Direct limit groups do not have small subgroups

Helge Glöckner

Abstract

We show that countable direct limits of finite-dimensional Lie groups do not have small subgroups. The same conclusion is obtained for suitable direct limits of infinite-dimensional Lie groups.

Introduction

The present investigation is related to an open problem in the theory of infinite-dimensional Lie groups, i.e., Lie groups modelled on locally convex spaces (as in [14]). Recall that a topological group G is said to *have small subgroups* if every identity neighbourhood $U \subseteq G$ contains a non-trivial subgroup of G. If every identity neighbourhood U contains a non-trivial torsion group, then G is said to *have small torsion subgroups*. The additive group of the Fréchet space $\mathbb{R}^{\mathbb{N}}$ is an example of a Lie group which has small subgroups. It is an open problem (formulated first in [16]) whether a Lie group modelled on a locally convex space can have small torsion subgroups. As a general proof for the non-existence of small torsion subgroups seems to be out of reach, it is natural to examine at least the main examples of infinite-dimensional Lie groups, and to rule out this pathology individually for each of them. The main examples comprise linear Lie groups, diffeomorphism groups, mapping groups, and *direct limit groups*, i.e., direct limits in the category of Lie groups of countable direct systems of finite-dimensional Lie groups, as constructed in [8] (see also [6], [13, Theorem 47.9] and [15] for special cases). We show that direct limit groups do not have small subgroups, thus ruling out the existence of small torsion subgroups in particular:

Theorem A. *Let $\mathcal{S} := ((G_n)_{n\in\mathbb{N}}, (i_{n,m})_{n\geq m})$ be a direct sequence of finite-dimensional real Lie groups G_n and smooth homomorphisms $i_{n,m}\colon G_m \to G_n$. Let $G = \varinjlim G_n$ be the direct limit of $\mathcal{S}$ in the category of Lie groups modelled on locally convex spaces. Then G does not have small subgroups.*

More generally, we can tackle direct limits of not necessarily finite-dimensional Lie groups.

Theorem B. *Let G be a Lie group modelled on a locally convex space which is the union of an ascending sequence $G_1 \leq G_2 \leq \cdots$ of Lie groups G_n modelled on locally convex spaces, such that the inclusion maps $i_{n,m}\colon G_m \to G_n$ for $m \leq n$ and $i_n\colon G_n \to G$ are smooth homomorphisms. Assume that at least one of the following conditions is satisfied:*

(i) *Each G_n is a Banach-Lie group, $L(i_{n,m})\colon L(G_m) \to L(G_n)$ is a compact operator for all positive integers $m < n$, and $G = \varinjlim G_n$ as a topological space; or:*

(ii) *G admits a direct limit chart, $L(G_n)$ is a k_ω-space admitting a continuous norm, and G_n has an exponential map which is a local homeomorphism at 0, for each $n \in \mathbb{N}$.*

Then G does not have small subgroups.

Remarks.

(a) All of the maps $L(i_{n,m})$ are injective in Theorem B, since $i_{n,m}$ is an injective smooth homomorphism and G_m has an exponential function (cf. [14, Lemma 7.1]).

(b) A Hausdorff topological space X is called a *k_ω-space* if there exists an ascending sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets of X such that $X = \bigcup_{n\in\mathbb{N}} K_n$ and $U \subseteq X$ is open if and only if $U \cap K_n$ is open in K_n, for each $n \in \mathbb{N}$ (i.e., $X = \varinjlim K_n$ as a topological space). Then $(K_n)_{n\in\mathbb{N}}$ is called a *k_ω-sequence* for X. For background information concerning k_ω-spaces with a view towards direct limit constructions, see [10] and the references therein.

(c) A locally convex space E is a *Silva space* (or (LS)-space) if it is the locally convex direct limit $E = \bigcup_{n\in\mathbb{N}} E_n = \varinjlim E_n$ of a sequence $E_1 \subseteq E_2 \subseteq \cdots$ of Banach spaces and each inclusion map $E_n \to E_{n+1}$ is a compact linear operator. Then $E = \varinjlim E_n$ as a topological space [4, §7.1, Satz], and E is a k_ω-space [9, Example 9.4]. It is also known that the dual space E' of any metrizable locally convex space E is a k_ω-space, when equipped with the topology of compact convergence (cf. [1, Corollary 4.7]).

(d) By definition, the existence of a *direct limit chart* means the following: $L(G) = \varinjlim L(G_n)$ as a locally convex space, and there exists a chart $\phi\colon U \to V \subseteq L(G)$ of G around 1, with the following properties: $U = \bigcup_{n\in\mathbb{N}} U_n$, $V = \bigcup_{n\in\mathbb{N}} V_n$ and $\phi = \bigcup_{n\in\mathbb{N}} \phi_n = \varinjlim \phi_n$ for certain charts $\phi_n\colon U_n \to V_n \subseteq L(G_n)$ of G_n around 1, satisfying $U_n \subseteq U_{n+1}$ and $\phi_{n+1}|_{U_n} = \phi_n$ for each $n \in \mathbb{N}$ (see [9] for further information).

(e) For example, every direct limit of an ascending sequence of finite-dimensional Lie groups admits a direct limit chart, by construction of the Lie group structure in [8]. In the situation of Theorem A, we may always assume that each $i_{n,m}$ (and hence also each limit map $i_n\colon G_n \to G$) is injective (see [8, Theorem 4.3]). Then $G = \varinjlim G_n$ as a topological space by [8, Theorem 4.3 (a)]. Thus Theorem A is a special case of Theorem B (i) and does not require a separate proof.

(f) If condition (ii) of Theorem B is satisfied, then $L(G)$ is a k_ω-space and $L(G) = \varinjlim L(G_n)$ as a topological space, by [10, Proposition 7.12]. If $\phi\colon U \to V \subseteq L(G)$ is a direct limit chart for G, with $U = \bigcup_{n\in\mathbb{N}} U_n$ and $V = \bigcup_{n\in\mathbb{N}} V_n$ as in (d), then $V = \varinjlim V_n$ as a topological space (see Lemma 1.1 (b) below) and hence $U = \varinjlim U_n$. Using translations, it easily follows that also $G = \varinjlim G_n$ as a topological space.

(g) Suppose that G_n is a Banach-Lie group in the situation of Theorem B, $L(i_{n,m})$ is a compact operator for $n > m$, and G admits a direct limit chart. Then $G = \varinjlim G_n$ as a topological space (since (c) allows us to repeat the argument from (f)), and thus condition (i) of Theorem B is satisfied. While the direct limit property required in (i) is somewhat elusive, the existence of a direct limit chart can frequently be checked in concrete situations.

Example. To illustrate the use of Theorem B (i), let H be a finite-dimensional complex Lie group and K be a non-empty compact subset of a finite-dimensional complex vector space X. Then the group $\Gamma(K,H)$ of germs of complex analytic H-valued maps on open neighbourhoods of K is a Lie group in a natural way. It is modelled on the locally convex direct limit $\Gamma(K,L(H)) = \varinjlim \mathrm{Hol}_b(U_n, L(H))$, where $U_1 \supseteq U_2 \supseteq \cdots$ is a fundamental sequence of open neighbourhoods of K with U_{n+1} relatively compact in U_n, for each $n \in \mathbb{N}$, and such that each connected component of U_n meets K. Furthermore, $\mathrm{Hol}_b(U_n, L(H))$ denotes the Banach space of bounded holomorphic functions from U_n to $L(H)$, equipped with the supremum norm. For the identity component, we have $G := \Gamma(K,H)_0 = \varinjlim G_n$ for certain Banach-Lie groups G_n satisfying condition (i) of Theorem B, and thus G does not have small subgroups (nor $\Gamma(K,H)$).

In fact, let $\mathrm{Hol}(U_n, H)$ be the group of all complex analytic H-valued maps on U_n. Since $\mathrm{Exp}_n\colon \mathrm{Hol}_b(U_n, L(H)) \to \mathrm{Hol}(U_n, H)$, $\mathrm{Exp}_n(\gamma) := \exp_H \circ \gamma$ is injective on a suitable 0-neighbourhood W in $\mathrm{Hol}_b(U_n, L(H))$ and a homomorphism of local groups with respect to the Baker-Campbell-Hausdorff multiplication on W, we deduce that the subgroup G_n of $\mathrm{Hol}(U_n, H)$ generated by $\mathrm{Exp}_n(\mathrm{Hol}_b(U_n, L(H)))$ can be made a Banach-Lie group with Lie algebra $\mathrm{Hol}_b(U_n, L(H))$. The restriction map $G_m \to G_n$, $\gamma \mapsto \gamma|_{U_n}$ is an injective, smooth homomorphism for $n > m$, and its differential $L(i_{n,m})\colon \mathrm{Hol}_b(U_m, L(H)) \to \mathrm{Hol}_b(U_n, L(H))$, $\gamma \mapsto \gamma|_{U_n}$ a compact operator. Also, G has a direct limit chart (see [7] and [9] for details).

We remark that, for a more restrictive class of Lie groups, there is a simple criterion for the non-existence of small subgroups (cf. [5, Lemma 2.23] and [16, Problem II.5]):

Proposition. *If a Lie group G has an exponential map which is a local homeomorphism at* 0, *then G does not have small torsion subgroups. Also, G does not have small subgroups if* (*and only if*) *$L(G)$ admits a continuous norm.* □

Combining Theorem B (i) and the preceding proposition, we see that every Silva space $E = \bigcup_{n\in\mathbb{N}} E_n$ does not have small additive subgroups and hence admits a continuous norm. Since $\Gamma(K,H)$ has an exponential function which is a local homeomorphism at 0 (see [7]) and $\Gamma(K,L(H))$ is a Silva space, applying the proposition again we get an alternative proof for the non-existence of small subgroups in $\Gamma(K,H)$.

The preceding proposition does not subsume Theorem A (although its hypotheses are satisfied by special cases of direct limit groups as in [13] or [15]). In fact, the exponential map of a direct limit group need not be injective on any 0-neighbourhood [6, Example 5.5].

1 Some preliminaries concerning direct limits

Background information concerning direct limits of topological groups, topological spaces and Lie groups can be found in [6], [8]–[12] and [17]. We recall: If $X_1 \subseteq X_2 \subseteq \cdots$ is an ascending sequence of topological spaces such that the inclusion maps $X_n \to X_{n+1}$

are continuous, then the final topology on $X := \bigcup_{n\in\mathbb{N}} X_n$ with respect to the inclusion maps $X_n \to X$ makes X the direct limit $\varinjlim X_n$ in the category of topological spaces and continuous maps. Thus, $S \subseteq X$ is open (resp., closed) if and only if $S \cap X_n$ is open (resp., closed) in X_n for each $n \in \mathbb{N}$. If each X_n is a locally convex real topological vector space here and each inclusion map $X_n \to X_{n+1}$ is continuous linear, then the *locally convex direct limit topology* on $X = \bigcup_{n\in\mathbb{N}} X_n$ is the finest locally convex vector topology making each inclusion map $X_n \to X$ continuous (see [2]). It is coarser then the direct limit topology, and can be properly coarser. For easy reference, let us compile various well-known facts:

Lemma 1.1 *Let $X_1 \subseteq X_2 \subseteq \cdots$ be an ascending sequence of topological spaces and $X := \bigcup_{n\in\mathbb{N}} X_n$, equipped with the direct limit topology.*

(a) *If $S \subseteq X$ is open or closed, then X induces on S the topology making S the direct limit $S = \varinjlim (S \cap X_n)$, where $S \cap X_n$ carries the topology induced by X_n.*

(b) *If $U_1 \subseteq U_2 \subseteq \cdots$ is an ascending sequence of open subsets $U_n \subseteq X_n$, then $U := \bigcup_{n\in\mathbb{N}} U_n$ is open in X and $U = \varinjlim U_n$ as a topological space.*

Proof. (a) is immediate from the definition of final topologies. (b) is [9, Lemma 1.7]. □

Given a topological space X and subset $Y \subseteq X$, we write Y^0 for its interior. A sequence $(U_k)_{k\in\mathbb{N}}$ of neighbourhoods of a point $x \in X$ is called a *fundamental sequence* if $U_k \supseteq U_{k+1}$ for each $k \in \mathbb{N}$ and $\{U_k \colon k \in \mathbb{N}\}$ is a basis of neighbourhoods for x.

2 Construction of neighbourhoods without subgroups

The following lemma is the technical backbone of our constructions. In the lemma, $\mathcal{K}$ denotes a set of subsets of the given topological group G, with the following properties:

(a) $\mathcal{K}$ is closed under finite unions; and

(b) For each compact subset $K \subseteq G$, the set $\mathcal{K}_K := \{S \in \mathcal{K} \colon S$ is a neighbourhood of $K\}$ is a basis of neighbourhoods of K in G.

Of main interest are the three cases where $\mathcal{K}$ is, respectively, the set of all closed subsets of G; the set of all compact subsets; and the set of all subsets $S \subseteq G$ such that $f(S)$ is compact, where $f \colon G \to H$ is a given continuous homomorphism to a topological group H, such that each $x \in G$ has a basis of neighbourhoods U with compact image $f(U)$.

Lemma 2.1 *Let G be a topological group without small subgroups and $K \subseteq G$ be a compact set that does not contain any non-trivial subgroup of G. If $1 \in K$, then there exists a neighbourhood W of K in G which does not contain any non-trivial subgroup of G, and such that $W \in \mathcal{K}_K$. Also, W can be chosen as a subset of any given neighbourhood X of K.*

Proof. We may assume that X is open. Let $V \subseteq X$ be an open identity neighbourhood such that V does not contain any non-trivial subgroup of G, and $Q \subseteq V$ be a closed identity neighbourhood of G. For each $x \in K \setminus Q^0$, there exists $k \in \mathbb{Z}$ such that $x^k \notin K$. Let J_x be a compact neighbourhood of x in K such that $I_x := \{y^k \colon y \in J_x\} \subseteq G \setminus K$. Choose a closed neighbourhood P_x of I_x in $G \setminus K$ and let A_x be a neighbourhood of J_x in G such that $y^k \in P_x$ for each $y \in A_x$. The set $K \setminus Q^0$ being compact, we find subsets $A_1, \ldots, A_m$ of G and compact subsets $J_1, \ldots, J_m$ of K such that $K \setminus Q^0 \subseteq \bigcup_{j=1}^m J_j$, closed subsets $P_1, \ldots, P_m$ of G disjoint from K and $k_1, \ldots, k_m \in \mathbb{Z}$ such that $J_j \subseteq A_j^0$ for each $j \in \{1, \ldots, m\}$ and $y^{k_j} \in P_j$ for each $y \in A_j$. Then $P := \bigcup_{j=1}^m P_j$ is a closed subset of G such that $P \cap K = \emptyset$. After replacing A_j with a neighbourhood $\tilde{A}_j \in \mathcal{K}_{J_j}$ of J_j contained in $X \cap (A_j^0 \setminus P)$ (which is an open neighbourhood of J_j) for each j, we may assume that $A := \bigcup_{j=1}^m A_j$ and P are disjoint, $A \subseteq X$, and $A \in \mathcal{K}$. Then $V \setminus P$ is a neighbourhood of the compact set $Q \cap K$, whence $B \subseteq V \setminus P$ for some $B \in \mathcal{K}_{Q \cap K}$. Then $W := A \cup B \in \mathcal{K}$. We now show that W does not contain any non-trivial subgroup of G. Let $1 \neq x \in W$. Case 1: If $x \in A$, then $x^k \in P$ for some $k \in \mathbb{Z}$ and thus $x^k \notin W$, since W and P are disjoint by construction. Hence $\langle x \rangle \not\subseteq W$. Case 2: If $x \in B \subseteq V$, then $\langle x \rangle \not\subseteq V$, whence there is $k \in \mathbb{Z}$ such that $x^k \notin V$. If $x^k \in A$, then $\langle x^k \rangle \not\subseteq W$ by Case 1 and hence $\langle x \rangle \not\subseteq W$ a fortiori. If $x^k \notin A$, then $x^k \notin W$ (as $x^k \notin B \subseteq V$ either) and thus $\langle x \rangle \not\subseteq W$. This completes the proof. □

Remark 2.2 The proof of Lemma 2.1 can easily be adapted to get further information. Namely, let $C_1, \ldots, C_M$ be compact subsets of $K \setminus \{1\}$ and $\ell_1, \ldots, \ell_M$ be integers such that $x^{\ell_j} \notin K$ for each $j \in \{1, \ldots, M\}$ and $x \in C_j$. Furthermore, let $R, T \subseteq G$ be closed subsets such that $T \cap K = \emptyset$ and $1 \notin R$. We then easily achieve that the following additional requirements are met in the proof of Lemma 2.1 (which will become vital later):

(a) $M \leq m$, $C_j \subseteq A_j^0$ and $k_j = \ell_j$ for each $j \in \{1, \ldots, M\}$;

(b) $W \cap T = \emptyset$ and $V \cap R = \emptyset$.

In fact, we can simply replace X by its intersection with the open set $G \setminus T$ and choose V as a subset of $G \setminus R$ to ensure (b). In the construction of k, J_x, P_x and A_x described at the beginning of the proof of Lemma 2.1, we can replace J_x with a compact neighbourhood J_j of C_j in K for $j \in \{1, \ldots, M\}$, such that $I_j := \{y^{\ell_j} \colon y \in J_j\} \subseteq G \setminus K$. After enlarging the chosen finite cover of $K \setminus Q^0$ by the preceding sets if necessary, we may assume that $m \geq M$ and $k_j = \ell_j$ as well as $C_j \subseteq A_j^0$, for all $j \in \{1, \ldots, M\}$.

Remark 2.3 It is a natural idea to try to prove, say, Theorem A for $G = \bigcup_{n \in \mathbb{N}} G_n$ in the following way: Start with a compact identity neighbourhood $W_1 \subseteq G_1$ without non-trivial subgroups, and use Lemma 2.1 recursively to obtain a sequence $(W_n)_{n \in \mathbb{N}}$ of compact subsets $W_n \subseteq G_n$ such that W_n has W_{n-1} in its interior and does not contain any non-trivial subgroup. Then $W := \bigcup_{n \in \mathbb{N}} W_n$ is an identity neighbourhood in G and is a candidate for an identity neighbourhood not containing non-trivial subgroups. But, unfortunately, it can happen that W does contain non-trivial subgroups, as the example $G_n := \mathbb{R}^n$, $G := \mathbb{R}^{(\mathbb{N})} = \varinjlim G_n$, $W_n := [-n,n]^n$, $W = \mathbb{R}^{(\mathbb{N})} = G$ shows. Therefore, this basic idea has to be refined, and each W_n has to be chosen in a much more restrictive way. The considerations from Remark 2.2 will provide the required additional control on the sets W_n. Further modifications will be necessary to adapt the basic idea to the (possibly) non-locally compact groups G_n in Theorem B.

3 Proof of Theorem B

We start with several lemmas which will help us to prove Theorem B. The first lemma is a well-known fact from the theory of Silva spaces, but it is useful to recall its proof here because details thereof are essential for subsequent arguments.

Lemma 3.1 *Let $E_1 \subseteq E_2 \subseteq \cdots$ be an ascending sequence of Banach spaces, such that the inclusion map $i_{n,m}\colon E_m \to E_n$ is a compact linear operator whenever $n > m$. Then there is an ascending sequence $E_1 \subseteq F_1 \subseteq E_2 \subseteq F_2 \subseteq \cdots$ of Banach spaces with continuous linear inclusion maps, such that, for each $n \in \mathbb{N}$, there exists a norm p_n on F_n which defines the topology of F_n and has the property that all closed p_n-balls $\overline{B}_r^{p_n}(x)$, $(r > 0,\ x \in F_n)$, are compact in F_{n+1}.*

Proof. Let B_n be the closed unit ball in E_n with respect to some norm defining its topology and K_n be the closure of B_n in E_{n+1}, which is compact by hypothesis. Let $F_n := (E_{n+1})_{K_n}$ be the vector subspace of E_{n+1} spanned by K_n and p_n be the Minkowski functional of K_n on F_n. Then F_n is a Banach space, by the corollary to Proposition 8 in [2, Chapter III, §1, no. 5]. The inclusion map $F_n \to E_{n+1}$ is continuous, and also the inclusion map $E_n \to F_n$, since $B_n \subseteq K_n = \overline{B}_1^{p_n}(0)$. As K_n is compact in E_{n+1} and the inclusion map $E_{n+1} \to F_{n+1}$ is continuous, K_n is compact in F_{n+1} (and hence also the image of any ball $\overline{B}_r^{p_n}(x)$). □

Lemma 3.2 *If each E_n is a Banach-Lie algebra in the situation of Lemma* 3.1 *and each $i_{n,m}$ also is a Lie algebra homomorphism, then F_n can be chosen as a Lie subalgebra of E_{n+1} and it can be achieved that p_n makes F_n a Banach-Lie algebra.*

Proof. Since $[B_n, B_n] \subseteq rB_n$ for some $r > 0$, we have $[K_n, K_n] \subseteq rK_n$, entailing that $F_n = \operatorname{span}(K_n)$ is a Lie subalgebra of E_{n+1} and the Lie bracket $F_n \times F_n \to F_n$ is a continuous bilinear map. □

Given a Banach-Lie group G, we let $\operatorname{Ad}^G\colon G \to \operatorname{Aut}(L(G))$, $x \mapsto \operatorname{Ad}^G_x$ be the adjoint homomorphism, $\operatorname{Ad}^G_x := L(c_x)$ with $c_x\colon G \to G$, $c_x(y) := xyx^{-1}$.

Lemma 3.3 *Let $G_1 \subseteq G_2 \subseteq \cdots$ be an ascending sequence of Banach-Lie groups, such that the inclusion maps $i_{n,m}\colon G_m \to G_n$ are smooth homomorphisms for $n \geq m$ and $L(i_{n,m})\colon L(G_m) \to L(G_n)$ is a compact linear operator whenever $n > m$. Then there is an ascending sequence $G_1 \subseteq H_1 \subseteq G_2 \subseteq H_2 \subseteq \cdots$ of Banach-Lie groups such that, for each $n \in \mathbb{N}$, there is a norm p_n on $L(H_n)$ which defines the topology of $L(H_n)$ and has the property that all closed p_n-balls $\overline{B}_r^{p_n}(x)$, $(r > 0,\ x \in L(H_n))$, are compact in $L(H_{n+1})$.*

Proof. We identify $L(G_n)$ with a Lie subalgebra of $L(G_{n+1})$ for each $n \in \mathbb{N}$. By Lemma 3.2, there is an ascending sequence

$$L(G_1) \subseteq F_1 \subseteq L(G_2) \subseteq F_2 \subseteq \cdots$$

of Banach-Lie algebras such that the inclusion maps are continuous Lie algebra homomorphisms, and such that, for each $n \in \mathbb{N}$, there exists a norm p_n on F_n defining its topology and such that all closed p_n-balls in F_n are compact subsets of F_{n+1}. As in the proofs of Lemmas 3.1 and 3.2, we may assume that the closed unit ball $K_n := \overline{B}_1^{p_n}(0)$ of F_n is the closure in $L(G_{n+1})$ of the closed unit ball B_n of $L(G_n)$. We give $S_n := \langle \exp_{G_{n+1}}(F_n)\rangle$ the Banach-Lie group structure making it an analytic subgroup of G_{n+1}, with Lie algebra F_n. For each $x \in G_n$, we have

$$\mathrm{Ad}_x^{G_{n+1}}(B_n) = \mathrm{Ad}_x^{G_n}(B_n) \subseteq rB_n$$

for some $r > 0$, whence $\mathrm{Ad}_x^{G_{n+1}}(K_n) \subseteq rK_n$ and hence $\mathrm{Ad}_x^{G_{n+1}}(F_n) \subseteq F_n$. Note that the linear automorphism of F_n induced by $\mathrm{Ad}_x^{G_{n+1}}$ is continuous, by the penultimate inclusion. As a consequence, the subgroup $H_n := \langle G_n \cup \exp_{G_{n+1}}(F_n)\rangle$ of G_{n+1} can be given a Banach-Lie group structure with S_n as an open subgroup (cf. Proposition 18 in [3, Chapter III, §1.9]). By construction, the Banach-Lie groups H_n have the desired properties. □

Lemma 3.4 *Let $f\colon G \to H$ be a smooth homomorphism between Banach-Lie groups such that, for some norm p on $L(G)$ defining its topology, $L(f)\colon L(G) \to L(H)$ takes closed balls in $L(G)$ to compact subsets of $L(H)$. Then each $x \in G$ has a basis of closed neighbourhoods U such that $f(U)$ is compact in H. Furthermore, every neighbourhood of a compact subset $K \subseteq G$ contains a closed neighbourhood A such that $f(A)$ is compact.*

Proof. Since G is a regular topological space and $\exp_G$ a local homeomorphism at 0, there is $R > 0$ such that $\exp_G|_{\overline{B}_R}$ is a homeomorphism onto its image and $V_r := \exp_G(\overline{B}_r)$ is closed in G for each $r \in]0, R]$, where $\overline{B}_r := \{x \in L(G)\colon p(x) \leq r\}$. Exploiting the naturality of exp and the hypothesis that $L(f).\overline{B}_r$ is compact in $L(H)$, we deduce that $f(V_r) = f(\exp_G(\overline{B}_r)) = \exp_H(L(f).\overline{B}_r)$ is compact in H, for each $r \in]0, R]$. Thus $\{V_r\colon r \in]0, R]\}$ is a basis of closed neighbourhoods of 1 in G with compact image under f. Then $\{xV_r\colon r \in]0, R]\}$ is a basis of closed neighbourhoods of $x \in G$ with compact image. The final assertion is an immediate consequence. □

Proof of Theorem B, assuming condition (i). We define Banach-Lie groups H_n as in Lemma 3.3. After replacing G_n with H_n for each $n \in \mathbb{N}$, we may assume that each point in G_n has a basis of neighbourhoods in G_n which are compact in G_{n+1} (see Lemma 3.4). We now construct, for each $n \in \mathbb{N}$:

- An identity neighbourhood $W_n \subseteq G_n$ such that W_n, when considered as a subset K_n of G_{n+1}, becomes compact;

- A fundamental sequence $(Y_k^{(n)})_{k\in\mathbb{N}}$ of open identity neighbourhoods in K_n;
- For some $m_n \in \mathbb{N}_0$, a family $(C_j^{(n)})_{j=1}^{m_n}$ of subsets $C_j^{(n)}$ of $W_n \setminus \{1\}$ which are compact in G_{n+1}; and
- A function $\kappa_n \colon \{1, \ldots, m_n\} \to \mathbb{Z}$,

with the following properties:

(a) If $n > 1$, then W_{n-1} is contained in the interior W_n^0 of W_n relative G_n;

(b) W_n does not contain any non-trivial subgroup of G_n;

(c) For each $j \in \{1, \ldots, m_n\}$ and $x \in C_j^{(n)}$, we have $x^{\kappa_n(j)} \not\in W_n$;

(d) If $n > 1$, then $m_n \geq m_{n-1}$ and $C_j^{(n-1)} \subseteq C_j^{(n)}$ as well as $\kappa_n(j) = \kappa_{n-1}(j)$, for all $j \in \{1, \ldots, m_{n-1}\}$;

(e) For all positive integers $\ell < n$, we have $K_\ell \setminus Y_n^{(\ell)} \subseteq \bigcup_{j=1}^{m_n} C_j^{(n)}$.

If this construction is possible, then $U := \bigcup_{n\in\mathbb{N}} W_n = \bigcup_{n\geq 2} W_n^0$ is an open identity neighbourhood in $G = \varinjlim G_n$, using (a) and Lemma 1.1 (b). Furthermore, U does not contain any non-trivial subgroup of G. In fact: If $1 \neq x \in U$, there is $m \in \mathbb{N}$ such that $x \in W_m$. Then $x \in K_m \setminus Y_n^{(m)}$ for some $n > m$, and thus $x \in C_j^{(n)}$ for some $j \in \{1, \ldots, m_n\}$, by (e). By (c) and (d), we have $x^{\kappa_n(j)} \not\in W_k$ for each $k \geq n$, whence $x^{\kappa_n(j)} \not\in U$ and thus $\langle x\rangle \not\subseteq U$.

It remains to carry out the construction. Since G_1 is a Banach-Lie group, it does not have small subgroups, whence we find an identity neighbourhood W_1 in G_1 which does not contain any non-trivial subgroup of G_1. By Lemma 3.4, after replacing W_1 be a smaller identity neighbourhood, we may assume that W_1, considered as subset K_1 of G_2, becomes compact. We set $m_1 := 0$, $\kappa_1 := \emptyset$, and choose any fundamental sequence $(Y_k^{(1)})_{k\in\mathbb{N}}$ of open identity neighbourhoods of K_1, which is possible because G_2 (and hence K_1) is metrizable.

Let N be an integer ≥ 2 now and suppose that W_n, $(Y_k^{(n)})_{k\in\mathbb{N}}$, $(C_j^{(n)})_{j=1}^{m_n}$ and κ_n have been constructed for $n \in \{1, \ldots, N-1\}$, such that (a)–(e) hold. Then $R := \bigcup_{\ell<N} K_\ell \setminus Y_N^{(\ell)}$ and $T := \bigcup_{j=1}^{m_{N-1}} \{x^{\kappa_{N-1}(j)} \colon x \in C_j^{(N-1)}\}$ are compact subsets of G_N such that $1 \not\in R$ and $T \cap W_{N-1} = \emptyset$. We now apply Lemma 2.1 to G_N and its compact subset $K := K_{N-1}$, with $\mathcal{K}$ the set of all subsets of G_N which are compact in G_{N+1}. Let $A_1, \ldots, A_m$, $k_1, \ldots, k_m$, V, A and $W_N := W \in \mathcal{K}$ be as described in Lemma 2.1 and its proof. As explained in Remark 2.2, we may assume that $V \cap R = \emptyset$, $W \cap T = \emptyset$, $m \geq m_{N-1}$, $C_j^{(N-1)} \subseteq A_j^0$ for each $j \in \{1, \ldots, m_{N-1}\}$, and $k_j = \kappa_{N-1}(j)$. Set $m_N := m$, $C_j^{(N)} := A_j$ for $j \in \{1, \ldots, m_N\}$, and $\kappa_N(j) := k_j$. Let $(Y_k^{(N)})_{k\in\mathbb{N}}$ be any fundamental sequence of open identity neighbourhoods in $K_N := W_N$, considered as a compact subset of G_{N+1}. If $\ell < N$, then $K_\ell \setminus Y_N^{(\ell)} \subseteq R$ and hence $(K_\ell \setminus Y_N^{(\ell)}) \cap V = \emptyset$, entailing that $K_\ell \setminus Y_N^{(\ell)} \subseteq W \setminus V \subseteq A = \bigcup_{j=1}^{m_N} C_j^{(N)}$. Thus (a)–(e) hold for all $n \in \{1, \ldots, N\}$.

Proof of Theorem B, assuming condition (ii). Let $\phi\colon \tilde{Z} \to \tilde{H} \subseteq L(G)$ be a direct limit chart of G around 1, such that $\phi(1) = 0$. Thus $\tilde{Z} = \bigcup_{n\in\mathbb{N}} Z_n$, $\tilde{H} = \bigcup_{n\in\mathbb{N}} H_n$, and $\phi = \bigcup_{n\in\mathbb{N}} \phi_n$ for certain charts $\phi_n\colon Z_n \to H_n$ of G_n, such that $Z_n \subseteq Z_{n+1}$ and $\phi_{n+1}|_{Z_n} = \phi_n$ for each $n \in \mathbb{N}$. By [10, Proposition 7.12], $L(G)$ is a k_ω-space and $L(G) = \varinjlim L(G_n)$ also as a topological space. By [10, Proposition 4.2 (g)], H_1 has an open 0-neighbourhood V_1 which is a k_ω-space. By Proposition 4.2 (g) and Lemma 4.3 in [10], V_1 has an open neighbourhood V_2 in H_2 which is a k_ω-space. Proceeding in this way, we find an ascending sequence $V_1 \subseteq V_2 \subseteq \cdots$ of open 0-neighbourhoods $V_n \subseteq H_n$, such that each V_n is a k_ω-space. By Lemma 1.1 (b), $V := \bigcup_{n\in\mathbb{N}} V_n \subseteq H$ is open in $L(G)$ and $V = \varinjlim V_n$ as a topological space, whence V is a k_ω-space by [10, Proposition 4.5]. For each $j \in \mathbb{N}$, choose a k_ω-sequence $(L_n^{(j)})_{n\in\mathbb{N}}$ for V_j. We may assume that $0 \in L_1^{(1)}$. After replacing $L_n^{(j)}$ with $\bigcup_{i=1}^{j} L_n^{(i)}$, we may assume that $L_n^{(i)} \subseteq L_n^{(j)}$ for all positive integers $i \leq j$ and n. Then $L_n^{(n)}$ is a k_ω-sequence for V (see the first half of the proof of Proposition 4.5 in [10]), and thus $K_n := \phi^{-1}(L_n^{(n)})$ defines a k_ω-sequence $(K_n)_{n\in\mathbb{N}}$ for the open identity neighbourhood $Z := \phi^{-1}(V) \subseteq G$. Note that $K_n = \phi_n^{-1}(L_n^{(n)})$ is a compact subset of G_n, and $1 \in K_1$. Because $L(G_n)$ admits a continuous norm, the compact set $L_n^{(n)}$ is metrizable and hence also K_n. We now construct, for each $n \in \mathbb{N}$:

- A compact identity neighbourhood W_n in K_n;
- A fundamental sequence $(Y_k^{(n)})_{k\in\mathbb{N}}$ of open identity neighbourhoods in W_n;
- For some $m_n \in \mathbb{N}_0$, a family $(C_j^{(n)})_{j=1}^{m_n}$ of subsets $C_j^{(n)} \subseteq W_n \setminus \{1\}$, and a function $\kappa_n\colon \{1, \ldots, m_n\} \to \mathbb{Z}$,

such that conditions (b)–(e) from the proof of Theorem B (i) are satisfied and also

(a)$'$ If $n > 1$, then W_{n-1} is contained in the interior W_n^0 of W_n relative K_n.

If this construction is possible, then $U := \bigcup_{n\in\mathbb{N}} W_n = \bigcup_{n\geq 2} W_n^0$ is an open identity neighbourhood in $Z = \varinjlim K_n$ (by (a)$'$ and Lemma 1.1 (b)), and hence in G. Furthermore, U does not contain any non-trivial subgroup of G, by the same argument as above.

To carry out the construction, we recall first that as G_n has an exponential map which is a local homeomorphism at 0 and $L(G_n)$ admits a continuous norm, G_n does not have small subgroups (by the proposition in the Introduction). In particular, we find a closed identity neighbourhood $\tilde{W}_1$ in G_1 which does not contain any non-trivial subgroup of G_1. Then $W_1 := \tilde{W}_1 \cap K_1$ is a compact identity neighbourhood in K_1. We set $m_1 := 0$, $\kappa_1 := \emptyset$, and choose any fundamental sequence $(Y_k^{(1)})_{k\in\mathbb{N}}$ of open identity neighbourhoods of W_1 (which is possible because K_1 is metrizable).

Let N be an integer ≥ 2 now and suppose that W_n, $(Y_k^{(n)})_{k\in\mathbb{N}}$, $(C_j^{(n)})_{j=1}^{m_n}$ and κ_n have been constructed for $n \in \{1, \ldots, N-1\}$ such that (a)$'$ and (b)–(e) hold. Then $R := \bigcup_{\ell<N} K_\ell \setminus Y_N^{(\ell)}$ and $T := \bigcup_{j=1}^{m_{N-1}} \{x^{\kappa_{N-1}(j)}\colon x \in C_j^{(N-1)}\}$ are compact subsets of G_N such

that $1 \notin R$ and $T \cap W_{N-1} = \emptyset$. We now apply Lemma 2.1 to G_N and its compact subset $K := K_{N-1}$, with $\mathcal{K}$ the set of all closed subsets of G_N. Let $A_1, \ldots, A_m$, $k_1, \ldots, k_m$, V, A and $W \in \mathcal{K}$ be as described in Lemma 2.1 and its proof. As explained in Remark 2.2, we may assume that $V \cap R = \emptyset$, $W \cap T = \emptyset$, $m \geq m_{N-1}$, $C_j^{(N-1)} \subseteq A_j^0$ for each $j \in \{1, \ldots, m_{N-1}\}$, and $k_j = \kappa_{N-1}(j)$. Set $W_N := W \cap K_N$, $m_N := m$, $C_j^{(N)} := A_j \cap K_N$ for $j \in \{1, \ldots, m\}$, and $\kappa_N(j) := k_j$. Let $(Y_k^{(N)})_{k \in \mathbb{N}}$ be any fundamental sequence of open identity neighbourhoods in W_N. If $\ell < N$, then $K_\ell \setminus Y_N^{(\ell)} \subseteq R$ and hence $(K_\ell \setminus Y_N^{(\ell)}) \cap V = \emptyset$, entailing that $K_\ell \setminus Y_N^{(\ell)} \subseteq K_N \cap (W \setminus V) \subseteq K_N \cap A = \bigcup_{j=1}^{m_N} C_j^{(N)}$. Thus (a)$'$ and (b)-(e) hold for all $n \in \{1, \ldots, N\}$. □

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References

[1] Außenhofer, L., *Contributions to the duality theory of Abelian topological groups and to the theory of nuclear groups*, Dissertationes Math. **384**, 1999.

[2] Bourbaki, N., "Topological Vector Spaces, Chapters 1–5," Springer, Berlin, 1987.

[3] Bourbaki, N., "Lie Groups and Lie Algebras, Chapters 1–3," Springer, Berlin, 1989.

[4] Floret, K., *Lokalkonvexe Sequenzen mit kompakten Abbildungen*, J. Reine Angew. Math. **247** (1971), 155–195.

[5] Glöckner, H., *Lie group structures on quotient groups and universal complexifications for infinite-dimensional Lie groups*, J. Funct. Anal. **194** (2002), 347–409.

[6] Glöckner, H. *Direct limit Lie groups and manifolds*, J. Math. Kyoto Univ. **43** (2003), 1–26.

[7] Glöckner, H., *Lie groups of germs of analytic mappings*, pp. 1–16 in: T. Wurzbacher (Ed.), "Infinite Dimensional Groups and Manifolds," IRMA Lecture Notes in Math. and Theor. Physics, de Gruyter, 2004.

[8] Glöckner, H. *Fundamentals of direct limit Lie theory*, Compos. Math. **141** (2005), 1551–1577.

[9] Glöckner, H., *Direct limits of infinite-dimensional Lie groups compared to direct limits in related categories*, arXiv:math.GR/0606078.

[10] Glöckner, H. and R. Gramlich, *Final group topologies, Phan systems and Pontryagin duality*, preprint, arXiv:math.GR/0603537.

[11] Hansen, V. L., *Some theorems on direct limits of expanding systems of manifolds*, Math. Scand. **29** (1971), 5–36.

[12] Hirai, T., H. Shimomura, N. Tatsuuma and E. Hirai, *Inductive limits of topologies, their direct products, and problems related to algebraic structures*, J. Math. Kyoto Univ. **41** (2001), 475–505.

[13] Kriegl, A. and P. W. Michor, "The Convenient Setting of Global Analysis," AMS, 1997.

[14] Milnor, J., *Remarks on infinite-dimensional Lie groups*, pp. 1007–1057 in: "Relativité, Groupes et Topologie II," B. DeWitt and R. Stora (Eds), North-Holland, Amsterdam, 1984.

[15] Natarajan, L., E. Rodríguez-Carrington and J. A. Wolf, *Differentiable structure for direct limit groups*, Letters Math. Phys. **23** (1991), 99–109.

[16] Neeb, K.-H., *Towards a Lie theory of locally convex groups*, TU Darmstadt Preprint **2459**, 2006.

[17] Tatsuuma, N., H. Shimomura, and T. Hirai, *On group topologies and unitary representations of inductive limits of topological groups and the case of the group of diffeomorphisms*, J. Math. Kyoto Univ. **38** (1998), 551–578.

Helge Glöckner, TU Darmstadt, FB Mathematik AG 5, Schlossgartenstr. 7, 64289 Darmstadt, Germany
E-Mail: `gloeckner@mathematik.tu-darmstadt.de`