Homogenization in viscoplasticity

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Abstract. In this work we present the justification of the formally derived homogenized problem for the quasistatic initial boundary value problem with internal variables, which models the deformation behavior of viscoplastic materials with a periodic microstructure.

Key words: homogenization, plasticity, viscoplasticity, two-scale convergence, maximal monotone operator, microstructure.

AMS 2000 subject classification: 74Q15, 74C05, 74C10, 74D10, 35J25, 34G20, 34G25, 47H04, 47H05

1 Introduction and statement of results

During the last decades the rigorous mathematical investigation of homogenization has brought appreciable success in determining the macroscopic behavior from the knowledge of the microstructure in many problems from different sciences. Among them are problems from the linear and nonlinear theory of elasticity, linear viscoelasticity and electrodynamics, hydrodynamics and porous media, see for example [5], [7], [9], [10], [19], [23], [24], [25], [26], [29], [30], [32]. The only rigorous results of homogenization related to problems from the theory of plasticity or viscoplasticity known to me are [4] and [20]. This is in contrast to the importance of homogenization in solid mechanics. This circumstance motivated the further study of such problems.

In this work I deal with the homogenization of the initial boundary value problem describing the deformation behavior of inelastic materials with a periodic microstructure, in particular for plastic and viscoplastic materials. The formulation of the problem is based on the assumption that only small strains occur: Let Ω be an open bounded set, the set of material points of the body, with C^1-boundary $\partial\Omega$. T_e denotes a positive number (time of existence) and for $0 \le t \le T_e$

$$\Omega_t = \Omega \times t.$$

Let $\mathcal{S}^3$ denote the set of symmetric 3×3-matrices, and let $u(x,t) \in \mathbb{R}^3$ be the unknown displacement of the material point x at time t , $T(x,t) \in \mathcal{S}^3$ is the

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unknown Cauchy stress tensor and $z(x,t) \in \mathbb{R}^3$ denotes the unknown vector of internal variables. The model equations of the problem (a microscopic problem) are

$$\begin{aligned} -\mathrm{div}_x T(x,t) &= b(x,t), & (1)\\ T(x,t) &= \mathcal{D}[\frac{x}{\eta}](\varepsilon(\nabla_x u(x,t)) - Bz(x,t)), & (2)\\ \frac{\partial}{\partial t} z(x,t) &\in g(\frac{x}{\eta}, -\nabla_z \psi(\varepsilon(\nabla_x u(x,t)), z(x,t))) & (3)\\ &= g(\frac{x}{\eta}, B^T T(x,t) - Lz(x,t)), \end{aligned}$$

which must hold for $x \in \Omega$ and $t \in [0,\infty)$. The initial value for $z(x,t)$ is taken in the form

$$z(x,0) = z^{(0)}(x, \frac{x}{\eta}), \tag{4}$$

which must hold for $x \in \Omega$. For simplicity we only consider the Dirichlet boundary condition

$$u(x,t) = \gamma(x,t), \tag{5}$$

which must be satisfied for $(x,t) \in \partial\Omega \times [0,\infty)$.

Here

$$\varepsilon(\nabla_x u(x,t)) = \frac{1}{2}(\nabla_x u(x,t) + (\nabla_x u(x,t))^T) \in \mathcal{S}^3,$$

is the strain tensor, $B : \mathbb{R}^{\mathrm{N}} \to \mathcal{S}^3$ is a linear mapping, which assigns to the vector $z(x,t)$ the plastic strain tensor $\varepsilon_p(x,t) = Bz(x,t)$. For every $y \in \mathbb{R}^3$ we denote by $\mathcal{D}[y] : \mathcal{S}^3 \to \mathcal{S}^3$ a linear, symmetric, positive definite mapping, the elasticity tensor. It is assumed that the mapping $y \to \mathcal{D}[y]$ is periodic with a rectangular periodicity cell $Y \subset \mathbb{R}^3$. $b : \Omega \times [0,\infty) \to \mathbb{R}^3$ is the volume force, $z^{(0)} : \Omega \to \mathbb{R}^{\mathrm{N}}$ is the initial value of the vector of internal variables, periodic in y with the same periodicity cell Y. The positive semi-definite quadratic form

$$\psi(y,\varepsilon,z) = \frac{1}{2}\mathcal{D}[y](\varepsilon - Bz)\cdot(\varepsilon - Bz) + \frac{1}{2}(Lz)\cdot z \tag{6}$$

represents the free energy (see Appendix [1]), and for all $y \in \mathbb{R}^3$ the function $z \to g(y,z) : \mathbb{R}^{\mathrm{N}} \to 2^{\mathbb{R}^{\mathrm{N}}}$ is monotone satisfying $0 \in g(y,0)$; $y \to g(y,z)$ is periodic with the rectangular periodicity cell $Y \subset \mathbb{R}^3$.

A function $g : D(g) \subseteq \mathbb{R}^{\mathrm{N}} \mapsto 2^{\mathbb{R}^{\mathrm{N}}}$ is called *monotone* if

$$(x_1 - x_2, y_1 - y_2) \geq 0$$

for any $y_i \in g(z_i)$, $i = 1,2$. A monotone function is said to be *maximal monotone* if it has no monotone extension. In other words, g is *maximal monotone* if and only if the inequality

$$(z_1 - z_2, y_1 - y_2) \geq 0 \qquad \text{for all } y_1 \in g(z_1)$$

implies $y_2 \in g(z_2)$.

The number $\eta > 0$ is the scaling parameter of the microstructure.

The differential inclusion (3) with a given function g and the equation (2) together define the material behavior. They are the constitutive relations which model the inelastic respond of the body, whereas (1) is the conservation law of linear momentum. (3) is called a constitutive relation (or equation) of monotone type which was firstly introduced in [1]. The class of constitutive relations of monotone type naturally generalizes the class of constitutive relations of generalized standard materials defined by B. Halphen and Nguyen Quoc Son, because in the last case the function g is the gradient or subdifferential of a convex function. For examples of models from engineering and for the study whether they are of monotone type we refer the reader to [1]. It must be said here that the classical models like the Prandtl-Reuss and the Norton-Hoff laws belong to the class of the constitutive equations of monotone type. However, this class is still too small to include all models used in engineering. Namely, all models describing the deformation behavior of inelastic bodies with infinitesimal strains can be written in the form (3), but often the function g is not monotone.[2].

The existence and uniqueness theory for (1) -(3) is well understood under additional assumptions: If the free energy is not only positive semi-definite but positive definite, equivalently if the $N \times N$-matrix L is positive definite; and additionally if the mapping $z \to g(y, z)$ is maximal monotone for all $y \in \mathbb{R}^3$, then the initial boundary value problem has a unique solution denoted by (u_η, T_η, z_η). See [3] or [1]. We remark that in many cases the free energy is not positive definite but positive semi-definite. For example the Prandtl-Reuss and the Norton-Hoff laws are constitutive equations with positive semi-definite free energy, whereas models with linear hardening have positive definite free energy. For problems with semi-definite free energy to my knowledge there are no general results till now. Yet, for some particular models with special choice of the function g, the existence and uniqueness theory is already available. See for example [16], [14], [13], [31], and the literature cited there concerning this side of the investigation.

To study the asymptotic behavior of (u_η, T_η, z_η) as η tends to 0 we postulate that this function is close to the function $(\hat{u}_\eta, \hat{T}_\eta, \hat{z}_\eta)$ defined by

$$\hat{u}_\eta(x, t) = u_0(x, t) + \eta u_1(x, \frac{x}{\eta}, t),$$

[2]To my knowledge all models proposed in engineering sciences belong to the problems of pre-monotone type. The problem (1) - (3) is of a pre-monotone type if the multifunction g in (3) satisfies the inequality

$$\forall\ z \in D(g),\ \forall\ z^* \in g(z) \quad z^* \cdot z \geq 0.$$

$$\hat{T}_\eta(x,t) = T_0(x, \frac{x}{\eta}, t),$$
$$\hat{z}_\eta(x,t) = z_0(x, \frac{x}{\eta}, t),$$

where (u_0, u_1, T_0, z_0) solves the homogenized initial boundary value problem (see [2]):

$$\begin{aligned}
-\mathrm{div}_x T_\infty(x,t) &= b(x,t), && (7)\\
T_\infty(x,t) &= \frac{1}{|Y|}\int_Y T_0(x,y,t)dy, && (8)\\
-\mathrm{div}_y T_0(x,y,t) &= 0, && (9)\\
T_0(x,y,t) &= \mathcal{D}[y](\varepsilon(\nabla_y u_1(x,y,t)) - Bz_0(x,y,t) && (10)\\
&\qquad +\varepsilon(\nabla_x u_0(x,t))), &&\\
\frac{\partial}{\partial t} z_0(x,y,t) &\in g(y, B^T T_0(x,y,t) - Lz_0(x,y,t)), && (11)\\
z_0(x,y,0) &= z_0^{(0)}(x,y), && (12)
\end{aligned}$$

which must hold for $(x,y,t) \in \Omega \times Y \times [0,\infty)$,

$$u_0(x,t) = \gamma(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty). \qquad (13)$$

Remark 1.1 For fixed x the equations (9)-(12) together with the periodicity assumption on $y \mapsto (u_1, T_0)(x,y,t)$, which can be considered to be a boundary condition, define an initial boundary problem, the cell problem, in $Y \times [0,\infty)$. u_0, u_1 can be interpreted as macro- and microdisplacement, T_0 as a microstress; the macrostress T_∞ is obtained by averaging of T_0 over the representative volume element. In Theorem 2 [4] it was shown that under some additional assumptions on the function g the homogenized initial boundary value problem (7)-(13) has a unique solution

$$(u_0, u_1, T_\infty) \in L^2(0, T_e; H^1(\Omega, \mathrm{IR}^3)) \times L^2(\Omega_{T_e}, H^1(Y, \mathrm{IR}^3)) \times L^2(\Omega_{T_e}, \mathcal{S}^3)$$

$$(T_0, z_0) \in L^2((\Omega \times Y)_{T_e}, \mathcal{S}^3) \times C([0, T_e]; L^2(\Omega \times Y, \mathrm{IR}^{\mathrm{N}})).$$

The main goal of this work is to prove that the solution of the microscopic problem (u_η, T_η, z_η) has as the asymptotics the function $(\hat{u}_\eta, \hat{T}_\eta, \hat{z}_\eta)$. The justification uses methods from the established homogenization theory for linear problems, but several difficulties arise not present in this theory.

Firstly, the existence and uniqueness theory for the homogenized initial boundary value problem as well as the justification procedure are more complicated due to the impossibility to decouple the homogenized problem (the first equation) and the so-called cell problem (the last three equations with the periodicity assumption, which can be considered as a boundary condition), unlike in linear elasticity, where the homogenized and the cell problem can be decoupled. This difficulty was successfully solved in [4].

Secondly, difficulties arise which are based on the fact that the solution of the homogenized problem is of low regularity because of the nonlinearity of the constitutive equation. One of the difficulties resulting from the low regularity of the solution of the homogenized problem is that the mapping $x \mapsto (T_0, z_0)(x, x/\eta, t)$ is not well defined because $x \mapsto (x, x/\eta)$ maps Ω onto a three-dimensional subspace of a six-dimensional space $\Omega \times \mathbb{R}^3$.

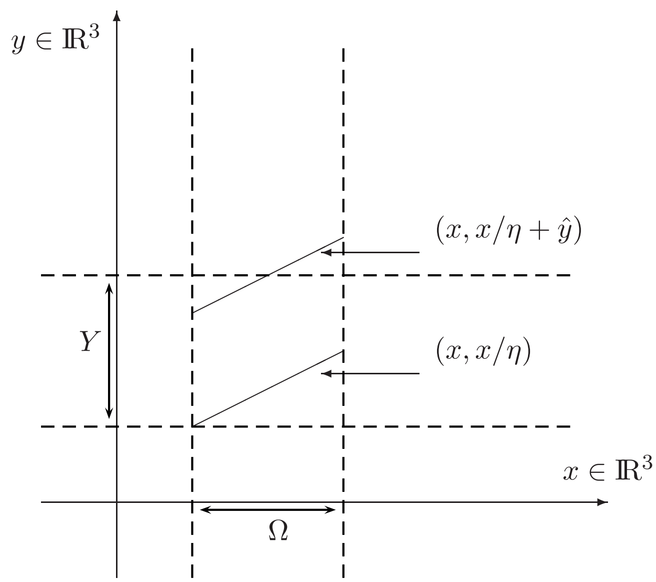

Figure 1.

Indeed, $x \mapsto (x, x/\eta)$ maps Ω onto a three-dimensional subspace of $\Omega \times \mathbb{R}^3$ (see Figure 1) and by virtue of Theorem 5.2.2 [12] the mapping $x \mapsto (T_0, z_0)(x, x/\eta, t)$ is not well defined. In other words, the function $(T_0, z_0)(t) \in L^2(\Omega \times Y, \mathcal{S}^3) \times L^2(\Omega \times Y, \mathbb{R}^{\mathrm{N}})$ has no trace on a three-dimensional subspace of a six-dimensional space.

In [4] this difficulty is overcome by imposing higher regularity on the given data in such a way that the solution of homogenized problem becomes smoother and the existence of the trace for $(T_0(t), z_0(t))$ on the three-dimensional subspace is an easy consequence of this obtained smoothness. The higher regularity of the solution of homogenized problem plays an essential role in this work also at another place: in order to apply an energy method of Tartar[3] in the justification the author needs that $\partial_t \mathrm{div}_x T_0(x, y, t) \mid_{y=x/\eta}$ and $\partial_t \mathrm{rot}_x \nabla_y u_1(x, y, t) \mid_{y=x/\eta}$ belong to compact subset of H^{-1}_{loc}. This is provided by the smoothness of $(T_0, u_1)(x, y, t)$. Unfortunately, one can not expect that the solution is of this higher regularity globally in time. Instead, after a certain finite time the solution is only of

[3]It is also called the *oscillating test function method* or the *compensated compactness method*.

lower regularity. Therefore in [4] the justification of the homogenized problem is only possible locally in time. In contract, here we can justify the homogenized problem globally in time without imposing additional smoothness on the data.

To avoid the difficulty with the trace I follow the idea[4], proposed in [2], of introducing an additional fast variable y, which is plugged into equations (1) - (5):

$$\begin{aligned} -\mathrm{div}_x T(x,y,t) &= b(x,t), && (14)\\ T(x,y,t) &= \mathcal{D}[\frac{x}{\eta}+y](\varepsilon(\bigtriangledown_x u(x,y,t)) - Bz(x,y,t)), && (15)\\ \frac{\partial}{\partial t} z(x,y,t) &\in g(\frac{x}{\eta}+y, B^T T(x,y,t) - Lz(x,y,t)), && (16)\\ z(x,y,0) &= z^{(0)}(x,\frac{x}{\eta}+y), && (17) \end{aligned}$$

which hold for $(x,y) \in \Omega \times Y$ and $t \in [0,\infty)$, and of the Dirichlet boundary condition

$$u(x,y,t) = \gamma(x,t), \qquad (18)$$

which holds for $(x,y) \in \partial\Omega \times Y$ and $t \in [0,\infty)$. The function $(u_\eta, T_\eta, z_\eta)(x,y,t)$ is periodic in y.

We give the definition of a **general solution** of the initial boundary value problem (14) - (18). $\eta > 0$ is fixed.

Definition 1.1 *Let*

$$(u_\eta, T_\eta, z_\eta) : \Omega \times \mathrm{I\!R}^3 \times \mathrm{I\!R}^+ \mapsto \mathrm{I\!R}^3 \times \mathcal{S}^3 \times \mathrm{I\!R}^{\mathrm{N}}$$

be a function which satisfies the initial condition (17) for a. e. $(x,y) \in \Omega \times \mathrm{I\!R}^3$ *and for which the function* $(x,y) \mapsto (u_\eta, T_\eta, z_\eta)(x,y,t)$ *is a solution of (14)-(18) for almost all* $y \in \mathrm{I\!R}^3$*. Then* (u_η, T_η, z_η) *is called a* family of solutions *of the initial boundary value problem (14)-(18) depending on the fast variable* y*.*

We assume now that for all $0 < \eta < \eta_0$ such a solution family (u_η, T_η, z_η) of the initial boundary value problem depending on the fast variable y exists and is close to

$$\begin{aligned} \hat{u}_\eta(x,y,t) &= u_0(x,t) + \eta u_1(x,\frac{x}{\eta}+y,t), && (19)\\ \hat{T}_\eta(x,y,t) &= T_0(x,\frac{x}{\eta}+y,t) && (20)\\ \hat{z}_\eta(x,y,t) &= z_0(x,\frac{x}{\eta}+y,t). && (21) \end{aligned}$$

[4] The idea of considering the family of shifted problems was also used in [17] to show that for some linear and nonlinear problems the averaging over the shifting y eliminates the rapid oscillations in the solution. For details we refer the reader to this work.

The functions $u_0(x,t), u_1(x,y,t), T_0(x,y,t), z_0(x,y,t)$, which are assumed to be periodic with respect to the y-argument with a rectangular periodicity cell $Y \subset \mathbb{R}^3$, solve the problem (7)-(13).

Now we can formulate the main result of the work.

Theorem 1.1 *Let $T_e > 0$. Assume that the $N \times N$-matrix L in (6) is positive definite and that the mapping $g : \mathbb{R}^3 \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ satisfies the following three conditions*

- $0 \in g(y,0)$,
- *$z \mapsto g(y,z)$ is maximal monotone,*
- *the mapping $y \mapsto j_\lambda(y,z) : \mathbb{R}^3 \to \mathbb{R}^N$, the inverse of $z \mapsto z + \lambda g(y,z)$*[5]*, is measurable for all $\lambda > 0$.*

Suppose that $b \in W^{2,1}(0,T_e; L^2(\Omega,\mathbb{R}^3))$ and $\gamma \in W^{2,1}(0,T_e; H^1(\Omega,\mathbb{R}^3))$. Assume that $z^{(0)} \in L^2(\Omega \times Y, \mathbb{R}^N)$ and there exists $\zeta \in L^2(\Omega \times Y, \mathbb{R}^N)$ such that

$$\zeta(x,y) \in g(y, B^T T^{(0)}(x,y) - Lz^{(0)}(x,y)), \quad \text{a.e. in } \Omega \times Y,$$

where $(u^{(0)}, T^{(0)})$ is a weak solution of the problem of linear elasticity theory (25) - (27) to the data $\hat{b} = b(0)$, $\hat{\varepsilon}_p = Bz^{(0)}$, $\hat{\gamma} = \gamma(0)$.

Assume additionally that the inequality

$$\|g_\lambda(\cdot/\eta + \cdot, B^T T^{(0)} - Lz^{(0)})\|_{\Omega\times Y} \le C\|B^T T^{(0)} - Lz^{(0)}\|_{\Omega\times Y} \tag{22}$$

with a constant $C = C(\lambda)$ independent on η holds, where g_λ is the Yosida approximation of g and $(u^{(0)}, T^{(0)})$ is the solution of (25) - (27) to the data $\hat{b} = b(0), \hat{\gamma} = \gamma(0)$ and $\hat{\varepsilon}_p = Bz^{(0)}$.

Then the solution (u_η, T_η, z_η) of the microscopic problem (14) - (18) with parameter y satisfies for all $0 \le t \le T_e$

$$\lim_{\eta\to 0}(\|u_0(t) - u_\eta(t)\|_{\Omega\times Y} + \|\hat{T}_\eta(t) - T_\eta(t)\|_{\Omega\times Y} + \|\hat{z}_\eta(t) - z_\eta(t)\|_{\Omega\times Y}) = 0. \tag{23}$$

Remark 1.2 For the future use we need one estimate obtained in Theorem 2 [4] for the time derivative of z_0. Define a function $h = -(B^T\mathcal{D}QB + L)z_0 + B^T\sigma_0$, where the operator Q is a projector in $L^2(\Omega \times Y, \mathcal{S}^3)$, and the function σ_0 solves an appropriate linear elasticity problem. Then the function h satisfies the inequality

$$||\frac{\partial}{\partial t}h(t)||_{\Omega\times Y} \le |Ch(0)| + ||B^T\sigma_{0,t}(0)||_{\Omega\times Y} + \int_0^t ||B^T\sigma_{0,tt}(s)||_{\Omega\times Y} ds, \tag{24}$$

with $|C\zeta| = \inf\{\|(B^T\mathcal{D}QB + L)\xi\|_{\Omega\times Y} \mid \xi(x,y) \in g(y,\zeta(x,y)) \text{ a.e.}\}$.

[5]The mapping $z \mapsto j_\lambda(y,z)$ is single valued and well-defined, since $z \mapsto g(y,z)$ is assumed to be maximal monotone.

Notations. Banach spaces $W^{m,p}(\Omega, \mathrm{IR}^{\mathrm{N}})$ are endowed with the norm $\|\cdot\|_{m,p,\Omega}$. $H^m(\Omega, \mathrm{IR}^{\mathrm{N}}) = W^{m,2}(\Omega, \mathrm{IR}^{\mathrm{N}})$ are Hilbert spaces with the usual scalar product on them and the norm $\|\cdot\|_{m,\Omega}$, $\|\cdot\|_{\Omega} = \|\cdot\|_{2,\Omega}$.

Define the space

$$W(Y) = \{v \in H^1(Y) \mid \frac{1}{|Y|}\int_Y v(y)dy = 0\},$$

which becomes a Banach space due to the Poincaré-Wirtinger inequality (Proposition 3.38 [18]) for the norm $\|u\|_{W(Y)} = \|\nabla u\|_Y$.

The symbol $\mathcal{D}_\eta$ denotes the mapping $\mathcal{D}[x/\eta + y]$, i.e. $\mathcal{D}_\eta := \mathcal{D}[x/\eta + y]$.

2 Justification of the homogenized model

2.1 Preliminaries

In this section we deal with a boundary value problem, a linear problem of elasticity theory with a parameter y, formed by the equations:

$$-\mathrm{div}T(x,y) = \hat{b}(x), \tag{25}$$

$$T(x,y) = \mathcal{D}[\frac{x}{\eta} + y](\varepsilon(\nabla_x u(x,y)) - \hat{\varepsilon}_p(x,y)), \tag{26}$$

$$u(x,y) = \hat{\gamma}(x), \qquad x \in \partial\Omega. \tag{27}$$

The solution of this problem is understood in the following sense: a function $(u,T) \in L^2(Y, H^1(\Omega, \mathrm{IR}^3)) \times L^2(\Omega \times Y, \mathcal{S}^3)$ is a solution of (25) - (27), if (26) is satisfied, and if for $\hat{b} \in L^2(\Omega, \mathrm{IR}^3)$, $\hat{\gamma} \in H^1(\Omega, \mathrm{IR}^3)$, $\hat{\varepsilon_p} \in L^2(\Omega \times Y, \mathcal{S}^3)$ and for a.e. $y \in Y$ the following identity

$$(\mathcal{D}[\frac{\cdot}{\eta} + y](\varepsilon(\nabla_x u(\cdot, y)) - \hat{\varepsilon}_p(\cdot, y)), \varepsilon(\nabla_x v(\cdot)))_\Omega = (\hat{b}, v)_\Omega. \tag{28}$$

holds for all $v \in H_0^1(\Omega, \mathrm{IR}^3)$, and if u can be represented in the form $u = \gamma + w$ with $w \in L^2(Y, H_0^1(\Omega, \mathrm{IR}^3))$.

By the well known theory for elliptic boundary value problems one gets immediately that to $\hat{b} \in L^2(\Omega, \mathrm{IR}^3)$, $\hat{\gamma} \in H^1(\Omega, \mathrm{IR}^3)$, $\hat{\varepsilon_p} \in L^2(\Omega \times Y, \mathcal{S}^3)$ and for a fixed $\eta > 0$ there is a unique weak solution (u, T) satisfying

$$\|u\|_{L^2(Y, H^1(\Omega, \mathrm{IR}^3))} + \|T\|_{\Omega \times Y} \le C(\|\hat{b}\|_\Omega + \|\hat{\varepsilon}_p\|_{\Omega \times Y} + \|\hat{\gamma}\|_{1,\Omega}), \tag{29}$$

with a constant C independent of η.

In the following we need a special projection operator.

Definition 2.1 *For every $\hat{\varepsilon}_p \in L^2(\Omega \times Y, \mathcal{S}^3)$ a linear operator $P : L^2(\Omega \times Y, \mathcal{S}^3) \mapsto L^2(\Omega \times Y, \mathcal{S}^3)$ can be defined by*

$$P_\eta \hat{\varepsilon}_p = \varepsilon(\nabla_x u),$$

where (u, T) is a unique solution of (25) - (27) to $\hat{b} = \hat{\gamma} = 0$. Furthermore we define the linear operator $Q_\eta = I - P_\eta$. Here I is identity operator.

It is immediate seen from the estimate (29) that the thus defined operator P_η is uniformly bounded. Other properties of P_η and Q_η are delivered by

Lemma 2.1 ***(i)*** *The operators P_η and Q_η are orthogonal projectors with respect to the scalar product $[\xi, \zeta]_{\Omega\times Y}$ on $L^2(\Omega \times Y, \mathcal{S}^3)$*
(ii) *The operator $B^T\mathcal{D}_\eta Q_\eta B : L^2(\Omega\times Y, \mathrm{I\!R}^{\mathrm{N}}) \mapsto L^2(\Omega\times Y, \mathrm{I\!R}^{\mathrm{N}})$ is selfajoint and non-negative with respect to the scalar product $(\xi, \zeta)_{\Omega\times Y}$. Moreover, there exists a constant $C > 0$ such that*

$$\|B^T\mathcal{D}_\eta Q_\eta B\xi\|_{\Omega\times Y} \leq C\|\xi\|_{\Omega\times Y}, \tag{30}$$

for all $\eta > 0$.

Proof of Lemma 2.1. Lemma 2.5 in [3]. ■

Since L is positive definite, it follows from Lemma 2.1 that the operator $L+B^T\mathcal{D}_\eta Q_\eta B$ is uniformly positive definite and bounded. This implies that

$$\langle\xi, \zeta\rangle_{\Omega\times Y,\eta} = ((L + B^T\mathcal{D}_\eta Q_\eta B)\xi, \zeta)_{\Omega\times Y}$$

defines a scalar product on $L^2(\Omega \times Y, \mathrm{I\!R}^{\mathrm{N}})$. Furthermore, the associated norm $||\xi||_{\Omega\times Y,\eta} = \langle\xi, \xi\rangle^{1/2}_{\Omega\times Y,\eta}$ is equivalent to the norm $||\cdot||_{\Omega\times Y}$.

2.2 Reduction to an evolution equation

The preparations made in the previous section enable us to reduce the initial-boundary value problem (14)-(18) to an evolution equation with a monotone operator.

Note that (15) yields

$$B^T T_\eta - Lz_\eta = B^T\mathcal{D}_\eta(\varepsilon(\nabla_x u_\eta) - Bz_\eta) - Lz_\eta. \tag{31}$$

Let (u_η, T_η, z_η) be a solution of the initial-boundary value problem (14)-(18). Now we fix t. If $z(t)$ is known, then (14), (15), (18) is a boundary value problem for the components $u_\eta(t), T_\eta(t)$ of the solution, the problem from linear elasticity theory with a parameter y. Due to linearity of such problems these functions are obtained in the form

$$(u_\eta(t), T_\eta(t)) = (\tilde{u}_\eta(t), \tilde{T}_\eta(t)) + (v_\eta(t), \sigma_\eta(t)),$$

with a solution $(v_\eta(t), \sigma_\eta(t))$ of the Dirichlet boundary value problem (25) - (27) to the data $\hat{b} = b(t)$, $\hat{\gamma} = \gamma(t)$, $\hat{\varepsilon}_p = 0$, and with a solution $(\tilde{u}_\eta(t), \tilde{T}_\eta(t))$

of the problem (25) - (27) to the data $\hat{b} = \hat{\gamma} = 0$, $\hat{\varepsilon}_p = Bz_\eta(t)$. Thus one obtains

$$\varepsilon((\nabla_x u_\eta)(t)) - Bz_\eta(t) = (P_\eta - I)Bz_\eta(t) + \varepsilon((\nabla_x v_\eta)(t)).$$

We insert this equation into (31) and obtain that (16) can be rewritten in the following form

$$\frac{\partial}{\partial t} z_\eta(t) \quad \in \quad G_\eta(-(B^T \mathcal{D}_\eta Q_\eta B + L)z_\eta(t) + B^T \sigma_\eta(t)), \tag{32}$$

with the mapping $G_\eta : L^2(\Omega \times Y, \mathrm{I\!R}^{\mathrm{N}}) \mapsto 2^{L^2(\Omega \times Y, \mathrm{I\!R}^{\mathrm{N}})}$ defined by

$$G_\eta(\xi) = \{\zeta \in L^2(\Omega \times Y, \mathrm{I\!R}^{\mathrm{N}}) |\ \zeta(x,y) \in g(x/\eta + y, \xi(x,y))\ a.e.\}.$$

The function σ_η can be determined from the boundary value problem (25) - (27) to the given data b, γ, $\hat{\varepsilon}_p = 0$ and can be considered as known.

The evolution equation (32) for z_η can be rewritten as a non-autonomous evolution equation in $L^2(\Omega \times Y, \mathrm{I\!R}^{\mathrm{N}})$

$$\frac{\partial}{\partial t} z_\eta(t) + A_\eta(t) z_\eta(t) \ni 0, \tag{33}$$

with the operator

$$A_\eta(t)z(t) = -G_\eta(-(B^T \mathcal{D}_\eta Q_\eta B + L)z_\eta(t) + B^T \sigma_\eta(t)).$$

It turns out that the operator $A_\eta(t)$ is maximal monotone as the next lemma shows.

Lemma 2.2 *Operator $A_\eta(t)$ is maximal monotone on $L^2(\Omega \times Y, \mathrm{I\!R}^{\mathrm{N}})$ with respect to the scalar product $\langle \xi, \zeta \rangle_{\Omega \times Y, \eta}$.*

Proof. Set for simplicity $M_\eta = B^T \mathcal{D}_\eta Q_\eta B + L$.

Monotonicity of $A_\eta(t)$ for all t and η with respect to the scalar product $\langle \xi, \zeta \rangle_{\Omega \times Y, \eta}$ is shown in Lemma [3].

Now we prove that the mapping $G_\eta : L^2(\Omega \times Y, \mathrm{I\!R}^{\mathrm{N}}) \mapsto 2^{L^2(\Omega \times Y, \mathrm{I\!R}^{\mathrm{N}})}$, defined through the maximal monotone function $g : Y \times \mathrm{I\!R}^{\mathrm{N}} \mapsto 2^{\mathrm{I\!R}^{\mathrm{N}}}$ with $g(y, 0) \ni 0$, is maximal monotone with respect to the scalar product $(\xi, \zeta)_{\Omega \times Y}$.

It is well known that G_η is maximal monotone if and only if $I + G_\eta$ is surjective. To show the surjectivity, we must prove that to every $q \in L^2(\Omega \times Y, \mathrm{I\!R}^{\mathrm{N}})$ the equation

$$q \in z + G_\eta z \tag{34}$$

has a solution $z \in L^2(\Omega \times Y, \mathrm{I\!R}^{\mathrm{N}})$. Since g is maximal monotone, for a.e. (x, y) the mapping $(I + g(x/\eta + y, \cdot)) : \mathrm{I\!R}^{\mathrm{N}} \mapsto 2^{\mathrm{I\!R}^{\mathrm{N}}}$ has an inverse $j(x/\eta + y, \cdot) : \mathrm{I\!R}^{\mathrm{N}} \mapsto \mathrm{I\!R}^{\mathrm{N}}$, which satisfies for a.e. (x, y) the inequality $|j(x/\eta + y, \xi) -$

$j(x/\eta + y, \zeta)| \le |\xi - \zeta|$ for all $\xi, \zeta \in \mathbb{R}^N$. This Lipschitz continuity together with the measurability of j with respect to the first argument yield that the function $j(x/\eta + y, q)$ is of Caratheodory type. Thus one can prove that the mapping $(x, y) \mapsto j(x/\eta + y, q(x, y))$ is measurable. From $g(\cdot, 0) \ni 0$ it follows that $j(x/\eta + y, 0) = 0$, whence

$$|j(x/\eta + y, \xi)| = |j(x/\eta + y, \xi) - j(x/\eta + y, 0)| \le |\xi|. \tag{35}$$

For $q \in L^2(\Omega \times Y, \mathbb{R}^N)$ we define $z(x, y) = j(x/\eta + y, q(x, y))$ for all $(x, y) \in \Omega \times Y$. Obviously, such defined z solves (34) if $z \in L^2(\Omega \times Y, \mathbb{R}^N)$. Yet, (35) yields that indeed $z \in L^2(\Omega \times Y, \mathbb{R}^N)$ and therefore we conclude that $I + G_\eta$ is surjective. Hence G_η is maximal monotone.

With this result it is easy to prove that $A_\eta(t)$ is maximal monotone for all t. In other words we have to show that for all $[z, \xi] \in L^2(\Omega \times Y, \mathbb{R}^N) \times L^2(\Omega \times Y, \mathbb{R}^N)$ and all $[\hat{z}, \hat{\xi}] \in GrA_\eta(t)$ such that

$$\langle z - \hat{z}, \xi - \hat{\xi} \rangle_{\Omega \times Y, \eta} \ge 0,$$

it follows that $[z, \xi] \in GrA_\eta(t)$.

Indeed,

$$\langle z - \hat{z}, \xi - \hat{\xi} \rangle_{\Omega \times Y, \eta} = ((-M_\eta z + B^T\sigma) - (-M_\eta \hat{z} + B^T\sigma), (-\xi) - (-\hat{\xi}))_{\Omega \times Y} \ge 0.$$

Since G_η is maximal monotone $[-M_\eta z + B^T\sigma, -\xi] \in GrG_\eta$, which means that $[z, \xi] \in GrA_\eta(t)$. ∎

2.3 Start of the justification

Now we can prove the main result of this work, Theorem 1.1.

Proof of Theorem 1.1. The approximate solution $(u_0, \hat{T}_\eta, \hat{z}_\eta)(x, y, t)$, determined from the homogenized problem, solves the same initial-boundary value problem as the exact solution, however with a special right hand side. Observing (7) - (13) we get by a simple computation that $(u_0, \hat{T}_\eta, \hat{z}_\eta)$ satisfies the equations

$$\begin{aligned}
-\mathrm{div}_x \hat{T}_\eta &= -\mathrm{div}_x T_0(x, \xi, t)|_{\xi = \frac{x}{\eta} + y} &(36)\\
\hat{T}_\eta &= \mathcal{D}[\frac{x}{\eta} + y](\varepsilon(\nabla_x u_0) - B\hat{z}_\eta + \varepsilon(\nabla_\xi u_1(x, \xi, t))|_{\xi = \frac{x}{\eta} + y}) &(37)\\
\frac{\partial}{\partial t}\hat{z}_\eta &\in g(\frac{x}{\eta} + y, B^T\hat{T}_\eta - L\hat{z}_\eta) &(38)\\
z_\eta(x, y, 0) &= z_0^{(0)}(x, \frac{x}{\eta} + y), \qquad (x, y) \in \Omega \times Y &(39)\\
u_0(x, t) &= \gamma(x, t) \qquad (x, t) \in \partial\Omega \times [0, \infty). &(40)
\end{aligned}$$

Since these equations have the same structure as the equations of the microscopic problem with a parameter y, we can again employ the procedure from the last section and obtain that if $(\hat{v}_\eta(t), \hat{\sigma}_\eta(t))$ is the solution of the linear boundary value problem (25) - (27) to the data

$$\hat{b}(x) = -\mathrm{div}_x T_0(x,\xi,t)|_{\xi=x/\eta+y} \quad (41)$$
$$\hat{\varepsilon}_p(x) = -\varepsilon(\nabla_\xi u_1(x,\xi,t))|_{\xi=x/\eta+y} \quad (42)$$
$$\hat{\gamma}(x) = \gamma(x,t), \quad (43)$$

then the special microscopic problem is equivalent to a non-autonomous evolution equation

$$\frac{\partial}{\partial t}\hat{z}_\eta(t) + \hat{A}_\eta(t)\hat{z}_\eta(t) \ni 0, \quad (44)$$

where

$$\hat{A}_\eta(t)v = -g(\frac{x}{\eta} + y, -(B^T\mathcal{D}_\eta Q_\eta B + L)v + B^T\hat{\sigma}_\eta).$$

It turns out that the operator $\hat{A}_\eta(t)$ is maximal monotone, namely

Lemma 2.3 *The operator $\hat{A}_\eta(t)$ is maximal monotone on $L^2(\Omega \times Y, \mathrm{I\!R}^{\mathrm{N}})$ with respect to the scalar product $\langle\xi,\xi\rangle_{\Omega\times Y,\eta}$.*

Proof of Lemma 2.3. The proof is the same as for Lemma 2.2 for $A_\eta(t)$. ∎

We are going to use the results of Lemma 2.2, Lemma 2.3 and a special distance between two maximal monotone operators to estimate the difference of solutions of the evolution inclusions (33), (44) by the norm of a function, which solves a linear elasticity problem to special data. This is a crucial step in the justification of the homogenized model because we are able to employ classical results from homogenization theory for linear problems. In the next section this estimate is given.

2.4 Main estimate

We define a special distance between two maximal monotone operators due to [36]. With its help we obtain an important estimate in the justification procedure of the homogenized model.

Let H be a Hilbert space.

Definition 2.2 *The distance between two maximal monotone operators on H is defined as*

$$\mathrm{dis}(A_1, A_2) = \sup\{\frac{(y_1 - y_2, x_2 - x_1)}{\|y_1\| + \|y_2\| + 1} \mid x_i \in D(A_i),\ y_i \in A_i x_i,\ i = 1,2\}$$

with the value possibly equal to $+\infty$.

The distance dis is not a metric because in a general case the triangle inequality is not fulfilled.

Concerning properties and application to the study of evolution equations in a Hilbert space the reader is referred to the original work [36].

The following lemma plays an important role in the proof of the convergence result, since it reduces the convergence problem for the nonlinear evolution equation to a convergence problem for the linear system of elasticity.

Lemma 2.4 *For the functions $\hat{z}_\eta(t)$ and $z_\eta(t)$ the following estimate with a constant C independent of η holds*

$$\|\hat{z}_\eta(t) - z_\eta(t)\|^2_{\Omega\times Y} \leq C \int_0^t \|\hat{\sigma}_\eta(s) - \sigma_\eta(s)\|_{\Omega\times Y} ds. \tag{45}$$

Proof. We use the operator distance introduced in Definition 2.2:

$$\operatorname{dis}(\hat{A}_\eta(t), A_\eta(t)) = \sup_{\substack{z_1\in D(\hat{A}_\eta),\ z_2\in D(A_\eta)\\ y_1\in\hat{A}_\eta z_1,\ y_2\in A_\eta z_2}} \frac{\langle y_1 - y_2, z_2 - z_1\rangle_{\Omega\times Y,\eta}}{1 + \|y_1\|_{\Omega\times Y,\eta} + \|y_2\|_{\Omega\times Y,\eta}},$$

which is well-defined, since $\hat{A}_\eta$ and A_η are maximal monotone operators.

We get

$$\langle y_1 - y_2, z_2 - z_1\rangle_{\Omega\times Y,\eta} = (M_\eta(y_1 - y_2), z_2 - z_1)_{\Omega\times Y} =$$

$$-(-y_2 + y_1, (-M_\eta z_2 + B^T\sigma_\eta(t)) - (-M_\eta z_1 + B^T\hat{\sigma}_\eta(t)))_{\Omega\times Y}$$

$$-(y_2 - y_1, B^T\sigma_\eta(t) - B^T\hat{\sigma}_\eta(t))_{\Omega\times Y} \leq$$

$$-(y_2 - y_1, B^T(\sigma_\eta(t) - \hat{\sigma}_\eta(t)))_{\Omega\times Y},$$

since the inclusion $-y_2+y_1 \in G_\eta(-M_\eta z_2+B^T\sigma_\eta(t))-G_\eta(-M_\eta z_1+B^T\hat{\sigma}_\eta(t))$ holds and the operator G_η is monotone.

Then we have

$$\operatorname{dis}(\hat{A}_\eta(t), A_\eta(t)) \leq \sup_{\substack{z_1\in D(\hat{A}_\eta),\ z_2\in D(A_\eta)\\ y_1\in\hat{A}_\eta z_1,\ y_2\in A_\eta z_2}} \frac{|(y_2 - y_1, B^T(\sigma_\eta(t) - \hat{\sigma}_\eta(t))_{\Omega\times Y}|}{1 + \|y_1\|_{\Omega\times Y,\eta} + \|y_2\|_{\Omega\times Y,\eta}}$$

$$\leq \sup_{\substack{z_1\in D(\hat{A}_\eta),\ z_2\in D(A_\eta)\\ y_1\in\hat{A}_\eta z_1,\ y_2\in A_\eta z_2}} \frac{(\|y_2\|_{\Omega\times Y,\eta} + \|y_1\|_{\Omega\times Y,\eta})\|B^T(\sigma_\eta(t) - \hat{\sigma}_\eta(t))\|_{\Omega\times Y}}{1 + \|y_1\|_{\Omega\times Y,\eta} + \|y_2\|_{\Omega\times Y,\eta}}$$

$$\leq C_1\|B^T(\sigma_\eta(t) - \hat{\sigma}_\eta(t))\|_{\Omega\times Y} \leq C_2\|\sigma_\eta(t) - \hat{\sigma}_\eta(t)\|_{\Omega\times Y}.$$

Now we use the last inequality to obtain the main estimate. If $\hat{z}_\eta(t)$ and $z_\eta(t)$ are, respectively, absolutely continuous solutions of the monotone evolution equations (33) and (44) with the same initial conditions, then one easily gets

$$\frac{d}{dt}\|\hat{z}_\eta(t)-z_\eta(t)\|^2_{\Omega\times Y,\eta} = 2\langle \hat{z}_{\eta;t}(t)-z_{\eta;t}(t), \hat{z}_\eta(t)-z_\eta(t)\rangle_{\Omega\times Y,\eta}$$

$$= 2\frac{\langle \hat{z}_{\eta;t}(t)-z_{\eta;t}(t), \hat{z}_\eta(t)-z_\eta(t)\rangle_{\Omega\times Y,\eta}}{1+\|\hat{z}_{\eta;t}(t)\|_{\Omega\times Y,\eta}+\|z_{\eta;t}(t)\|_{\Omega\times Y,\eta}}(1+\|\hat{z}_{\eta;t}(t)\|_{\Omega\times Y,\eta}+\|z_{\eta;t}(t)\|_{\Omega\times Y,\eta})$$

$$\leq \mathrm{dis}(\hat{A}_\eta(t), A_\eta(t))(1+\|\hat{z}_{\eta;t}(t)\|_{\Omega\times Y,\eta}+\|z_{\eta;t}(t)\|_{\Omega\times Y,\eta})$$

$$\leq C_2\|\bar{\sigma}_\eta(t)\|_{\Omega\times Y}(1+\|\hat{z}_{\eta;t}(t)\|_{\Omega\times Y}+\|z_{\eta;t}(t)\|_{\Omega\times Y}),$$

since $-\hat{z}_{\eta;t}(t)\in \hat{A}_\eta\hat{z}_\eta(t)$, $-z_{\eta;t}(t)\in A_\eta z_\eta(t)$ a. e.. Here $\bar{\sigma}_\eta(t)=\sigma_\eta(t)-\hat{\sigma}_\eta(t)$.

As a result of all calculations:

$$\|\hat{z}_\eta(T_e)-z_\eta(T_e)\|^2_{\Omega\times Y}\leq 2C_2\int_0^{T_e}\|\bar{\sigma}_\eta(t)\|_{\Omega\times Y}(1+\|\hat{z}_{\eta;t}(t)\|_{\Omega\times Y}+\|z_{\eta;t}(t)\|_{\Omega\times Y})dt.$$

We have to show that $\|\hat{z}_{\eta;t}(t)\|_{\Omega\times Y}$ and $\|z_{\eta;t}(t)\|_{\Omega\times Y}$ are uniformly bounded with respect to η.

We can transform the equation (33) to an autonomous equation by inserting

$$h_\eta = -(B^T\mathcal{D}_\eta Q_\eta B+L)z_\eta + B^T\sigma_\eta$$

into (32). This autonomous equation is

$$\frac{d}{dt}h_\eta(t)+C_\eta h(t)\ni B^T\sigma_{\eta,t}(t)$$

with the operator $C_\eta : L^2(\Omega\times Y,\mathrm{I\!R}^N) \to 2^{L^2(\Omega\times Y,\mathrm{I\!R}^N)}$ defined by $C_\eta = (B^T\mathcal{D}_\eta Q_\eta B+L)G_\eta$.

The estimate (81) then implies

$$||\frac{\partial}{\partial t}h_\eta(t)||_{\Omega\times Y}\leq ||G_\lambda h(0)||_{\Omega\times Y}+||B^T\sigma_{\eta,t}(0)||_{\Omega\times Y}+\int_0^t||B^T\sigma_{\eta,tt}(s)||_{\Omega\times Y}ds.$$

From the estimate (29) with $\hat{\varepsilon}_p=0$ we conclude that the $L^2(\Omega\times Y)$-norm of σ_η is uniformly bounded with respect to η. By virtue of the assumptions made on b,γ, we can differentiate the equations (25) - (27) with respect to t and apply the existence theory for elliptic problems to the obtained system. It results in the inequality

$$\|v_{\eta,t}\|_{L^2(Y,H^1(\Omega,\mathrm{I\!R}^3))}+\|\sigma_{\eta,t}\|_{\Omega\times Y}\leq C(\|\hat{b}_t\|_\Omega+\|\hat{\gamma}_t\|_{1,\Omega}), \tag{46}$$

with a constant C independent of η. (46) yields that the $L^2(\Omega \times Y)$-norm of $\sigma_{\eta,t}$ is uniformly bounded with respect to η. Similarly we conclude the same result for $\sigma_{\eta,tt}$. From (22) and the inequality

$$\|B^T T^{(0)} - Lz^{(0)}\|_{\Omega\times Y} \le C\|z^{(0)}\|_{\Omega\times Y} \le C_1,$$

which holds with the constants C and C_1 independent of η, it follows that $||G_\lambda h(0)||_{\Omega\times Y}$ is also uniformly bounded with respect to η (G_λ is Yosida approximation of G). These imply that the function $z_{\eta,t}$ is uniformly bounded[6].

We notice that $\|\hat{z}_{\eta,t}\|_{\Omega\times Y} = \|z_{0,t}\|_{\Omega\times Y}$, where $z_0(t)$ is a solution of the homogenized problem. Thus the required result is obtained from the estimate (24).

This completes the proof of Lemma. ■

Remark 2.1 Using a special distance between two maximal monotone operators the difference of solutions of the evolution inclusions can be estimated by the norm a function which solves a linear elasticity problem to a special data. It is a crucial step in the justification of the homogenized model. Instead of treating to evolution equations (inclusions) with in general nonlinear multivalued operators the problem is reduced to a linear elasticity case. It enables us with possibility to use standard methods which perfectly work for linear problems.

2.5 End of the justification

Thus, we can estimate $\hat{z}_\eta - z_\eta$ by the difference $\sigma_\eta - \hat{\sigma}_\eta$. In the next section we present an estimate for the function $\sigma_\eta - \hat{\sigma}_\eta$. Both, σ_η and $\hat{\sigma}_\eta$ solve the same boundary value problem, the problem of linear elasticity theory, but to different data. Here we only state the estimate and refer to the next section for the proof.

Lemma 2.5 *Let $(v_\eta(t), \sigma_\eta(t))$ be a solution of the boundary value problem (25) - (27) to the data*

$$\hat{b} = b(t), \qquad \hat{\varepsilon}_p = 0, \qquad \hat{\gamma} = \gamma(t),$$

and let $(\hat{v}_\eta(t), \hat{\sigma}_\eta(t))$ be a solution of the problem (25) - (27) to the data

$$\hat{b} = -\mathrm{div}_x T_0(x, \frac{x}{\eta} + y, t), \qquad \hat{\varepsilon}_p = -\varepsilon(\nabla_y u_1(x, \frac{x}{\eta} + y, t)), \qquad \hat{\gamma} = \gamma(t).$$

Then for all $t \in [0, T_e]$

$$\|v_\eta(t) - \hat{v}_\eta(t)\|_{\Omega\times Y} + \|\sigma_\eta(t) - \hat{\sigma}_\eta(t)\|_{\Omega\times Y} \to 0, \qquad \text{as } \eta \to 0. \qquad (47)$$

[6]Remember also that the operator $(B^T \mathcal{D}_\eta Q_\eta B + L)$ is uniformly bounded and invertible.

Moreover, there exists a constant C independent of η such that for all $t \in [0, T_e]$ and all $\eta > 0$

$$\|\sigma_\eta(t) - \hat{\sigma}_\eta(t)\|_{\Omega\times Y} \leq C. \tag{48}$$

With this lemma the proof of Theorem 1.1 can be finished. Lemma 2.5 and the inequality (45) yield together with Lebesgue's convergence theorem that for all $t \in [0, T_e]$

$$\lim_{\eta\to 0} \|\hat{z}_\eta(t) - z_\eta(t)\|_{\Omega\times Y} = 0. \tag{49}$$

We observe also that the equations (14), (15), (18) form the boundary value problem for (u_η, T_η) and the equations (36), (37), (40) form a boundary value problem for $(u_0, \hat{T}_\eta)$. The Definition 2.1 of P_η and the definitions of $(\hat{v}_\eta(t), \hat{\sigma}_\eta(t))$ and $(v_\eta(t), \sigma_\eta(t))$ thus yield the decomposition

$$u_\eta = w_\eta + v_\eta, \qquad T_\eta = \mathcal{D}[\frac{\cdot}{\eta} + y](P_\eta - I)Bz_\eta + \sigma_\eta,$$

$$u_0 = \hat{w}_\eta + \hat{v}_\eta, \qquad \hat{T}_\eta = \mathcal{D}[\frac{\cdot}{\eta} + y](P_\eta - I)B\hat{z}_\eta + \hat{\sigma}_\eta,$$

where $w_\eta(t), \hat{w}_\eta(t) \in L^2(Y, H_0^1(\Omega, \mathbb{R}^3))$ are the unique functions from Definition 2.1 which satisfy $\varepsilon(\nabla_x w_\eta(t)) = P_\eta B z_\eta(t)$ and $\varepsilon(\nabla_x \hat{w}_\eta(t)) = P_\eta B\hat{z}_\eta(t)$. We thus have

$$\varepsilon(\nabla_x(w_\eta - \hat{w}_\eta)) = P_\eta B(z_\eta - \hat{z}_\eta), \tag{50}$$

$$T_\eta - \hat{T}_\eta = -\mathcal{D}[\frac{\cdot}{\eta} + y]Q_\eta B(z_\eta - \hat{z}_\eta) + (\sigma_\eta - \hat{\sigma}_\eta), \tag{51}$$

$$u_0 - u_\eta = (w_\eta - \hat{w}_\eta) + (v_\eta - \hat{v}_\eta). \tag{52}$$

From Lemma 2.5, (49), (51) and the uniform boundedness of $\mathcal{D}_\eta Q_\eta B$ we infer that

$$\lim_{\eta\to 0} \|\hat{T}_\eta(t) - T_\eta(t)\|_{\Omega\times Y} = 0.$$

Since for a.e. $y \in Y$ the function $(w_\eta - \hat{w}_\eta)$ belongs to $H_0^1(\Omega, \mathbb{R}^3)$, we conclude from the first Korn's inequality for a. e. y $\|(w_\eta - \hat{w}_\eta)(t, y)\|_{1,\Omega} \leq C\|\varepsilon(\nabla_x(w_\eta - \hat{w}_\eta)(t, y))\|_\Omega$ and from (49), (50) that $\|w_\eta(t) - \hat{w}_\eta(t)\|_{\Omega\times Y} \to 0$ as $\eta \to 0$; from Lemma 2.5 and (52) we thus conclude

$$\lim_{\eta\to 0} \|u_0(t) - u_\eta(t)\|_{\Omega\times Y} = 0.$$

for all $t \in [0, T_e]$. These two relations and (49) together yield

$$\lim_{\eta\to 0}(\|u_0(t) - u_\eta(t)\|_{\Omega\times Y} + \|\hat{T}_\eta(t) - T_\eta(t)\|_{\Omega\times Y} + \|\hat{z}_\eta(t) - z_\eta(t)\|_{\Omega\times Y}) = 0$$

. This completes the proof of Theorem 1.1. ■

To finish the proof of Theorem 1.1 it thus remains to verify Lemma 2.5. The next section is devoted to the proof of this lemma.

3 Convergence results based on the two-scale convergence method

In this section we present the convergence result stated in Lemma 2.5. To do this an auxiliary function is taken into consideration. We define it to be a solution of a Dirichlet boundary value problem of linear elasticity theory to special data. This data is chosen to be smooth enough such that the two-scale convergence method can be applied. The direct application of the method seems not possible to me because of the low regularity of the functions $T_0(x,y,t)$ and $u_1(x,y,t)$, the solutions of the homogenized problem, which are now considered as the data to an elasticity problem.

Now we give the definition of the so-called two-scale converged sequence. We assume that $|Y| = 1$.

Definition 3.1 *A sequence of functions u_η in $L^2(\Omega, \mathbb{R}^3)$ is said to two-scale converge to a limit $u_0(x,y)$ belonging to $L^2(\Omega \times Y, \mathbb{R}^3)$ if, for any test function $\psi(x,y)$ in $L^2(\Omega, C(Y, \mathbb{R}^3))$, one has*

$$\lim_{\eta \to 0} \int_\Omega u_\eta(x)\psi(x, \frac{x}{\eta})dx = \int_\Omega \int_Y u_0(x,y)\psi(x,y)dxdy. \tag{53}$$

Remark 3.1 This definition makes sense for every bounded sequence u_η in $L^2(\Omega, \mathbb{R}^3)$. As it is shown in Theorem 1.2 [6] for such a sequence u_η there exists a limit $u_0 \in L^2(\Omega \times Y, \mathbb{R}^3)$ such that, with possible expanse of extracting a subsequence, $u_\eta(x)$ two-scale converges to $u_0(x,y)$.

Remark 3.2 The two-scale convergence method [28], [6], which is an alternative to the compensated compactness method [27], applied for partial differential equations with periodically oscillating coefficients. We only briefly recall the definition of a two-scale converging sequence and the main properties, which we use in this work. For a deeper understanding of the subject the reader is referred to the work [6].

Let the boundary value problem be given:

$$-\mathrm{div}_x T(x,y) = \hat{b}(x), \tag{54}$$

$$T(x,y) = \mathcal{D}[\frac{x}{\eta} + y](\varepsilon(\nabla_x u(x,y)) - \hat{\varepsilon}_p(x,y)), \tag{55}$$

$$u(x,y) = \hat{\gamma}(x), \qquad x \in \partial\Omega, \tag{56}$$

with given functions $\hat{b} : \Omega \mapsto \mathbb{R}^3$, $\hat{\gamma} : \partial\Omega \mapsto \mathbb{R}^3$, $\hat{\varepsilon}_p : \Omega \mapsto \mathcal{S}^3$, a given number $\eta > 0$ and fixed $y \in Y$. This is the linear problem of elasticity with a parameter y.

We recall that for a.e. $y \in Y$ the function $(v_\eta(t), \sigma_\eta(t))$ solves the boundary value problem (54) - (56) to the data

$$\hat{b} = b(t), \qquad \hat{\varepsilon}_p = 0, \qquad \hat{\gamma} = \gamma(t),$$

and $(\hat{v}_\eta(t), \hat{\sigma}_\eta(t))$ solves the problem (54) - (56) to

$$\hat{b} = -\mathrm{div}_x T_0(x, \frac{x}{\eta} + y, t), \qquad \hat{\varepsilon}_p = -\varepsilon(\nabla_y u_1(x, \frac{x}{\eta} + y, t)), \qquad \hat{\gamma} = \gamma(t).$$

My goal now is to show that for almost all $y \in Y$ and all $t \in [0, T_e]$

$$\lim_{\eta \to 0} \|v_\eta(\cdot, y, t) - \hat{v}_\eta(\cdot, y, t)\|_\Omega = 0,$$
$$\lim_{\eta \to 0} \|\sigma_\eta(\cdot, y, t) - \hat{\sigma}_\eta(\cdot, y, t)\|_\Omega = 0.$$

These relations follow from two auxiliary lemmas proved in the next section.

3.1 Two auxiliary lemmas

Lemma 3.1 *Let the function $\tau \in L^2(\Omega \times Y, \mathcal{S}^3)$ have the property $\mathrm{div}_y \tau(x, y) = 0$, and let the family $\{\tau_{\eta,n}(x) = \tau_n(x, x/\eta)\}_{\eta,n}$ with $\tau_n \in L^2(\Omega, C(Y, \mathcal{S}^3))$ be such that the sequence $\tau_n(x, y)$ converges strongly to $\tau(x, y)$ in $L^2(\Omega \times Y, \mathcal{S}^3)$. Denote*

$$\tau_{n,\infty}(x) := \frac{1}{|Y|} \int_Y \tau_n(x, y) dy \quad \textit{and} \quad \tau_\infty(x) = \frac{1}{|Y|} \int_Y \tau(x, y) dy.$$

Then $\tau_{n,\infty}(x)$ converges strongly to $\tau_\infty(x)$ in $L^2(\Omega, \mathcal{S}^3)$.

Let $(v_{\eta,n}, \sigma_{\eta,n}) \in H^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)$ be the weak solution of the boundary value problem formed by the equations

$$-\mathrm{div}\ \sigma_{\eta,n} = b + \mathrm{div}_x \tau_{\eta,n}, \tag{57}$$
$$\sigma_{\eta,n} = \mathcal{D}[\frac{\cdot}{\eta}]\ \varepsilon(\nabla_x v_{\eta,n}), \tag{58}$$

which must hold in Ω, and by the boundary condition

$$v_{\eta,n}(x) = 0, \qquad x \in \partial\Omega. \tag{59}$$

If additionally the function τ_∞ satisfies

$$-\mathrm{div} \tau_\infty = b$$

for $b \in L^2(\Omega, \mathbb{R}^3)$, then

$$\lim_{n \to \infty} \lim_{\eta \to 0} (\|v_{\eta,n}\|_\Omega + \|\sigma_{\eta,n}\|_\Omega) = 0.$$

Proof of Lemma 3.1. Firstly we observe that the symmetry of the matrices $\sigma_{\eta,n}$, $\tau_{\eta,n}$ and the equations (57) - (59) yield

$$\begin{aligned} c\|\sigma_{\eta,n}\|^2_\Omega &\le \int_\Omega \mathcal{D}^{-1}[\frac{x}{\eta}]\ \sigma_{\eta,n}(x)\cdot\sigma_{\eta,n}(x)dx = \int_\Omega \varepsilon(\nabla_x v_{\eta,n}(x))\cdot\sigma_{\eta,n}(x)dx \\ &= \int_\Omega v_{\eta,n}(x)\cdot b(x) + \mathrm{div}_x\tau_n(x,\frac{x}{\eta})\cdot v_{\eta,n}(x)dx \qquad (60) \\ &= (v_{\eta,n}, b)_\Omega - (\tau_n(\cdot,\frac{\cdot}{\eta}), \nabla_x v_{\eta,n})_\Omega. \end{aligned}$$

Now we notice that for a fixed n the function $\tau_n(x, x/\eta)$ can be considered as a test function in the definition of the two-scale convergence (see Definition 3.1), and by properties of such functions

$$\|\psi(\cdot,\frac{\cdot}{\eta})\|_\Omega \le \|\psi(\cdot,\cdot)\|_{L^2(\Omega,C(Y,\mathbb{R}^3))} \equiv (\int_\Omega \sup_{y\in Y}|\psi(x,y)|^2dx)^{1/2} \qquad (61)$$

we can easily conclude using the standard estimates for elliptic boundary value problems that the sequence $v_{\eta,n}$ is uniformly bounded in $H^1_0(\Omega,\mathbb{R}^3)$ for a fixed n. Then by virtue of the property (i) of Proposition 1.14 in [6] one gets the following result

$$\begin{aligned} &(v_{\eta,n}, b)_\Omega - (\tau_n(\cdot,\frac{\cdot}{\eta}), \nabla_x v_{\eta,n})_\Omega \to (v_{0,n}, b)_\Omega - \int_{\Omega\times Y}\tau_n(x,y)\cdot\nabla_x v_{0,n}(x)dxdy \\ &- \int_{\Omega\times Y}\tau_n(x,y)\cdot\nabla_y w_{1,n}(x,y)dxdy = (v_{0,n}, b)_\Omega - (\tau_{n,\infty}, \nabla_x v_{0,n})_\Omega \qquad (62) \\ &- \int_{\Omega\times Y}\tau_n(x,y)\cdot\nabla_y w_{1,n}(x,y)dydx, \end{aligned}$$

where the function $(v_{0,n}, w_{1,n}) \in H^1_0(\Omega,\mathbb{R}^3)\times L^2(\Omega, W(Y,\mathbb{R}^3))$ solves the problem written in the variational form

$$\begin{aligned} &\int_Y\int_\Omega \mathcal{D}[y]\varepsilon(\nabla v_{0,n}(x) + \nabla_y w_{1,n}(x,y))\varepsilon(\nabla\psi(x) + \nabla_y\psi_1(x,y))dxdy \\ &= (b + \mathrm{div}_x\tau_{n,\infty}, \psi)_\Omega \qquad (63) \end{aligned}$$

with a function $(\psi,\psi_1) \in H^1_0(\Omega,\mathbb{R}^3)\times L^2(\Omega, W(Y,\mathbb{R}^3))$. Here $v_{0,n}(x)$ is a weak limit of $v_{\eta,n}(x)$ in $H^1_0(\Omega,\mathbb{R}^3)$. The equation (63) is obtained in the same way as (91) in the proof of Theorem 4.2.

The existence and uniqueness of the solution for this problem is obtained in Theorem 4.2. If in (63) we choose $\psi = v_{0,n}$ and $\psi_1 = w_{1,n}$ then we obtain easily the following estimate for $(v_{0,n}(x), w_{1,n}(x,y))$ with constants C, C_1 independent of n

$$\|v_{0,n}\|^2_{1,\Omega} + \|w_{1,n}\|^2_{L^2(\Omega,W(Y,\mathbb{R}^3))} \le C(\|b\|_\Omega + \|\tau_{n,\infty}\|_\Omega) \le C_1. \qquad (64)$$

As a consequence of the last estimate one can extract subsequences of $v_{0,n}(x)$ and $w_{1,n}(x,y)$, which converge weakly to $v_{0,\infty}(x)$ in $H_0^1(\Omega, \mathbb{R}^3)$ and to $w_{1,\infty}(x,y)$ in $L^2(\Omega,\ W(Y, \mathbb{R}^3))$, respectively. Taking into account this fact and the properties of τ_∞ and τ we finally obtain after passage to the limit in (62) as $n \to \infty$

$$(v_{0,\infty}, b)_\Omega - (\tau_\infty, \nabla_x v_{0,\infty})_\Omega - (\tau, \nabla_y w_{1,\infty})_{\Omega\times Y}$$

$$= (b + \mathrm{div}_x \tau_\infty, v_{0,\infty})_\Omega + (\mathrm{div}_y \tau, w_{1,\infty})_{\Omega\times Y} = 0.$$

This means that $\|\sigma_{\eta,n}\|_\Omega \to 0$ as $\eta \to 0$ and $n \to \infty$. Together with Korn's inequality it follows from this that

$$\lim_{n\to\infty} \lim_{\eta\to 0} (\|\sigma_{\eta,n}\|_\Omega + \|v_{\eta,n}\|_\Omega) = 0.$$

This ends the proof of Lemma 3.1. ■

Lemma 3.2 *Let $\kappa_\eta(x) := \kappa(x, x/\eta)$ be a sequence of functions in $L^2(\Omega, \mathbb{R}^{3\times 3})$, where $\kappa \in L^2(\Omega, C(Y, \mathbb{R}^{3\times 3}))$ satisfies the relation $\kappa = \nabla_y \vartheta$ with a suitable Y-periodic in y function $\vartheta(x,y)$. Suppose that $(v_\eta, \sigma_\eta) \in H^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)$ be a weak solution of the boundary value problem formed by the equations*

$$-\mathrm{div}\ \sigma_\eta = 0 \tag{65}$$

$$\sigma_\eta = \mathcal{D}[\frac{\cdot}{\eta}]\ (\varepsilon(\nabla_x v_\eta) + \varepsilon(\kappa_\eta)), \tag{66}$$

which must hold in Ω, and by the boundary condition

$$v_\eta(x) = 0, \qquad x \in \partial\Omega. \tag{67}$$

Then

$$\lim_{\eta\to 0} (\|v_\eta\|_\Omega + \|\sigma_\eta\|_\Omega) = 0.$$

Proof of Lemma 3.2. Similarly as in the proof of Lemma 3.1 the symmetry of σ_η and equations (65) - (67) yield:

$$c\|\sigma_\eta\|_\Omega^2 \le \int_\Omega \kappa(x, \frac{x}{\eta}) \cdot \sigma_\eta(x) dx + \int_\Omega \nabla_x v_\eta \cdot \sigma_\eta(x) dx = (\kappa_\eta, \sigma_\eta)_\Omega \tag{68}$$

Notice that due to the regularity assumption for the function $\kappa(x, x/\eta)$ it can be taken as a test function in the sense of the definition of two-scale convergence (see Definition 3.1). Moreover, from the estimate

$$\|\psi(x, \frac{x}{\eta})\|_{L^2(\Omega, \mathbb{R}^{3\times 3})} \le \|\psi(x,y)\|_{L^2(\Omega, C(Y, \mathbb{R}^{3\times 3}))},$$

which holds for every $\psi \in L^2(\Omega, C(Y, \mathbb{R}^{3\times3}))$ and from (68) we obtain

$$c\|\sigma_\eta\|_\Omega^2 \le \|\kappa_\eta\|_{L^2(\Omega,C(Y,\mathbb{R}^{3\times3}))}\ \|\sigma_\eta\|_\Omega,$$

which implies that $\sigma_\eta(x)$ is uniformly bounded in $L^2(\Omega, \mathcal{S}^3)$. Simultaneously one gets a uniform bound for v_η in $H_0^1(\Omega, \mathbb{R}^3)$. The boundness of σ_η allows us to pass to the limit in the inequality (68) as $\eta \to 0$. Using property (iii) of Proposition 1.14 in [6] we obtain

$$\int_\Omega \kappa(x, \frac{x}{\eta}) \cdot \sigma_\eta dx \to \int_{\Omega\times Y} \kappa(x,y)\cdot\sigma_0(x,y)dxdy = -(\vartheta, \mathrm{div}_y\sigma_0)_{\Omega\times Y} = 0,$$

where $\sigma_0(x,y)$ is a two-scale limit of the sequence $\sigma_\eta(x)$. From the last limit relation and the inequality (68) we can conclude that σ_η converges strongly to 0 in $L^2(\Omega, \mathcal{S}^3)$.

Now we observe that the uniform boundness of v_η in $H_0^1(\Omega, \mathbb{R}^3)$ gives us the possibility to extract a subsequence that converges weakly to a function v_0 in $H_0^1(\Omega, \mathbb{R}^3)$, or , by the compactness result, converges strongly to the same function v_0 in $L^2(\Omega, \mathbb{R}^3)$. To finish the proof of Lemma 3.2 we have to show that $v_0(x) = 0$. Notice first that well known properties[7]of functions from $L^2(\Omega, C(Y, \mathbb{R}^{3\times3}))$ yield weak convergence of κ_η to 0 in $L^2(\Omega, \mathbb{R}^{3\times3})$. Indeed, from the weak convergence of $\kappa(\cdot, \cdot/\eta)$ and periodicity of $\vartheta(x,y)$ with respect to the second variable it follows that for $\psi \in L^2(\Omega, \mathbb{R}^{3\times3})$:

$$\int_\Omega \kappa(x, \frac{x}{\eta}) \cdot \psi(x)dx \to \int_\Omega\int_Y \nabla_y\vartheta(x,y)dy\ \psi(x)dx = 0.$$

Taking into account the last result we get from the equality

$$(\varepsilon(\nabla_x v_\eta), \psi)_\Omega + (\kappa_\eta, \psi)_\Omega = \int_\Omega \mathcal{D}^{-1}[\frac{x}{\eta}]\sigma_\eta(x)\cdot\psi(x)dx,$$

where $\psi \in L^2(\Omega, \mathcal{S}^3)$, that $\varepsilon(\nabla_x v_\eta(x))$ weakly converges to 0 in $L^2(\Omega, \mathcal{S}^3)$[8].

Since $\nabla_x v_\eta(x) \rightharpoonup \nabla_x v_0(x)$ weakly in $L^2(\Omega, \mathbb{R}^{3\times3})$ it follows that $\varepsilon(\nabla_x v_0(x)) = 0$. Using that $v_0 \in H_0^1(\Omega, \mathbb{R}^3)$ we conclude from Korn's first inequality that $v_0(x) = 0$.

Therefore

$$\|\sigma_\eta\|_\Omega + \|v_\eta\|_\Omega \to 0 \qquad \text{as} \quad \eta \to 0.$$

This completes the proof of Lemma 3.2. ■

[7]We use the fact (Lemma 9.1, [18]) that for any $\psi \in L^2(\Omega, C(Y, \mathbb{R}^{3\times3}))$ the sequence

$$\psi(\cdot, \frac{\cdot}{\eta}) \rightharpoonup \int_Y \psi(\cdot, y)dy \quad \text{weakly in } L^2(\Omega, \mathbb{R}^{3\times3}).$$

[8]From the strong convergence of σ_η to 0 and the inequality $\|\mathcal{D}^{-1}[\cdot/\eta]\sigma_\eta\|_\Omega \le C\|\sigma_\eta\|_\Omega$ we conclude that also the sequence $\mathcal{D}^{-1}[\cdot/\eta]\sigma_\eta$ converges strongly to 0 in $L^2(\Omega, \mathcal{S}^3)$. The constant C is independent of η.

3.2 Proof of Lemma 2.5

Now we are well prepared to prove Lemma 2.5. The crucial point in the proof of Lemma 2.5 is to introduce an auxiliary function, which solves a linear elasticity problem to smoothed data. This smoothed data satisfies all requirements of Lemma 3.1 and Lemma 3.2. The rest of the proof is a consequence of these two lemmas.

Proof of Lemma 2.5. Let $T_{0,n}(x,y,t)$ be a sequence of smooth functions in $C_0^\infty(\Omega, C(Y, \mathcal{S}^3))$ that converges strongly to $T_0(x,y,t)$ in $L^2(\Omega \times Y, \mathcal{S}^3)$ for all t, let $u_{1,n}(t)$ be another sequence of smooth functions in $C_0^\infty(\Omega, C(Y, \mathbb{R}^3))$ that converges strongly to $u_1(t)$ in $L^2(\Omega, H^1(Y, \mathbb{R}^3))$ for all t. We notice that an approximation sequence for $T_\infty(t)$ in the strong topology of $L^2(\Omega, \mathcal{S}^3)$ necessary has to be of the form $T_{\infty,n}(x,t) = \frac{1}{|Y|}\int_Y T_{0,n}(x,y,t)dy$.

Now we fix t and introduce an auxiliary function[9] $(v_{\eta,n}, \sigma_{\eta,n})$. We define it as a unique solution of an elasticity problem to the data determined by the smooth functions $T_{0,n}(x,y,t)$ and $u_{1,n}(x,y,t)$:

$$-\mathrm{div}\sigma_{\eta,n}(x) = -\mathrm{div}_x T_{0,n}(x, \frac{x}{\eta} + y, t), \tag{69}$$

$$\sigma_{\eta,n}(x) = \mathcal{D}[\frac{x}{\eta} + y](\varepsilon(\nabla_x v_{\eta,n}(x)) + \varepsilon(\nabla_y u_{1,n}(x, \frac{x}{\eta} + y, t))), \tag{70}$$

$$v_{\eta,n}(x) = \gamma(x,t), \qquad x \in \partial\Omega. \tag{71}$$

The existence and uniqueness of the function $(v_{\eta,n}, \sigma_{\eta,n})$ as a solution of the boundary value problem (69) - (71) can be obtained by the well known theory for elliptic problems. Details are omitted.

We obviously have that

$$\sigma_\eta(x,t) - \hat{\sigma}_\eta(x,t) = (\sigma_\eta(x,t) - \sigma_{\eta,n}(x)) + (\sigma_{\eta,n}(x) - \hat{\sigma}_\eta(x,t)),$$

$$v_\eta(x,t) - \hat{v}_\eta(x,t) = (v_\eta(x,t) - v_{\eta,n}(x)) + (v_{\eta,n}(x) - \hat{v}_\eta(x,t)).$$

The proof of the convergence result will be separated in two steps. In the first step we show that the sequences $(\bar{v}_{\eta,n}(x),\ \bar{\sigma}_{\eta,n}(x))$ with $\bar{\sigma}_{\eta,n}(x) := \sigma_\eta(x,t) - \sigma_{\eta,n}(x)$ and $\bar{v}_{\eta,n}(x) := v_\eta(x,t) - v_{\eta,n}(x)$ converge to 0 in appropriate strong topologies. Then the same result will be shown for sequences $(\hat{v}_{\eta,n}(x),\ \hat{\sigma}_{\eta,n}(x))$, where $\hat{\sigma}_{\eta,n}(x) := \sigma_{\eta,n}(x) - \hat{\sigma}_\eta(x,t)$ and $\hat{v}_{\eta,n}(x) := v_{\eta,n}(x) - \hat{v}_\eta(x,t)$.

First step. By definition, $(\bar{v}_{\eta,n}(x),\ \bar{\sigma}_{\eta,n}(x))$ is a weak solution of the boundary value problem (54) - (56) to the data:

$$\hat{b}(x) = b(x,t) + \mathrm{div}_x T_{0,n}(x, \frac{x}{\eta} + y, t) \tag{72}$$

$$\hat{\varepsilon}_p(x,y) = \varepsilon(\nabla_y u_{1,n}(x, \frac{x}{\eta} + y, t)) \tag{73}$$

$$\hat{\gamma}(x) = 0. \tag{74}$$

[9] A similar idea was used in [8].

By linearity of the problem (54) - (56) to the data (72) - (74) the required convergence of $(\bar{v}_{\eta,n}(x),\, \bar{\sigma}_{\eta,n}(x))$ is an easy consequence of Lemma 3.1 and Lemma 3.2, which are applied for a.e. value $y = \hat{y} \in Y$.

Indeed, to see that we set for fixed t and a.e. $\hat{y}$

$$\tau_{\eta,n}(x) = T_{0,n}(x, \frac{x}{\eta} + \hat{y}, t), \quad \tau(x,y) = T_0(x, y + \hat{y}, t), \quad \tau_\infty(x) = T_\infty(x,t),$$
$$\kappa_{\eta,n}(x) = -\nabla_y u_{1,n}(x, \frac{x}{\eta} + \hat{y}, t), \quad \tau_{\infty,n}(x) = T_{\infty,n}(x,t), \quad b = b(t).$$

Note first that periodicity of $T_{0,n}, T_0$ and the choice of $T_{0,n}$ yield

$$\int_{\Omega\times Y} (T_{0,n}(x, y+\hat{y}, t) - T_0(x, y+\hat{y}, t))^2 dxdy$$
$$= \int_{\Omega\times Y} (T_{0,n}(x,y,t) - T_0(x,y,t))^2 dxdy \to 0 \ \text{ as } n \to \infty.$$

From (9) one immediately gets

$$\operatorname{div}_y \tau(x,y) = \operatorname{div}_y T_0(x, y+\hat{y}, t) = 0$$

and

$$\operatorname{div}_x \tau_{\eta,n}(x) = \operatorname{div}_x T_{0,n}(x, \frac{x}{\eta} + \hat{y}, t).$$

Due to periodicity of $T_0(x,y,t)$ we obtain also that

$$\int_Y T_0(x, y+\hat{y}, t) dy = \int_Y T_0(x,y,t) dy = T_\infty(x,t) = \tau_\infty(x),$$

and from strong convergence of $T_{0,n}$ to T_0 we deduce that the sequence $\tau_{\infty,n}(x) = T_{\infty,n}(x,t)$ converges strongly to $\tau_\infty(x) = T_\infty(x,t)$ in $L^2(\Omega, \mathcal{S}^3)$. One needs to apply a Hölder's inequality.

Moreover, (7) implies that

$$b + \operatorname{div}_x \tau_\infty(x) = b(t) + \operatorname{div}_x T_\infty(t) = 0.$$

Since by definition $(\bar{v}_{\eta,n}(x),\ \bar{\sigma}_{\eta,n}(x))$ is a weak solution of the boundary value problem (54) - (56) to the data

$$\begin{aligned}
\hat{b}(x) &= b(x,t) + \operatorname{div}_x T_{0,n}(x, \frac{x}{\eta} + \hat{y}, t) = b + \operatorname{div}_x \tau_{\eta,n}, \\
\hat{\varepsilon}_p(x) &= \varepsilon(\nabla_y u_{1,n}(x,y,t))|_{y = \frac{x}{\eta} + \hat{y}} = -\varepsilon(\kappa_\eta) \\
\hat{\gamma}(x) &= 0,
\end{aligned}$$

we can use the linearity of the problem (54) - (56) to write the function $(\bar{v}_{\eta,n}(x),\ \bar{\sigma}_{\eta,n}(x))$ as a sum of two functions

$$(\bar{v}_{\eta,n}, \bar{\sigma}_{\eta,n})(x) = (\bar{v}, \bar{\sigma})(x) + (\hat{v}, \hat{\sigma})(x),$$

where the function $(\bar{v}, \bar{\sigma})(x)$ solves the boundary problem (57) - (59) and the function $(\hat{v}, \hat{\sigma})(x)$ solves the problem (65) - (67). Then the application of Lemma 3.1 and Lemma 3.2 to $(\bar{v}, \bar{\sigma})$ and $(\hat{v}, \hat{\sigma})$ gives for every t and a.e. $\hat{y} \in Y$

$$\lim_{n\to\infty} \lim_{\eta\to 0} (\|\bar{v}_{\eta,n}(\cdot, \hat{y}, t)\|_\Omega + \|\bar{\sigma}_{\eta,n}(\cdot, \hat{y}, t)\|_\Omega) = 0. \tag{75}$$

Second step. We fix t and y. $(\hat{v}_{\eta,n}(x),\ \hat{\sigma}_{\eta,n}(x))$ is a weak solution of the boundary value problem (54)-(56) to the data:

$$\hat{b}(x) = -\mathrm{div}_x(T_{0,n}(x, \frac{x}{\eta} + y, t) - T_0(x, \frac{x}{\eta} + y, t)) \tag{76}$$

$$\hat{\varepsilon}_p(x) = -\varepsilon(\nabla_y(u_{1,n}(x, \frac{x}{\eta} + y, t) - u_1(x, \frac{x}{\eta} + y, t))), \tag{77}$$

$$\hat{\gamma}(x) = 0. \tag{78}$$

Using the properties of $\mathcal{D}$, we obtain elliptic theory applied to the boundary value problem (54) - (56) to the data (76) - (78), observing that $\hat{b} \in H^{-1}(\Omega, \mathrm{I\!R}^3)$, $\hat{\varepsilon}_p \in L^2(\Omega, \mathcal{S}^3)$, $\hat{\gamma} \in H^1(\Omega, \mathrm{I\!R}^3)$ and $\eta > 0$, $y \in Y$, that there is a constant C independent of η, y, t and n, such that

$$\| \hat{\sigma}_{\eta,n}(t)\|^2_\Omega \le C \ [\ \|T_{0,n}(\cdot, \frac{\cdot}{\eta} + y, t) - T_0(\cdot, \frac{\cdot}{\eta} + y, t)\|^2_\Omega$$

$$+\|\nabla_y u_{1,n}(\cdot, \frac{\cdot}{\eta} + y, t) - \nabla_y u_1(\cdot, \frac{\cdot}{\eta} + y, t)\|^2_\Omega \ ]$$

We integrate the right hand side of this inequality with respect to the parameter y over Y. As result we have

$$\int_\Omega \int_Y \ |\ T_{0,n}(x, \frac{x}{\eta} + y, t) - T_0(x, \frac{x}{\eta} + y, t)\ |^2$$

$$+|\ \nabla_y u_{1,n}(x, \frac{x}{\eta} + y, t) - \nabla_y u_1(x, \frac{x}{\eta} + y, t)\ |^2 dydx$$

$$= \int_\Omega \int_{\frac{x}{\eta}+Y} \ |\ T_{0,n}(x, y, t) - T_0(x, y, t)\ |^2$$

$$+|\ \nabla_y u_{1,n}(x, y, t) - \nabla_y u_1(x, y, t)\ |^2 dydx$$

$$= \|\ T_{0,n}(t) - T_0(t)\ \|^2_{\Omega\times Y} + \|\ \nabla_y u_{1,n}(t) - \nabla_y u_1(t)\ \|^2_{\Omega\times Y}.$$

Thus the function $\sigma_\eta(t) - \hat{\sigma}_\eta(t)$ must satisfy the following inequality

$$\|\sigma_\eta(t) - \hat{\sigma}_\eta(t)\|^2_{\Omega\times Y} \le C\ (\int_Y \ \|\bar{\sigma}_{\eta,n}(y, t)\|^2_\Omega dy$$

$$+\ \|T_{0,n}(t) - T_0(t)\|^2_{\Omega\times Y} + \|\nabla_y u_{1,n}(t) - \nabla_y u_1(t)\|^2_{\Omega\times Y}).$$

We notice that the $L^2(\Omega)$-norm of $\bar{\sigma}_{\eta,n}$ is uniformly bounded with respect to η and y. Indeed, $(\bar{v}_{\eta,n}(x),\ \bar{\sigma}_{\eta,n}(x))$ is the solution of (54) - (56) to the data (72) - (74). Applying the standard existence theory for linear elliptic problems we obtain that

$$\|\bar{\sigma}_{\eta,n}(y,t)\|_\Omega \le C\ (\|b(t)\|_\Omega + \|T_{0,n}(\cdot,\frac{\cdot}{\eta}+y,t)\|_\Omega + \|\nabla_y u_{1,n}(\cdot,\frac{\cdot}{\eta}+y,t)\|_\Omega).$$

The functions $T_{0,n}$, $\nabla_y u_{1,n}$ can be considered as an "admissible" test function in the definition of the two-scale convergence. By the properties of these function we get that

$$\|\bar{\sigma}_{\eta,n}(y,t)\|_\Omega \le C\ (\|b(t)\|_\Omega + \|T_{0,n}(t)\|_{L^2(\Omega,C(Y,\mathcal{S}^3))} + \|\nabla_y u_{1,n}(t)\|_{L^2(\Omega,C(Y,\mathbb{R}^{3\times 3}))}).$$

Thus we can use Lebesgue's convergence theorem to interchange the passage to the limit and the integration in the lines below.

Let us now pass to the limit as $\eta \to 0$:

$$\lim_{\eta\to 0}\|\sigma_\eta(t)-\hat{\sigma}_\eta(t)\|^2_{\Omega\times Y} \le C\ \ \lim_{\eta\to 0}(\int_Y \|\bar{\sigma}_{\eta,n}(y,t)\|^2_\Omega dy$$

$$+\|T_{0,n}(t)-T_0(t)\|^2_{\Omega\times Y} + \|\nabla_y u_{1,n}(t)-\nabla_y u_1(t)\|^2_{\Omega\times Y})$$

$$= C\ (\int_Y \lim_{\eta\to 0}\|\bar{\sigma}_{\eta,n}(y,t)\|^2_\Omega dy + \|T_{0,n}(t)-T_0(t)\|^2_{\Omega\times Y}$$

$$+\|\nabla_y u_{1,n}(t)-\nabla_y u_1(t)\|^2_{\Omega\times Y}).$$

Lemma 3.1 and Lemma 3.2 imply, if we set $\tau_{\eta,n}(x) = T_{0,n}(x,x/\eta+y,t)$ and $\kappa_{\eta,n}(x) = -\nabla_y u_{1,n}(x,x/\eta+y,t)$, that the function $\lim_{\eta\to 0}\|\bar{\sigma}_{\eta,n}(y,t)\|^2_\Omega$ is uniformly bounded with respect to n and y. It follows from (60), (62) and (64). We can pass now to the limit as $n\to\infty$:

$$\lim_{\eta\to 0}\|\sigma_\eta(t)-\hat{\sigma}_\eta(t)\|^2_{\Omega\times Y} \le \lim_{n\to\infty} C\ (\int_Y \lim_{\eta\to 0}\|\bar{\sigma}_{\eta,n}(y,t)\|^2_{2,\Omega}dy$$

$$+\ \lim_{n\to\infty}\|T_{0,n}(t)-T_0(t)\|^2_{\Omega\times Y} + \lim_{n\to\infty}\|\nabla_y u_{1,n}(t)-\nabla_y u_1(t)\|^2_{\Omega\times Y})$$

$$= C\ (\int_Y \lim_{n\to\infty}\lim_{\eta\to 0}\|\bar{\sigma}_{\eta,n}(y,t)\|^2_{2,\Omega}dy + \lim_{n\to\infty}\|T_{0,n}(t)-T_0(t)\|^2_{\Omega\times Y}$$

$$+\lim_{n\to\infty}\|\nabla_y u_{1,n}(t)-\nabla_y u_1(t)\|^2_{\Omega\times Y}) = 0.$$

In exactly the same way we get

$$\lim_{\eta\to 0}\|v_\eta(t)-\hat{v}_\eta(t)\|^2_{\Omega\times Y} \le C\ (\int_Y \lim_{n\to\infty}\lim_{\eta\to 0}\|\bar{v}_{\eta,n}(y)\|^2_{2,\Omega}dy$$

$$+\ \lim_{n\to\infty}\|T_{0,n}(t)-T_0(t)\|^2_{\Omega\times Y} + \lim_{n\to\infty}\|\nabla_y u_{1,n}(t)-\nabla_y u_1(t)\|^2_{\Omega\times Y}) = 0.$$

This ends the proof of Theorem 1.1. ∎

4 Appendix

4.1 Existence of solutions for the reduced equation

In this subsection we are going to show the existence result[10] in a Hilbert space H for the following Cauchy problem

$$\frac{d}{dt}u(t) + A(u(t)) \ni f(t), \tag{79}$$

$$u(0) = u_0 \tag{80}$$

with $A = MG$, where M is a linear, bounded, positive definite, selfadjoint operator, G is a maximal monotone operator with respect to the usual scalar product $(\cdot, \cdot)$. It is already shown in Theorem 3.3 [3] that A is maximal monotone with respect to the scalar product $\langle \cdot, \cdot \rangle = (M^{-1}\cdot, \cdot)$.

Theorem 4.1 *Let G_λ be a Yosida approximation of G. Assume that $u_0 \in D(A)$ and $f \in W^{1,1}(0, T_e; H)$. Then the Cauchy problem has a unique solution $u \in W^{1,\infty}(0, T_e; H)$. The solution satisfies the inequality*

$$\|\frac{d}{dt}u(t)\|_H \leq C\|G_\lambda u_0\|_H + \|f(0)\|_H + \int_0^t \|f^{'}(s)\|_H ds \tag{81}$$

with a constant C independent of λ.

Proof. The uniqueness follows directly from the monotonicity of A.

For each $\lambda > 0$ let u_λ be the solution of

$$\frac{d}{dt}u_\lambda(t) + A_\lambda(u_\lambda(t)) \ni f(t), \tag{82}$$

$$u_\lambda(0) = u_0 \tag{83}$$

with a maximal monotone (with respect to $\langle \cdot, \cdot \rangle = (M^{-1}\cdot, \cdot)$) operator $A_\lambda = MG_\lambda$.

Then similarly as in Theorem IV.4.1 [33] we get the estimate for $u_\lambda^{'}$

$$\begin{aligned} \|u_\lambda^{'}(t)\|_H &\leq \|A_\lambda u_0\|_H + \|f(0)\|_H + \int_0^t \|f^{'}(s)\|_H ds \\ &\leq C\|G_\lambda u_0\|_H + \|f(0)\|_H + \int_0^t \|f^{'}(s)\|_H ds \\ &\leq C|G^0 u_0| + \|f(0)\|_H + \int_0^t \|f^{'}(s)\|_H ds. \end{aligned} \tag{84}$$

From (82) and (84) it follows that $u_\lambda^{'}$, u_λ and $A_\lambda u_\lambda$ are uniformly bounded in $C([0, T_e], H)$.

u_λ is a Cauchy sequence in $C([0, T_e], H)$. To see that let $\lambda, \mu > 0$ and use (82) to obtain

$$\frac{1}{2}\frac{d}{dt}\|u_\lambda(t) - u_\mu(t)\|^2 = -\langle A_\lambda(u_\lambda(t)) - A_\mu(u_\mu(t)), u_\lambda(t) - u_\mu(t)\rangle.$$

[10] Actually in [11], [33] it is already shown that the problem (79) has a unique solution, but in one place in the justification proof we need other estimates for solutions than this theory delivers. We slightly modify the classical proof in order to obtain the mentioned estimates. More details can be found in [11], [33].

With $u_\lambda = \lambda G_\lambda u_\lambda + J_\lambda u_\lambda$ and $u_\mu = \mu G_\mu u_\mu + J_\mu u_\mu$ (J_λ is a resolvent of G) we obtain as in Theorem IV.4.1 [33]

$$\|u_\lambda(t) - u_\mu(t)\|^2 \leq \frac{\lambda+\mu}{2}K^2, \qquad 0 \leq t \leq T_e,$$

where $K = \sup\{\|G_\lambda(u_\lambda(t))\| \mid 0 \leq t \leq T_e,\ \lambda > 0\}$, so u_λ is a Cauchy sequence in $C([0,T_e], H)$ with

$$\|u_\lambda(t) - u(t)\|_H \leq \sqrt{\frac{\lambda}{2}}C(|G^0 u_0| + \|f(0)\|_H + \|f\|_{C([0,T_e],H)} + \int_0^{T_e} \|f^{'}(s)\|_H ds). \quad (85)$$

The proof ends similarly as in Theorem IV.4.1 [33]. ■

4.2 Homogenization of linear elasticity system

Now we show how to apply the two-scale convergence method to the homogenization of linear elasticity systems with periodically oscillating coefficients. This example is of great importance in the rigorous justification procedure because of the frequent use of estimates obtained for the sequence of solutions of the linear elasticity problem as well as of its homogenized problem. Therefore the proof, actually a rephrasing of Theorem 2.3 [6] for the case of linear elasticity with a slight modification, is given in all details.

Consider the following problem

$$\begin{aligned} -\text{div}\ \mathcal{D}[\frac{x}{\eta}]\varepsilon\left(\nabla_x u(x)\right) &= b(x), \quad x \in \Omega, && (86)\\ u(x) &= 0, \qquad x \in \partial\Omega, && (87) \end{aligned}$$

with a given function $b \in L^2(\Omega, \mathbb{R}^3)$ and an Y-periodic linear positive definite mapping $\mathcal{D}[y] : \mathcal{S}^3 \mapsto \mathcal{S}^3$, the elasticity tensor. $\mathcal{D}[y]$ is such that there exist two positive constants $0 < \alpha \leq \beta$ satisfying

$$\alpha|\xi|^2 \leq \mathcal{D}_{ijkl}[y]\xi_{kl}\xi_{ij} \leq \beta|\xi|^2 \quad \text{for any } \xi \in \mathcal{S}^3. \quad (88)$$

The last assumption (88) implies that the mapping $\mathcal{D}[y]$ belongs to $L^\infty(Y, \mathcal{S}^3)$ and consequently in virtue of the first Korn's inequality (for example, [29]) that the problem (86)-(87) admits a unique solution u_η in $H_0^1(\Omega, \mathbb{R}^3)$, which satisfies the estimate

$$\|u_\eta\|_{1,\Omega} \leq C\|b\|_2, \quad (89)$$

where C is a positive constant that depends on Ω and α, and not on η.

Theorem 4.2 *The sequence u_η of solutions of the problem (86)-(87) converges weakly to $u(x)$ in $H_0^1(\Omega, \mathbb{R}^3)$, and the sequence ∇u_η two-scale converges to $\nabla u(x) + \nabla_y u_1(x,y)$, where $(u, u_1) \in H_0^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, W(Y, \mathbb{R}^3))$ is the unique solution of the following two-scale homogenized system:*

$$\begin{aligned} -\text{div}_y[\mathcal{D}[y]\varepsilon(\nabla u(x) + \nabla_y u_1(x,y))] &= 0 \quad \textit{in } \Omega \times Y,\\ -\text{div}_x[\int_Y \mathcal{D}[y]\varepsilon(\nabla u(x) + \nabla_y u_1(x,y))dy] &= b(x) \quad \textit{in } \Omega,\\ u(x) &= 0 \quad \textit{on } \partial\Omega,\\ y &\mapsto u_1(x,y) \quad Y-\textit{periodic}. \end{aligned}$$

Proof. By virtue of the estimate (89) and Proposition 1.14 in [6] the sequence u_η and the sequence of its gradient ∇u_η, up to a subsequence, have the weak limit $u \in H_0^1(\Omega, \mathbb{R}^3)$ and the two-scale limit $\nabla u(x) + \nabla_y u_1(x,y)$, respectively, with $u_1 \in L^2(\Omega, W(Y, \mathbb{R}^3))$.

Multiply (86) by a test function $\psi(x) + \eta\psi_1(x, x/\eta)$, with $\psi \in C_0^\infty(\Omega, \mathbb{R}^3)$ and $\psi_1 \in C_0^\infty(\Omega, C^\infty(Y, \mathbb{R}^3))$. Integration by parts of the resulting equation and some rewriting imply

$$\int_\Omega \nabla u_\eta(x) \mathcal{D}[\frac{x}{\eta}] \varepsilon[\nabla\psi(x) + \nabla_y \psi_1(x, \frac{x}{\eta})] dx \tag{90}$$
$$+\eta \int_\Omega \mathcal{D}[\frac{x}{\eta}] \varepsilon(\nabla u_\eta(x)) \varepsilon(\nabla_x \psi_1(x, \frac{x}{\eta})) dx = \int_\Omega b(x)(\psi(x) + \eta\psi_1(x, \frac{x}{\eta})) dx.$$

Here the symmetry of $\mathcal{D}$ was used in the first term.
It is easily seen now that applying consecutively the limit relation in the definition of the admissible test functions and Theorem 1.8 [6] justifies the passage to the two-scale limit in (90):

$$\int_Y \int_\Omega \mathcal{D}[y] \varepsilon(\nabla u(x) + \nabla_y u_1(x,y)) \varepsilon(\nabla\psi(x) + \nabla_y \psi_1(x,y)) dx dy$$
$$= \int_\Omega b(x)\psi(x) dx. \tag{91}$$

(91) holds true for any function $(\psi, \psi_1) \in H_0^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, W(Y, \mathbb{R}^3))$. (91) is variational formulation associated to the two-scale homogenized problem stated above. Due to the first Korn's inequality, and to Korn's inequality for periodic functions ([29]), the Hilbert space $\mathcal{H} = H_0^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, W(Y, \mathbb{R}^3))$ can be endowed with following norm

$$\|\Psi\|_{\mathcal{H}}^2 = \|\psi\|_{1,\Omega}^2 + \|\varepsilon(\nabla_y \psi_1)\|_{\Omega \times Y}^2.$$

Then application of the Lax-Milgram lemma shows that there exists a unique solution of the two-scale homogenized problem. Consequently, the entire sequences $u_\eta(x)$ and $\nabla u_\eta(x)$ converge to $u(x)$ and $\nabla u(x) + \nabla_y u_1(x,y)$. ■

4.3 Homogenization of the second order elliptic operators with non-uniformly oscillating coefficients

For another interesting application of a parameter y consider the following problem with non-uniformly oscillating coefficients (chapter 1, section 6, [9])

$$-\text{div}\, \mathcal{D}[x, \frac{x}{\eta}] (\nabla_x u_\eta(x)) = b(x), \quad x \in \Omega, \tag{92}$$
$$u_\eta(x) = 0, \quad x \in \partial\Omega, \tag{93}$$

with a given function $b \in L^2(\Omega)$ and Y-periodic in y a matrix $\mathcal{D}[x,y]$ such that there exist two positive constants $0 < \alpha \leq \beta$ satisfying

$$\alpha|\xi|^2 \leq \mathcal{D}_{ij}[y]\xi_i\xi_j \leq \beta|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^{\mathrm{N}}. \tag{94}$$

The last assumption (94) implies that the mapping $\mathcal{D}[x,y]$ belongs to $L^\infty(\Omega \times Y, \mathbb{R}^{\mathrm{N}\times\mathrm{N}})$. But it is not enough to ensure that the mapping $x \mapsto \mathcal{D}[x, x/\eta]$ is measurable, so in [9] it is assumed that $\mathcal{D} \in C(\Omega, L^\infty(Y, \mathbb{R}^{N\times N}))$ to obtain a solution of (92)-(93). Instead of increasing the regularity of $\mathcal{D}[x,y]$ we consider a family of shifted problems

$$-\mathrm{div}_x\, \mathcal{D}[x, \frac{x}{\eta} + y]\,(\nabla_x u_\eta(x,y)) = b(x), \qquad (x,y) \in \Omega \times Y, \tag{95}$$

$$u_\eta(x,y) = 0, \qquad (x,y) \in \partial\Omega \times Y. \tag{96}$$

Now the problem (95)-(96) admits a unique solution u_η in $L^2(Y, H_0^1(\Omega))$, which satisfies the estimate

$$\|u_\eta\|_{L^2(Y,H_0^1(\Omega))} \le C\|b\|_\Omega,$$

where C is a positive constant independent of η. For a.e. fixed $y \in Y$ the existence of the solution is provided by the well known result for second order elliptic operators and the integrability with respect to y is then the easy consequence of it.

It is convenient now to write (95)-(96) in the form

$$-\mathrm{div}_x \sigma_\eta(x,y) = b(x), \qquad (x,y) \in \Omega \times Y, \tag{97}$$

$$\sigma_\eta(x,y) = \mathcal{D}[x, \frac{x}{\eta} + y]\,(\nabla_x u_\eta(x,y))\,, \quad (x,y) \in \Omega \times Y, \tag{98}$$

$$u_\eta(x,y) = 0, \qquad (x,y) \in \partial\Omega \times Y. \tag{99}$$

Now the solution of (97)-(99) is a function $(u_\eta, \sigma_\eta) \in L^2(Y, H_0^1(\Omega)) \times L^2(\Omega \times Y, \mathbb{R}^N)$.

Then inserting the formal ansatz for the solution u_η

$$\hat{u}_\eta(x,y) = u_0(x) + \eta u_1(x, \frac{x}{\eta} + y) + \eta^2 u_2(x, \frac{x}{\eta} + y) + ...$$

into the boundary problem (95)-(96) and identifying powers of η lead to the homogenized problem

$$\begin{aligned} -\mathrm{div}_x \sigma_\infty(x) &= b(x), \\ \sigma_\infty(x) &= \frac{1}{|Y|}\int_Y \sigma_0(x,y)dy \\ -\mathrm{div}_y \sigma_0(x,y) &= 0, \\ \sigma_0(x,y,t) &= \mathcal{D}[x,y](\nabla_y u_1(x,y) + \nabla_x u_0(x)), \end{aligned}$$

which must hold for $(x,y) \in \Omega \times Y$

$$u_0(x) = 0,$$

which must hold for $x \in \partial\Omega$. This form of the homogenized problem is equivalent to the already obtained one in [9]. A slight modification of the proof of Lemma 2.5 yields the following convergence result

$$\lim_{\eta\to 0}(\|u_0 - u_\eta\|_{\Omega\times Y} + \|\sigma_0 - \sigma_\eta\|_{\Omega\times Y}) = 0, \tag{100}$$

where (u_0, u_1, σ_0) is the solution of the homogenized problem and (u_η, σ_η) is the solution of the problem (97)-(99) with a parameter y.

(100) holds without imposing additional regularity on $\mathcal{D}[x,y], b(x)$ and $\partial\Omega$.

Acknowledgement. The author thanks Hans-Dieter Alber for helpful suggestions.

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