_0,T_1;L^2(\Omega))}$ and $\|\nabla u\|_{L^2(T_0,T_1;L^2(\Omega))}$.

2000 Mathematics Subject Classification: 35Q30; 76D05, 35B65

Keywords: Instationary Navier-Stokes equations, local in time regularity, Serrin's condition

1 Introduction and main results

Let $\Omega \subseteq \mathbb{R}^3$ be a domain with smooth boundary $\partial\Omega$ in the sense that $\partial\Omega$ is uniformly of class $C^{2,1}$, let $[0,T)$ be a time interval with $0 < T \leq \infty$, and let $u_0 \in L^2_\sigma(\Omega)$ be some initial value. Then we consider the Navier-Stokes system

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, &\quad \operatorname{div} u = 0 \\ u\big|_{\partial\Omega} = 0, &\quad u\big|_{t=0} = u_0 \end{aligned} \tag{1.1}$$

with vanishing external force; for notational convenience the coefficient of viscosity has been set to 1. Then we are interested in weak solutions u of this system defined as follows.

Definition 1.1 *A vector field*

$$u \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2_{\mathrm{loc}}([0,T); W_0^{1,2}(\Omega)) \tag{1.2}$$

is called a weak solution *of the system* (1.1) *with initial value* $u_0 \in L^2_\sigma(\Omega)$ *if the relation*

$$-\langle u, v_t\rangle_{\Omega,T} + \langle \nabla u, \nabla v\rangle_{\Omega,T} - \langle uu, \nabla v\rangle_{\Omega,T} = \langle u_0, v(0)\rangle_\Omega \tag{1.3}$$

is satisfied for all test functions $v \in C_0^\infty([0,T); C^\infty_{0,\sigma}(\Omega))$.

Here we use the following notations: $\langle \cdot,\cdot\rangle_\Omega$ means the usual pairing of functions on Ω, $\langle \cdot,\cdot\rangle_{\Omega,T}$ means the corresponding pairing on $\Omega \times [0,T)$, $L^2_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2}$ with $C^\infty_{0,\sigma}(\Omega) = \{v \in C_0^\infty(\Omega); \operatorname{div} v = 0\}$ and $W_0^{1,2}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,2}}}$. Moreover, $uu = (u_i u_j)^3_{i,j=1}$ for $u = (u_1, u_2, u_3)$ yielding $u\cdot\nabla u = (u\cdot\nabla)u = \operatorname{div}(uu)$ when $\operatorname{div} u = 0$.

Without loss of generality we may assume in the following that

$$u : [0,T) \to L^2_\sigma(\Omega) \quad \text{is weakly continuous} \tag{1.4}$$

in Definition 1.1, with $u(0) = u_0$. Further, there exists a distribution p, called an *associated pressure*, such that

$$u_t - \Delta u + u\cdot\nabla u + \nabla p = 0 \tag{1.5}$$

holds in the sense of distributions, see [15], Chapter V.1. Conversely, if u satisfies (1.2), (1.4), $u(0) = u_0$, and if (1.5) holds with some p in the sense of distributions, then u is a weak solution in the sense of Definition 1.1.

We will use Definition 1.1 with obvious modifications if the interval $[0,T)$ is replaced by any other interval $[t_0, T)$ with $0 < t_0 < T$, and with $u|_{t=t_0} = u_0$.

A weak solution u in Definition 1.1 is uniquely determined if *Serrin's condition*

$$u \in L^s(0,T; L^q(\Omega)), \quad 2 < s < \infty, \quad 3 < q < \infty, \quad \frac{2}{s} + \frac{3}{q} = 1 \tag{1.6}$$

is satisfied, see [14], [15], i.e.,

$$\|u\|_{L^s(0,T;L^q(\Omega))} = \|u\|_{q,s} = \left(\int_0^T \|u\|_q^s \, dt\right)^{\frac{1}{s}} < \infty, \tag{1.7}$$

where $\|u\|_q = \|u(t)\|_{L^q(\Omega)} = (\int_\Omega |u(x,t)|^q \, dx)^{1/q}$.

Moreover, if u in Definition 1.1 satisfies (1.6), then u is regular in the sense that

$$u \in C^\infty(\overline{\Omega} \times (0,T)), \ p \in C^\infty(\overline{\Omega} \times (0,T)), \tag{1.8}$$

provided $\partial\Omega$ is of class C^∞, see [15], Theorem V.1.8.2. Hence a weak solution u satisfying (1.6) is called a *strong solution*.

Further we know, see [5], that there exists a weak solution u as in Definition 1.1 which additionally satisfies the *strong energy inequality*

$$\frac{1}{2}\|u(t)\|_2^2 + \int_\sigma^t \|\nabla u\|_2^2\, d\tau \le \frac{1}{2}\|u(\sigma)\|_2^2 \tag{1.9}$$

for almost all $\sigma \in [0,T)$, including $\sigma = 0$, and all $t \in [\sigma, T)$. This energy inequality is needed for the local in time identification of u with strong solutions. To prove the existence of u satisfying (1.9) for general unbounded domains, we need only that the boundary is uniform of class C^2, see [5]. However, for simplicity we suppose uniform $C^{2,1}$-regularity of $\partial\Omega$ since this is needed in our main Theorem 1.2 below for bounded domains.

Each weak solution u of (1.1) with $u_0 \in L^2_\sigma(\Omega)$ satisfies the condition

$$u \in L^r(0,T; L^q(\Omega)) \tag{1.10}$$

for all r, q satisfying

$$2 \le q \le 6, \quad \frac{2}{r} + \frac{3}{q} = 1 + \alpha, \quad \frac{1}{2} \le \alpha < \frac{3}{2}. \tag{1.11}$$

The proof is based on the energy inequality (1.9) for $\sigma = 0$ and an estimate of the norm $\|u\|_{q,r}$ by the energy quantities on the left hand side of (1.9), see [15], Theorem V.1.6.2.

Now our first main results read as follows:

Theorem 1.2 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, $0 < T \le \infty$, let u be a weak solution of the Navier-Stokes system* (1.1) *with $u_0 \in L^2_\sigma(\Omega)$ satisfying the general energy inequality* (1.9)*, and let $4 < s < \infty$, $3 < q < 6$ with $\frac{2}{s} + \frac{3}{q} = 1$.*

(i) Assume $u_0 \in L^q_\sigma(\Omega)$. Given $r \in [1, s)$ such that $\frac{2}{r} + \frac{3}{q} = 1 + \alpha$, $0 \le \alpha \le 2\big(1 - \frac{1}{s}\big)$, there is a constant $C = C(u_0, \Omega, r, s) > 0$ with the following property: If

$$\|u\|_{L^r(0,T;L^q(\Omega))} \le C, \tag{1.12}$$

then u is regular in the sense that $u \in L^s(0,T; L^q(\Omega))$.

(ii) Suppose for each $T_1 \in (0,T)$ there is some $0 < \delta = \delta(T_1) < T_1$ such that u satisfies the leftward L^s–L^q–condition

$$u \in L^s(T_1 - \delta, T_1; L^q(\Omega)). \tag{1.13}$$

Then u is regular in the sense that $u \in L^s_{\text{loc}}((0,T); L^q(\Omega))$.

The proof is based on the following theorem yielding local in time regularity results.

Theorem 1.3 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, $0 < T \le \infty$, let u be a weak solution of the Navier-Stokes system (1.1) with $u_0 \in L^2_\sigma(\Omega)$ satisfying the general energy inequality (1.9), and let $4 < s < \infty$, $3 < q < 6$ with $\frac{2}{s} + \frac{3}{q} = 1$. Then there is a constant $C = C(\Omega, q) > 0$ independent of u_0 and u with the following property:*

If $0 < T_0 < T_1 < T$, $0 \le \alpha \le 2(1 - \frac{1}{s})$, and if one of the following smallness conditions is satisfied,

$$(i) \qquad \int_{T_0}^{T_1} \|u(t)\|_q^r \, dt \le C(T_1 - T_0) \quad \text{with} \quad \frac{2}{r} + \frac{3}{q} = 1 + \alpha,\ 1 \le r \le s, \tag{1.14}$$

or, when $T < \infty$,

$$(ii) \quad \int_{T_0}^{T_1} (T-t)^{\frac{r}{s}} \|u(t)\|_q^r \, dt \le C(T_1 - T_0) \quad \text{with} \quad \frac{2}{r} + \frac{3}{q} = 1 + \alpha,\ 1 \le r \le s, \tag{1.15}$$

then u is regular on the interval (T_1, T) in the sense that Serrin's condition

$$u \in L^s(T_1, T; L^q(\Omega)) \tag{1.16}$$

is satisfied.

Remark 1.4 The time exponent r in (1.14) is uniquely determined by $0 \le \alpha \le 2(1 - \frac{1}{s})$ and by $\frac{2}{r} + \frac{3}{q} = 1 + \alpha$, it holds $\alpha = 2(\frac{1}{r} - \frac{1}{s})$.

Using (1.11) we see that each weak solution u in Theorem 1.3 satisfies (1.10) if α is restricted by $\frac{1}{2} \le \alpha \le 2(1 - \frac{1}{s})$ such that $1 \le r \le (\frac{1}{4} + \frac{1}{s})^{-1}$. Of course, if $0 \le \alpha < \frac{1}{2}$, $\alpha = 2(\frac{1}{r} - \frac{1}{s})$, then the condition (1.14) means the following: $\int_{T_0}^{T_1} \|u(t)\|_q^r \, dt$ is well defined and bounded by $C(T_1 - T_0)$. An analogous interpretation holds for (1.15).

Corollary 1.5 *Let u be a weak solution in $\Omega \times [0, T)$ as in Theorem 1.3, and let s, q be exponents with $\frac{2}{s} + \frac{3}{q} = 1$.*

(i) Let $T = \infty$, $\frac{1}{2} \le \alpha \le 2(1 - \frac{1}{s})$, $1 \le r < s$, $\frac{2}{r} + \frac{3}{q} = 1 + \alpha$ such that (1.10) is satisfied. Then u is regular for $t > T_1$ with

$$T_1 > C^{-1} \|u\|_{L^r(0,\infty; L^q(\Omega))}, \quad C \text{ as in } (1.14), \tag{1.17}$$

in the sense that $u \in L^s\big(T_1, \infty; L^q(\Omega)\big)$. In particular the choice of $T_1 > 0$ in (1.17) only depends on C and the norm $\|u\|_{q,r}$.

(ii) Let $0 < T_1 < T \le \infty$, choose $1 \le r \le s$ as in (i) such that $u \in L^r(0, T; L^q(\Omega))$, and assume that

$$C_1 := \liminf_{\delta \to 0} \frac{1}{\delta} \int_{T_1 - \delta}^{T_1} \|u(t)\|_q^r \, dt < \infty. \tag{1.18}$$

Then there exists $T' = T'(C_1)$, $T_1 < T' \le T$, such that u is regular on (T_1, T') in the sense

$$u \in L^s\big(T_1, T'; L^q(\Omega)\big).$$

In particular, this condition is satisfied if $T_1 \in (0,T)$ is a Lebesgue point of $t \mapsto \|u(t)\|_q^r$, $t \in (0,T)$, in the sense that

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{T_1-\delta}^{T_1} \|u(t)\|_q^r \, dt = \|u(T_1)\|_q^r. \tag{1.19}$$

Conversely, if $T_1 \in (0,T)$ is a singular point in the sense that there is no $T' > T_1$ such that u is contained in $L^s\big(T_1, T'; L^q(\Omega)\big)$, then

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{T_1-\delta}^{T_1} \|u(t)\|_q^r \, dt = \infty. \tag{1.20}$$

The set of such singular points (is empty or) has Lebesgue measure zero. The condition (1.18) is sufficient that T_1 is not a singular point.

In the next theorem the domain $\Omega \subset \mathbb{R}^3$ need not be bounded. In this case $A_2 : D(A_2) \to L^2_\sigma(\Omega)$, $D(A_2) \subseteq L^2_\sigma(\Omega)$, denotes the usual Stokes operator well defined in the L^2-approach, see the next section. The case that $\Omega = \mathbb{R}^3$, $\partial\Omega = \emptyset$, is included; in this case the condition "$u|_{\partial\Omega} = 0$" in (1.1) is omitted. See [5] concerning the uniformity condition of $\partial\Omega$.

In this case our result reads as follows:

Theorem 1.6 *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain with boundary $\partial\Omega$ uniformly of class $C^{2,1}$, $0 < T \le \infty$, and let u be a weak solution of the Navier-Stokes system* (1.1) *with $u_0 \in L^2_\sigma(\Omega)$ satisfying the strong energy inequality* (1.9)*. Assume that the map $t \mapsto \|\nabla u(t)\|_2$ is locally left bounded in the following sense: For each $T_1 \in (0,T)$ there is some $0 < \delta = \delta(T_1) < T_1$ such that*

$$\|\nabla u(\cdot)\|_2 \in L^\infty(T_1 - \delta, T_1).$$

Then u is regular in the sense that

$$u \in L^8\big(T_1, T; L^4(\Omega)\big).$$

The proof of this theorem is based on the following more general local in time regularity results:

Theorem 1.7 *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain with boundary $\partial\Omega$ uniformly of class $C^{2,1}$, $0 < T \le \infty$, and let u be a weak solution of the Navier-Stokes system* (1.1) *with $u_0 \in L^2_\sigma(\Omega)$ satisfying the strong energy inequality* (1.9)*. Then there is an absolute constant $C > 0$, not depending on Ω and u_0, with the following property:*

If $0 < T_0 < T_1 < T$, and if one of the following smallness conditions is satisfied,

$$\text{(i)} \quad \int_{T_0}^{T_1} \|A_2^{\frac{1}{4}} u(t)\|_2 \, dt \le C(T_1 - T_0), \tag{1.21}$$

$$\text{(ii)} \quad \int_{T_0}^{T_1} \|\nabla u(t)\|_2 \, \|u(t)\|_2 \, dt \le C(T_1 - T_0) \tag{1.22}$$

or

$$\text{(iii)} \quad \big(\sup_{T_0 \le t \le T_1} \|u(t)\|_2^2 \big) \int_{T_0}^{T_1} \|\nabla u(t)\|_2^2 \, dt \le C(T_1 - T_0), \tag{1.23}$$

then u is regular on the interval (T_1, T) in the sense that u satisfies Serrin's condition

$$u \in L^8\big(T_1, T; L^4(\Omega)\big), \tag{1.24}$$

and has the properties

$$\nabla u \in L^4\big(T_1, T; L^4(\Omega)\big), \quad uu \in L^4\big(T_1, T; L^2(\Omega)\big). \tag{1.25}$$

Note that, due to the energy inequality (1.9) with $\sigma = 0$, the expressions on the left hand side of (i), (ii), (iii) are well defined, see (3.13) below.

Corollary 1.8 *Let u be a weak solution in $\Omega \times [0, \infty)$ with initial value $u_0 \in L^2_\sigma(\Omega)$ as in Theorem 1.7 with $T = \infty$, let C be the constant in (1.23), and let*

$$T_1 > C^{-1} \big(\sup_{0 \le t < \infty} \|u(t)\|_2^2 \big) \int_0^\infty \|\nabla u\|_2^2 \, dt \, . \tag{1.26}$$

Then the weak solution u is regular for $t > T_1$ in the sense of (1.24) and (1.25) with $T = \infty$. In particular, if

$$T_1 > \frac{1}{2} C^{-1} \|u_0\|_2^4, \tag{1.27}$$

then u is regular in this sense for $t > T_1$. Therefore, the smaller $\|u_0\|_2$, the smaller the time $T_1 > 0$ such that u is regular for $t > T_1$.

The next result enables us to construct a regularity interval of the form (T_1, T') with $0 < T_1 < T' \le T$.

Theorem 1.9 *Let u be a weak solution in $\Omega \times [0, T)$ with initial value $u_0 \in L^2_\sigma(\Omega)$ as in Theorem 1.7, and let $0 < \varepsilon < \frac{1}{4}$, $0 < T < \infty$. Then there is a constant $C_\varepsilon > 0$, not depending on Ω and u_0, with the following property:*

If $0 < T_0 < T_1 < T$, and if the smallness condition

$$\frac{1}{T_1 - T_0} \int_{T_0}^{T_1} \big(\|\nabla u\|_2^2 + \|u\|_2^2 \big) dt \le C_\varepsilon (T - T_0)^{-\frac{\varepsilon}{4}} \tag{1.28}$$

is satisfied, then u is regular on the interval (T_1, T) in the sense that (1.24) and (1.25) are valid.

Corollary 1.10 *Let Ω, u, u_0 be as in Theorem 1.7, let $0 < T_1 < T \leq \infty$, and assume that*

$$C_1 := \liminf_{\delta \to 0} \frac{1}{\delta} \int_{T_1-\delta}^{T_1} \big(\|\nabla u\|_2^2 + \|u\|_2^2 \big) dt < \infty \,. \tag{1.29}$$

Then there exists $T' = T'(C_1) \in (T_1, T]$ such that u is regular on (T_1, T') in the sense that (1.24) *and* (1.25) *are valid with (T_1, T) replaced by (T_1, T').*

2 Some preliminaries

Given a domain $\Omega \subseteq \mathbb{R}^3$ we use the well known spaces $L^q(\Omega)$, $1 < q < \infty$, with norm $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and pairing $\langle v, w\rangle = \langle v, w\rangle_\Omega = \int_\Omega v \cdot w \, dx$ for $v \in L^q(\Omega)$, $w \in L^{q'}(\Omega)$, $q' = \frac{q}{q-1}$. Moreover, given $0 < T \leq \infty$, we need the Bochner spaces $L^s(0,T;L^q(\Omega))$, $1 < s < \infty$, with norm $\|\cdot\|_{L^s(0,T;L^q(\Omega))} = \|\cdot\|_{q,s} = (\int_0^T \|\cdot\|_q^s \, dt)^{1/s}$ and the corresponding pairing $\langle \cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\Omega,T}$ on $L^s(0,T;L^q(\Omega)) \times L^{s'}(0,T;L^{q'}(\Omega))$, $s' = \frac{s}{s-1}$. Furthermore, we will use the smooth function spaces $C_0^\infty(\Omega)$, $C_{0,\sigma}^\infty(\Omega) = \{v \in C_0^\infty(\Omega); \operatorname{div} v = 0\}$ and the spaces $L^q_\sigma(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_q}$.

In the general case of an unbounded domain $\Omega \subset \mathbb{R}^3$ as in Theorem 1.6, the Stokes operator $A_2 = -P_2\Delta : D(A_2) \to L^2_\sigma(\Omega)$, $D(A_2) \subseteq L^2_\sigma(\Omega)$, is defined in the usual way by the Hilbert space approach in $L^2_\sigma(\Omega)$, together with the Helmholtz projection $P_2 : L^2(\Omega) \to L^2_\sigma(\Omega)$. We collect some well known properties for A_2, its fractional powers A_2^α, $0 \leq \alpha \leq 1$, and the corresponding analytic semigroup e^{-tA_2}, $t \geq 0$. In particular we need the following estimates, see [15], III. 2.1 – 2.6, IV. 1.5:

$$\|A_2^\alpha v\|_2 \leq \|A_2 v\|_2^\alpha \|v\|_2^{1-\alpha} \text{ for all } v \in D(A_2),\ 0 \leq \alpha \leq 1, \tag{2.1}$$

$$\|v\|_q \leq C\|A_2^\alpha v\|_2 \text{ for all } v \in D(A_2^\alpha),\ 0 \leq \alpha \leq \frac{1}{2},\ 2 \leq q < \infty, \tag{2.2}$$

$$\text{where } 2\alpha + \frac{3}{q} = \frac{3}{2} \text{ and } C = C(\alpha, q) > 0,$$

$$\|A_2^\alpha e^{-tA_2} v\|_2 \leq t^{-\alpha}\|v\|_2 \text{ for all } v \in L^2_\sigma(\Omega),\ 0 \leq \alpha \leq 1, t > 0, \tag{2.3}$$

$$\|A_2^{-\frac{1}{2}} P_2 \operatorname{div} v\|_2 \leq \|v\|_2 \text{ for all } v = (v_{ij})_{i,j=1}^3 \in L^2_\sigma(\Omega), \tag{2.4}$$

$$\|\nabla E\|_{L^2(0,T;L^2(\Omega))} = \|A_2^{\frac{1}{2}} E\|_{L^2(0,T;L^2(\Omega))} \leq \|v\|_2 \text{ for all } v \in L^2_\sigma(\Omega), \tag{2.5}$$

$$\text{where } E(t) = e^{-tA_2} v.$$

Note that $\operatorname{div} v = (\sum_{i=1}^3 D_i v_{ij})_{j=1}^3$ in (2.5) and that $\|A_2^{1/2} v\|_2 = \|\nabla v\|_2$ for $v \in D(A_2^{1/2})$.

If $\Omega \subseteq \mathbb{R}^3$ is a smooth bounded domain as in Theorem 1.2, then we use the Stokes operator $A_q = -P_q\Delta : D(A_q) \to L^q_\sigma(\Omega)$, $D(A_q) \subseteq L^q_\sigma(\Omega)$, and the

Helmholtz projection $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$ in L^q-spaces; see, e.g., [1], [3] – [8], concerning these operators. In particular the following estimates hold, see [4]:

$$\|v\|_\gamma \le C\|A_q^\alpha v\|_q \ \text{ for all } v \in D(A_q^\alpha),\ 1 < q \le \gamma,\ 0 \le \alpha \le 1, \qquad (2.6)$$
$$\text{where } 2\alpha + \frac{3}{\gamma} = \frac{3}{q},$$

$$\|A_q^\alpha e^{-tA_q} v\|_q \le Ce^{-\delta t} t^{-\alpha}\|v\|_q \ \text{ for all } v \in L^q_\sigma(\Omega),\ t > 0, \qquad (2.7)$$
$$\text{where } \delta = \delta(\Omega, q) > 0 \ \text{ and } \ 0 \le \alpha \le 1,$$

$$\|A_q^{-\frac{1}{2}} P_q \operatorname{div} v\|_q \le C\|v\|_q \ \text{ for all } v = (v_{ij})_{i,j=1}^3 \in L^q_\sigma(\Omega), \qquad (2.8)$$

$$\|v\|_{L^s(0,T;L^q(\Omega))} \le C\|f\|_{L^s(0,T;L^q(\Omega))} \ \text{ for all } f \in L^s(0,T;L^q(\Omega)), \qquad (2.9)$$
$$\text{where } v(t) = A_q \int_0^t e^{-(t-\tau)A_q} f(\tau)\, d\tau.$$

The constants C in (2.6)–(2.9) depend on Ω and q, s, α, but are independent of v. Further note that the norms $\|A_q^{1/2} v\|_q$ and $\|\nabla v\|_q$ are equivalent for $v \in D(A_q^{1/2})$.

To prove our main results we have to identify the given weak solution u locally in time with strong solutions, i.e. with weak solutions satisfying Serrin's regularity condition. There are many results on the existence of such solutions for some given interval $[0, T)$, $0 < T \le \infty$, if the initial value u_0 satisfies a certain smallness condition, see, e.g., [9] – [13], [16]. However, we need some particular weak assumption on u_0 and will apply Theorem 1 in [4] for bounded domains, and Theorem 4.2.2, V, [15] for the general case. The restriction $4 \le q < 6$ in Lemma 2.1, needed for technical reasons in the proof, is not important for our application.

Lemma 2.1 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, $4 < s < \infty$, $3 < q < 6$, $\frac{2}{s} + \frac{3}{q} = 1$, and let $u_0 \in L^q_\sigma(\Omega)$. Then there is a constant $C = C(\Omega, q) > 0$ independent of u_0 with the following property: If*

$$\int_0^T \|e^{-tA_q} u_0\|_q^s \, dt \le C \qquad (2.10)$$

for some $T \in (0, \infty]$, then there exists a unique weak solution u in $\Omega \times [0, T)$ of the Navier-Stokes system (1.1) *satisfying Serrin's condition*

$$u \in L^s\big(0, T; L^q(\Omega)\big) \qquad (2.11)$$

and the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 \, d\tau \le \frac{1}{2}\|u_0\|_2^2,\ 0 \le t < T. \qquad (2.12)$$

Proof. The existence result of Theorem 1 in [4] yields – under the smallness condition (2.10), see [4], p. 133, (4.23) – a unique solution u in the following (so-called very weak) sense: It holds (2.11) and the relation

$$-\langle u, v_t\rangle_{\Omega,T} - \langle u, \Delta v\rangle_{\Omega,T} - \langle uu, \nabla v\rangle_{\Omega,T} = \langle u_0, v(0)\rangle_\Omega \tag{2.13}$$

for all $v \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$. In order to prove that u is a weak solution satisfying (2.11) we have to show several regularity properties.

We start with the case that $4 < s \le 8$ and hence $4 \le q < 6$. Due to the proof in [4], p. 132, (4.19), we know that u satisfies the relation

$$\tilde{u}(t) \equiv u(t) - E(t) = -\int_0^t A_q^{\frac{1}{2}} e^{-(t-\tau)A_q} A_q^{-\frac{1}{2}} P_q \operatorname{div}(uu)\, d\tau,\ 0 \le t < T, \tag{2.14}$$

with $E(t) = e^{-tA_q} u_0$. Using (2.8) and Hölder's inequality we obtain that

$$\|A_{q/2}^{-\frac{1}{2}} P_{q/2} \operatorname{div}(uu)\|_{q/2} \le C_1 \|uu\|_{q/2} \le C_2 \|u\|_q^2 \tag{2.15}$$

with $C_j = C_j(\Omega, q) > 0$, $j = 1, 2$. By (2.14)

$$A_q^{\frac{1}{2}} \tilde{u}(t) = -A_q \int_0^t e^{-(t-\tau)A_q} A_q^{-\frac{1}{2}} P_q \operatorname{div}(uu)\, d\tau,\ 0 \le t < T, \tag{2.16}$$

and using (2.9) we get the estimate

$$\|\nabla \tilde{u}\|_{\frac{q}{2},\frac{s}{2}} \le C_3 \|A_{q/2}^{\frac{1}{2}} \tilde{u}\|_{\frac{q}{2},\frac{s}{2}} \le C_4 \|uu\|_{\frac{q}{2},\frac{s}{2}} \le C_5 \|u\|_{q,s}^2 < \infty, \tag{2.17}$$

$C_j = C_j(\Omega, q) > 0$, $j = 3, 4, 5$. This shows that

$$\nabla \tilde{u} \in L^{s/2}\big(0, T; L^{q/2}(\Omega)\big) \tag{2.18}$$

and, since $4 \le q < 6$, $4 < s \le 8$, that

$$\nabla \tilde{u} \in L^2_{\mathrm{loc}}\big([0,T); L^2(\Omega)\big),\ \tilde{u} \in L^2_{\mathrm{loc}}\big([0,T); W_0^{1,2}(\Omega)\big). \tag{2.19}$$

Applying (2.7) to (2.14), using (2.8), Hölder's inequality and the properties of q and s we obtain from (2.14) the estimate

$$\begin{aligned} \|\tilde{u}(t)\|_2 &\le C_6 \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\, e^{-\delta(t-\tau)} \|uu\|_2\, d\tau \\ &\le C_7 \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\, e^{-\delta(t-\tau)} \|uu\|_{\frac{q}{2}}\, d\tau \\ &\le C_8 \|uu\|_{\frac{q}{2},\frac{s}{2}} \le C_9 \|u\|_{q,s}^2 \end{aligned} \tag{2.20}$$

with $C_j = C_j(\Omega, q) > 0$, $j = 6, \dots, 9$. Hence (2.19) and (2.20) imply that

$$\tilde{u} \in L^\infty\big([0,T); L^2_\sigma(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0,T); W^{1,2}_0(\Omega)\big). \tag{2.21}$$

Next we use (2.3) with $\alpha = 0$ and (2.5) to obtain that

$$\|E(t)\|_2 \le \|u_0\|_2, \ \ \|\nabla E\|_{2,2} = \|A_2^{\frac{1}{2}} E\|_{2,2} \le \|u_0\|_2. \tag{2.22}$$

With the help of (2.21) and (2.22) we conclude that

$$u \in L^\infty\big([0,T); L^2_\sigma(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0,T); W^{1,2}_0(\Omega)\big). \tag{2.23}$$

Since $u \in L^{s_1}(0,T'; L^{q_1}(\Omega))$ for all $0 < T' < T$, Hölder's inequality yields

$$uu \in L^2_{\mathrm{loc}}\big([0,T); L^2(\Omega)\big), \tag{2.24}$$

cf. [15], p. 275. Using (2.23) and (2.24), a calculation shows that (2.13) implies (1.3), and that the energy inequality (2.12) is satisfied; see also [15], Theorem V.1.4.1, concerning the last property. Consequently u is a weak solution of (1.1) satisfying (2.11) and (2.12). Hence it is also a strong solution. The uniqueness of u with these properties follows from Serrin's uniqueness argument, see [14], [15]. This completes the proof in the case that $4 < s \le 8$.

In the second case we assume that $8 < s < \infty$ and $3 < q < 4$. Now we need several steps. First let $s_1 = s$, $q_1 = q$. Then we get as in (2.14)–(2.18) that $\nabla\tilde{u} \in L^{s_1/2}\big(0,T; L^{q_1/2}(\Omega)\big)$. Defining $s_2 = \frac{s_1}{2}$ and $q_2 > q_1$ such that $\frac{1}{3} + \frac{1}{q_2} = \frac{1}{q_1/2}$, $\frac{2}{s_2} + \frac{3}{q_2} = 1$, we obtain by Sobolev's embedding theorem that $\tilde{u} \in L^{s_2}\big(0,T; L^{q_2}(\Omega)\big)$. Moreover, using (2.6), (2.7) we see that $E \in L^{s_2}\big(0,T; L^{q_2}(\Omega)\big)$ which leads to $u \in L^{s_2}\big(0,T; L^{q_2}(\Omega)\big)$. Proceeding in the same way, let $s_k = \frac{s_{k-1}}{2}$ and $q_k > q_{k-1}$ such that $\frac{1}{3} + \frac{1}{q_k} = \frac{1}{q_{k-1}/2}$, $\frac{2}{s_k} + \frac{3}{q_k} = 1$, for $k \in \mathbb{N}$. Since $\frac{1}{3} - \frac{1}{q_k} = 2^{k-1}\big(\frac{1}{3} - \frac{1}{q_1}\big)$, we choose $k \in \mathbb{N}$ such that $\frac{1}{3} - \frac{1}{q_{k-1}} < \frac{1}{12} \le \frac{1}{3} - \frac{1}{q_k}$, leading to $4 \le q_k < 6$, $4 < s_k \le 8$. Now $q_k/2 \ge 2$, and using (2.17), (2.20) with q, s replaced by q_k, s_k, we obtain the properties (2.19), (2.21). This yields the result in the same way as in the first case. Now the proof of the lemma is complete. ∎

Corollary 2.2 *Let $u_0 \in L^q_\sigma(\Omega)$ and q, s be given as in Lemma 2.1, and let $T = \infty$. Then there is a constant $C = C(\Omega, q) > 0$ with the following property: If*

$$\|u_0\|_q \le C, \tag{2.25}$$

then there exists a unique weak solution u in $\Omega \times [0,\infty)$ of the Navier-Stokes system (1.1) *satisfying* (2.11) *and* (2.12) *with $T = \infty$.*

Proof. Using (2.7) with $\alpha = 0$ we obtain that

$$\int_0^\infty \|e^{-tA_q} u_0\|_q^s \, dt \le C_1 \|u_0\|_q^s \int_0^\infty e^{-\delta t} \, dt \le \frac{C_1}{\delta} \|u_0\|_q^s$$

with $C_1 = C_1(\Omega, q) > 0$. Now the result follows from Lemma 2.1. ■

The next lemma is totally based on the Hilbert space approach in $L^2_\sigma(\Omega)$. This is needed since $\Omega \subseteq \mathbb{R}^3$ may be a completely general domain. Note that the smallness constant $C > 0$ in the next lemma does *not* depend on Ω and u_0. Thus we obtain the same existence interval $[0, T)$, $0 < T \leq \infty$, for all domains Ω and all initial values $u_0 = u_0(\Omega) \in L^2_\sigma(\Omega)$ if the smallness condition (2.26) below holds. Put $e^{-2TA_2} = 0$ in (2.26) if $T = \infty$.

Lemma 2.3 *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, i.e. a connected open subset, let $0 < T \leq \infty$, and $u_0 \in D(A_2^{1/4})$. Then there is an absolute constant $C > 0$, not depending on Ω, T and u_0 with the following property: If*

$$\|(I - e^{-2TA_2})A_2^{\frac{1}{4}} u_0\|_2^{\frac{1}{8}} \, \|A_2^{\frac{1}{4}} u_0\|_2^{\frac{7}{8}} \leq C, \tag{2.26}$$

then the Navier-Stokes system (1.1) *possesses a unique weak solution u on the interval $[0, T)$ satisfying Serrin's condition*

$$u \in L^8\big(0, T; L^4(\Omega)\big), \tag{2.27}$$

the properties

$$\nabla u \in L^4\big(0, T; L^2(\Omega)\big), \; uu \in L^4\big(0, T; L^2(\Omega)\big) \tag{2.28}$$

and the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 \, d\tau \leq \frac{1}{2}\|u_0\|_2^2, \; 0 \leq t < T. \tag{2.29}$$

Proof. The existence result together with (2.27) and (2.28) is a special case of Theorem V.4.2.2 in [15]. Then the energy inequality (2.29) follows, using (2.28), from Theorem V.1.4.1 in [15]. This proves the lemma. ■

Corollary 2.4 *Let $\Omega, T, u_0 \in D(A_2^{1/4})$ be as in Lemma 2.3. Then there is an absolute constant $C > 0$, not depending on Ω, T and u_0, with the following property: If*

$$\|A_2^{\frac{1}{4}} u_0\|_2 \leq C, \tag{2.30}$$

then the Navier-Stokes system (1.1) *possesses a unique weak solution u in $\Omega \times [0, T)$ satisfying* (2.27), (2.28), *and* (2.29).

Proof. The result follows from Lemma 2.3 since

$$\|(I - e^{-2TA_2})A_2^{\frac{1}{4}} u_0\|_2^{\frac{1}{8}} \, \|A_2^{\frac{1}{4}} u_0\|_2^{\frac{7}{8}} \leq 2^{\frac{1}{8}} \|A_2^{\frac{1}{4}} u_0\|_2^{\frac{1}{8}} \, \|A_2^{\frac{1}{4}} u_0\|_2^{\frac{7}{8}} = 2^{\frac{1}{8}} \|A_2^{\frac{1}{4}} u_0\|_2,$$

see (2.3) with $\alpha = 0$. This completes the proof. ■

3 Proof of the theorems

First we have to prove Theorem 1.3.

Proof of Theorem 1.3. Given the bounded domain $\Omega \subset \mathbb{R}^3$, $0 < T_0 < T_1 < T$ and u, q, r, s as in this theorem, we have to prove the existence of some constant $C = C(\Omega, q) > 0$ yielding regularity of u on (T_1, T) if (i) or (ii) is satisfied.

Using the weak continuity of the weak solution $u : [0, T) \to L^2_\sigma(\Omega)$, see (1.4), we know that $u(t_0) \in L^2_\sigma(\Omega)$ is well defined for all $t_0 \in [0, T)$. Since $\nabla u \in L^2(0, T; L^2(\Omega))$, see (1.9) for $\sigma = 0$, and since $3 < q < 6$, the embedding inequality $\|u(t)\|_q \le C_1\|u(t)\|_6 \le C_2\|\nabla u(t)\|_2$ with $C_j = C_j(\Omega, q) > 0$, $j = 1, 2$, implies that $u \in L^2(0, T; L^q_\sigma(\Omega))$. Then the Lebesgue point argument shows that there is a null set $N \subseteq (0, T)$ such that $\|u(t_0)\|_q$ is well defined by the property

$$\lim_{\delta\to 0} \frac{1}{2\delta} \int_{t_0-\delta}^{t_0+\delta} \|u(t)\|_q^2 \, dt = \|u(t_0)\|_q^2, \ 0 < t_0 < T, \tag{3.1}$$

for all $t_0 \in (0, T)\backslash N$. Moreover, since the energy inequality (1.9) holds for a.a. $\sigma \in [0, T)$, we may assume in the following that the null set $N \subseteq (0, T)$ is chosen in such a way that both (3.1) and the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^{t} \|\nabla u\|_2^2 \, d\tau \le \frac{1}{2}\|u(t_0)\|_2^2, \ t_0 \le t < T, \tag{3.2}$$

hold for all $t_0 \in (0, T)\backslash N$.

Let $t_0 \in (0, T)\backslash N$. Then $u(t_0) \in L^q_\sigma(\Omega)$, and we are able to apply the local existence results of Lemma 2.1 and Corollary 2.2, replacing the existence interval $[0, T)$ by the interval $[t_0, T)$, and using $u(t_0)$ as initial value. Hence, if the smallness condition

$$\int_0^{T-t_0} \|e^{-\tau A_q} u(t_0)\|_q^s \, d\tau \le C \tag{3.3}$$

or the condition

$$\|u(t_0)\|_q \le C \tag{3.4}$$

is satisfied with C as in Lemma 2.1 or in Corollary 2.2, respectively, then we obtain a unique weak solution $\tilde{u}$ on the interval $[t_0, T)$, corresponding to Definition 1.1, of the Navier-Stokes system

$$\begin{aligned} \tilde{u}_t - \Delta\tilde{u} + \tilde{u}\cdot\nabla\tilde{u} + \nabla\tilde{p} = 0, \quad & \operatorname{div}\tilde{u} = 0, \\ \tilde{u}\big|_{\partial\Omega} = 0, \quad & \tilde{u}\big|_{t=t_0} = u(t_0), \end{aligned} \tag{3.5}$$

satisfying

$$\begin{aligned} &\tilde{u} \in L^\infty\big(t_0, T; L^2_\sigma(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0, T); W^{1,2}_0(\Omega)\big), \\ &\tilde{u} \in L^s\big(t_0, T; L^q(\Omega)\big), \end{aligned} \tag{3.6}$$

and the energy inequality

$$\frac{1}{2}\|\tilde{u}(t)\|_2^2 + \int_{t_0}^{t} \|\nabla \tilde{u}\|_2^2 \, d\tau \le \frac{1}{2}\|u(t_0)\|_2^2 , \ t_0 \le t < T. \tag{3.7}$$

By Serrin's uniqueness argument, see [14], [15], V, Theorem 1.5.1, we obtain that

$$u = \tilde{u} \text{ on } [t_0, T).$$

This yields the properties (3.6) with $\tilde{u}$ replaced by u, and we get the desired result of Theorem 1.3. Thus it remains to prove the existence of some $t_0 \in (0,T)\backslash N$ as above with $T_0 \le t_0 \le T_1$ such that (3.3) or (3.4) is satisfied.

First suppose that the condition (i) in Theorem 1.3 holds with any constant $C_1 > 0$. Then there is at least one $t_0 \in (T_0, T_1)\backslash N$ such that

$$\|u(t_0)\|_q^r \le \frac{1}{T_1 - T_0} \int_{T_0}^{T_1} \|u(t)\|_q^r \, dt \le C_1. \tag{3.8}$$

Therefore, setting $C_1 = C^r$ with C as in Corollary 2.2, the condition (3.4) is satisfied and we obtain the desired result of Theorem 1.3.

Next assume that condition (ii) is satisfied, i.e.,

$$\int_{T_0}^{T_1} (T-t)^{\frac{r}{s}} \, \|u(t)\|_q^r \, dt \le C_1(T_1 - T_0) \tag{3.9}$$

with any constant $C_1 > 0$. Then we find at least one $t_0 \in (T_0, T_1)\backslash N$ such that

$$(T-t_0)^{\frac{r}{s}} \, \|u(t_0)\|_q^r \le \frac{1}{T_1 - T_0} \int_{T_0}^{T_1} (T-t)^{\frac{r}{s}} \, \|u(t)\|_q^r \le C_1. \tag{3.10}$$

Further we obtain, using (2.7) with $\alpha = 0$ and (3.9), (3.10), that

$$\int_0^{T-t_0} \|e^{\tau A_q} u(t_0)\|_q^s \, d\tau \le C_2 (T-t_0) \, \|u(t_0)\|_q^s \le C_2 C_1^{\frac{s}{r}}$$

holds with some constant $C_2 = C_2(\Omega, q) > 0$. Setting $C_1 = (C/C_2)^{r/s}$ with C from Lemma 2.1 we see that (3.3) is satisfied. This completes the proof. ■

Proof of Theorem 1.2 (i) By Lemma 2.1 there exists some $\delta = \delta(u_0, \Omega, s) > 0$ such that $u \in L^s(0, \delta; L^q(\Omega))$. Next we choose $0 < T_0 < T_1 < T$ with $T_0 < \delta$, $T_1 = T_0 + \frac{\delta - T_0}{2}$, and assume that $\|u\|^r_{L^r(0,T;L^q(\Omega))} \le \frac{\delta}{2} C$ is satisfied with C from (1.14). Using Theorem 1.3 (i), we conclude that $u \in L^s(T_1, T; L^q(\Omega))$ and hence $u \in L^s(0, T; L^q(\Omega))$.

(ii) In this case we use Theorem 1.3 (ii) with $\alpha = 0$, $r = s$. Let $T_1 \in (0,T)$ and choose $0 < \delta < T_1$ such that $u \in L^s(T_1 - \delta, T_1; L^q(\Omega))$ and $2\|u\|_{L^r(T_1 - \delta, T_1; L^q(\Omega))} \le C$ with C from (1.15). Moreover, let $T' = T_1 + \delta$, $T_0 = T_1 - \delta$. Then

$$\frac{1}{T_1 - T_0} \int_{T_0}^{T_1} (T' - t) \|u(t)\|_q^s \, dt \le 2 \int_{T_0}^{T_1} \|u(t)\|_q^s \, dt \le C.$$

Now Theorem 1.3 (ii), implies that $u \in L^s(T_1-\delta, T_1+\delta; L^q(\Omega))$. Since we find such a $\delta > 0$ for each $T_1 \in (0,T)$, we obtain the result. ∎

Proof of Corollary 1.5 (i) Condition (1.17) implies (1.14) for some sufficiently small $T_0 > 0$.

(ii) Assume that (1.18) holds. Then there is a sequence $(\delta_j) \subset (0,\infty)$ such that $\lim_{j\to\infty} \delta_j = 0$ and

$$\lim_{j\to\infty} \frac{1}{\delta_j} \int_{T_1-\delta_j}^{T_1} \|u(t)\|_q^r \, dt = C_1. \tag{3.11}$$

Moreover, for given $\varepsilon > 0$ and $T_1 < T' \le T$, $T' < \infty$, there is some $k \in \mathbb{N}$ satisfying

$$\begin{aligned} &\frac{1}{\delta_j} \int_{T_1-\delta_j}^{T_1} (T'-t)^{\frac{r}{s}} \|u(t)\|_q^r \, dt \\ &\quad \le \big((T'-(T_1-\delta_j)\big)^{\frac{r}{s}}(C_1+\varepsilon) \le (T'-T_1+\varepsilon)^{\frac{r}{s}}(C_1+\varepsilon) \end{aligned}$$

for $j > k$. Therefore, choosing $\varepsilon > 0$ sufficiently small and $k \in \mathbb{N}$ sufficiently large, we find $T' = T'(C_1)$, $T_1 < T' \le T$, such that

$$(T'-T_1+\varepsilon)^{\frac{r}{s}}(C_1+\varepsilon) \le C$$

with C from (1.15). Now (1.15) is satisfied with T replaced by T', and Theorem 1.3 (ii) shows that u is regular on (T_1, T'), i.e., $u \in L^s\big(T_1, T'; L^q(\Omega)\big)$. ∎

Proof of Theorem 1.7. Let $\Omega \subset \mathbb{R}^3$, $0 < T_0 < T_1 < T \le \infty$ and u_0, u be as in Theorem 1.7. We have to prove the existence of some absolute constant $C > 0$ such that each of the conditions (1.21), (1.22), (1.23) implies the result. In principle we argue as in the proof of Theorem 1.3 for bounded domains, using now Corollary 2.4 instead of Corollary 2.2.

Since $u : [0,T) \to L^2_\sigma(\Omega)$ is weakly continuous, $u(t_0) \in L^2_\sigma(\Omega)$ is well defined for all $t_0 \in [0,T)$, and there is a null set $N \subseteq (0,T)$ such that

$$\lim_{\delta\to 0} \frac{1}{2\delta} \int_{t_0-\delta}^{t_0+\delta} \|\nabla u(t)\|_2^2 \, dt = \|\nabla u(t_0)\|_2^2 \tag{3.12}$$

is well defined for all $t_0 \in (0,T)\backslash N$. Further we can choose the null set $N \subseteq (0,T)$ in such a way that both (3.12) and the energy inequality (3.2) are satisfied for all $t_0 \in (0,T)\backslash N$.

Let $t_0 \in (0,T)\backslash N$. Using the interpolation inequality (2.1) with A_2 replaced by $A_2^{1/2}$ we obtain since $A_2^{1/4}u(t_0) = (A_2^{1/2})^{1/2}u(t_0)$ that

$$\|A_2^{\frac14}u(t_0)\|_2 \le \|A_2^{\frac12}u(t_0)\|_2^{\frac12} \, \|u(t_0)\|_2^{\frac12} = \|\nabla u(t_0)\|_2^{\frac12} \, \|u(t_0)\|_2^{\frac12}. \tag{3.13}$$

Therefore, $A_2^{1/4}u(t_0) \in L^2_\sigma(\Omega)$ is well defined. Thus if the smallness condition

$$\|A_2^{\frac{1}{4}}u(t_0)\|_2 \le C \tag{3.14}$$

is satisfied with C from (2.30), we obtain a weak solution $\tilde{u}$ of the system (3.5) with initial value $\tilde{u}(t_0) = u(t_0) \in D(A_2^{1/4})$, satisfying (3.6), Serrin's condition $\tilde{u} \in L^8\big(t_0, T; L^4(\Omega)\big)$, and the conditions $\nabla\tilde{u} \in L^4\big(t_0, T; L^2(\Omega)\big)$, $\tilde{u}\tilde{u} \in L^4\big(t_0, T; L^2(\Omega)\big)$ and (3.7). Using Serrin's uniqueness argument as in the proof of Theorem 1.3 we conclude that $u = \tilde{u}$ on $[t_0, T)$. Hence, since $t_0 \le T_1$, the conditions (1.24), (1.25) are satisfied.

Now assume the smallness condition

$$\int_{T_0}^{T_1} \|A_2^{\frac{1}{4}}u(t)\|_2\, dt \le C(T_1 - T_0) \tag{3.15}$$

with the same constant C as in (3.14). Then we argue as in (3.8) and obtain at least one $t_0 \in (0,T)\backslash N$, $T_0 \le t_0 \le T_1$, such that

$$\|A_2^{\frac{1}{4}}u(t_0)\|_2 \le \frac{1}{T_1 - T_0}\int_{T_0}^{T_1} \|A_2^{\frac{1}{4}}u(t)\|_2\, dt \le C. \tag{3.16}$$

Thus (3.14) holds, and Theorem 1.7 is proved in the case (i) with C from (2.30).

Using (3.13) and the inequality of Cauchy-Schwarz we obtain the estimates

$$\int_{T_0}^{T_1} \|A_2^{\frac{1}{4}}u\|_2\, dt \le \int_{T_0}^{T_1} \|\nabla u\|_2^{\frac{1}{2}}\, \|u\|_2^{\frac{1}{2}}\, dt \le (T_1 - T_0)^{\frac{1}{2}}\Big(\int_{T_0}^{T_1} \|\nabla u\|_2\, \|u\|_2\, dt\Big)^{\frac{1}{2}} \tag{3.17}$$

and

$$\begin{aligned} \int_{T_0}^{T_1} \|\nabla u\|_2\, \|u\|_2\, dt \;&\le\; (T_1 - T_0)^{\frac{1}{2}}\Big(\int_{T_0}^{T_1} \|\nabla u\|_2^2\, \|u\|_2^2\, dt\Big)^{\frac{1}{2}} \\ &\le\; (T_1 - T_0)^{\frac{1}{2}} \sup_{T_0 \le t \le T_1} \|u\|_2 \,\Big(\int_{T_0}^{T_1} \|\nabla u\|_2^2\, dt\Big)^{\frac{1}{2}}. \end{aligned} \tag{3.18}$$

Assume that

$$\int_{T_0}^{T_1} \|\nabla u\|_2\, \|u\|_2\, dt \le C^2(T_1 - T_0) \tag{3.19}$$

holds with C from (2.30). Then (3.17) implies (3.15), the smallness condition (i) is satisfied, and it follows the desired result.

Finally assume that

$$\sup_{T_0 \le t \le T_1} \|u\|_2^2 \,\Big(\int_{T_0}^{T_1} \|\nabla u\|_2^2\, dt\Big) \le C^4(T_1 - T_0) \tag{3.20}$$

is satisfied with C from (2.30). Then (3.18) implies (3.19), and it follows again the desired result.

Now the proof is complete. ■

Proof of Corollary 1.8 Choosing T_0 with $0 < T_0 < T_1$ sufficiently small, the result follows from Theorem 1.7 (iii). Note that (1.27) implies (1.26) which can be shown by (1.9) with $\sigma = 0$. ■

Proof of Theorem 1.9. Consider $u_0 \in L^2_\sigma(\Omega)$, a weak solution u on $\Omega \times [0,T)$ with $T < \infty$ and $0 < \varepsilon < \frac{1}{4}$ as in this theorem. Then we have to find some constant $C_\varepsilon > 0$ such that the smallness condition (1.28) implies the desired result. In this case we will apply directly Lemma 2.3. For this purpose we have to prepare several estimates.

First we use the same argument as in (3.16), with $\|A_2^{1/4}u(t)\|_2$ replaced by $\|(I - e^{-2(T-t)A_2})A_2^{1/4}u(t)\|_2^{1/8}\,\|A_2^{1/4}u(t)\|_2^{7/8}$, and obtain at least one $t_0 \in (0,T)\backslash N$, $T_0 \le t_0 \le T_1$, with N as in (3.16), satisfying the estimate

$$\begin{aligned}&\|(I - e^{-2(T-t_0)A_2})A_2^{\frac{1}{4}}u(t_0)\|_2^{\frac{1}{8}}\,\|A_2^{\frac{1}{4}}u(t_0)\|_2^{\frac{7}{8}} \\ &\quad\le \frac{1}{T_1 - T_0}\int_{T_0}^{T_1} \|(I - e^{-2(T-t)A_2})A_2^{\frac{1}{4}}u(t)\|_2^{\frac{1}{8}}\,\|A_2^{\frac{1}{4}}u(t)\|_2^{\frac{7}{8}}\,dt.\end{aligned} \tag{3.21}$$

Since $\frac{d}{d\tau}\,e^{-2(T-\tau)A_2} = 2A_2 e^{-2(T-\tau)A_2}$ we get that

$$(I - e^{-2(T-t)A_2})A_2^{\frac{1}{4}}u(t) = 2\int_t^T A_2^{1-\varepsilon}e^{-2(T-\tau)A_2}A_2^{\varepsilon+1/4}u(t)\,d\tau,$$

and using (2.3) with $\alpha = 1-\varepsilon$ we conclude that

$$\begin{aligned}\|(I - e^{-2(T-t)A_2})A_2^{\frac{1}{4}}u(t)\|_2 &\le 2\int_t^T \|A_2^{1-\varepsilon}e^{-2(T-\tau)A_2}A_2^{\varepsilon+1/4}u(t)\|_2\,d\tau \\ \le 2\Big(\int_t^T \big(2(T-\tau)\big)^{-(1-\varepsilon)}\,d\tau\Big)&\|A_2^{\varepsilon+1/4}u(t)\|_2 \le \frac{2^\varepsilon}{\varepsilon}(T-t)^\varepsilon\|A_2^{\varepsilon+1/4}u(t)\|_2\end{aligned} \tag{3.22}$$

Next we use (2.1) with A_2 replaced by $A_2^{\frac{1}{2}}$ and obtain the interpolation inequality

$$\begin{aligned}&\|A_2^{\varepsilon+1/4}u(t)\|_2 = \|(A_2^{\frac{1}{2}})^{\frac{1}{2}+\varepsilon'}u(t)\|_2 \\ &\quad\le \|A_2^{\frac{1}{2}}u(t)\|_2^{\frac{1}{2}+\varepsilon'}\,\|u(t)\|_2^{\frac{1}{2}-\varepsilon'} \le C(\varepsilon)\big(\|A_2^{\frac{1}{2}}u(t)\|_2 + \|u(t)\|_2\big)\end{aligned} \tag{3.23}$$

with $\varepsilon' = 2\varepsilon$; by analogy a similar result holds when $\varepsilon = \varepsilon' = 0$.

By (3.21) - (3.23) there are constants $C_1(\varepsilon),\dots,C_4(\varepsilon) > 0$ such that the following inequalities hold:

$$
\begin{aligned}
&\|(I - e^{-2(T-t_0)A_2})A_2^{\frac{1}{4}}u(t_0)\|_2^{\frac{1}{8}}\, \|A_2^{\frac{1}{4}}u(t)\|_2^{\frac{7}{8}} \\
\leq\ & C_1(\varepsilon)(T_1 - T_0)^{-1} \int_{T_0}^{T_1} (T-t)^{\frac{\varepsilon}{8}} \|A_2^{\varepsilon+\frac{1}{4}}u(t)\|_2^{\frac{1}{8}}\, \|A_2^{\frac{1}{4}}u(t)\|_2^{\frac{7}{8}}\, dt \\
\leq\ & C_2(\varepsilon)(T_1 - T_0)^{-1}(T-T_0)^{\frac{\varepsilon}{8}} \int_{T_0}^{T_1} \big(\|A_2^{\frac{1}{2}}u(t)\|_2 + \|u(t)\|_2\big)\, dt \\
\leq\ & C_3(\varepsilon)(T_1 - T_0)^{-\frac{1}{2}}(T-T_0)^{\frac{\varepsilon}{8}} \Big(\int_{T_0}^{T_1} \big(\|(A_2^{\frac{1}{2}}u(t)\|_2^2 + \|u(t)\|_2^2\big) dt\Big)^{\frac{1}{2}} \\
=\ & C_3(\varepsilon)(T_1 - T_0)^{-\frac{1}{2}}(T-T_0)^{\frac{\varepsilon}{8}} \Big(\int_{T_0}^{T_1} \big(\|\nabla u(t)\|_2^2 + \|u(t)\|_2^2\big) dt\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Therefore, if $C_\varepsilon = C^2\big(C_3(\varepsilon)\big)^{-2}$ in (1.28) with C from Lemma 2.3, then we conclude from the last estimates that the smallness condition

$$\|(I - e^{-2(T-t_0)A_2})A_2^{\frac{1}{4}}u(t)\|_2^{\frac{1}{8}} \|A_2^{\frac{1}{4}}u(t)\|_2^{\frac{7}{8}} \leq C \tag{3.24}$$

is satisfied.

Similarly as explained in the proof of Theorem 1.3 concerning the Lemma 2.1, we can apply Lemma 2.3 also to the case that the existence interval $[0, T)$ and the initial value u_0 are replaced by $[t_0, T)$ and $u(t_0)$. Then T and u_0 in the smallness condition (2.26) have to be replaced by $T - t_0$ and $u(t_0)$, respectively, yielding the condition (3.24). Therefore, applying Lemma 2.3 in this situation we obtain a unique weak solution $\tilde{u}$ of the system (3.5) satisfying (3.6), and the properties $\tilde{u} \in L^8(t_0, T; L^4(\Omega))$, $\nabla\tilde{u} \in L^4(t_0, T; L^2(\Omega))$, $\tilde{u}\tilde{u} \in L^4(t_0, T; L^2(\Omega))$, and (3.7). Then, by Serrin's argument, $u = \tilde{u}$ and u satisfies (1.24), (1.25). This completes the proof. ■

Proof of Corollary 1.10 Replacing T in (1.28) by a sufficiently small $T' = T'(C_1) \in (T_1, T)$ we get the assertion from Theorem 1.9. ■

Proof of Theorem 1.6. Choose $0 < T_0 < T_1 < T' < T$ such that $\|\nabla u(\cdot)\|_2 \in L^\infty(T_0, T_1)$ and that

$$\operatorname{ess\,sup}_{T_0 \leq t \leq T_1} \big(\|\nabla u(t)\|_2 + \|u(t)\|_2\big) \leq C_\varepsilon (T' - T_0)^{-\frac{\varepsilon}{4}}$$

holds with C_ε from (1.28). Then we obtain the estimate

$$\frac{1}{T_1 - T_0} \int_{T_0}^{T_1} \big\|\nabla u\|_2 + \|u\|_2\big)\, dt \leq C(T' - T_0)^{-\frac{\varepsilon}{4}},$$

and from Theorem 1.9 we conclude that $u \in L^8(T_1, T'; L^4(\Omega))$. A well known embedding argument implies that $u \in L^8(T_0, T'; L^4(\Omega))$. Thus for each $T_1 \in (0, T)$ we find some $0 < \delta < T_1$ with $u \in L^8(T_1 - \delta, T_1 + \delta; L^4(\Omega))$. Now the theorem is proved. ■

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